On a Reduction-Procedure for Full First Order Classical Natural Deduction

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0.1 Introduction

Gentzen's Hauptsatz [3] is one of the most fundamental theorems in proof theory. Its intentional meaning is that; if a derivation of first order predicate logic is given, then we can remove the roundabouts in the derivation. In the synopsis of [3], Gentzen explained the progress to get his Hauptsatz as follows: (1) He developed a new formal system called natural deduction, which is close to actual mathematical reasoning. (2) In consequence of an investigation of natural deduction, he obtained the Hauptsatz. But his system of natural deduction is not suitable to represent the Hauptsatz in the case of classical logic. (3) In order to represent and prove the Hauptsatz both in the case of intuitionistic and classical logic, he developed another formal system called sequent-calculus.

Since primarily the system of natural deduction is more close to actual mathematical reasoning in comparison with the system of sequent-calculus, it stands to reason that one wants to prove Hauptsatz in natural deduction. The obstacle for representing Hauptsatz in Gentzen's system of classical natural deduction, named **NK**, is caused by the fact that **NK** contains the axioms for the law of the excluded middle. It is difficult to represent the roundabouts included in an axiom for the law of the excluded middle.

Prawitz resolved this problem by using his system of classical natural deduction which is obtained from the natural deduction system of the minimal logic by adding the classical absurdity rule [5][6]. In this system, roundabouts in derivations are represented as maximum formulae. If a derivation contains no maximum formulae, then the derivation has the subformula property, that is, the derivation is built-up only from the notions included in the assumptions or the conclusion of the derivation. Therefore, the representation of Hauptsatz in natural deduction is the following normalization theorem: In the system of classical natural deduction which is obtained from the natural deduction system of the minimal logic by adding the classical absurdity rule, if an arbitrary derivation is given, then we can transform it to a normal one, i.e. a derivation which contains no maximum formulae.

There is one difficulty to prove the normalization theorem above. That is how to define a reduction-step which removes a maximum formula introduced a classical absurdity rule. Prawitz avoided this problem. He restricted the system of classical natural deduction to the fragment {& (and), $\supset$ (implies), $\forall$ (for all) }. In such restricted system, formulae applied to the classical absurdity rule can be restricted to atomic ones. Hence the normalization-procedure of such system is easily defined.

But the aim of investigating the normalization theorem of natural deduction is to represent Hauptsatz in a more natural form, i.e. in a form as close to actual mathematical reasoning as possible. Therefore, it occurs a question: Is there a *natural* normalization-procedure for the Prawitz's system of classical natural deduction with full logical symbols? The answer is "Yes". A natural normalization-procedure for such system was given by the author [1], and it is explained in Chapter 1 in this dissertation. Our reduction used in that normalization-procedure is very simple compared with ones in the preceding works by Seldin [7][8] or by Stålmarck [9]. The main point of our reduction is to regard the classical absurdity rule as a structural rule. More precisely, the classical absurdity rule is regarded as a set of rules corresponding with some exchanges of right and some contractions of right in the sequent-calculus. By using our normalization-procedure, we can prove the normalization theorem for intuitionistic and for classical natural deduction simultaneously.

By virtue of simplicity of our reduction, we can prove that Church-Rosser property of our reduction holds [2]. That is explained in Chapter 2 in this dissertation. Under the condition that the normalization theorem concerning our reduction holds, Church-Rosser property is equivalent to the condition that the normal derivations obtained from one derivation by applying our reduction-steps are identical, that is, the uniqueness of the normal form.

This dissertation is organized as follows: Chapter 1 is devoted to the proof of normalization theorem. In section 1.1, we introduce our system and some notational conventions. Our system is the classical natural deduction system obtained from the natural deduction system of the minimal logic by adding the classical absurdity rule, and contains all logical symbols including the symbol for disjunction and the one for existential quantifier. In section 1.2, we define our reduction which is a natural extension of Prawitz's reduction for intuitionistic logic. In section 1.3, we extend the notion *segment* defined by Prawitz to our system. In section 1.4, we prove Theorem 1, i.e. the normalization theorem concerning our reduction. Our normalization theorem is one of the so called weak normalization theorem.

Chapter 2 is devoted to prove that Church-Rosser property of our reduction holds. In section 2.1, we define the notions *segment-tree* and *segment-wood* in order to represent transformations which consist of continuous contractions along segments. Such transformations, named *structural reductions*, are defined in section 2.2, where the notion *substitution-sequence* plays an central role. In section 2.3, we define 1-reduction by using structural reduction, and state the main lemma from which we have easily Theorem 2, i.e. Church-Rosser property of our reduction. The proof of the main lemma is given in section 2.4.

In appendix A, we give another proof of normalization theorem of our system. It is obtained by using Hauptsatz for sequent-calculus, i.e. cut-elimination theorem. If we do not want to define *natural* reduction *in* natural deduction, the normalization theorem is immediately obtained by such method. In appendix B, we explain another formulation of classical natural deduction which is obtained from intuitionistic natural deduction system by adding the Peirce's law. In this system, we can define a simple normalization-procedure similarly to the system in our main issue.

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Chapter 1

Our reduction and its normalization theorem

1.1 Basic definitions and notations

1.1.1 System

In this paper, we investigate the natural deduction system for the first order classical logic. Our system contains all logical symbols, that is; & (and), $\vee$ (or), $\supset$ (implies), $\neg$ (not), $\forall$ (for all), and $\exists$ (there exists). The inference rules are the introduction and elimination rules for each logical symbol, and the classical absurdity rule. They are shown by the following schemata.

Introduction rules

$$\frac{A_1 \quad A_2}{A_1 \,\&\, A_2}\ (\&I) \qquad \frac{A_1}{A_1 \vee A_2}\ (\vee I_1) \qquad \frac{A_2}{A_1 \vee A_2}\ (\vee I_2)$$

$$\frac{\begin{matrix}[A]\\ B\end{matrix}}{A \supset B}\ (\supset I) \qquad \frac{\begin{matrix}[A]\\ \bot\end{matrix}}{\neg A}\ (\neg I) \qquad \frac{F(a)}{\forall x F(x)}\ (\forall I) \qquad \frac{F(t)}{\exists x F(x)}\ (\exists I)$$

Elimination rules

$$\frac{A_1 \,\&\, A_2}{A_1}\ (\&E_1) \qquad \frac{A_1 \,\&\, A_2}{A_2}\ (\&E_2) \qquad \frac{A_1 \vee A_2 \quad \begin{matrix}[A_1]\\ C\end{matrix} \quad \begin{matrix}[A_2]\\ C\end{matrix}}{C}\ (\vee E)$$

$$\frac{A \supset B \quad A}{B}\ (\supset E) \qquad \frac{\neg A \quad A}{\bot}\ (\neg E) \qquad \frac{\forall x F(x)}{F(t)}\ (\forall E) \qquad \frac{\exists x F(x) \quad \begin{matrix}[F(a)]\\ C\end{matrix}}{C}\ (\exists E)$$

Classical absurdity rule

$$\frac{\begin{matrix}[\neg A]\\ \bot\end{matrix}}{A}\ (\bot_c)$$

$(\forall I)$ and $(\exists E)$ are subject to the restriction of eigenvariables [3].

1.1.1.1 Regularity of eigenvariables.

It is assumed that the eigenvariable of an application of $(\forall I)$ (or $(\exists E)$) in a derivation occurs only in the subderivation of the premiss (or minor premiss respectively) of the application. If a derivation Π is transformed to another derivation Σ and Σ does not satisfy the regularity of eigenvariables mentioned above, then we make Σ regular by changing some eigenvariables in Σ properly.

1.1.2 Regularity of $(\bot_c)$.

1.1.2.1 Definition ($(\bot_c)$-regular)

In a derivation, an assumption-formula discharged by an application of $(\bot_c)$ is $(\bot_c)$-regular iff it is the major premiss of an application of $(\neg E)$. A derivation is $(\bot_c)$-regular iff any assumption-formula discharged by any application of $(\bot_c)$ in the derivation is $(\bot_c)$-regular.

1.1.2.2 Fact

Let Π be a given derivation. If Π is not $(\bot_c)$-regular, then we can transform it to a $(\bot_c)$-regular derivation.

Proof. Let α be an occurrence of a formula $\neg A$ in Π which is discharged by an application I of $(\bot_c)$ and is not $(\bot_c)$-regular. Then, transform Π by replacing α with the following subderivation: $\dfrac{\dfrac{\neg A \quad A}{\bot}\,K}{\neg A}\,J$ where K is an application of $(\neg E)$, and J is an application of $(\neg I)$ which discharges the minor premiss of K. Moreover, discharge the major premiss of K by I. Then, the major premiss of K is $(\bot_c)$-regular. By applying this transformation for all non $(\bot_c)$-regular assumption-formulae of all applications of $(\bot_c)$ in Π, we get a $(\bot_c)$-regular derivation.□

In the rest of this paper, we treat only $(\bot_c)$-regular derivations. That is inessential restriction, because the previous fact holds. By definition of our reduction which will be stated in 1.2, it will easily be verified that; if Π' is the derivation obtained by our reduction from a $(\bot_c)$-regular derivation Π, then Π' is also $(\bot_c)$-regular.

1.1.3 Notational conventions

(1) Small Greek letters α, β, ... are used as syntactical variables for formula-occurrences in derivations. If α is an formula-occurrence of a formula A, $Form(\alpha)$ denotes the formula A. We make a distinction between inference rules and applications of inference rules in derivations. If I is an application of an inference rule in a derivation, $Inf(I)$ denotes the inference rule applied at I. For example, if I is an application of $(\vee E)$ in a derivation, then $Inf(I)$ is the inference rule $(\vee E)$. When I is an application of an inference rule in a derivation, we call I a D-inference [3] (in [10]).

(2) Let Π be a derivation. $FO(\Pi)$ denotes the set of all formula-occurrences in Π. Notations $oa(\Pi)$, $OA(\Pi)$, $end(\Pi)$, $END(\Pi)$, $li(\Pi)$, and $LI(\Pi)$ are defined by the following:

$oa(\Pi) = \{\alpha \in FO(\Pi) \mid \alpha \text{ is an open assumption of } \Pi\}$
$OA(\Pi) = \{Form(\alpha) \mid \alpha \in oa(\Pi)\}$
$end(\Pi)$ is the end formula-occurrence of Π.
$END(\Pi) = Form(end(\Pi))$
$li(\Pi)$ is the last D-inference of Π.
Namely, $li(\Pi)$ is the D-inference whose conclusion is $end(\Pi)$.
$LI(\Pi) = Inf(li(\Pi))$

$li(\Pi)$ and $LI(\Pi)$ are defined in the case that the length of Π is greater than 1, that is, there is at least one D-inference in Π. For a formula-occurrence α in Π, $sbd(\alpha)$ denotes the subderivation of Π satisfying $end(sbd(\alpha)) = \alpha$. Let I be an D-inference in Π. Notations $pm(I)$, $cl(I)$, and $dc(I)$ are defined by the following:

$pm(I) = \{\alpha \in FO(\Pi) \mid \alpha \text{ is a premiss of } I\}$
$cl(I)$ is the conclusion of I.
$dc(I) = \{\alpha \in FO(\Pi) \mid \alpha \text{ is discharged by } I\}$

Moreover, in the case that $Inf(I)$ is an elimination rule, notations $mj(I)$, $MJ(I)$, and $mn(I)$ are defined by the following:

$mj(I)$ is the major premiss of I.
$MJ(I) = Form(mj(I))$
$mn(I) = \{\alpha \in FO(\Pi) \mid \alpha \text{ is a minor premiss of } I\}$

(3) Let Π, a, and t be a derivation, a free variable, and a term respectively. If the figure obtained by substituting t for all occurrences a in Π is a derivation, we denote the derivation by $\Pi(t/a)$. Let A be a formula. The notation $\begin{matrix}[A]\\ \Pi\end{matrix}$ is used in the following situation, that is, $[A]$ in $\begin{matrix}[A]\\ \Pi\end{matrix}$ denotes a subset, say O, of $oa(\Pi)$ satisfying that $Form(\alpha) = A$ holds for all α in Π. Let Σ be a derivation satisfying $END(\Sigma) = A$. If the figure obtained by substituting Σ for all elements of the subset of $oa(\Pi)$ denoted by $[A]$ in $\begin{matrix}[A]\\ \Pi\end{matrix}$ is a derivation, we denote the derivation by $\begin{matrix}\Sigma\\ [A]\\ \Pi\end{matrix}$. When a derivation Π is denoted by $\dfrac{\Pi_0 \quad (\Pi_1 \quad \Pi_2)}{A}$, it means that Π equals to $\dfrac{\Pi_0}{A}$, $\dfrac{\Pi_0 \quad \Pi_1}{A}$, or $\dfrac{\Pi_1 \quad \Pi_1 \quad \Pi_2}{A}$ if the cardinality of $pm(li(\Pi))$ is 1, 2, or 3 respectively. The notation $\dfrac{\Pi_0 \quad (\Pi_1)}{A}$ is used similarly.

(4) $\mathcal{Z}$, $\mathcal{N}^0$, and $\mathcal{N}^+$ denote the set of all integers, the set of all non-negative integers, and the set of all positive integers respectively. For a finite set S, $Card(S)$ denotes the cardinality of S. We use $\sqcup$ and $\bigsqcup$ to denote disjoint sums.

1.2 Reduction

In this section, we define our reduction and state theorems about it. The aim of the reduction is to remove maximum formulae in a derivation and to obtain a normal derivation. Maximum formulae and normal derivations are defined as follows.

1.2.1 Definition (Maximum formula)

Let Π be a derivation. A formula-occurrence μ in Π is a maximum formula in Π iff it satisfies the following conditions.

(1) μ is the conclusion of an application of an introduction rule, $(\vee E)$, $(\exists E)$, or $(\bot_c)$.

(2) μ is the major premiss of an application of an elimination rule.

1.2.2 Definition (Normal derivation)

A derivation Π is normal iff it contains no maximum formula.

1.2.3 Definition (Contraction)

To define our reduction, first we define the contraction of Π where Π is a derivation satisfying that $mj(li(\Pi))$ is a maximum formula. Let I be the D-inference in Π satisfying $cl(I) = mj(li(\Pi))$. The contraction of Π is defined according to $Inf(I)$. In the case that $Inf(I) \neq (\bot_c)$, the contraction is the same with Prawitz's reduction for the intuitionistic logic [5][6].

1.2.3.1 $\&_i$-contraction ($i = 1$, or 2)

If $\Pi = \dfrac{\dfrac{\Pi_1 \quad \Pi_2}{A_1 \& A_2}\, I}{A_i}\, K$ where $Inf(I) = (\& I)$ and $Inf(K) = (\& E_i)$; then Π contracts to the derivation Π_i .

1.2.3.2 $\vee_i$-contraction ($i = 1$, or 2)

If $\Pi = \dfrac{\dfrac{\Pi_0}{A_1 \vee A_2}\, I \quad \begin{matrix}[A_1] \\ \Pi_1\end{matrix} \quad \begin{matrix}[A_2] \\ \Pi_2\end{matrix}}{C}\, K$ where $Inf(I) = (\vee I_i)$, $Inf(K) = (\vee E)$, $[A_p]$ in $\begin{matrix}[A_p] \\ \Pi_p\end{matrix}$ denotes $dc(K) \cap FO(\Pi_p)$ for each $p \in \{1, 2\}$; then Π contracts to the derivation $\begin{matrix}\Pi_0 \\ [A_i] \\ \Pi_i\end{matrix}$.

1.2.3.3 $\supset$-contraction

If $\Pi = \dfrac{\dfrac{\begin{matrix}[A]\\ \Pi_0\end{matrix}}{A \supset B}\, I \quad \Pi_1}{B}\, K$ where $Inf(I) = (\supset I)$, $Inf(K) = (\supset E)$, and $[A]$ in $\begin{matrix}[A]\\ \Pi_0\end{matrix}$ denotes $dc(I)$; then Π contracts to the derivation $\begin{matrix}\Pi_1\\ [A]\\ \Pi_0\end{matrix}$.

1.2.3.4 $\neg$-contraction

If $\Pi = \dfrac{\dfrac{\begin{matrix}[A]\\ \Pi_0\end{matrix}}{\neg A}\, I \quad \Pi_1}{B}\, K$ where $Inf(I) = (\neg I)$, $Inf(K) = (\neg E)$, and $[A]$ in $\begin{matrix}[A]\\ \Pi_0\end{matrix}$ denotes $dc(I)$; then Π contracts to the derivation $\begin{matrix}\Pi_1\\ [A]\\ \Pi_0\end{matrix}$.

1.2.3.5 $\forall$-contraction

If $\Pi = \dfrac{\dfrac{\Pi_0}{\forall x F(x)}\, I}{F(t)}\, K$ where $Inf(I) = (\forall I)$, $Inf(K) = (\forall E)$,and the eigenvariable of I is a; then Π contracts to the derivation $\Pi_0(t/a)$.

1.2.3.6 $\exists$-contraction

If $\Pi = \dfrac{\dfrac{\Pi_0}{\exists x F(x)}\, I \quad \begin{matrix}[F(a)]\\ \Pi_1\end{matrix}}{C}\, K$ where $Inf(I) = (\exists I)$, $Inf(K) = (\exists E)$, $END(\Pi_0) = F(t)$, and $[F(a)]$ in $\begin{matrix}[F(a)]\\ \Pi_1\end{matrix}$ denotes $dc(K)$; then Π contracts to the derivation $\begin{matrix}\Pi_0\\ [F(t)]\\ \Pi_1(t/a)\end{matrix}$ where $[F(t)]$ in $\begin{matrix}[F(t)]\\ \Pi_1(t/a)\end{matrix}$ denotes the subset of $oa(\Pi_1(t/a))$ which naturally corresponds with $[F(a)]$ in $\begin{matrix}[F(a)]\\ \Pi_1\end{matrix}$.

1.2.3.7 $\vee E$-contraction

If $\Pi = \dfrac{\dfrac{\Pi_0 \quad \Pi_1 \quad \Pi_2}{M}\, I \quad (\Pi_3 \quad \Pi_4)}{C}\, K$ where $Inf(I) = (\vee E)$ and $Inf(K)$ is an elimination rule; then Π contracts to the derivation

$$\dfrac{\Pi_0 \qquad \dfrac{\Pi_1 \quad (\Pi_3 \quad \Pi_4)}{C}\, K_1 \qquad \dfrac{\Pi_2 \quad (\Pi_3 \quad \Pi_4)}{C}\, K_2}{C}\, I'$$

where for each $p \in \{1, 2\}$, $Inf(K_p) = Inf(K)$ and $dc(K_p)$ is defined naturally according to $dc(K)$; and where $Inf(I') = (\vee E)$ and $dc(I')$ is defined naturally according to $dc(I)$.

1.2.3.8 $\exists E$-contraction

If $\Pi = \dfrac{\dfrac{\Pi_0 \quad \Pi_1}{M}\, I \qquad (\Pi_2 \quad \Pi_3)}{C}\, K$ where $Inf(I) = (\exists E)$ and $Inf(K)$ is an elimination rule; then Π contracts to the derivation $\dfrac{\Pi_0 \qquad \dfrac{\Pi_1 \quad (\Pi_2 \quad \Pi_3)}{C}\, K'}{C}\, I'$ where $Inf(K') = Inf(K)$ and $dc(K')$ is defined naturally according to $dc(K)$; and where $Inf(I') = (\exists E)$ and $dc(I')$ is defined naturally according to $dc(I)$.

1.2.3.9 $\bot_c$-contraction

Let $\Pi = \dfrac{\dfrac{\begin{matrix}[\neg M]\\ \Pi_0\end{matrix}}{M}\, I \qquad (\Pi_1 \quad \Pi_2)}{C}\, K$ where $Inf(I) = (\bot_c)$, $Inf(K)$ is an elimination rule, and $[\neg M]$ in $\begin{matrix}[\neg M]\\ \Pi_0\end{matrix}$ denotes $dc(I)$. Since Π satisfies the regularity of $(\bot_c)$, any element of $dc(I)$ is the major premiss of an application of $(\neg E)$. Let $J_1, \ldots, J_n$ be all the applications of $(\neg E)$ whose major premiss is discharged by I, if they exists. Let Π_0' be the derivation obtained from Π_0 by the transformation represented by the following diagram:

$$\dfrac{\neg M \quad M}{\bot}\, J_p \quad \longrightarrow \quad \dfrac{\neg C \qquad \dfrac{M \quad (\Pi_1 \quad \Pi_2)}{C}\, K_p'}{\bot}\, J_p'$$

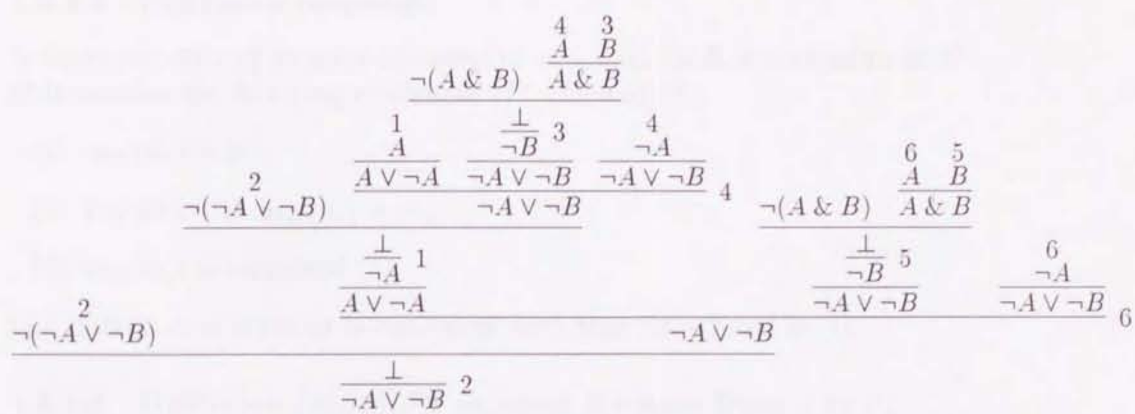
where $Inf(K_p') = Inf(K)$, $Inf(J_p') = (\neg E)$, and $dc(K_p')$ is defined naturally according to $dc(K)$. These replacements are done simultaneously for all $p \in \{1, \ldots, n\}$. We denote by $[\neg C]$ in $\begin{matrix}[\neg C]\\ \Pi_0'\end{matrix}$ the set $\{mj(J_1'), \ldots, mj(J_n')\}$.

Then Π contracts to $\dfrac{\begin{matrix}[\neg C]\\ \Pi_0'\end{matrix}}{C}\, I'$ where $Inf(I') = (\bot_c)$ and $dc(I')$ is $[\neg C]$ in $\begin{matrix}[\neg C]\\ \Pi_0'\end{matrix}$.

Example of $\bot_c$-contraction

$$\dfrac{\dfrac{\dfrac{\overset{2}{\neg(A \vee \neg A)} \qquad \dfrac{\dfrac{\dfrac{\overset{2}{\neg(A \vee \neg A)} \quad \dfrac{\overset{1}{A}}{A \vee \neg A}}{\bot}}{\neg A}\,1}{A \vee \neg A}}{\bot}}{A \vee \neg A}\,2 \qquad \dfrac{\dfrac{\dfrac{\neg(A \,\&\, B) \quad \dfrac{\overset{4}{A} \quad \overset{3}{B}}{A \,\&\, B}}{\bot}}{\neg B}\,3}{\neg A \vee \neg B} \qquad \dfrac{\overset{4}{\neg A}}{\neg A \vee \neg B}}{\neg A \vee \neg B}\,4$$

contracts to



In the figures above, the formula-occurrences indexed by a natural number are discharged by the D-inference indexed by the same number.

1.2.4 Immediate reducibility and Reduction sequence

We say that a derivation Π is immediately reduced to a derivation Π' iff Π' is the derivation obtained from Π by replacing a subderivation, say Γ, of Π by the derivation to which Γ contracts. A sequence Π_1, Π_2, ... is called a reduction sequence iff for all i, Π_i is immediately reduced to Π_{i+1}.

1.3 Segment

1.3.1 Segment

We extend the definition of segment introduced by Prawitz [5] in order to treat $(\bot_c)$-contraction. Let Π be a derivation.

1.3.1.1 Definition($ss_\Pi(\alpha)$: segment successor of α)

A partial function on $FO(\Pi)$ denoted by ss_Π is defined as follows. Let α be a formula-occurrence in Π.

(1) If α is a minor premiss of an application, say I, of $(\vee E)$ or $(\exists E)$; then $ss_\Pi(\alpha)$ is the conclusion of I.

(2) If α is the minor premiss of an application of $(\neg E)$ whose major premiss is discharged by an application, say I, of $(\bot_c)$; then $ss_\Pi(\alpha)$ is the conclusion of I.

(3) Otherwise, $ss_\Pi(\alpha)$ is undefined.

Clearly it holds that $Form(ss_\Pi(\alpha)) = Form(\alpha)$ if $ss_\Pi(\alpha)$ is defined.

1.3.1.2 Definition ($sp_\Pi(\alpha)$: segment predecessor of α)

Let α be a formula-occurrence in Π. $sp_\Pi(\alpha)$ is the subset of $FO(\Pi)$ defined by

$$sp_\Pi(\alpha) = \{\beta \in FO(\Pi) \mid ss_\Pi(\beta) = \alpha\}.$$

1.3.1.3 Definition (segment)

A finite sequence of formula-occurrences $\alpha_1, \ldots, \alpha_n$ in Π is a segment in Π iff it satisfies the following conditions (1), (2), and (3).

(1) $sp_\Pi(\alpha_1) = \phi$

(2) For all $i < n$, $ss_\Pi(\alpha_i) = \alpha_{i+1}$

(3) $ss_\Pi(\alpha_n)$ is undefined.

Our definition of segment is equivalent with that introduced in [1].

1.3.1.4 Definition ($sd_\Pi(\alpha, \beta)$: segment distance from α to β)

sd_Π is a function from $FO(\Pi) \times FO(\Pi)$ to $\mathcal{Z} \cup \{\infty\}$ defined as follows. Let α and β be formula-occurrences in Π.

(1) If there exists a segment $\delta_1, \ldots \delta_n$ in Π satisfying $\{\alpha, \beta\} \subset \{\delta_1, \ldots, \delta_n\}$, then $sd_\Pi(\alpha, \beta) = y - x$ where $\alpha = \delta_x$ and $\beta = \delta_y$.

(2) Otherwise, $sd_\Pi(\alpha, \beta) = \infty$.

Note that sd_Π is well-defined. Because if two segments $\delta_1, \ldots, \delta_n$ and $\tau_1, \ldots, \tau_m$ include the same formula-occurrence, say $\delta_p = \tau_q$, then the sequences $\delta_p, \ldots, \delta_n$ and $\tau_q, \ldots, \tau_m$ are identical.

1.4 Normalization theorem

1.4.1 Notations

Let α be a maximum formula in a derivation. By $g(\alpha)$ we denote the number of the logical symbols occurring in α. By $r(\alpha)$ we denote the maximum length of the segments whose last formula is α. By $l(\alpha)$ we denote the number of inferences below α in the derivation.

1.4.2 Definition (degree of a maximum formula)

Let α be a maximum formula in a derivation. The degree of α, denoted by $d(\alpha)$, is the ordered pair defined as follows:

$$d(\alpha) = \langle g(\alpha), r(\alpha) \rangle$$

Degrees of maximum formulae are compared by lexicographical order.

1.4.3 Notations

Let Π be a derivation. Notations $M(\Pi)$ and $E(\Pi)$ are defined as follows:

$$M(\Pi) = \begin{cases} \langle 0, 0 \rangle, & \text{if } \Pi \text{ is normal,} \\ \max\{d(\alpha) \mid \alpha \text{ is a maximum formula in } \Pi\}, & \text{otherwise.} \end{cases}$$

$$E(\Pi) = \{\alpha : \text{a maximum formula in } \Pi \mid d(\alpha) = M(\Pi)\}.$$

1.4.4 Definition (degree of a derivation)

Let Π be a derivation. The degree of Π, denoted by $d(\Pi)$, is the ordered triple defined as follows:

$$d(\Pi) = \langle M(\Pi), Card(E(\Pi)), \sum_{\alpha \in E(\Pi)} l(\alpha) \rangle$$

where in the case of $E(\Pi)$ is empty, by $\sum_{\alpha \in E(\Pi)} l(\alpha)$ we mean 0. Degrees of derivations are compared by lexicographical order.

1.4.5 Definition (side-set formula)

We call a formula-occurrence α a side-set formula of a formula-occurrence β, if α is one of the minor premisses of the D-inference whose major premiss is β.

1.4.6 Lemma

Let Π be a given derivation. If Π is not normal, we can find in it a formula-occurrence α which satisfies the following conditions.

(1) $\alpha \in E(\Pi)$.

(2) If $\beta \in E(\Pi)$; and if $\mathcal{S}$ is a segment in Π, whose last formula is α; then β is not above the first formula of $\mathcal{S}$.

(3) If $\beta \in E(\Pi)$; and if $\mathcal{S}$ is a segment in Π, whose last formula is β; then the first formula of $\mathcal{S}$ is not above nor equal to any of the side-set formulae of α.

Proof. Construct a sequence $\alpha_1, \alpha_2, \ldots$ of maximum formulae in Π by the following manner. Take α_1 from the maximum formulae satisfying the condition (1) and (2). If α_1 also satisfies the condition (3), terminate the sequence at it. If not, take α_2 from the maximum formulae destroying the condition (3) for α_1 and satisfying the condition (1) and (2). By iterating this construction, we obtain the sequence $\alpha_1, \alpha_2, \ldots$. It holds that if $m < n$ then $\alpha_m \neq \alpha_n$, by induction on $n - m$. Therefore, the sequence $\alpha_1, \alpha_2, \ldots$ is finite. Then, the last formula of the sequence satisfies all the conditions for α. □

1.4.7 Fact

Let α be a formula-occurrence in a derivation Π. If α satisfies the conditions of lemma 1.4.6, then it also satisfies the following condition.

(3') If $\beta \in E(\Pi)$, then β is not above nor equal to any of the side-set formulae of α.

Proof. Clear.□

1.4.8 Theorem 1 (Normalization theorem)

For every derivation Π, we can construct a finite reduction sequence from Π to a normal derivation.

Proof. We prove this theorem by induction on the degree of Π. If Π is not normal, we can find in Π a formula-occurrence, say μ, which is one of the maximum formulae satisfying the conditions for α of lemma 1.4.6. Let J be the D-inference satisfying $mj(J) = \mu$, and let $\Gamma = sbd(cl(J))$. Let Π' be the derivation obtained from Π by replacing the subderivation Γ by the derivation to which Γ contracts. Then, the degree of Π', is lower than that of Π. In the following we show this fact according to $Inf(I)$ where I is the D-inference satisfying $cl(I) = \mu$.

Case 1. If $Inf(I) = (\&I)$ or $(\forall I)$: Because μ satisfies the condition (1) for α of lemma 1.4.6, it holds that

$$\langle M(\Pi), Card(E(\Pi))\rangle > \langle M(\Pi'), Card(E(\Pi'))\rangle$$

This leads $d(\Pi) > d(\Pi')$.

Case 2. If $Inf(I) = (\vee I_i)$ or $(\exists I)$ ($i = 1$ or 2): Because μ satisfies the conditions (1) and (2) for α of lemma 1.4.6, it holds that

$$\langle M(\Pi), Card(E(\Pi))\rangle > \langle M(\Pi'), Card(E(\Pi'))\rangle$$

This leads $d(\Pi) > d(\Pi')$.

Case 3. If $Inf(I) = (\supset I)$ or $(\neg I)$: Because μ satisfies the conditions (1) and (3') for α of lemma 1.4.6 and fact 1.4.7, it holds that

$$\langle M(\Pi), Card(E(\Pi))\rangle > \langle M(\Pi'), Card(E(\Pi'))\rangle$$

This leads $d(\Pi) > d(\Pi')$.

Case 4. If $Inf(I) = (\vee E)$, $(\exists E)$, or $(\bot_c)$: Let δ_1 be the formula-occurrence in Π which is the conclusion of J. Let δ_0 be the last formula of a segment in Π which includes δ_1 as its member.

Case 4-1. If δ_0 is not a maximum formula in Π: Because μ satisfies the conditions (1) and (3') for α of lemma 1.4.6 and fact 1.4.7, it holds that

$$\langle M(\Pi), Card(E(\Pi))\rangle > \langle M(\Pi'), Card(E(\Pi'))\rangle$$

This leads $d(\Pi) > d(\Pi')$.

Case 4-2. If δ_0 is a maximum formula in Π: It holds that $d(\delta_0) < M(\Pi)$, since;

(a) If $Inf(J) = (\vee E)$ or $(\exists E)$, then there exists a segment in Π whose first formula is above or equal to one of the side-set formulae of μ and whose last formula is δ_0. This leads $d(\delta_0) < M(\Pi)$, because μ satisfies the condition (3) for α of lemma 1.4.6.

(b) Otherwise, it holds that $g(\delta_1) < g(\mu)$. This leads $d(\delta_0) < d(\mu) = M(\Pi)$.

Let $\widetilde{\delta_0}$ be the maximum formula in Π' which corresponds with δ_0. Then it holds that $d(\widetilde{\delta_0}) \leq M(\Pi)$, since $g(\widetilde{\delta_0}) = g(\delta_0)$ and $r(\widetilde{\delta_0}) \leq r(\delta_0) + 1$.

Case 4-2-1. If $d(\widetilde{\delta_0}) < M(\Pi)$: Because μ satisfies the conditions (1) and (3') for α of lemma 1.4.6 and fact 1.4.7, it holds that

$$\langle M(\Pi), Card(E(\Pi))\rangle > \langle M(\Pi'), Card(E(\Pi'))\rangle$$

This leads $d(\Pi) > d(\Pi')$.

Case 4-2-2 . If $d(\widetilde{\delta_0}) = M(\Pi)$: For each ψ in $E(\Pi)$, we define a maximum formula ψ' in Π' as follows:

(a) If ψ is μ, then ψ' is $\widetilde{\delta_0}$.

(b) Otherwise, ψ' is the maximum formula in Π' which corresponds with ψ. (Since μ satisfies the condition (3') for α of fact 1.4.7, exactly one formula-occurrence in Π' corresponds with ψ.)

For $d(\widetilde{\delta_0}) = M(\Pi)$, it holds that $E(\Pi') = \{\psi' \mid \psi \in E(\Pi)\}$. Therefore,

(i) $$\langle M(\Pi), Card(E(\Pi))\rangle = \langle M(\Pi'), Card(E(\Pi'))\rangle.$$

Next, we compare $l(\psi')$ with $l(\psi)$. If ψ is μ, then $l(\psi) > l(\psi')$; since μ is above δ_0. Otherwise, $l(\psi) \geq l(\psi')$; since μ satisfies the condition (3') for α of fact 1.4.7. Therefore,

(ii) $$\sum_{\alpha \in E(\Pi)} l(\alpha) > \sum_{\alpha \in E(\Pi')} l(\alpha).$$

From (i) and (ii), we obtain that $d(\Pi) > d(\Pi')$. □

Chapter 2

Church-Rosser property of our reduction

2.1 Segment-tree and segment-wood

2.1.1 Segment-tree

To prove theorem 2 in 2.3, i.e. the Church-Rosser property of our reduction, we will introduce in the next section an extended reduction (i.e. the structural reduction) which consists of $\vee E$-, $\exists E$-, or $\bot_c$-contractions applied continually for a *tree* of formula-occurrences in a derivation. Next we give the precise definition for the notion *tree* mentioned above.

2.1.1.1 Notation ($FO^*(\Pi)$)

We denote the set $FO(\Pi) \times \{0,1\}$ by $FO^*(\Pi)$.

2.1.1.2 Definition (sgt : segment-tree)

Let α be a formula-occurrence in Π, and T a subset of $FO^*(\Pi)$. The relation "*T is a segment-tree at α in Π* " holds iff one of the following conditions (a), (b), or (c) holds. It is defined by induction on the number of formula-occurrences above α.

(a) $T = \{< \alpha, 0 >\}$

(b) $sp_{\Pi}(\alpha) = \{\beta_1, \ldots, \beta_n\} \neq \phi$ where $\beta_i \neq \beta_j$ if $i \neq j$; and $T = \{< \alpha, 0 >\} \cup \bigcup_{1 \leq p \leq n} T_p$ where T_p is a segment-tree at β_p in Π for each $p \in \{1, \ldots, n\}$.

(c) α is the conclusion of an application of $(\bot_c)$; $sp_{\Pi}(\alpha) = \phi$; and $T = \{< \alpha, 0 >, < \alpha, 1 >\}$.

We use the notation sgt for the abbreviation of segment-tree.

2.1.1.3 Facts

Let T be a sgt at α in Π. The following facts (1),...,(8) are easily verified.

(1) $<\alpha, 0>\in T$

(2) *Exactly one of the conditions (a), (b), or (c) in definition 2.1.1.2 holds.*

(3) $<\beta, 1>\in T$ *implies* $<\beta, 0>\in T$.

(4) $<\beta, 0>\in T$ *implies* $sd_{\Pi}(\beta, \alpha) \in \mathcal{N}^0$.

(5) *If* $<\beta, 0>\in T$, $sd_{\Pi}(\beta, \gamma) \in \mathcal{N}^0$, *and* $sd_{\Pi}(\gamma, \alpha) \in \mathcal{N}^0$; *then* $<\gamma, 0>\in T$.

(6) *If* U *is a sgt at* β *in* Π *and* $T \cap U \neq \phi$; *then* $<\alpha, 0>\in U$ *or* $<\beta, 0>\in T$.

(7) *If* Γ *is a subderivation of* Π *satisfying* $\alpha \in FO(\Gamma)$, *then* T *is a sgt at* α *in* Γ.

(8) *If* $T_1, \ldots, T_n$ *are sgt's at a formula-occurrence* α *in* Π, *then* $\bigcup_{1\leq p\leq n} T_p$ *is also a sgt at* α *in* Π.

2.1.1.4 Some definitions

If T is a sgt at α in Π, then the construction of T is uniquely determined. Namely, first by fact (2) of 2.1.1.3 we can determine which condition (a), (b), or (c) holds in the definition 2.1.1.2; and second by fact (6) of 2.1.1.3, in the case of (b), we can determine T_p's uniquely except for their order.

Let T be a sgt at α in Π. We define two subsets of $FO(\Pi)$ denoted by $top(T)$ and $nf(T)$, and also define a natural number denoted by $len(T)$; by induction on the construction of T. In the following definitions of $top(T)$, $nf(T)$, and $len(T)$; (a), (b), and (c) means respectively (a), (b), and (c) in the Definition 2.1.1.2.

Definition ($top(T)$: tops of T)

Case (a): $top(T) = \{\alpha\}$

Case (b): $top(T) = \bigcup_{1\leq p\leq n} top(T_p)$

Case (c): $top(T) = \phi$

Definition ($nf(T)$: negation-friends of T)

Case (a): $nf(T) = \phi$

Case (b): Let I be the D-inference satisfying $cl(I) = \alpha$.

$$nf(T) = \begin{cases} \bigcup_{1\leq p\leq n} nf(T_p) & \text{if } Inf(I) = (\vee E) \text{ or } (\exists E) \\ dc(I) \cup \bigcup_{1\leq p\leq n} nf(T_p) & \text{if } Inf(I) = (\bot_c) \end{cases}$$

Case (c): $nf(T) = \phi$

Definition ($len(T)$: length of T)

Case (a): $len(T) = 1$

Case (b): $len(T) = 1 + \max_{1 \le p \le n} len(T_p)$

Case (c): $len(T) = 2$

2.1.2 Segment-wood

We will introduce a notion *segment-wood*. This is used for the inductive definition of the continual reduction for a sgt at a maximum formula in a derivation.

2.1.2.1 Definition (connectable formula-occurrence)

A formula-occurrence α in Π is connectable in Π iff it satisfies one of the following conditions (1) or (2).

(1) $\alpha = end(\Pi)$

(2) There exists a D-inference I in Π; such that $Inf(I) = (\neg E)$, $mn(I) = \{\alpha\}$, and $mj(I) \in oa(\Pi)$.

2.1.2.2 Definition (sgw: segment-wood)

Let W be a subset of $FO^*(\Pi)$. W is a segment-wood in Π iff it satisfies one of the following conditions (a) or (b).

(a) $W = \phi$

(b) There exists mutually distinct formula-occurrences $\alpha_1, \ldots, \alpha_n$ in Π and subsets $T_1, \ldots, T_n$ of $FO^*(\Pi)$ such that:

(b1) for all $p, q \in \{1, \ldots, n\}$, $Form(\alpha_p) = Form(\alpha_q)$;

(b2) for all $p \in \{1, \ldots, n\}$, α_p is connectable in Π, and T_p is a sgt at α_p in Π;

and (b3) $W = \bigcup_{1 \le p \le n} T_p$.

We use the notation sgw for the abbreviation of segment-wood.

2.1.2.3 Definition ($cmp(W)$: component of W)

For a sgw W in Π, $cmp(W)$ is the finite set of formulae defined by

$$cmp(W) = \{Form(\alpha) \mid \text{There exists } k \in \{0, 1\} \text{ such that } <\alpha, k> \in W\}$$

2.1.2.4 Definition ($rt(W)$: roots of W)

For a sgw W in Π, $rt(W)$ is the subset of $FO(\Pi)$ defined by

$$rt(W) = \{\alpha \in FO(\Pi) \mid <\alpha, 0> \in W \text{ and } \alpha \text{ is connectable in } \Pi\}$$

2.1.2.5 Facts

Let W be a sgw in Π. The following facts (1),...,(6) are easily verified by using the facts 2.1.1.3.

(1)

$$Card(cmp(W)) = \begin{cases} 0, & if\ W = \phi, \\ 1, & otherwise. \end{cases}$$

(2) *If $cmp(W) = \{A\}$, and if I is an application of $(\forall I)$ or $(\exists E)$ in Π; then the eigenvariable of I does not occur in A.*

(3) *Let α be a connectable formula-occurrence in Π and T a sgt at β in Π. Then, $<\alpha, 0> \in T$ implies $\alpha = \beta$.*

(4) *If $W \neq \phi$; then the formula-occurrences $\alpha_1, \ldots, \alpha_n$ and the subsets $T_1, \ldots, T_n$ of $FO^*(\Pi)$ in (b) of definition 2.1.2.2 are uniquely determined except for their order.*

(5) *Let $W_1, \ldots, W_n$ be sgw's in Π satisfying that $Card(\bigcup_{1 \leq p \leq n} cmp(W_p)) \leq 1$. Then, $\bigcup_{1 \leq p \leq n} W_p$ is a sgw in Π and $cmp(\bigcup_{1 \leq p \leq n} W_p) = \bigcup_{1 \leq p \leq n} cmp(W_p)$.*

(6) *If α is a connectable formula-occurrence in Π, and if Γ is a subderivation of Π satisfying $\alpha \in FO(\Gamma)$; then α is connectable in Γ.*

2.1.2.6 Definition ($W\lceil_\Gamma$)

Let W be a sgw in Π and Γ a subderivation of Π. $W\lceil_\Gamma$ is the subset of $FO^*(\Pi)$ defined by $W\lceil_\Gamma = W \cap FO^*(\Gamma)$.

2.1.2.7 Fact

Let T be a sgt at α in Π where α is a connectable formula-occurrence in Π (so, T is a sgw), and let Γ be a subderivation of Π. Then, $T\lceil_\Gamma$ is a sgw in Γ and $cmp(T\lceil_\Gamma) \subset cmp(T)$.

Proof. We prove this fact by induction on the construction of T.
Case 1. If $\alpha \in FO(\Gamma)$: Using fact (7) of 2.1.1.3 and fact (6) of 2.1.2.5, we have $T\lceil_\Gamma$(i.e. T) is a sgw in Γ and $cmp(T\lceil_\Gamma) = cmp(T)$.
Case 2. If $\alpha \notin FO(\Gamma)$:
Case 2-1. If $T \subset \{<\alpha, 0>, <\alpha, 1>\}$: It holds that $T\lceil_\Gamma = \phi$.
Case 2-2. If $T = \{<\alpha, 0>\} \cup \bigcup_{1 \leq p \leq n} T_p$ where $sp_\Pi(\alpha) = \{\beta_1, \ldots, \beta_n\} \neq \phi$, $\beta_i \neq \beta_j$ if $i \neq j$, and T_p is a sgt at β_p in Π for all $p \in \{1, \ldots, n\}$: Suppose $sbd(\alpha) = \dfrac{\Pi_0 \quad (\Pi_1 \quad \Pi_2)}{A}$.
Case 2-2-1. If for all $x \in \{0, 1, 2\}$, $end(\Gamma) \notin FO(\Pi_x)$: It holds that $T\lceil_\Gamma = \phi$.
Case 2-2-2. If not: Suppose $end(\Gamma) \in FO(\Pi_i)$, and let $P = \{p \in \{1, \ldots, n\} \mid \beta_p \in FO(\Pi_i)\}$. Then, $T\lceil_\Gamma = \bigcup_{p \in P} T_p\lceil_\Gamma$ holds. By fact (7) of 2.1.1.3 and fact (6) of 2.1.2.5, and by induction hypothesis; we have for all $p \in P$, $T\lceil_\Gamma$ is a sgw in Γ and $cmp(T_p\lceil_\Gamma) \subset cmp(T_p)(= cmp(T))$. Therefore,

by fact (5) of 2.1.2.5, we have $T\lceil_\Gamma$ is a sgw in Γ and $cmp(T\lceil_\Gamma) \subset cmp(T)$. □

2.1.2.8 Fact

Let W be a sgw in Π and Γ a subderivation of Π. Then, $W\lceil_\Gamma$ is a sgw in Γ and $cmp(W\lceil_\Gamma) \subset cmp(W)$.

Proof. By fact 2.1.2.7 and (5) of fact 2.1.2.5.

2.1.2.9 Some definitions

Let W be a sgw in Π. We define three subsets of $FO(\Pi)$ denoted by $top(W)$, $on(W)$, and $nf(W)$. In the following definitions of $top(W)$, $on(W)$, and $nf(W)$; (a) and (b) means respectively (a) and (b) in the definition 2.1.2.2.

Definition ($top(W)$: tops of W)

Case (a): $top(W) = \phi$

Case (b): $top(W) = \bigcup_{1 \leq p \leq n} top(T_p)$

Definition ($on(W)$: open negation of W)

Case (a): $on(W) = \phi$

Case (b): For any $\beta \in FO(\Pi)$, $\beta \in on(W)$ is equivalent to the following condition. That is, there exists $\alpha \in rt(W) \setminus \{end(\Pi)\}$ such that $\beta = mj(I)$ where I is the D-inference satisfying $mn(I) = \{\alpha\}$.

Definition ($nf(W)$: negation-friends of W)

Case (a): $nf(W) = \phi$

Case (b): $nf(W) = on(W) \cup \bigcup_{1 \leq p \leq n} nf(T_p)$

2.2 Structural reduction

In this section, we define the structural reduction. It is applied for a sgt T at a maximum formula in a derivation where $len(T) > 1$. The structural reduction is an extension of $\vee E$-, $\exists E$-, and $\perp_c$-contractions in the following meaning. One application of $\vee E$-, $\exists E$-, or $\perp_c$-contraction removes a maximum formula μ in a derivation Π up to the elements of $sp_\Pi(\mu)$. The structural reduction for a sgt T at a maximum formula μ in a derivation where $len(T) > 1$ removes μ up to the elements of $top(T)$. In order to define the structural reduction, we introduce a method to *substitute* a derivation for a sgw in a derivation.

2.2.1 Substitution-sequence

2.2.1.1 Definition (substitution-sequence)

Let Π and Θ be derivations and W a sgw in Π. We call the sequence $< \Pi, W, \Theta >$ a substitution-sequence iff it satisfies the following conditions (a), (b), and (c).

(a) Any eigenvariable occurring in one of the derivations Π and Θ does not occur in the other.

(b) $LI(\Theta)$ is an elimination rule, and $mj(li(\Theta)) \in oa(\Theta)$.

(c) $cmp(W) \subset \{MJ(li(\Theta))\}$

2.2.1.2 Definition ($\mathcal{P}_S$, $\mathcal{E}_S^1$, $\mathcal{E}_S^2$, $\mathcal{F}_S^U$, $\mathcal{F}_S^D$)

Let S be a substitution-sequence $< \Pi, W, \Theta >$. By the following clauses from Case 0 to Case 2, we define a derivation denoted by $\mathcal{P}_S$; two subsets of $FO(\mathcal{P}_S)$ denoted by $\mathcal{E}_S^1$ and $\mathcal{E}_S^2$; and two injection from $FO(\Pi)$ to $FO(\mathcal{P}_S)$ denoted by $\mathcal{F}_S^U$ and $\mathcal{F}_S^D$; where they satisfy the following conditions (a), (b), (c), and (d). Suppose $\Theta = \dfrac{MJ(li(\Theta)) \quad (\Theta_1 \quad \Theta_2)}{END(\Theta)}$, and let $Q = Card(mn(li(\Theta)))$.

(a)

$$END(\mathcal{P}_S) = \begin{cases} END(\Theta), & \text{if } < end(\Pi), 0 > \in W, \\ END(\Pi), & \text{otherwise.} \end{cases}$$

(b) If $Q \geq 1$, then for all $\alpha \in \mathcal{E}_S^1$ it holds that $sbd(\alpha)$ is identical with Θ_1; otherwise, $\mathcal{E}_S^1 = \phi$. If $Q = 2$, then for all $\beta \in \mathcal{E}_S^2$ it holds that $sbd(\beta)$ is identical with Θ_2; otherwise, $\mathcal{E}_S^2 = \phi$.

(c) For all $\alpha \in oa(\Pi)$, $Form(\mathcal{F}_S^U(\alpha)) = \begin{cases} \neg(END(\Theta)), & \text{if } \alpha \in on(W), \\ Form(\alpha), & \text{otherwise.} \end{cases}$

(d) $oa(\mathcal{P}_S) = \{\mathcal{F}_S^U(\alpha) \mid \alpha \in oa(\Pi)\} \cup \bigcup_{l \in \{1,2\}} \bigcup_{\alpha \in \mathcal{E}_S^l} oa(sbd(\alpha))$

$\mathcal{P}_S$, $\mathcal{E}_S^1$, $\mathcal{E}_S^2$, $\mathcal{F}_S^U$, and $\mathcal{F}_S^D$ are defined by induction on the length of Π.

Case 0. If $W = \phi$:

$$\mathcal{P}_S = \Pi.$$

$$\mathcal{E}_S^1 = \mathcal{E}_S^2 = \phi.$$

$\mathcal{F}_S^U$ and $\mathcal{F}_S^D$ are the identity mapping on $FO(\Pi)$.

Case 1. If $W \neq \phi$ and the length of Π is 1:

$$\mathcal{P}_S = \Theta.$$

$$\begin{cases} \mathcal{E}_S^1 = \mathcal{E}_S^2 = \phi, & \text{if } Q = 0, \\ \mathcal{E}_S^1 = \{end(\Theta_1)\} \text{ and } \mathcal{E}_S^2 = \phi, & \text{if } Q = 1, \\ \mathcal{E}_S^1 = \{end(\Theta_1)\} \text{ and } \mathcal{E}_S^2 = \{end(\Theta_2)\}, & \text{if } Q = 2. \end{cases}$$

$$\mathcal{F}^U_S(end(\Pi)) = mj(li(\mathcal{P}_S)).$$
$$\mathcal{F}^D_S(end(\Pi)) = end(\mathcal{P}_S).$$

Case 2. If $W \neq \phi$ and the length of Π is greater than 1:

Suppose $\Pi = \dfrac{\Pi_0 \quad (\Pi_1 \quad \Pi_2)}{END(\Pi)}$. Let S_r be the substitution-sequence defined by $S_r =< \Pi_r, W\lceil_{\Pi_r}, \Theta >$ for each $r \in \{0, 1, 2\}$.

Case 2-1. If $< end(\Pi), 0 > \notin W$:

Case 2-1-1. If $end(\Pi_0) \notin on(W)$:

$$\mathcal{P}_S = \frac{\mathcal{P}_{S_0} \quad (\mathcal{P}_{S_1} \quad \mathcal{P}_{S_2})}{END(\Pi)}\, K$$

where $Inf(K) = LI(\Pi)$ and

$$dc(K) = \bigcup_{0 \leq r \leq 2} \{\mathcal{F}^U_{S_r}(\alpha) \mid \alpha \in dc(li(\Pi)) \cap FO(\Pi_r)\}.$$

$$\text{For all } l \in \{1, 2\}, \mathcal{E}^l_S = \bigcup_{0 \leq r \leq 2} \mathcal{E}^l_{S_r}.$$

$$\begin{cases} \mathcal{F}^U_S(end(\Pi)) = \mathcal{F}^D_S(end(\Pi)) = end(\mathcal{P}_S). \\ \text{For all } r \in \{0, 1, 2\}, \text{ and for all } \alpha \in FO(\Pi_r); \\ \quad \mathcal{F}^U_S(\alpha) = \mathcal{F}^U_{S_r}(\alpha) \text{ and } \mathcal{F}^D_S(\alpha) = \mathcal{F}^D_{S_r}(\alpha). \end{cases}$$

Case 2-1-2. If $end(\Pi_0) \in on(W)$:

$$\mathcal{P}_S = \frac{\neg(END(\Theta)) \quad \mathcal{P}_{S_1}}{\bot}\, K$$

where $Inf(K) = (\neg E)$.

$$\text{For all } l \in \{1, 2\}, \mathcal{E}^l_S = \mathcal{E}^l_{S_1}.$$

$$\begin{cases} \mathcal{F}^U_S(end(\Pi)) = \mathcal{F}^D_S(end(\Pi)) = end(\mathcal{P}_S). \\ \mathcal{F}^U_S(end(\Pi_0)) = \mathcal{F}^D_S(end(\Pi_0)) = mj(K). \\ \text{For all } \alpha \in FO(\Pi_1), \mathcal{F}^U_S(\alpha) = \mathcal{F}^U_{S_1}(\alpha) \text{ and } \mathcal{F}^D_S(\alpha) = \mathcal{F}^D_{S_1}(\alpha). \end{cases}$$

Case 2-2. If $< end(\Pi), 0 > \in W$:

Case 2-2-1. If $end(\Pi) \notin top(W)$:

$$\mathcal{P}_S = \frac{\mathcal{P}_{S_0} \quad (\mathcal{P}_{S_1} \quad \mathcal{P}_{S_2})}{END(\Theta)}\, K$$

where $Inf(K) = LI(\Pi)$ and

$$dc(K) = \bigcup_{0 \leq r \leq 2} \{\mathcal{F}^U_{S_r}(\alpha) \mid \alpha \in dc(li(\Pi)) \cap FO(\Pi_r)\}.$$

$$\text{For all } l \in \{1, 2\}, \mathcal{E}^l_S = \bigcup_{0 \leq r \leq 2} \mathcal{E}^l_{S_r}.$$

$$\begin{cases} \mathcal{F}^U_S(end(\Pi)) = \mathcal{F}^D_S(end(\Pi)) = end(\mathcal{P}_S). \\ \text{For all } r \in \{0, 1, 2\}, \text{ and for all } \alpha \in FO(\Pi_r); \\ \quad \mathcal{F}^U_S(\alpha) = \mathcal{F}^U_{S_r}(\alpha) \text{ and } \mathcal{F}^D_S(\alpha) = \mathcal{F}^D_{S_r}(\alpha). \end{cases}$$

Case 2-2-2. If $end(\Pi) \in top(W)$:

$$\mathcal{P}_S = \cfrac{\cfrac{\mathcal{P}_{S_0} \quad (\mathcal{P}_{S_1} \quad \mathcal{P}_{S_2})}{END(\Pi)} K \qquad (\Theta_1 \quad \Theta_2)}{END(\Theta)} I$$

where $Inf(K) = LI(\Pi)$.

$$dc(K) = \bigcup_{0 \leq r \leq 2} \{\mathcal{F}^U_{S_r}(\alpha) \mid \alpha \in dc(li(\Pi)) \cap FO(\Pi_r)\},$$

$Inf(I) = LI(\Theta)$, and $dc(I)$ is identical with $dc(li(\Theta))$ as the subset of $\bigcup_{1 \leq q \leq Q} FO(\Theta_q)$.

$$\begin{cases} \mathcal{E}^1_S = \mathcal{E}^2_S = \phi, & \text{if } Q = 0, \\ \mathcal{E}^1_S = mn(I) \cup \bigcup_{0 \leq r \leq 2} \mathcal{E}^1_{S_r} \text{ and } \mathcal{E}^2_S = \phi, & \text{if } Q = 1, \\ \text{For all } l \in \{1, 2\}, \mathcal{E}^l_S = \{\alpha_l\} \cup \bigcup_{0 \leq r \leq 2} \mathcal{E}^l_{S_r}, & \text{if } Q = 2, \end{cases}$$

where, in the case of $Q = 2$, α_1 and α_2 are formula-occurrences of $\mathcal{P}_S$ satisfying that $mn(I) = \{\alpha_1, \alpha_2\}$ and α_1 stands on the left hand of α_2.

$$\begin{cases} \mathcal{F}^U_S(end(\Pi)) = mj(I), \ \mathcal{F}^D_S(end(\Pi)) = end(\mathcal{P}_S). \\ \text{For all } r \in \{0, 1, 2\}, \text{ and for all } \alpha \in FO(\Pi_r); \\ \qquad \mathcal{F}^U_S(\alpha) = \mathcal{F}^U_{S_r}(\alpha) \text{ and } \mathcal{F}^D_S(\alpha) = \mathcal{F}^D_{S_r}(\alpha). \end{cases}$$

2.2.2 Structural reduction

2.2.2.1 Definition (structural reduction)

Let Π be a derivation satisfying that $mj(li(\Pi))$ is a maximum formula in Π, and let T be a sgt at $mj(li(\Pi))$ in Π satisfying $len(T) \geq 2$. Then, the structural reduction of Π with T is the transformation of Π to the derivation $\mathcal{P}_S$ where the substitution-sequence S is defined by the following. Suppose $\Pi = \dfrac{\Pi_0 \quad (\Pi_1 \quad \Pi_2)}{END(\Pi)} K$. Let Θ be a derivation defined by $\Theta = \dfrac{END(\Pi_0) \quad (\Pi_1 \quad \Pi_2)}{END(\Pi)} K'$ where $Inf(K') = Inf(K)$, and $dc(K')$ is identical with $dc(K)$ as a subset of $FO(\Pi_1) \cup FO(\Pi_2)$. Then, the substitution-sequence S is defined by $S = < \Pi_0, T, \Theta >$. We call this substitution-sequence the *accompanying* substitution-sequence of the structural reduction of Π with T.

2.2.2.2 Notation

$\Pi \overset{SR(T)}{\longrightarrow} \Pi'$ denotes the fact that the derivation Π' is obtained by the structural reduction of Π with T.

2.2.2.3 Facts

We have the following facts (1) and (2) by definition.

(1) *Let α be a formula-occurrence in a derivation Π satisfying that α is the conclusion of an application of* $(\vee E)$, $(\exists E)$, *or* $(\bot_c)$. *Then, there exists exactly one sgt T at α in Π such that* $len(T) = 2$.

(2) *Let Π be a derivation satisfying that $mj(li(\Pi))$ is a maximum formula and is the conclusion of an application of* $(\vee E)$, $(\exists E)$, *or* $(\bot_c)$. *Suppose Π contracts to Π'. Then, it holds that* $\Pi \overset{SR(T)}{\longrightarrow} \Pi'$ *where T is the sgt at $mj(li(\Pi))$ in Π satisfying* $len(T) = 2$.

At the end of this section, we will state the fact that; if $\Pi \overset{SR(T)}{\longrightarrow} \Pi'$ holds, then there exists a reduction sequence from Π to Π' consisting of $\vee E$-, $\exists E$-, and $\bot_c$ -contractions (for subderivations).

2.2.2.4 Notation

For a derivation Π, we denote the set of all sgw's in Π by $SGW(\Pi)$.

2.2.3 Mappings

When $\Pi \overset{SR(T)}{\longrightarrow} \Pi'$ holds, we often need to use the *natural* mappings from $SGW(\Pi)$ to $SGW(\Pi')$ and from $oa(\Pi)$ to $oa(\Pi')$. In order to represent such mappings, we define the mappings CS^1_S, OS^1_S, CS^2_S, and OS^2_S for a substitution-sequence S.

2.2.3.1 Definition (CS^1_S, OS^1_S, CS^2_S, OS^2_S)

Let S be a substitution-sequence $< \Pi, W, \Theta >$. For $U \in SGW(\Pi)$ satisfying $U \cap W = \phi$, $CS^1_S(U)$ is the subset of $FO^*(\mathcal{P}_S)$ defined by

$$CS^1_S(U) = \{< \mathcal{F}^D_S(\theta), k > | < \theta, k > \in U\}$$

For $\alpha \in oa(\Pi) \setminus on(W)$, $OS^1_S(\alpha)$ is the subset of $oa(\mathcal{P}_S)$ defined by $OS^1_S(\alpha) = \{\mathcal{F}^U_S(\alpha)\}$. For $V \in SGW(\Theta)$, $CS^2_S(V)$ is the subset of $FO^*(\mathcal{P}_S)$ defined by

$$CS^2_S(V) = \begin{cases} \bigcup_{l \in \{1,2\}} \bigcup_{\lambda \in \mathcal{E}^l_S} \{< i_\lambda(\theta), k > | < \theta, k > \in V\lceil_{\Theta_l}\} \\ \quad \cup \{< \mathcal{F}^D_S(\theta), k > | < \theta, k > \in W\}, & \text{if } < end(\Theta), 0 > \in V, \\ \\ \bigcup_{l \in \{1,2\}} \bigcup_{\lambda \in \mathcal{E}^l_S} \{< i_\lambda(\theta), k > | < \theta, k > \in V\lceil_{\Theta_l}\}, & \text{otherwise,} \end{cases}$$

where for each $l \in \{1, 2\}$ and for each $\lambda \in \mathcal{E}^l_S$, i_λ is the canonical bijection from $FO(\Theta_l)$ ($\subset FO(\Theta)$) to $FO(sbd(\lambda))$ ($\subset FO(\mathcal{P}_S)$). For $\beta \in oa(\Theta) \setminus \{mj(li(\Theta))\}$, $OS^2_S(\beta)$ is the subset of $oa(\mathcal{P}_S)$ defined by

$$OS^2_S(\beta) = \begin{cases} \bigcup_{\lambda \in \mathcal{E}^1_S} \{i_\lambda(\beta)\}, & \text{if } \beta \in FO(\Theta_1), \\ \bigcup_{\lambda \in \mathcal{E}^2_S} \{i_\lambda(\beta)\}, & \text{if } \beta \in FO(\Theta_2), \end{cases}$$

where i_λ is defined as above.

2.2.3.2 Facts

Let S, U, α, V, and β be given as above. Then, we have the following facts (1),...,(4) by definition.

(1) *If* $\alpha' \in OS^1_S(\alpha)$, *then* $Form(\alpha') = Form(\alpha)$. *If* $\beta' \in OS^2_S(\beta)$, *then* $Form(\beta') = Form(\beta)$.

(2)
$$oa(\mathcal{P}_S) = \bigsqcup_{\alpha\in A} OS^1_S(\alpha) \sqcup \bigsqcup_{\beta\in B} OS^2_S(\beta) \sqcup \bigsqcup_{\gamma\in on(W)} \{\mathcal{F}^U_S(\gamma)\}$$
where $A = oa(\Pi) \setminus on(W)$ *and* $B = oa(\Theta) \setminus \{mj(li(\Theta))\}$.

(3) $CS^1_S(U)$ *and* $CS^2_S(V)$ *are sgw's in* $\mathcal{P}_S$. *Moreover, the following facts hold.*
$$cmp(CS^1_S(U)) = cmp(U).$$
$$on(CS^1_S(U)) = \bigcup_{\theta\in on(U)} OS^1_S(\theta).$$
$$cmp(CS^2_S(V)) \subset cmp(V).$$
$$on(CS^2_S(V)) = \begin{cases} \bigcup_{\theta\in on(V)} OS^2_S(\theta) \cup \bigcup_{\theta\in on(W)} \{\mathcal{F}^U_S(\theta)\}, & \textit{if} < end(\Theta), 0 >\in V, \\ \bigcup_{\theta\in on(V)} OS^2_S(\theta), & \textit{otherwise}. \end{cases}$$

(4)
$$< end(\mathcal{P}_S), 0 >\in CS^1_S(U) \quad \textit{iff} \quad < end(\Pi), 0 >\in U.$$
$$< end(\mathcal{P}_S), 0 >\in CS^2_S(V) \quad \textit{iff} \quad \Big(< end(\Theta), 0 >\in V \ \textit{and} \ < end(\Pi), 0 >\in W\Big).$$

2.2.3.3 Definition ($CS_{\Pi,T}$, $OS_{\Pi,T}$)

Let Π' be the derivation obtained from a derivation Π by the structural reduction of Π with T, i.e. $\Pi \overset{SR(T)}{\longrightarrow} \Pi'$. Suppose $\Pi = \dfrac{\Pi_0 \quad (\Pi_1 \quad \Pi_2)}{END(\Pi)}$, and let S be the accompanying substitution-sequence of the structural reduction of Π with T. For $W \in SGW(\Pi)$, $CS_{\Pi,T}(W)$ is the subset of $FO^*(\Pi')$ defined by
$$CS_{\Pi,T}(W) = CS^1_S(W\lceil_{\Pi_0}) \cup CS^2_S(W \setminus W\lceil_{\Pi_0}).$$
For $\alpha \in oa(\Pi)$, $OS_{\Pi,T}(\alpha)$ is the subset of $oa(\Pi')$ defined by
$$OS_{\Pi,T}(\alpha) = \begin{cases} OS^1_S(\alpha), & \text{if } \alpha \in FO(\Pi_0), \\ OS^2_S(\alpha), & \text{otherwise}. \end{cases}$$

2.2.3.4 Facts

Let Π, T, Π', W, and α be given as above. Then, we have the following facts (1),...,(4) by the previous facts 2.2.3.2 and by definition.

(1) *If* $\alpha' \in OS_{\Pi,T}(\alpha)$, *then* $Form(\alpha') = Form(\alpha)$.

(2)
$$oa(\Pi') = \bigsqcup_{\alpha \in oa(\Pi)} OS_{\Pi,T}(\alpha).$$

(3) $CS_{\Pi,T}(W)$ *is a sgw in* Π'. *Moreover, the following facts hold.*

$$cmp(CS_{\Pi,T}(W)) = cmp(W).$$
$$on(CS_{\Pi,T}(W)) = \bigcup_{\theta \in on(W)} OS_{\Pi,T}(\theta).$$

(4)
$$< end(\Pi'), 0 > \in CS_{\Pi,T}(W) \quad \textit{iff} \quad < end(\Pi), 0 > \in W.$$

2.2.4 Relationship between structural reductions and contractions

2.2.4.1 Fact

Let S *be a substitution-sequence* $< \Pi, W, \Theta >$. *Let* V_1 *and* V_2 *be sgw's in* Π *satisfying* $V_1 \cup V_2 = W$ *and* $V_1 \cap V_2 = \phi$. *Let* S^1 *and* S^2 *be the substitution-sequences defined by* $S^1 =< \Pi, V_1, \Theta >$ *and* $S^2 =< \mathcal{P}_{S^1}, CS^1_{S^1}(V_2), \Theta >$. *Then, it holds that* $\mathcal{P}_S = \mathcal{P}_{S^2}$.

Proof. By induction on the length of Π.□

2.2.4.2 Definition ($supp(W)$: support of W)

Let W be a sgw in Π. $supp(W)$ is the sgw in Π defined by

$$supp(W) = \{< \alpha, 0 > \in FO^*(\Pi) \mid \alpha \in rt(W)\}$$

2.2.4.3 Fact

Let S *be a substitution-sequence* $< \Pi, W, \Theta >$. *If* S' *is the substitution-sequence defined by* $S' =< \Pi, supp(W), \Theta >$; *then, it holds that there exists a reduction sequence from* $\mathcal{P}_{S'}$ *to* $\mathcal{P}_S$ *consisting of* $\vee E$-, $\exists E$-, *and* $\perp_c$-*contractions (for subderivations).*

Proof. By induction on $Card(W)$. Suppose $\Theta = \dfrac{MJ(li(\Theta)) \quad (\Theta_1 \quad \Theta_2)}{END(\Theta)}$.

Case 0. If $W = \phi$: Clear.

Case 1. If $Card(rt(W)) = 1$: Without loss of generality, we can assume that $rt(W) = \{end(\Pi)\}$.

Case 1-1. If $supp(W) = W$: Clear.

Case 1-2. If $supp(W) \neq W$, $li(\Pi) = (\bot_c)$, and $< end(\Pi), 1 > \notin W$: Suppose $\Pi = \dfrac{\Pi_0}{END(\Pi)}$. Then, there exists a sgw W_0 in Π_0, such that $W = \{< end(\Pi), 0 >\} \cup W_0$. Let S_0 and S_0' be the substitution-sequences defined by $S_0 =< \Pi_0, W_0, \Theta >$ and $S_0' =< \Pi_0, supp(W_0), \Theta >$. Now, $\mathcal{P}_{S'}$ is of the form $\dfrac{\dfrac{\Pi_0}{END(\Pi)} \quad (\Theta_1 \quad \Theta_2)}{END(\Theta)}$. Let Π' be the derivation obtained from $\mathcal{P}_{S'}$ by $(\bot_c)$-contraction. Then, Π' is of the form $\dfrac{\mathcal{P}_{S_0'}}{END(\Theta)}$, and by induction hypothesis, there exists a reduction sequence from Π' to the derivation $\dfrac{\mathcal{P}_{S_0}}{END(\Theta)} = \mathcal{P}_S$, consisting of $\vee E$-, $\exists E$-, and $\bot_c$-contractions.

Case 1-3. If $supp(W) \neq W$, $li(\Pi) = (\bot_c)$, and $< end(\Pi), 1 > \in W$; i.e. if $W = \{< end(\Pi), 0 >, < end(\Pi), 1 >\}$: Easy.

Case 1-4. If $supp(W) \neq W$ and $li(\Pi) = (\vee E)$ or $(\exists E)$: Similarly to the case 1-2.

Case 2. If $Card(rt(W)) > 1$: Take two sgw's in Π, say V_1 and V_2, satisfying that $W = V_1 \cup V_2$, $V_1 \cap V_2 = \phi$, $V_1 \neq \phi$, and $V_2 \neq \phi$. Let X be the substitution-sequence defined by $X =< \Pi, V_1 \cup supp(V_2), \Theta >$. Let Y_1, Y_2, and Y_3 be the substitution-sequence defined by

$$Y_1 =< \Pi, supp(V_2), \Theta >, \quad Y_2 =< \mathcal{P}_{Y_1}, CS^1_{Y_1}(supp(V_1)), \Theta >,$$

and

$$Y_3 =< \mathcal{P}_{Y_1}, CS^1_{Y_1}(V_1), \Theta > .$$

Using fact 2.2.4.1, we have $\mathcal{P}_{S'} = \mathcal{P}_{Y_2}$ and $\mathcal{P}_X = \mathcal{P}_{Y_3}$. It holds that $Card(CS^1_{Y_1}(V_1)) = Card(V_1)$ and that $supp(CS^1_{Y_1}(V_1)) = CS^1_{Y_1}(supp(V_1))$. Hence, by induction hypothesis, there exists a reduction sequence from $\mathcal{P}_{Y_2}$ to $\mathcal{P}_{Y_3}$, i.e. from $\mathcal{P}_{S'}$ to $\mathcal{P}_X$, consisting of $\vee E$-, $\exists E$-, and $\bot_c$-contractions. Similarly, we have the existance of a reduction sequence from $\mathcal{P}_X$ to $\mathcal{P}_S$, consisting of $\vee E$-, $\exists E$-, and $\bot_c$-contractions. This leads the result.□

2.2.4.4 Fact

Let Π' be the derivation obtained from a derivation Π by the structural reduction of Π with T, i.e. $\Pi \stackrel{SR(T)}{\longrightarrow} \Pi'$. Then, there exists a reduction sequence from Π to Π' consisting of $\vee E$-, $\exists E$-, and $\bot_c$-contractions (for subderivations).

Proof. By fact 2.2.4.3.□

2.3 1-reduction and Church-Rosser property

In this section, we define 1-reduction to prove the Church-Rosser property of our reduction. The definition of 1-reduction is an extension of that of Girard [4, pp135].

2.3.1 Mappings for essential reduction

2.3.1.1 Notation ($\Pi \stackrel{ER}{\longrightarrow} \Pi'$)

When a derivation Π' is obtained from a derivation Π by $\&_1$-, $\&_2$-, $\vee_1$-, $\vee_2$-, $\supset$-, $\neg$-, $\forall$-, or $\exists$-contraction; we denote the fact by $\Pi \stackrel{ER}{\longrightarrow} \Pi'$.

2.3.1.2 Definition (CE_Π, OE_Π)

Let Π and Π' be derivations satisfying $\Pi \stackrel{ER}{\longrightarrow} \Pi'$. For $W \in SGW(\Pi)$ and for $\alpha \in oa(\Pi)$, $CE_\Pi(W)$ and $OE_\Pi(\alpha)$ are the subset of $FO^*(\Pi')$ and the subset of $oa(\Pi')$ respectively, defined by the following clauses (1),...,(6).

(1) If Π' is obtained from Π by $\&_l$-contraction ($l = 1$ or 2): Suppose $\Pi = \dfrac{\dfrac{\Pi_1 \quad \Pi_2}{A_1 \,\&\, A_2}}{A_l}$. Then, $\Pi' = \Pi_l$. Let i be the canonical bijection from $FO(\Pi_l)$ (as a subset of $FO(\Pi)$) to $FO(\Pi')$. Then, $CE_\Pi(W)$ and $OE_\Pi(\alpha)$ are defined as follows.

$$CE_\Pi(W) = \begin{cases} \{<i(\theta),k> \mid <\theta,k> \in W\lceil_{\Pi_l}\} \\ \quad \cup\{<end(\Pi'),0>\}, & \text{if } <end(\Pi),0> \in W, \\ \\ \{<i(\theta),k> \mid <\theta,k> \in W\lceil_{\Pi_l}\}, & \text{otherwise.} \end{cases}$$

$$OE_\Pi(\alpha) = \begin{cases} \{i(\alpha)\}, & \text{if } \alpha \in FO(\Pi_l), \\ \phi, & \text{otherwise.} \end{cases}$$

(2) If Π' is obtained from Π by $\vee_l$-contraction ($l = 1$ or 2): Suppose $\Pi = \dfrac{\dfrac{\Pi_0}{A_1 \vee A_2} \quad \begin{matrix}[A_1] \\ \Pi_1\end{matrix} \quad \begin{matrix}[A_2] \\ \Pi_2\end{matrix}}{C}$. Then, $\Pi' = \begin{matrix}\Pi_0 \\ [A_l] \\ \Pi_l\end{matrix}$. Let i be the canonical bijection from $FO(\Pi_l)$ (as a subset of $FO(\Pi)$) to $FO(\Pi_l)$ (as a subset of $FO(\Pi')$). Let Λ be the subset of $FO(\Pi')$ defined by

$$\Lambda = \{i(\theta) \mid \theta \in dc(li(\Pi)) \cap FO(\Pi_l)\}.$$

For each $\lambda \in \Lambda$, let i_λ be the canonical bijection from $FO(\Pi_0)$ (as a subset of $FO(\Pi)$) to $FO(sbd(\lambda))$. Then, $CE_\Pi(W)$ and $OE_\Pi(\alpha)$ are defined as follows.

$$CE_\Pi(W) = \begin{cases} \{<i(\theta),k> \mid <\theta,k> \in W\lceil_{\Pi_l}\} \cup \bigcup_{\lambda\in\Lambda}\{<i_\lambda(\theta),k> \mid <\theta,k> \in W\lceil_{\Pi_0}\} \\ \quad \cup\{<end(\Pi'),0>\}, \qquad \text{if } <end(\Pi),0> \in W, \\ \{<i(\theta),k> \mid <\theta,k> \in W\lceil_{\Pi_l}\} \cup \bigcup_{\lambda\in\Lambda}\{<i_\lambda(\theta),k> \mid <\theta,k> \in W\lceil_{\Pi_0}\}, \\ \qquad \text{otherwise.} \end{cases}$$

$$OE_\Pi(\alpha) = \begin{cases} \{i(\alpha)\}, & \text{if } \alpha \in FO(\Pi_l), \\ \phi, & \text{if } \alpha \in FO(\Pi_m) \text{ where } \{l,m\} = \{1,2\}, \\ \bigcup_{\lambda\in\Lambda}\{i_\lambda(\alpha)\}, & \text{if } \alpha \in FO(\Pi_0). \end{cases}$$

(3) If Π' is obtained from Π by $\supset$-contraction: Suppose $\Pi = \dfrac{\dfrac{\begin{matrix}[A]\\ \Pi_0\end{matrix}}{A \supset B}\, I \qquad \Pi_1}{B}$.

Then, $\Pi' = \begin{matrix}\Pi_1\\ [A]\\ \Pi_0\end{matrix}$. Let i be the canonical bijection from $FO(\Pi_0)$ (as a subset of $FO(\Pi)$) to $FO(\Pi_0)$ (as a subset of $FO(\Pi')$). Let Λ be the subset of $FO(\Pi')$ defined by $\Lambda = \{i(\theta) \mid \theta \in dc(I)\}$. For each $\lambda \in \Lambda$, let i_λ be the canonical bijection from $FO(\Pi_1)$ (as a subset of $FO(\Pi)$) to $FO(sbd(\lambda))$. Then, $CE_\Pi(W)$ and $OE_\Pi(\alpha)$ is defined as follows.

$$CE_\Pi(W) = \begin{cases} \{<i(\theta),k> \mid <\theta,k> \in W\lceil_{\Pi_0}\} \cup \bigcup_{\lambda\in\Lambda}\{<i_\lambda(\theta),k> \mid <\theta,k> \in W\lceil_{\Pi_1}\} \\ \quad \cup\{<end(\Pi'),0>\}, \qquad \text{if } <end(\Pi),0> \in W, \\ \{<i(\theta),k> \mid <\theta,k> \in W\lceil_{\Pi_0}\} \cup \bigcup_{\lambda\in\Lambda}\{<i_\lambda(\theta),k> \mid <\theta,k> \in W\lceil_{\Pi_1}\}, \\ \qquad \text{otherwise.} \end{cases}$$

$$OE_\Pi(\alpha) = \begin{cases} \{i(\alpha)\}, & \text{if } \alpha \in FO(\Pi_0), \\ \bigcup_{\lambda\in\Lambda}\{i_\lambda(\alpha)\}, & \text{if } \alpha \in FO(\Pi_1). \end{cases}$$

(4) If Π' is obtained from Π by $\neg$-contraction: Similarly to the case (3).

(5) If Π' is obtained from Π by $\forall$-contraction: Similarly to the case (1).

(6) If Π' is obtained from Π by $\exists$-contraction: Similarly to the case (2).

2.3.1.3 Facts

Let Π, Π', W, and α be given as above. Then, we have the following facts (1),...,(4) by definition.

(1) *If $\alpha' \in OE_\Pi(\alpha)$, then $Form(\alpha') = Form(\alpha)$.*

(2)
$$oa(\Pi') = \bigsqcup_{\alpha \in oa(\Pi)} OE_\Pi(\alpha).$$

(3) *$CE_\Pi(W)$ is a sgw in Π'. Moreover, the following facts hold.*

$$cmp(CE_\Pi(W)) \subset cmp(W).$$
$$on(CE_\Pi(W)) = \bigcup_{\alpha \in on(W)} OE_\Pi(\alpha).$$

(4)
$$< end(\Pi'), 0 > \in CE_\Pi(W) \quad \textit{iff} \quad < end(\Pi), 0 > \in W.$$

2.3.2 1-reduction

2.3.2.1 Definition (1-reduction)

Let Π and Π' be derivations satisfying $END(\Pi') = END(\Pi)$ and $OA(\Pi') \subset OA(\Pi)$. The transformation of Π to Π' is called 1-reduction iff it satisfies one of the conditions (1), (2), (3), or (4) below. We denote by $\Pi \stackrel{1}{\longrightarrow} \Pi'$ the fact that the transformation of Π to Π' is a 1-reduction. 1-reduction is defined inductively with a mapping from $SGW(\Pi)$ to $SGW(\Pi')$, denoted by $C_\Pi^{\Pi'}$, and with a mapping from $oa(\Pi)$ to the power set of $oa(\Pi')$, denoted by $O_\Pi^{\Pi'}$; where $C_\Pi^{\Pi'}$ and $O_\Pi^{\Pi'}$ satisfy the following conditions (a), (b), and (c).

(a) For all $\alpha \in oa(\Pi)$ and for all $\beta \in O_\Pi^{\Pi'}(\alpha)$, $Form(\alpha) = Form(\beta)$ holds.

(b)
$$oa(\Pi') = \bigsqcup_{\alpha \in oa(\Pi)} O_\Pi^{\Pi'}(\alpha)$$

(c) For all $W \in SGW(\Pi)$, $cmp(C_\Pi^{\Pi'}(W)) \subset cmp(W)$ and $on(C_\Pi^{\Pi'}(W)) = \bigcup_{\alpha \in on(W)} O_\Pi^{\Pi'}(\alpha)$ hold.

(1) Π and Π' are identical. In this case, $C_\Pi^{\Pi'}$ and $O_\Pi^{\Pi'}$ are defined as follows.
For each $W \in SGW(\Pi)$, $C_\Pi^{\Pi'}(W) = W$.
For each $\alpha \in oa(\Pi)$, $O_\Pi^{\Pi'}(\alpha) = \{\alpha\}$.

(2) Π and Π' are of the form $\dfrac{\Pi_0 \quad (\Pi_1 \quad \Pi_2)}{A}\,K$ and $\dfrac{\Pi'_0 \quad (\Pi'_1 \quad \Pi'_2)}{A}\,K'$ respectively, where $\Pi_p \stackrel{1}{\longrightarrow} \Pi'_p$ (for all $p \in \{0,1,2\}$), $Inf(K') = Inf(K)$, and $dc(K') = \bigcup_{0 \le p \le 2} \bigcup_{\alpha \in dc(K) \cap FO(\Pi_p)} O_{\Pi_p}^{\Pi'_p}(\alpha)$. In this case, $C_\Pi^{\Pi'}$ and $O_\Pi^{\Pi'}$ are defined as follows.
For each $W \in SGW(\Pi)$,

$$C_\Pi^{\Pi'}(W) = \bigcup_{0 \le p \le 2} C_{\Pi_p}^{\Pi'_p}(W\lceil_{\Pi_p}) \cup \{< end(\Pi'), k > | < end(\Pi), k > \in W\} \cup E$$

where

$$E = \begin{cases} \{< end(\Pi'), 1 >\}, & \text{if } Inf(K) = (\bot_c),\ dc(K) \cap nf(W) \neq \phi, \text{ and } dc(K') = \phi, \\ \phi, & \text{otherwise.} \end{cases}$$

For each $p \in \{0, 1, 2\}$ and for each $\alpha \in oa(\Pi) \cap FO(\Pi_p)$, $O_{\Pi}^{\Pi'}(\alpha) = O_{\Pi_p}^{\Pi'_p}(\alpha)$.

(3) Π is of the form $\dfrac{\dfrac{\Pi_0 \quad (\Pi_1)}{M}\, I \qquad (\Pi_2 \quad \Pi_3)}{A}\, K$ where $Inf(I)$ is an introduction rule and $Inf(K)$ is an elimination rule; and

$$\dfrac{\dfrac{\Pi'_0 \quad (\Pi'_1)}{M}\, I' \qquad (\Pi'_2 \quad \Pi'_3)}{A}\, K' \quad \stackrel{ER}{\longrightarrow} \quad \Pi'$$

where $\Pi_p \stackrel{1}{\longrightarrow} \Pi'_p$ (for all $p \in \{0, \ldots, 3\}$), $Inf(I') = Inf(I)$, $dc(I') = \bigcup_{\alpha \in dc(I)} O_{\Pi_0}^{\Pi'_0}(\alpha)$, $Inf(K') = Inf(K)$, and $dc(K') = \bigcup_{2 \leq p \leq 3} \bigcup_{\alpha \in dc(K) \cap FO(\Pi_p)} O_{\Pi_p}^{\Pi'_p}(\alpha)$. In this case, $C_{\Pi}^{\Pi'}$ and $O_{\Pi}^{\Pi'}$ are defined as follows. Let Δ be the derivation $\dfrac{\dfrac{\Pi'_0 \quad (\Pi'_1)}{M}\, I' \qquad (\Pi'_2 \quad \Pi'_3)}{A}\, K'$. For each $W \in SGW(\Pi)$, $C_{\Pi}^{\Pi'}(W) = CE_{\Delta}(W')$ where

$$W' = \begin{cases} \bigcup_{0 \leq p \leq 3} C_{\Pi_p}^{\Pi'_p}(W \lceil_{\Pi_p}) \cup \{< end(\Delta), 0 >\}, & \text{if } < end(\Pi), 0 > \in W, \\ \bigcup_{0 \leq p \leq 3} C_{\Pi_p}^{\Pi'_p}(W \lceil_{\Pi_p}), & \text{otherwise.} \end{cases}$$

For each $p \in \{0, \ldots, 3\}$ and for each $\alpha \in oa(\Pi) \cap FO(\Pi_p)$, $O_{\Pi}^{\Pi'}(\alpha) = \bigcup_{\theta \in O_{\Pi_p}^{\Pi'_p}(\alpha)} OE_{\Delta}(\theta)$.

(4) Π is of the form $\dfrac{\Pi_0 \quad (\Pi_1 \quad \Pi_2)}{A}\, K$ where $Inf(K)$ is an elimination rule and $LI(\Pi_0)$ is $(\vee E)$, $(\exists E)$, or $(\bot_c)$; and

$$\dfrac{\Pi'_0 \quad (\Pi'_1 \quad \Pi'_2)}{A}\, K' \quad \stackrel{SR(C_{\Pi_0}^{\Pi'_0}(T))}{\longrightarrow} \quad \Pi'$$

where $\Pi_p \stackrel{1}{\longrightarrow} \Pi'_p$ (for all $p \in \{0, 1, 2\}$), $Inf(K') = Inf(K)$, $dc(K') = \bigcup_{1 \leq p \leq 2} \bigcup_{\alpha \in dc(K) \cap FO(\Pi_p)} O_{\Pi_p}^{\Pi'_p}(\alpha)$, and T is a sgt at $end(\Pi_0)$ in Π_0 satisfying $len(T) > 1$ and $len(C_{\Pi_0}^{\Pi'_0}(T)) > 1$. In this case, $C_{\Pi}^{\Pi'}$ and $O_{\Pi}^{\Pi'}$ are defined as follows. Let Δ be the derivation $\dfrac{\Pi'_0 \quad (\Pi'_1 \quad \Pi'_2)}{A}\, K'$ and T' the sgt $C_{\Pi_0}^{\Pi'_0}(T)$ at $end(\Pi'_0)$ in Π'_0. For each $W \in SGW(\Pi)$, $C_{\Pi}^{\Pi'}(W) = CS_{\Delta, T'}(W')$ where

$$W' = \begin{cases} \bigcup_{0 \leq p \leq 2} C_{\Pi_p}^{\Pi'_p}(W \lceil_{\Pi_p}) \cup \{< end(\Delta), 0 >\}, & \text{if } < end(\Pi), 0 > \in W, \\ \bigcup_{0 \leq p \leq 2} C_{\Pi_p}^{\Pi'_p}(W \lceil_{\Pi_p}), & \text{otherwise.} \end{cases}$$

For each $p \in \{0, 1, 2\}$ and for each $\alpha \in oa(\Pi) \cap FO(\Pi_p)$, $O_{\Pi}^{\Pi'}(\alpha) = \bigcup_{\theta \in O_{\Pi_p}^{\Pi'_p}(\alpha)} OS_{\Delta, T'}(\theta)$.

2.3.2.2 Notice

When derivations Π and Π' satisfying $\Pi \xrightarrow{1} \Pi'$ are given; it is assumed that the construction of $\Pi \xrightarrow{1} \Pi'$ is also given, and so, the number of the clauses in definition 2.3.2.1 used in the construction of $\Pi \xrightarrow{1} \Pi'$ is uniquely determined.

2.3.2.3 Notation ($|\Pi \xrightarrow{1} \Pi'|$, $LC(\Pi \xrightarrow{1} \Pi')$)

Let Π and Π' be derivations satisfying $\Pi \xrightarrow{1} \Pi'$. We denote by $|\Pi \xrightarrow{1} \Pi'|$ the number of the clauses in definition 2.3.2.1 used in the construction of $\Pi \xrightarrow{1} \Pi'$. Also we denote by $LC(\Pi \xrightarrow{1} \Pi')$ the last clause in definition 2.3.2.1 used in the construction of $\Pi \xrightarrow{1} \Pi'$.

2.3.2.4 Fact

If a derivation Π is immediately reduced to a derivation Π', then it holds that $\Pi \xrightarrow{1} \Pi'$.

Proof. By fact (2) of 2.2.2.3.

2.3.2.5 Fact

If a derivation Π is 1-reduced to a derivation Π', i.e. $\Pi \xrightarrow{1} \Pi'$, then there exists a reduction sequence from Π to Π'.

Proof. By fact 2.2.4.4.

2.3.2.6 Notation

Let Π, Π', and Π'' be derivations. For a mapping f from $SGW(\Pi)$ to $SGW(\Pi')$ and a mapping g from $SGW(\Pi')$ to $SGW(\Pi'')$, $g \circ f$ denotes the mapping from $SGW(\Pi)$ to $SGW(\Pi'')$ defined by $g \circ f(W) = g(f(W))$. Also, for a mapping F from $oa(\Pi)$ to the power set of $oa(\Pi')$ and a mapping G from $oa(\Pi')$ to the power set of $oa(\Pi'')$, $G \circ F$ denotes the mapping from $oa(\Pi)$ to the power set of $oa(\Pi'')$ defined by $G \circ F(\alpha) = \bigcup_{\theta \in F(\alpha)} G(\theta)$. We use these notations also in the case of partial mappings.

2.3.2.7 Main Lemma.

If $\Pi \xrightarrow{1} \Pi'$ and $\Pi \xrightarrow{1} \Pi''$ hold, then there exists a derivation Π''' such that $\Pi' \xrightarrow{1} \Pi'''$, $\Pi'' \xrightarrow{1} \Pi'''$, $C_{\Pi'}^{\Pi'''} \circ C_{\Pi}^{\Pi'} = C_{\Pi''}^{\Pi'''} \circ C_{\Pi}^{\Pi''}$, and $O_{\Pi'}^{\Pi'''} \circ O_{\Pi}^{\Pi'} = O_{\Pi''}^{\Pi'''} \circ O_{\Pi}^{\Pi''}$.

Main Lemma will be proved in the next section. The following theorem, i.e. the Church-Rosser property of our reduction, can be easily proved using fact 2.3.2.4, fact 2.3.2.5, and Main Lemma.

Theorem 2. (Church-Rosser property) *If two finite reduction sequences $\varPi, \ldots, \varSigma$ and $\varPi, \ldots, \varSigma'$ are given, then we can construct two finite reduction sequences $\varSigma, \ldots, \varDelta$ and $\varSigma', \ldots, \varDelta$ for some derivation $\varDelta$.*

2.4 Proof of Main Lemma

2.4.1 Lemmata

It now remains for us to establish the proof of Main Lemma. The essential parts of the proof are obtained from Lemma A (2.4.1.2) and Lemma B (2.4.1.3).

2.4.1.1 Notation ($W \prec V$)

Let W and V be sgw's in a derivation. We denote by $W \prec V$ the fact that $W \subset V$ and $rt(W) = rt(V)$ hold.

2.4.1.2 Lemma A

If $\varPi \xrightarrow{1} \varPi'$, $LC(\varPi \xrightarrow{1} \varPi')$ is (2), $\varPi \xrightarrow{ER} \varSigma$, and $\varPi' \xrightarrow{ER} \varSigma'$ hold; then, $\varSigma \xrightarrow{1} \varSigma'$, $C_{\varSigma}^{\varSigma'} \circ CE_{\varPi} = CE_{\varPi'} \circ C_{\varPi}^{\varPi'}$, and $O_{\varSigma}^{\varSigma'} \circ OE_{\varPi} = OE_{\varPi'} \circ O_{\varPi}^{\varPi'}$ hold.

2.4.1.3 Lemma B

Let S be a substitution-sequence $< \varPi, W, \varTheta >$, and let V be a sgw in $\varPi$ satisfying $W \prec V$. If $\varPi \xrightarrow{1} \varPi'$ and $\varTheta \xrightarrow{1} \varTheta'$ hold, and let S' be the substitution-sequence defined by $S' =< \varPi', V', \varTheta' >$ where $V' = C_{\varPi}^{\varPi'}(V)$; then, the following facts (a),...,(e) hold.

(a) $\mathcal{P}_S \xrightarrow{1} \mathcal{P}_{S'}$

(b) For all $U \in SGW(\varPi)$ satisfying $U \cap V = \phi$, it holds that $C_{\mathcal{P}_S}^{\mathcal{P}_{S'}} \circ CS_S^1(U) = CS_{S'}^1 \circ C_{\varPi}^{\varPi'}(U)$.

(c) For all $\alpha \in oa(\varPi) \backslash on(V)$, it holds that $O_{\mathcal{P}_S}^{\mathcal{P}_{S'}} \circ OS_S^1(\alpha) = OS_{S'}^1 \circ O_{\varPi}^{\varPi'}(\alpha)$.

(d) $C_{\mathcal{P}_S}^{\mathcal{P}_{S'}} \circ CS_S^2 = CS_{S'}^2 \circ C_{\varTheta}^{\varTheta'}$

(e) For all $\alpha \in oa(\varTheta) \setminus \{mj(li(\varTheta))\}$, it holds that $O_{\mathcal{P}_S}^{\mathcal{P}_{S'}} \circ OS_S^2(\alpha) = OS_{S'}^2 \circ O_{\varTheta}^{\varTheta'}(\alpha)$.

2.4.1.4 Remark

Lemma A and Lemma B are proved using some facts stated in the following. We state these facts in an abbreviated form. Namely, the commutativity of mappings on sgw's and on open assumptions (e.g. (b), (c), (d), and (e) in Lemma B) is not represented in these statement. But all these facts stated in the following are hold with such commutativity.

2.4.2 Some facts

2.4.2.1 Fact

If $\Pi \xrightarrow{1} \Pi'$ holds and let a and t be a free variable and a term respectively satisfying $\Pi(t/a)$ becomes a derivation: then $\Pi'(t/a)$ is a derivation, and $\Pi(t/a) \xrightarrow{1} \Pi'(t/a)$ holds.

Proof. By induction on $|\Pi \xrightarrow{1} \Pi'|$.□

2.4.2.2 Fact

Let Σ and $\begin{matrix}[A]\\ \Pi\end{matrix}$ be derivations satisfying $END(\Sigma) = A$. Let P be the subset of $oa(\Pi)$ denoted by $[A]$ in $\begin{matrix}[A]\\ \Pi\end{matrix}$. Suppose that $\Sigma \xrightarrow{1} \Sigma'$ and $\begin{matrix}[A]\\ \Pi\end{matrix} \xrightarrow{1} \begin{matrix}[A]\\ \Pi'\end{matrix}$ hold where $[A]$ in $\begin{matrix}[A]\\ \Pi'\end{matrix}$ denotes the subset of $oa(\Pi')$, say P', defined by $P' = \bigcup_{\alpha \in P} O_{\Pi}^{\Pi'}(\alpha)$. Then, we have $\begin{matrix}\Sigma\\ [A]\\ \Pi\end{matrix} \xrightarrow{1} \begin{matrix}\Sigma'\\ [A]\\ \Pi'\end{matrix}$.

Proof. By induction on $|\Pi \xrightarrow{1} \Pi'|$. In the case that $LC(\Pi \xrightarrow{1} \Pi')$ is (4), we use the following fact. That is, if S and X are substitution-sequence defined by $S =< \begin{matrix}[B]\\ \Gamma\end{matrix}, W, \begin{matrix}[B]\\ \Theta\end{matrix} >$ and $X =< \begin{matrix}\Delta\\ [B]\\ \Gamma\end{matrix}, W, \begin{matrix}\Delta\\ [B]\\ \Theta\end{matrix} >$, then it holds that $\mathcal{P}_X = \begin{matrix}\Sigma\\ [B]\\ \mathcal{P}_S\end{matrix}$ where we define $[B]$ in $\begin{matrix}[B]\\ \mathcal{P}_S\end{matrix}$ using OS_S^1 and OS_S^2.□

2.4.2.3 Fact

Let S be a substitution-sequence $< \Pi, W, \Theta >$, and let V be a sgw in Π satisfying $W \prec V$. If $\Theta \xrightarrow{1} \Theta'$ holds, and let S' be a substitution-sequence defined by $S' =< \Pi, V, \Theta' >$; then, $\mathcal{P}_S \xrightarrow{1} \mathcal{P}_{S'}$ holds.

Proof. By induction on the length of Π. We prove this fact in the case that $end(\Pi) \in top(W)$ and $end(\Pi) \notin top(V)$ hold, since other cases are straight-forward. Now we assume that. Suppose Π and Θ are of the form $\dfrac{\Pi_0 \quad (\Pi_1 \quad \Pi_2)}{A}\, I$ and $\dfrac{A \quad (\Theta_1 \quad \Theta_2)}{B}\, K$ respectively. Then, $\mathcal{P}_S$ and $\mathcal{P}_{S'}$ are of the form $\dfrac{\dfrac{\mathcal{P}_{S_0} \quad (\mathcal{P}_{S_1} \quad \mathcal{P}_{S_2})}{A} \quad (\Theta_1 \quad \Theta_2)}{B}$ and $\dfrac{\mathcal{P}_{S'_0} \quad (\mathcal{P}_{S'_1} \quad \mathcal{P}_{S'_2})}{B}$ respectively, where $S_p =< \Pi_p, W\lceil_{\Pi_p}, \Theta >$ and $S'_p =< \Pi_p, V\lceil_{\Pi_p}, \Theta' >$ for each $p \in \{0, 1, 2\}$. Let V_0 and V_1 be sgw's in Π satisfying that $V = V_0 \cup V_1$, $V_0 \cap V_1 = \phi$, and $rt(V_0) = \{end(\Pi)\}$. Define substitution-sequences X and X_p for each $p \in \{0, 1, 2\}$ by $X =< \Pi, V_1, \Theta' >$ and $X_p =< \Pi_p, V_1\lceil_{\Pi_p}, \Theta' >$ for each $p \in \{0, 1, 2\}$. Denote $sbd(mj(li(\mathcal{P}_S)))$ by Δ_0. From the condition $end(\Pi) \in top(W)$ and the definition of V_0 and V_1, we have $W\lceil_{\Pi_p} \prec V_1\lceil_{\Pi_p}$ for each $p \in \{0, 1, 2\}$. Hence, by induction hypothesis, we have $\mathcal{P}_{S_p} \overset{1}{\longrightarrow} \mathcal{P}_{X_p}$ for each $p \in \{0, 1, 2\}$. Therefore, we have $\Delta_0 \overset{1}{\longrightarrow} \mathcal{P}_X$ using the clause (2) for $LC(\Delta_0 \overset{1}{\longrightarrow} \mathcal{P}_X)$, since $\mathcal{P}_X$ is of the form $\dfrac{\mathcal{P}_{X_0} \quad (\mathcal{P}_{X_1} \quad \mathcal{P}_{X_2})}{A}$. Let T be the sgt at $end(\Delta_0)$ in Δ_0 defined by

$$T = \{< end(\Delta_0), k > | < end(\Pi), k > \in V_0\} \cup \bigcup_{0 \le p \le 2} CS^1_{S_p}(V_0\lceil_{\Pi_p}),$$

and let T' be the sgt at $end(\mathcal{P}_X)$ in $\mathcal{P}_X$ defined by $T' = C^{\mathcal{P}_X}_{\Delta_0}(T)$. Define a substitution-sequence Y by $Y =< \mathcal{P}_X, T', \Theta' >$. By induction hypothesis (about commutativity of mappings) for Π_p, we have $T' = CS^1_X(V_0)$. Hence, by fact 2.2.4.1, $\mathcal{P}_{S'} = \mathcal{P}_Y$ holds. On the other hand, we have $\mathcal{P}_S \overset{1}{\longrightarrow} \mathcal{P}_Y$ because $\dfrac{\mathcal{P}_X \quad (\Theta'_1 \quad \Theta'_2)}{B} \overset{SR(T')}{\longrightarrow} \mathcal{P}_X$ holds where we suppose $\Theta' = \dfrac{A \quad (\Theta'_1 \quad \Theta'_2)}{B}$. Therefore, $\mathcal{P}_S \overset{1}{\longrightarrow} \mathcal{P}_{S'}$ holds.□

2.4.2.4 Fact

Let Π and Σ be derivations satisfying $\Pi \overset{ER}{\longrightarrow} \Sigma$. Let S be a substitution-sequence $< \Pi, W, \Theta >$, and X the substitution-sequence defined by $X =< \Sigma, CE_\Pi(W), \Theta >$. Then, $\mathcal{P}_S \overset{ER}{\longrightarrow} \mathcal{P}_X$ holds.

Proof. By definition of CE_Π.□

2.4.2.5 Fact

Let S, X, and Y be substitution-sequences $< \Pi, W, \Theta >$, $< \Pi, V_1, \Delta >$, and $< \Theta, V_2, \Delta >$ respectively; satisfying $W \cap V_1 = \phi$. Let $\widetilde{S}$ and $\widetilde{X}$ be the substitution-sequences defined by $\widetilde{S} =< \mathcal{P}_X, CS^1_X(W), \mathcal{P}_Y >$ and $\widetilde{X} =< \mathcal{P}_S, CS^1_S(V_1) \cup CS^2_S(V_2), \Delta >$. Then, $\mathcal{P}_{\widetilde{S}} = \mathcal{P}_{\widetilde{X}}$ holds.

Proof. By induction on the length of Π.□

2.4.3 Proof of lemmata

Now we prove Lemma A, Lemma B, and Main Lemma.

2.4.3.1 Proof of Lemma A

Since $\Pi \xrightarrow{ER} \Sigma$, Π is of the form $\dfrac{\dfrac{\Pi_0 \quad (\Pi_1)}{M}\,I \qquad (\Pi_2 \quad \Pi_3)}{A}\,K$ where $Inf(I)$ is an introduction rule and $Inf(K)$ is an elimination rule. Then, Π' is of the form $\dfrac{\dfrac{\Pi'_0 \quad (\Pi'_1)}{M} \qquad (\Pi'_2 \quad \Pi'_3)}{A}$ where $\Pi_p \xrightarrow{1} \Pi'_p$ for each $p \in \{0, \ldots, 3\}$, because $LC(\Pi \xrightarrow{1} \Pi')$ is (2) and $Inf(I)$ is an introduction rule. Then, using fact 2.4.2.1 and fact 2.4.2.2, we have the result.□

2.4.3.2 Proof of Lemma B

By induction on $|\Pi \xrightarrow{1} \Pi'|$.

Case 1. $LC(\Pi \xrightarrow{1} \Pi')$ is (1): Use fact 2.4.2.3.

Case 2. $LC(\Pi \xrightarrow{1} \Pi')$ is (2): Similarly with the proof of fact 2.4.2.3.

Case 3. $LC(\Pi \xrightarrow{1} \Pi')$ is (3): Use fact 2.4.2.4.

Case 4. $LC(\Pi \xrightarrow{1} \Pi')$ is (4): Use fact 2.4.2.5.□

2.4.3.3 Proof of Main Lemma.

By induction on $|\Pi \xrightarrow{1} \Pi'| + |\Pi \xrightarrow{1} \Pi''|$.

Case 1. $LC(\Pi \xrightarrow{1} \Pi')$ is (1): Take Π'' for Π'''.

Case 1'. $LC(\Pi \xrightarrow{1} \Pi'')$ is (1): Similarly to the case 1.

Case 2. $LC(\Pi \xrightarrow{1} \Pi')$ and $LC(\Pi \xrightarrow{1} \Pi'')$ are (2): Suppose Π, Π', and Π'' are of the form $\dfrac{\Pi_0 \quad (\Pi_1 \quad \Pi_2)}{A}$, $\dfrac{\Pi'_0 \quad (\Pi'_1 \quad \Pi'_2)}{A}$, and $\dfrac{\Pi''_0 \quad (\Pi''_1 \quad \Pi''_2)}{A}$ respectively, where for each $p \in \{0, 1, 2\}$, $\Pi_p \xrightarrow{1} \Pi'_p$ and $\Pi_p \xrightarrow{1} \Pi''_p$ hold. Then by induction hypothesis, for each $p \in \{0, 1, 2\}$ there exists a derivation Π'''_p such that $\Pi'_p \xrightarrow{1} \Pi'''_p$, $\Pi''_p \xrightarrow{1} \Pi'''_p$, $C^{\Pi'''_p}_{\Pi'_p} \circ C^{\Pi'_p}_{\Pi_p} = C^{\Pi'''_p}_{\Pi''_p} \circ C^{\Pi''_p}_{\Pi_p}$, and $O^{\Pi'''_p}_{\Pi'_p} \circ O^{\Pi'_p}_{\Pi_p} = O^{\Pi'''_p}_{\Pi''_p} \circ O^{\Pi''_p}_{\Pi_p}$ hold. Let Π''' be the derivation of the form $\dfrac{\Pi'''_0 \quad (\Pi'''_1 \quad \Pi'''_2)}{A}$. Then, the result holds for this Π'''.

Case 3. $LC(\Pi \xrightarrow{1} \Pi')$ and $LC(\Pi \xrightarrow{1} \Pi'')$ are (3): Suppose Π is of the form $\dfrac{\dfrac{\Pi_0 \quad (\Pi_1)}{M} \qquad (\Pi_2 \quad \Pi_3)}{A}$, and suppose Π' and Π'' satisfy that

$$\dfrac{\dfrac{\Pi'_0 \quad (\Pi'_1)}{M} \qquad (\Pi'_2 \quad \Pi'_3)}{A} \xrightarrow{ER} \Pi' \text{ and } \dfrac{\dfrac{\Pi''_0 \quad (\Pi''_1)}{M} \qquad (\Pi''_2 \quad \Pi''_3)}{A} \xrightarrow{ER} \Pi'',$$

where for each $p \in \{0, \ldots, 3\}$, $\Pi_p \xrightarrow{1} \Pi'_p$ and $\Pi_p \xrightarrow{1} \Pi''_p$ hold. Then by induction hypothesis, for each $p \in \{0, \ldots, 3\}$ there exists a derivation Π'''_p such that $\Pi'_p \xrightarrow{1} \Pi'''_p$, $\Pi''_p \xrightarrow{1} \Pi'''_p$, $C^{\Pi'''_p}_{\Pi'_p} \circ C^{\Pi'_p}_{\Pi_p} = C^{\Pi'''_p}_{\Pi''_p} \circ C^{\Pi''_p}_{\Pi_p}$, and $O^{\Pi'''_p}_{\Pi'_p} \circ O^{\Pi'_p}_{\Pi_p} = O^{\Pi'''_p}_{\Pi''_p} \circ O^{\Pi''_p}_{\Pi_p}$ hold. Let Π''' be the derivation satisfying

$$\cfrac{\cfrac{\Pi'''_0 \quad (\Pi'''_1)}{M} \qquad (\Pi'''_2 \quad \Pi'''_3)}{A} \quad \xrightarrow{ER} \quad \Pi'''$$

Then, by Lemma A (2.4.1.2), the result holds for this Π'''.

Case 4. One of the $LC(\Pi \xrightarrow{1} \Pi')$ and $LC(\Pi \xrightarrow{1} \Pi'')$ is (2) and the other is (3): Similarly to the case 3.

Case 5. $LC(\Pi \xrightarrow{1} \Pi')$ and $LC(\Pi \xrightarrow{1} \Pi'')$ are (4): Suppose Π is of the form $\dfrac{\Pi_0 \quad (\Pi_1 \quad \Pi_2)}{A}$ and suppose Π' and Π'' satisfy that

$$\frac{\Pi'_0 \quad (\Pi'_1 \quad \Pi'_2)}{A} \xrightarrow{SR(C^{\Pi'_0}_{\Pi_0}(T_1))} \Pi'$$

and

$$\frac{\Pi''_0 \quad (\Pi''_1 \quad \Pi''_2)}{A} \xrightarrow{SR(C^{\Pi''_0}_{\Pi_0}(T_2))} \Pi''$$

; where for each $p \in \{0, 1, 2\}$, $\Pi_p \xrightarrow{1} \Pi'_p$ and $\Pi_p \xrightarrow{1} \Pi''_p$ hold, and where T_1 and T_2 are sgt's at $end(\Pi_0)$ in Π_0 satisfying $len(T_1) > 1$, $len(C^{\Pi'_0}_{\Pi_0}(T_1)) > 1$, $len(T_2) > 1$, and $len(C^{\Pi''_0}_{\Pi_0}(T_2)) > 1$. Then, by induction hypothesis, for all $p \in \{0, 1, 2\}$ there exists a derivation Π'''_p such that $\Pi'_p \xrightarrow{1} \Pi'''_p$, $\Pi''_p \xrightarrow{1} \Pi'''_p$, $C^{\Pi'''_p}_{\Pi'_p} \circ C^{\Pi'_p}_{\Pi_p} = C^{\Pi'''_p}_{\Pi''_p} \circ C^{\Pi''_p}_{\Pi_p}$, and $O^{\Pi'''_p}_{\Pi'_p} \circ O^{\Pi'_p}_{\Pi_p} = O^{\Pi'''_p}_{\Pi''_p} \circ O^{\Pi''_p}_{\Pi_p}$ hold. Let T be the sgt at $end(\Pi_0)$ in Π_0 defined by $T = T_1 \cup T_2$, and let T''' be the sgt at $end(\Pi'''_0)$ in Π'''_0 defined by

$$T''' = C^{\Pi'''_0}_{\Pi'_0} \circ C^{\Pi'_0}_{\Pi_0}(T) = C^{\Pi'''_0}_{\Pi''_0} \circ C^{\Pi''_0}_{\Pi_0}(T)$$

Let Θ''' be the derivation of the form $\dfrac{END(\Pi'''_0) \quad (\Pi'''_1 \quad \Pi'''_2)}{A}$, and S the substitution-sequence defined by $S = < \Pi'''_0, T''', \Theta''' >$. Let Π''' be the derivation $\mathcal{P}_S$. Then by Lemma B (2.4.1.3), the result holds for this Π'''.

Case 6. One of the $LC(\Pi \xrightarrow{1} \Pi')$ and $LC(\Pi \xrightarrow{1} \Pi'')$ is (2) and the other is (4): Similarly to the case 5.□

Appendix A

A remark on normalization theorem

Our main subject of Chapter 1 was how to define, *in* the system of classical natural deduction, a simple reduction-procedure which removes maximum formulae in derivations. But to remove maximum formulae in a derivation of our system of classical natural deduction, it is suffice to use Gentzen's cut-elimination theorem for the sequent-calculus **LK** [3]. Notice that the following fact holds.

Fact. *If a cut-free* **LK***-derivation* $\mathcal{P}$ *whose end-sequent* $\mathcal{S}$ *is of the form* $A_1, \ldots, A_m \to B_1, \ldots, B_n$ *is given, then we can construct a derivation* Π *of our system of classical natural deduction satisfying the following conditions (1), (2), and (3).*

(1) $OA(\Pi) = \{A_1, \ldots, A_m, \neg B_1, \ldots, \neg B_{n-1}\}$.

(2) $END(\Pi) = \begin{cases} \bot, & \text{if the succedent of the sequent } \mathcal{S} \text{ is empty,} \\ B_n, & \text{otherwise.} \end{cases}$

(3) Π *is normal.*

Proof. By induction on the length of $\mathcal{P}$. □

By using the fact above, we obtain the normalization theorem of our system of classical natural deduction in the following form:

Theorem (Normalization theorem). *If a derivation* Π *is given, then we can construct a normal derivation* Π' *such that* $OA(\Pi') = OA(\Pi)$ *and* $END(\Pi') = END(\Pi)$.

Proof. Suppose $OA(\Pi) = \{A_1, \ldots, A_n\}$ and $END(\Pi) = B$. First, we can construct a **LK**-derivation $\mathcal{P}$ whose end-sequent is of the form $A_1, \ldots, A_n \to B$; where in the case that $B = \bot$, the sequent $A_1, \ldots, A_n \to B$ stands for the sequent $A_1, \ldots, A_n \rightarrow$. Next, by using the cut-elimination theorem for **LK**, we can construct a cut-free **LK**-derivation $\mathcal{P}'$ whose end-sequent is the same with that of $\mathcal{P}$. Then, by using the previous fact, we can construct a normal derivation Π' such that $OA(\Pi') = OA(\Pi)$ and $END(\Pi') = END(\Pi)$. □

Appendix B

A remark on Peirce's law

The classical natural deduction for which Seldin proved the normalization theorem in [8] is obtained from the natural deduction system of the intuitionistic logic by adding Peirce's law. To that system, our reduction-procedure can be applied. The regularity of Peirce's law is defined similarly with that of $(\bot_c)$ 1.1.2. We represent the contraction for Peirce's law briefly by the following diagram:

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{M \supset A \quad \begin{array}{c}\vdots \\ M\end{array}}{A} \\ \vdots \\ M\end{array}}{M}\, I \qquad (\Theta_1 \quad \Theta_2)}{D}\, K \qquad \text{contracts to} \qquad \dfrac{\dfrac{\begin{array}{c}\dfrac{D \supset A \quad \dfrac{\begin{array}{c}\vdots \\ M\end{array} \quad (\Theta_1 \quad \Theta_2)}{D}}{A} \\ \vdots \\ M\end{array} \qquad (\Theta_1 \quad \Theta_2)}{D}}{D}$$

where $Inf(I)$ is Peirce's law and $Inf(K)$ is an elimination rule. With the appropriate definition of segments, the normalization theorem can be proved similarly to that of Chapter 1.

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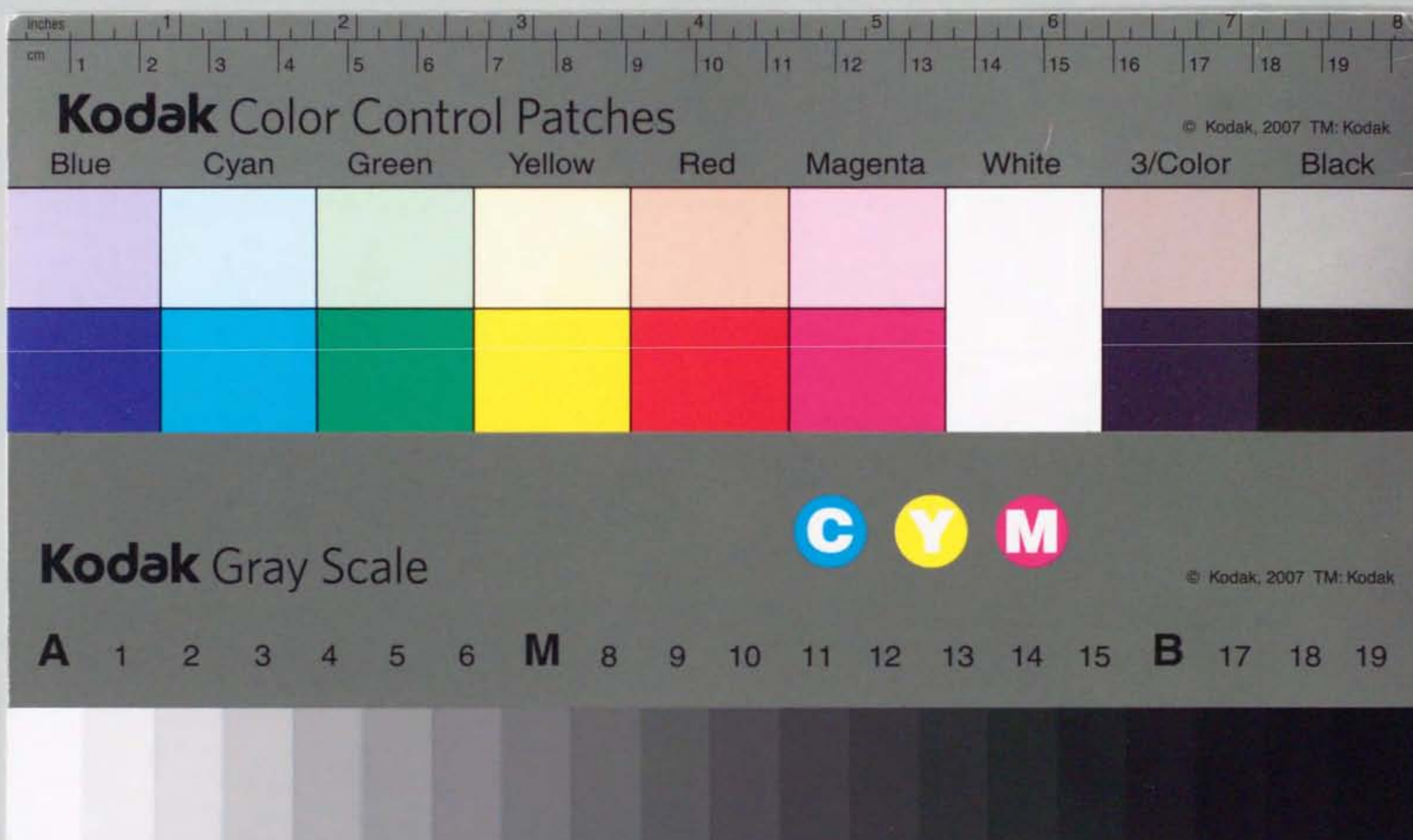

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