Coxeter Groups and Their Boundaries

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Contents

Introduction

The purpose of this paper is to study finitely generated infinite Coxeter groups and their boundaries. A *Coxeter group* is a group W having a presentation

$$\langle\, S \mid (st)^{m(s,t)} = 1 \text{ for } s, t \in S \,\rangle,$$

where S is a finite set and $m : S \times S \to \mathbb{N} \cup \{\infty\}$ is a function satisfying the following conditions:

(1) $m(s,t) = m(t,s)$ for all $s, t \in S$,

(2) $m(s,s) = 1$ for all $s \in S$, and

(3) $m(s,t) \geq 2$ for all $s \neq t \in S$.

The pair (W, S) is called a *Coxeter system*. If $m(s,t) = 2$ or ∞ for all $s \neq t \in S$, then (W, S) is said to be *right-angled*. Let (W, S) be a Coxeter system. For a subset $T \subset S$, W_T is defined as the subgroup of W generated by T, and called a *parabolic subgroup*. It is known that the pair (W_T, T) is also a Coxeter system ([Bo]). If T is the empty set, then W_T is the trivial group.

In recent years, M. Bestvina, M. W. Davis and A. N. Dranishnikov have obtained some interesting results on the boundaries of Coxeter groups. They constructed and studied a certain simplicial complex $L(W, S)$ and a certain CAT(0) space $\Sigma(W, S)$ from a Coxeter system (W, S) instead of a direct algebraic investigation. Let $\mathcal{S}^f(W,S)$ be the family of subsets T of S such that W_T is finite. The simplicial complex $L(W,S)$ is defined by the following conditions:

(1) the vertex set of $L(W,S)$ is S, and

(2) for each nonempty subset T of S, T spans a simplex of $L(W,S)$ if and only if $T \in \mathcal{S}^f(W,S)$.

For each nonempty subset T of S, $L(W_T,T)$ is a subcomplex of $L(W,S)$. For a Coxeter system (W,S), there exists a natural CAT(0) space $\Sigma(W,S)$ on which W acts properly discontinuously and cocompactly as isometries. The definition of $\Sigma(W,S)$ is given in Section 2.2. The CAT(0) space $\Sigma(W,S)$ can be compactified by adding its "ideal boundary" $\partial\Sigma(W,S)$ (cf. Section 1.4). The boundary $\partial\Sigma(W,S)$ is called a *boundary* of the Coxeter group W. A boundary of a Coxeter group is determined by a Coxeter system. It is still unknown whether the following conjecture holds.

Rigidity Conjecture (Dranishnikov [Dr4]**).** Isomorphic Coxeter groups have homeomorphic boundaries.

For each subset $T \subset S$, $\Sigma(W_T,T)$ is a subspace of $\Sigma(W,S)$ and $\partial\Sigma(W_T,T) \subset \partial\Sigma(W,S)$.

It is known that there exist the following isomorphisms:

$$H^*(W;\mathbb{Z}W) \cong H^*_c(\Sigma(W,S)) \cong \check{H}^{*-1}(\partial\Sigma(W,S))$$

([D3]), where H^*_c and $\check{H}^*$ denote the cohomology with compact supports and the reduced Čech cohomology, respectively. Also the following theorem is known.

Theorem 1 (Bestvina and Mess [BM], [B2]**).** *Let (W,S) be a Coxeter system and R a commutative ring with identity. Then there exists the formula*

$$\text{c-dim}_R\,\partial\Sigma(W,S) = \text{vcd}_R\,W - 1,$$

where c-$\dim_R \partial\Sigma(W,S)$ *is the cohomological dimension of* $\partial\Sigma(W,S)$ *over* R *and* $\mathrm{vcd}_R\, W$ *is the virtual cohomological dimension of* W *over* R.

After some preliminaries in Chapters 1 and 2, we study the virtual cohomological dimension of Coxeter groups and the cohomological dimension of their boundaries in Chapter 3. In [D1], M. W. Davis showed the inequality $\mathrm{vcd}_{\mathbb{Z}}\, W \leq \dim L(W,S)$, and M. Bestvina constructed a finite simplicial complex B_R with $\mathrm{vcd}_R\, W = \dim B_R$ in [B1]. Using these results, Dranishnikov gave a formula for the virtual cohomological dimension of Coxeter groups in terms of the cohomologies of subcomplexes of $L(W,S)$, and proved the following theorem as an application of the formula in [Dr3].

Theorem 2 (Dranishnikov [Dr3]**).** *A Coxeter group* W *has the following properties*:

(a) $\mathrm{vcd}_{\mathbb{Q}}\, W \leq \mathrm{vcd}_R\, W \leq \mathrm{vcd}_{\mathbb{Z}}\, W$ *for each principal ideal domain* R.

(b) $\mathrm{vcd}_{\mathbb{Z}_p}\, W = \mathrm{vcd}_{\mathbb{Q}}\, W$ *for all but finite primes* p.

(c) *There exists a prime* p *such that* $\mathrm{vcd}_{\mathbb{Z}_p}\, W = \mathrm{vcd}_{\mathbb{Z}}\, W$.

(d) $\mathrm{vcd}_{\mathbb{Z}}\, W \times W = 2\,\mathrm{vcd}_{\mathbb{Z}}\, W$.

In Section 3.2, we extend this theorem as follows:

Theorem 3 ([HY]**).** *Let* W *be a Coxeter group and* R *a principal ideal domain. Then* W *has the following properties*:

(a) $\mathrm{vcd}_{\mathbb{Q}}\, W \leq \mathrm{vcd}_{R/I}\, W \leq \mathrm{vcd}_R\, W \leq \mathrm{vcd}_{\mathbb{Z}}\, W$ *for each prime ideal* I *in* R.

(b) $\mathrm{vcd}_{R/I}\, W = \mathrm{vcd}_{\mathbb{Q}}\, W$ *for all but finite prime ideals* I *in* R, *if* R *is not a field.*

(c) *There exists a non-trivial prime ideal* I *in* R *such that* $\mathrm{vcd}_{R/I}\, W = \mathrm{vcd}_R\, W$, *if* R *is not a field.*

(d) $\mathrm{vcd}_R\, W \times W = 2\,\mathrm{vcd}_R\, W$.

Using the Dranishnikov formula for $\operatorname{vcd}_R W$, we also prove the following theorem in Section 3.3.

Theorem 4 ([HY]). *Let (W, S) be a right-angled Coxeter system with $\operatorname{vcd}_R W = n$, where R is a principal ideal domain. Then there exists a sequence $T_0 \subset T_1 \subset \cdots \subset T_{n-1} \subset S$ such that $\operatorname{vcd}_R W_{T_i} = i$ for each $i = 0, \ldots, n-1$.*

By Theorem 1, we obtain the following corollary.

Corollary 5 ([HY]). *Let (W, S) be a right-angled Coxeter system and $n = \text{c-dim}_R\, \partial\Sigma(W, S)$, where R is a principal ideal domain. Then there exists a sequence $\partial\Sigma(W_{T_0}, T_0) \subset \partial\Sigma(W_{T_1}, T_1) \subset \cdots \subset \partial\Sigma(W_{T_{n-1}}, T_{n-1})$ of the boundaries of parabolic subgroups of (W, S) such that $\text{c-dim}_R\, \partial\Sigma(W_{T_i}, T_i) = i$ for each $i = 0, 1, \ldots, n-1$.*

In Chapter 4, we study the cohomology of Coxeter groups. By calculating the cohomology $H_c^*(\Sigma(W, S))$, M. W. Davis gave the following formula for the cohomology of Coxeter groups in [D3].

Theorem 6 (Davis [D3]). *For a Coxeter system (W, S), there exist the following isomorphisms:*

$$\begin{aligned} H^*(W; \mathbb{Z}W) &\cong H_c^*(\Sigma(W, S)) \cong \check{H}^{*-1}(\partial\Sigma(W, S)) \\ &\cong \bigoplus_{T \in S^f(W,S)} \left(\mathbb{Z}(W^T) \otimes \tilde{H}^{*-1}(L(W_{S\setminus T}, S \setminus T))\right), \end{aligned}$$

where $\tilde{H}^*$ denotes the reduced cohomology and $\mathbb{Z}(W^T)$ is the free abelian group on W^T.

Here W^T is defined as follows: For each $w \in W$, we first define a subset $S(w)$ of S as $S(w) = \{s \in S \mid \ell(ws) < \ell(w)\}$, where $\ell(w)$ is the minimum length of word in S which represents w. For each subset T of S, we define the subset W^T of W as $W^T = \{w \in W \mid S(w) = T\}$.

For a given Coxeter system (W,S) and each $T \in \mathcal{S}^f(W,S)$, it is difficult to calculate the number of elements of W^T. The purpose of Chapter 4 is to simplify the Davis formula. We first give definitions: A Coxeter system (W,S) is said to be *irreducible* if, for any nonempty and proper subset T of S, W does not decompose as the direct product of W_T and $W_{S\setminus T}$. For a Coxeter system (W,S), there exists a unique decomposition $\{S_1,\dots,S_r\}$ of S such that W is the direct product of the parabolic subgroups $W_{S_1},\dots,W_{S_r}$ and each Coxeter system (W_{S_i},S_i) is irreducible (cf. [Bo], [Hu, p.30]). We define a subset $\tilde{S}$ of S as $\tilde{S} = \bigcup\{S_i \,|\, S_i \notin \mathcal{S}^f(W,S)\}$. In Section 4.3, we show that for a member $T \in \mathcal{S}^f(W,S)$, if $L(W_{S\setminus T}, S\setminus T)$ is not acyclic and W^T is finite, then $T = S\setminus\tilde{S}$ and W^T is a singleton. We also show that if $S\setminus\tilde{S} \not\subset T \in \mathcal{S}^f(W,S)$, then $L(W_{S\setminus T}, S\setminus T)$ is contractible. Using these results, we can reformulate Theorem 6 as follows:

Theorem 7 ([H1]). *For a Coxeter system (W,S), there exists the following isomorphism*:

$$H^*(W;\mathbb{Z}W) \cong \tilde{H}^{*-1}(L(W_{\tilde{S}},\tilde{S})) \oplus \left(\bigoplus_{\substack{T\in\mathcal{S}^f(W,S)\\ S\setminus\tilde{S}\subsetneq T}} \bigoplus_{\mathbb{Z}} \tilde{H}^{*-1}(L(W_{S\setminus T}, S\setminus T)) \right).$$

Theorem 7 implies the following corollary.

Corollary 8 ([H1]). *The following statements are equivalent*:

(i) $H^i(W;\mathbb{Z}W)$ *is finitely generated*;

(ii) $H^i(W;\mathbb{Z}W)$ *is isomorphic to* $\tilde{H}^{i-1}(L(W_{\tilde{S}},\tilde{S}))$;

(iii) $\tilde{H}^{i-1}(L(W_U,U)) = 0$ *for every proper subset U of $\tilde{S}$ such that $W_{S\setminus U}$ is finite.*

In Chapter 5, we study geometrically finite groups acting on Busemann spaces. Every CAT(0) space is a Busemann space. For a Coxeter system (W,S), the

Coxeter group W acting on $\Sigma(W,S)$ is an example of "geometrically finite groups acting on Busemann spaces". In Section 5.2, we introduce a definition of geometrically finite groups acting on hyperbolic spaces (in the sense of Gromov), and we define geometrically finite groups acting on Busemann spaces by analogy in Section 5.4. For a hyperbolic and Busemann space (X,d), a group G acting on X is hyperbolic-geometrically finite if and only if G is Busemann-geometricaly finite. Let (X,d) be a hyperbolic or Busemann space, let ∂X be the hyperbolic or Busemann boundary of X (cf. Sections 5.1 and 5.3), and let Γ be a group which acts properly discontinuously on X. The *limit set of* Γ (*with respect to* X) is defined as $\partial\Gamma = \mathrm{cl}_{X\cup\partial X}(\Gamma x_0) \cap \partial X$, where $\mathrm{cl}_{X\cup\partial X}$ means the closure in $X \cup \partial X$, and x_0 is a point in X. The limit set $\partial\Gamma$ is independent of the choice of the point $x_0 \in X$. For a Coxeter system (W,S), every paraboloc subgroup W_T is geometrically finite with respect to $\Sigma(W,S)$ and the equality $\partial W_T = \partial\Sigma(W_T,T)$ holds for each $T \subset S$. In Section 5.4, we prove the following theorem which is a Busemann space-analogue of results proved by A. Ranjbar-Motlagh in [R] for geometrically finite groups acting on hyperbolic spaces.

Theorem 9 ([H2]). *Let X be a proper Busemann space and Γ a group which acts properly discontinuously on X.*

(i) *Suppose that $H \subset G$ are two subgroups of Γ and H is geometrically finite. Then, $\partial G = \partial H$ if and only if $[G:H] < \infty$.*

(ii) *Let G be a subgroup of finite index in Γ. Then Γ is geometrically finite if and only if G is geometrically finite.*

(iii) *Suppose that G is a subgroup of Γ and $\gamma \in \Gamma$ such that $\gamma G \gamma^{-1} \subset G$. If $H := \bigcap_{i=1}^{\infty} \gamma^i G \gamma^{-i}$ is geometrically finite and if $\partial H = \bigcap_{i=1}^{\infty} \partial(\gamma^i G \gamma^{-i})$, then $\gamma(\partial G) = \partial G$ and $G = \gamma G \gamma^{-1}$.*

(iv) *If G_1 and G_2 are two geometrically finite subgroups of Γ, then $G_1 \cap G_2$ is* *also geometrically finite and* $\partial(G_1 \cap G_2) = \partial G_1 \cap \partial G_2$.

Chapter 1

Preliminaries

In this chapter, we introduce some definitions and some basic results.

§1.1. General definitions and notation

The standard sets and spaces are denoted as follows:

(1) $\mathbb{N}$: the set of natural numbers,

(2) $\mathbb{Z}$: the set of integers,

(3) $\mathbb{Q}$: the set of rational numbers,

(4) $\mathbb{R} = (-\infty, \infty)$: the real line with usual metric,

(5) $\mathbb{R}^n$: the n-dimensional Euclidean space with usual metric.

Let S be a set. Then,

(6) $|S|$: the cardinality of S.

Let (X, d) be a metric space, $A, B \subset X$ and $\epsilon > 0$. We use the following notation:

(7) $\mathrm{cl}_X A$: the closure of A in X,

(8) $\mathrm{int}_X A$: the interior of A in X,

(9) $\operatorname{diam} A$: the diameter of A,

(10) $d(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}$,

(11) $B(A, \epsilon) = \{x \in X \mid d(x, A) < \epsilon\}$: the open ϵ-ball about A (we denote $B(A, \epsilon) = B(a, \epsilon)$ if $A = \{a\}$).

Let K and L be simplicial complexes, and let σ and τ be simplexes of K.

(12) $K * L$: the simplicial join of K and L,

(13) $\operatorname{sd} K$: the barycentric subdivision of K,

(14) $K^{(n)}$: the n-skeleton of K,

(15) $\sigma^{(n)}$: the union of n-faces of σ,

(16) $\sigma \prec \tau$ means that σ is a proper face of τ.

Let F and G be groups and H a subgroup of G.

(17) $F \times G$: the direct product of F and G,

(18) $F * G$: the free product of F and G,

(19) $[G : H]$: the index of H in G.

Let X be a compact metric space and G a group.

(20) $\dim X$: the covering dimension of X,

(21) $\text{c-dim}_G X$: the cohomological dimension of X over G,

(22) $H^i(X; G)$: the i-th cohomology of X over G,

(23) $\tilde{H}^i(X; G)$: the i-th reduced cohomology of X over G,

(24) $\check{H}^i(X; G)$: the i-th reduced Čech cohomology of X over G.

Let X be a noncompact metric space and G a group.

(25) $H^i_c(X; G)$: the i-th cohomology with compact supports of X over G.

§1.2. The cohomology and the cohomological dimension of groups

In this section, we give definitions of the cohomology and the (virtual) cohomological dimension of groups, and we introduce some basic properties. Details are found in [Br1].

Definition 1.2.1 ([Br1, Chapter III §1]). Let Γ be a group, R a commutative ring with identity, and

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P_{-1} = R$$

a projective resolution of R over $R\Gamma$-module. For an $R\Gamma$-module M, the *cohomology of* Γ *over* M is defined as

$$H^*(\Gamma; M) = H^*(\mathrm{Hom}_{R\Gamma}(P_*, M)).$$

Definition 1.2.2 ([Br1, Chapter VIII §2]). Let Γ be a group and R a commutative ring with identity. The *cohomological dimension of* Γ *over* R is defined as

$$\mathrm{cd}_R \Gamma = \sup\{i \mid H^i(\Gamma; M) \neq 0 \text{ for some } R\Gamma\text{-module } M\}.$$

If $R = \mathbb{Z}$ then $\mathrm{cd}_{\mathbb{Z}} \Gamma$ is simply called the cohomological dimension of Γ, and denoted $\mathrm{cd}\, \Gamma$.

Remark. It is obvious that $\mathrm{cd}_R \Gamma \leq \mathrm{cd}\, \Gamma$ for each commutative ring R with identity. It is known that $\mathrm{cd}\, \Gamma = \infty$ if Γ is not torsion-free ([Br1, Corollary VIII.2.5]).

Definition 1.2.3. Let Γ be a torsion-free group. Then Γ is said to be of *type FP* if $\mathbb{Z}$ admits a finitely generated projective resolution of finite length over $\mathbb{Z}\Gamma$. Also Γ is of *type FL* if $\mathbb{Z}$ admits a finitely generated free resolution of finite length over $\mathbb{Z}\Gamma$.

Remark. It is known that Γ is of type FL if there exists an acyclic CW complex on which Γ acts freely, cellularly and cocompactly (cf. [Br1]).

The following results are known.

Proposition 1.2.4 ([Br1, Proposition VIII.6.7]). *If a group Γ is of type FP then for a commutative ring R with identity,*

$$\operatorname{cd}_R \Gamma = \max\{i \mid H^i(\Gamma; R\Gamma) \neq 0\}.$$

Proposition 1.2.5 ([Br1, Proposition VIII.7.5]). *Let Γ be a group and X a contractible, free Γ-complex with compact quotient X/Γ. Then for a commutative ring R with identity, there exists an isomorphism*

$$H^*(\Gamma; R\Gamma) \cong H_c^*(X; R),$$

where $H_c^*(X; R)$ is the cohomology with compact supports of X over R.

We obtain the following corollary from Proposition 1.2.4 and Proposition 1.2.5.

Corollary 1.2.6 ([Br1, Proposition VIII.7.6]). *Let Γ be a group of type FP and X a contractible, free Γ-complex with compact quotient X/Γ. Then for a commutative ring R with identity,*

$$\operatorname{cd}_R \Gamma = \max\{i \mid H_c^i(X; R) \neq 0\}.$$

We give a definition of the virtual cohomological dimension of groups.

Definition 1.2.7 ([Br1, Chapter VIII §11]). A group Γ is said to be *virtually torsion-free* if Γ has a torsion-free subgroup of finite index.

For a virtually torsion-free group Γ, the *virtual cohomological dimension of* Γ *over a commutative ring* R is defined as $\operatorname{cd}_R \Gamma'$, where Γ' is a torsion-free subgroup of Γ of finite index, and denoted $\operatorname{vcd}_R \Gamma$. If $R = \mathbb{Z}$ then $\operatorname{vcd}_{\mathbb{Z}} \Gamma$ is simply called the virtual cohomological dimension of Γ, and denoted $\operatorname{vcd} \Gamma$.

The definition above is well-defined by Serre's Theorem: if G is a torsion-free group and G' is a subgroup of finite index, then $\mathrm{cd}_R G' = \mathrm{cd}_R G$ (cf. [Br1, Theorem VIII.3.1]).

Next we define duality groups and (virtual) Poincaré duality groups.

Definition 1.2.8 ([Br1], [F]). A torsion-free group Γ of type FP is an *n-dimensional duality group*, if $H^i(\Gamma; \mathbb{Z}\Gamma) = 0$ for each $i \neq n$. If, in addition, $H^n(\Gamma; \mathbb{Z}\Gamma)$ is infinite cyclic, then Γ is called an *n-dimensional Poincaré duality group*. We note that the trivial group is a 0-dimensional Poincaré duality group.

A group G is a *virtual Poincaré duality group* if G contains a torsion-free subgroup Γ of finite index such that Γ is a Poincaré duality group.

The following theorem was proved by F. T. Farrell.

Theorem 1.2.9 (Farrell [F, Theorem 3]**).** *Suppose that Γ is a finitely presented group of type FP, and let n be the smallest integer such that $H^n(\Gamma; \mathbb{Z}\Gamma) \neq 0$. If $H^n(\Gamma; \mathbb{Z}\Gamma)$ is a finitely generated abelian group, then Γ is an n-dimensional Poincaré duality group.*

§1.3. The cohomological dimension of compact metric spaces

In this section, we give a definition of the cohomological dimension of compact metric spaces and introduce some basic properties. Details are found in [Dr1] and [K].

Definition 1.3.1. Let X be a compact metric space and G an abelian group. The *cohomological dimension of X over G* is defined as

$$\text{c-dim}_G X = \sup\{i \mid \check{H}^i(X, A; G) \neq 0 \text{ for some closed set } A \subset X\},$$

where $\check{H}^i(X, A; G)$ is the Čech cohomology of (X, A) over G.

The following result is known.

Proposition 1.3.2 (cf. [K, Remark 3]**).** *Let X be a compact metric space and G an abelian group. Then* $\text{c-dim}_G X \leq \text{c-dim}_{\mathbb{Z}} X \leq \dim X$.

The following theorem is proved by P. S. Aleksandrov.

Theorem 1.3.3 (cf. [K, Remark 4]**).** *Let X be a finite dimensional compact metric space. Then the equality* $\dim X = \text{c-dim}_{\mathbb{Z}} X$ *holds.*

§1.4. CAT(0) SPACES AND THEIR BOUNDARIES

In this section, we introduce definitions and some basic properties of CAT(0) spaces and their boundaries. Details of CAT(0) spaces and their boundaries are found in [GH], [BH] and [D2].

We first define geodesic spaces and proper spaces.

Definition 1.4.1. We say that a metric space (X, d) is a *geodesic space* if for each $x, y \in X$, there exists an isometry $\xi : [0, d(x, y)] \to X$ such that $\xi(0) = x$ and $\xi(d(x, y)) = y$ (such ξ is called a *geodesic*). Also a metric space (X, d) is said to be *proper* if every closed metric ball is compact.

Definition 1.4.2. Let (X, d) be a geodesic space. Let T be a geodesic triangle in X. A *comparison triangle* for T is a geodesic triangle T' in the Euclidean plain $\mathbb{R}^2$ with same edge lengths as T. Choose two points x and y in T. Let x' and y' denote the corresponding points in T'. Then the inequality

$$d(x, y) \leq d_0(x', y')$$

is called the *CAT(0)-inequality*, where d_0 is the natural metric on $\mathbb{R}^2$. A geodesic space (X, d) is called a *CAT(0) space* if the CAT(0)-inequality holds for all geodesic triangles T and for all choices of two points x and y in T.

Definition 1.4.3. Let (X, d) be a geodesic space. Two geodesic rays $\xi, \zeta : [0, \infty) \to X$ are said to be *asymptotic* if there exists a constant N such that $d(\xi(t), \zeta(t)) \leq N$ for each $t \geq 0$.

The following proposition is known.

Proposition 1.4.4 (cf. [BH], [GH], [D2]**).** *Let (X, d) be a proper CAT(0) space.*

(1) *For each two points $x, y \in X$, there exists a unique geodesic segment between x and y in X.*

(2) *X is contractible.*

(3) *For each geodesic ray ξ in X and each point $x_0 \in X$, there exists a unique geodesic ray ξ' issuing from x_0 such that ξ and ξ' are asymptotic.*

Let (X, d) be a proper CAT(0) space and $x_0 \in X$. The *boundary of X with respect to x_0*, denoted by $\partial_{x_0} X$, is defined as the set of all geodesic rays issuing from x_0. Then $X \cup \partial_{x_0} X$ has a natural topology, in which X is an open subspace, and a neighborhood basis for each point $\xi \in \partial_{x_0} X$ is given by the sets

$$U(\xi; r, \epsilon) = \{x \in X \cup \partial X \mid x \notin B(x_0, r),\ d(\xi(r), \xi_x(r)) < \epsilon\},$$

where $r, \epsilon > 0$ and $\xi_x : [0, d(x_0, x)] \to X$ is the geodesic from x_0 to x ($\xi_x = x$ if $x \in \partial_{x_0} X$). This is called the *cone topology* on $X \cup \partial_{x_0} X$. It is known that $X \cup \partial_{x_0} X$ is a metrizable compactification of X ([BH], [GH]).

Let x_0 and x_1 be two points of a proper CAT(0) space X. By Proposition 1.4.4 (3), there exists a unique bijection $\Phi : \partial_{x_0} X \to \partial_{x_1} X$ such that ξ and $\Phi(\xi)$ are asymptotic for each $\xi \in \partial_{x_0} X$. It is known that $\Phi : \partial_{x_0} X \to \partial_{x_1} X$ is a homeomorphism ([BH], [GH]).

Let X be a proper CAT(0) space. The asymptotic relation is an equivalence relation in the set of all geodesic rays in X. The *(ideal) boundary of* X, denoted by ∂X, is defined as the set of asymptotic equivalence classes of geodesic rays. The equivalence class of a geodesic ray ξ is denoted by $\xi(\infty)$. By Proposition 1.4.4 (3), for each $x_0 \in X$ and each $\alpha \in \partial X$, there exists a unique element $\xi \in \partial_{x_0} X$ with $\xi(\infty) = \alpha$. Thus we may identify ∂X with $\partial_{x_0} X$ for each $x_0 \in X$.

Let (X, d) be a proper CAT(0) space and Γ a group which acts on X by isometries. For each element $\gamma \in \Gamma$ and each geodesic ray $\xi : [0, \infty) \to X$, a map $\gamma\xi : [0, \infty) \to X$ defined by $(\gamma\xi)(t) := \gamma(\xi(t))$ is also a geodesic ray. If geodesic rays ξ and ξ' are asymptotic, then $\gamma\xi$ and $\gamma\xi'$ are also asymptotic. Thus γ induces a homeomorphism of ∂X and Γ acts on ∂X.

CHAPTER 2

Spaces associated with Coxeter systems

In this chapter, we introduce definitions and some properties of spaces associated with Coxeter systems.

§2.1. COXETER SYSTEMS

In this section, we introduce definitions and some properties of Coxeter groups, Coxeter systems and parabolic subgroups.

We first give definitions of Coxeter groups and Coxeter systems.

Definition 2.1.1. A *Coxeter group* is a group W having a presentation

$$\langle\, S \mid (st)^{m(s,t)} = 1 \text{ for } s,t \in S \,\rangle,$$

where S is a finite set and $m : S \times S \to \mathbb{N} \cup \{\infty\}$ is a function satisfying the following conditions:

(1) $m(s,t) = m(t,s)$ for all $s,t \in S$,

(2) $m(s,s) = 1$ for all $s \in S$, and

(3) $m(s,t) \geq 2$ for all $s \neq t \in S$.

The pair (W, S) is called a *Coxeter system.* If, in addition,

(4) $m(s,t) = 2$ or ∞ for all $s \neq t \in S$,

then (W, S) is said to be *right-angled.*

By the condition (2), we see that s is an involution (i.e., $s^2 = 1$) for each $s \in S$, and $s^{-1} = s$.

Next, we give a definition of parabolic subgroups of Coxeter groups.

Definition 2.1.2. Let (W, S) be a Coxeter system. For a subset $T \subset S$, W_T is defined as the subgroup of W generated by T, and called a *parabolic subgroup.* It is known that the pair (W_T, T) is also a Coxeter system ([B1]). If T is the empty set, then W_T is the trivial group.

Example 2.1.3. Let (W_1, S_1) and (W_2, S_2) be Coxeter systems defined by functions $m_1 : S_1 \times S_1 \to \mathbb{N} \cup \{\infty\}$ and $m_2 : S_2 \times S_2 \to \mathbb{N} \cup \{\infty\}$, respectively. Then the map $m : (S_1 \cup S_2) \times (S_1 \cup S_2) \to \mathbb{N} \cup \{\infty\}$ defined by

$$m(s,t) = \begin{cases} m_1(s,t) & \text{if } s,t \in S_1, \\ m_2(s,t) & \text{if } s,t \in S_2, \\ 2 & \text{otherwise,} \end{cases}$$

induces the Coxeter system $(W_1 \times W_2, S_1 \cup S_2)$. Also the map $m' : (S_1 \cup S_2) \times (S_1 \cup S_2) \to \mathbb{N} \cup \{\infty\}$ defined by

$$m'(s,t) = \begin{cases} m_1(s,t) & \text{if } s,t \in S_1, \\ m_2(s,t) & \text{if } s,t \in S_2, \\ \infty & \text{otherwise,} \end{cases}$$

induces the Coxeter system $(W_1 * W_2, S_1 \cup S_2)$, where $W_1 * W_2$ is the free product of W_1 and W_2.

We recall some properties of Coxeter groups needed later. We first define sets $S(w)$, A_T, C_T and W^T as follows:

Definition 2.1.4. Let (W, S) be a Coxeter system. For each $w \in W$, we define a subset $S(w)$ of S as

$$S(w) := \{s \in S \mid \ell(ws) < \ell(w)\},$$

where $\ell(w)$ is the minimum length of word in S which represents w. For each subset T of S, we define the following subsets of W:

$$A_T := \{w \in W \mid \ell(wt) > \ell(w), \text{ for all } t \in T\} = \{w \in W \mid T \subset S \setminus S(w)\},$$

$$C_T := \{w \in W \mid \ell(wt) < \ell(w), \text{ for all } t \in T\} = \{w \in W \mid T \subset S(w)\}, \text{ and}$$

$$W^T := \{w \in W \mid S(w) = T\} = C_T \cap A_{S \setminus T}.$$

Definition 2.1.5. Let (W, S) be a Coxeter system and $w \in W$. A representation $w = s_1 \cdots s_l$ ($s_i \in S$) is said to be *reduced*, if $\ell(w) = l$.

The following lemma is known.

Lemma 2.1.6 ([Bo], [D3]). *Let (W, S) be a Coxeter system.*

(i) *$S(w)$ is empty if and only if $w = 1$, i.e., $W^\emptyset = \{1\}$.*

(ii) *If a representation $w = s_1 \cdots s_l$ is not reduced, then $w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_l$ for some $i < j$.*

(iii) *For each $w \in W$ and $s \in S$, $\ell(ws)$ equals either $\ell(w) + 1$ or $\ell(w) - 1$, and $\ell(sw)$ also equals either $\ell(w) + 1$ or $\ell(w) - 1$.*

(iv) *For each $T \subset S$ and $w \in W_T$, $\ell_T(w) = \ell(w)$, where $\ell_T(w)$ is the length of w in W_T.*

(v) *Let $T \subset S$, $w \in W_T$ and $s \in S \setminus T$. Then $\ell(ws) = \ell(sw) = \ell(w) + 1$.*

Lemma 2.1.7 ([Bo, p.37, Exercise 3], [D3, Lemma 1.3]). *Let (W, S) be a Coxeter system, $w \in W$ and $T \subset S$. Then there exists a unique element of shortest length in the coset wW_T. Moreover, the following statements are equivalent:*

(i) *w is the element of shortest length in the coset wW_T;*

(ii) $w \in A_T$;

(iii) $\ell(wu) = \ell(w) + \ell(u)$ *for each* $u \in W_T$.

Proof. Let x be an element of wW_T such that x has shortest length in wW_T. We show that $\ell(xu) = \ell(x) + \ell(u)$ for each $u \in W_T$. Let $u \in W_T$ and let $x = s_1 \cdots s_k$ and $u = t_1 \cdots t_l$ be reduced representations. Suppose that $\ell(xu) < \ell(x) + \ell(u)$. By Lemma 2.1.6 (ii), there exist numbers i and j such that

$$xu = (s_1 \cdots s_k)(t_1 \cdots t_l) = (s_1 \cdots \hat{s}_i \cdots s_k)(t_1 \cdots \hat{t}_j \cdots t_l).$$

Let $x' := s_1 \cdots \hat{s}_i \cdots s_k$ and $u' := t_1 \cdots \hat{t}_j \cdots t_l$. Then $\ell(x') < \ell(x)$ and

$$x' = (xu)(u')^{-1} = x(u(u')^{-1}) \in xW_T = wW_T.$$

This contradicts the definition of x. Hence $\ell(xu) = \ell(x) + \ell(u)$ for each $u \in W_T$. This means that x is a unique element of shortest length in $xW_T = wW_T$.

The above argument implies that (i) and (iii) are equivalent.

We show that (iii) implies (ii). Suppose that (iii) holds. Let $t \in T$. Since $t \in W_T$, $\ell(wt) = \ell(w) + \ell(t) = \ell(w) + 1 > \ell(w)$ by (iii). Hence $w \in A_T$.

We show that (ii) implies (i). Let x be the element of shortest length in wW_T and suppose that $w \neq x$. Since $w \in wW_T = xW_T$, $w = xu$ for some $u \in W_T$. We note that $u \neq 1$. Then $\ell(w) = \ell(xu) = \ell(x) + \ell(u)$, because (i) and (iii) are equivalent. Let $x = s_1 \cdots s_k$ and $u = t_1 \cdots t_l$ ($t_i \in T$) be reduced representations. Then $w = xu = (s_1 \cdots s_k)(t_1 \cdots t_l)$ is reduced and $\ell(wt_l) < \ell(w)$, i.e., $w \notin A_T$. $\square$

Lemma 2.1.8 (cf. [D3, Lemma 1.5]**).** *Let* (W, S) *be a Coxeter system,* $w \in W$ *and* $T \subset S(w)$*. Then* $\ell(wu) = \ell(w) - \ell(u)$ *for each* $u \in W_T$.

Proof. Let $w = s_1 \cdots s_k$ and $u = t_1 \cdots t_l$ ($t_i \in T$) be reduced representations. Since $t_l \in T \subset S(w)$, $\ell(wt_l) < \ell(w)$. By Lemma 2.1.6 (ii), there exists a number

i such that

$$wt_l = (s_1 \cdots s_k)t_l = s_1 \cdots \hat{s}_i \cdots s_k.$$

Then $w = (s_1 \cdots \hat{s}_i \cdots s_k)t_l$ is reduced. Suppose that $w = (s'_1 \cdots s'_{k-n})(t_l \cdots t_{l-n+1})$ is a reduced representation for some $s'_1, \ldots, s'_{k-n} \in S$. Since $t_{l-n} \in T \subset S(w)$, $\ell(wt_{l-n}) < \ell(w)$. By Lemma 2.1.6 (ii), there exists a number i' such that

$$wt_{l-n} = (s'_1 \cdots s'_{k-n})(t_l \cdots t_{l-n+1})t_{l-n} = (s'_1 \cdots \hat{s'_{i'}} \cdots s'_{k-n})(t_l \cdots t_{l-n+1}),$$

because $t_l \cdots t_{l-n+1}t_{l-n}$ is reduced. Then $w = (s'_1 \cdots \hat{s'_{i'}} \cdots s'_{k-n})(t_l \cdots t_{l-n+1}t_{l-n})$ is reduced. By induction, there exist $s''_1, \ldots, s''_{k-l} \in S$ such that $w = (s''_1 \cdots s''_{k-l})(t_l \cdots t_1)$ is reduced. Then

$$\begin{aligned} \ell(wu) &= \ell((s''_1 \cdots s''_{k-l})(t_l \cdots t_1)(t_1 \cdots t_1)) \\ &= \ell(s''_1 \cdots s''_{k-l}) = k - l = \ell(w) - \ell(u). \end{aligned}$$

□

Lemma 2.1.9 ([Bo, p.43, Exercise 22], [D3, Lemma 1.4]). *Let (W, S) be a Coxeter system and $T \in \mathcal{S}^f$. Then there exists a unique element w_T of longest length in W_T. Moreover,*

(i) *w_T is an involution, i.e., $(w_T)^2 = 1$.*

(ii) *For each $x \in W_T$, $x = w_T$ if and only if $S(x) = T$.*

(iii) *For each $x \in W_T$, $\ell(w_T x) = \ell(w_T) - \ell(x)$.*

Proof. Let w_T be an element of longest length in W_T. Then $\ell(w_T t) < \ell(w_T)$ for each $t \in T$, i.e., $T \subset S(w_T)$. By Lemma 2.1.8, $\ell(w_T u) = \ell(w_T) - \ell(u)$ for each $u \in W_T$. This implies (iii). For each element x of longest length in W_T, $\ell(w_T x) = \ell(w_T) - \ell(x) = 0$, i.e., $w_T = x$. Hence w_T is a unique element of longest length in W_T.

(i) Since $\ell(w_T w_T) = \ell(w_T) - \ell(w_T) = 0$, $(w_T)^2 = 1$.

(ii) Lemma 2.1.8 implies that if $S(x) = T$ then $x = w_T$. We show that $S(w_T) = T$. By the above argument, $T \subset S(w_T)$. For each $s \in S \setminus T$, by Lemma 2.1.6 (v), $\ell(w_T s) = \ell(w_T) + 1$, i.e., $s \in S \setminus S(w_T)$. Hence $S \setminus T \subset S \setminus S(w_T)$, that is, $S(w_T) \subset T$. $\square$

Lemma 2.1.10 ([D3, Lemma 1.6]). *Let (W, S) be a Coxeter system, $T \in \mathcal{S}^f$ and $w \in W$. Then there exists a unique element of longest length in wW_T. Moreover, the following statements are equivalent:*

(i) *w is the element of longest length in wW_T;*

(ii) *$w = uw_T$ for some $u \in A_T$, where w_T is the element of longest length in W_T;*

(iii) *$T \subset S(w)$.*

§2.2. Spaces associated with Coxeter systems

In this section, we define a certain simplicial complex $L(W, S)$ and a certain CAT(0) cell complex $\Sigma(W, S)$ induced by a Coxeter system (W, S).

We first define $L(W, S)$ as follows:

Definition 2.2.1. Let (W, S) be a Coxeter system and let $\mathcal{S}^f(W, S)$ be the family of subsets T of S such that W_T is finite. We note that the empty set is a member of $\mathcal{S}^f(W, S)$. We define a simplicial complex $L(W, S)$ by the following conditions:

(1) the vertex set of $L(W, S)$ is S, and

(2) for each nonempty subset T of S, T spans a simplex of $L(W, S)$ if and only if $T \in \mathcal{S}^f(W, S)$.

For each nonempty subset T of S, $L(W_T, T)$ is a subcomplex of $L(W, S)$. In this paper, $\mathcal{S}^f(W, S)$, $L(W, S)$ and $L(W_T, T)$ are abbreviated to $\mathcal{S}^f$, L and L_T, respectively.

Example 2.2.2. Let (W_1, S_1) and (W_2, S_2) be Coxeter systems. Then $(W_1 \times W_2, S_1 \cup S_2)$ and $(W_1 * W_2, S_1 \cup S_2)$ are also Coxeter systems (cf. Example 2.1.3), and

$$L(W_1 \times W_2, S_1 \cup S_2) = L(W_1, S_1) * L(W_2, S_2) \quad \text{(simplicial join) and}$$
$$L(W_1 * W_2, S_1 \cup S_2) = L(W_1, S_1) \cup L(W_2, S_2) \quad \text{(disjoint union).}$$

Next we define $\Sigma(W, S)$ as follows:

Definition 2.2.3 ([D2, §8, §9]). Let (W, S) be a Coxeter system such that W is finite. The canonical representation shows that W can be represented as an orthogonal linear reflection group on $\mathbb{R}^n$, where $n = |S|$. The hyperplanes of the reflections divide $\mathbb{R}^n$ into *chambers*, each of which is a simplicial cone (see [Bo, p.85]). Let x be a point in the interior of some chamber such that x is of distance $1/2$ from each supporting hyperplane. Define $\Sigma(W, S)$ to be the convex hull of Wx (the orbit of x). $\Sigma(W, S)$ is called the *Coxeter cell of type* (W, S). Then the 1-skeleton of $\Sigma(W, S)$ is the Cayley graph of W with respect to S with unit edges.

Let (W, S) be a Coxeter system such that W is infinite. A cell complex $\Sigma(W, S)$ is defined as follows. The vertex set of $\Sigma(W, S)$ is W. Take the Coxeter cell of type (W_T, T) for each coset wW_T, with $w \in W$ and $T \in \mathcal{S}^f$. Identify the vertexes of this Coxeter cell with the element of wW_T. Identify two faces of two Coxeter cells if they have the same set of vertexes. This completes the definition of $\Sigma(W, S)$ as a cell complex. The set of cells in $\Sigma(W, S)$ is $\{w\Sigma(W_T, T) \mid w \in W,\ T \in \mathcal{S}^f\}$, where $w\Sigma(W_\emptyset, \emptyset) = w$. The 1-skeleton of $\Sigma(W, S)$ is the Cayley graph of W with respect to S with unit edges. The piecewise Euclidean cell complex $\Sigma(W, S)$ has a natural metric.

In [M], G. Moussong proved the following theorem.

Theorem 2.2.4 (Moussong [M], cf. [D2, Theorem 7.8]). *The piecewise Euclidean cell complex $\Sigma(W, S)$ is a CAT(0) space for every Coxeter system (W, S).*

The geometric realization of a partially ordered set is defined as follows:

Definition 2.2.5 ([D3]). Let P be a partially ordered set. A simplicial complex $\mathrm{geom}(P)$ which is called the *geometric realization* of P is defined as follows:

(1) The vertex set of $\mathrm{geom}(P)$ is P.

(2) For each nonempty subset T of P, T spans a simplex of $\mathrm{geom}(P)$ if and only if T is a finite chain, i.e., $T = \{t_1, \dots, t_n\}$ for some $t_1 < t_2 < \cdots < t_n$.

Let (W, S) be a Coxeter system and let $W\mathcal{S}^f$ be the set of all cosets of the form wW_T, with $w \in W$ and $T \in \mathcal{S}^f$. The sets $\mathcal{S}^f$ and $W\mathcal{S}^f$ are partially ordered by inclusion. Contractible simplicial complexes $K(W, S)$ and $A(W, S)$ are defined as the geometric realizations of the partially ordered sets $\mathcal{S}^f$ and $W\mathcal{S}^f$, respectively ([D3, §3], [D1]). Here $K(W, S)$ is the cone on the barycentric subdivision of $L(W, S)$. The natural embedding $\mathcal{S}^f \to W\mathcal{S}^f$ defined by $T \mapsto W_T$ induces an embedding $K(W, S) \to A(W, S)$. It is known that the barycentric subdivision of $\Sigma(W, S)$ is just equal to $A(W, S)$ ([D2, §9]). Hence there exists the natural embedding $K(W, S) \to \Sigma(W, S)$ which we regard as an inclusion.

For each subset $T \subset S$, $\Sigma(W_T, T)$ is a subcomplex of $\Sigma(W, S)$. In this paper, $\Sigma(W, S)$ and $\Sigma(W_T, T)$ are abbreviated to Σ and Σ_T, respectively.

We note that $\Sigma = WK(W, S)$ and $\Sigma/W \cong K(W, S)$ ([D1], [D3]). For each $w \in W$, $wK(W, S)$ is called a *chamber* of Σ. If W is infinite, then Σ is noncompact. Hence, if W is infinite, Σ can be compactified by adding its ideal boundary $\partial\Sigma$ (cf. Section 1.4). We note that the natural action of W on Σ is properly discontinuous and cocompact ([D1], [D2]).

Example 2.2.6 (cf. [D3], [BH]). Let (W_1, S_1) and (W_2, S_2) be Coxeter systems.

Then

$$\Sigma(W_1 \times W_2, S_1 \cup S_2) = \Sigma(W_1, S_1) \times \Sigma(W_2, S_2) \text{ and}$$

$$\partial\Sigma(W_1 \times W_2, S_1 \cup S_2) = \partial\Sigma(W_1, S_1) * \partial\Sigma(W_2, S_2) \quad \text{(join)}.$$

Every Coxeter group has a torsion-free subgroup of finite index (cf. [D1, Corollary 5.2]). The following is known (cf. Proposition 1.2.5 and [Br1, p.209 Exercise 4]).

Proposition 2.2.7 ([D3]). *Let (W, S) be a Coxeter system and Γ a torsion-free subgroup of finite index in W. Then there exist the following isomorphisms*:

$$H^*(W; RW) \cong H^*(\Gamma; R\Gamma) \cong H_c^*(\Sigma; R) \cong \check{H}^{*-1}(\partial\Sigma; R).$$

Chapter 3

The virtual cohomological dimension of Coxeter groups

In this chapter, we study the virtual cohomological dimension of Coxeter groups. In Section 3.1, we introduce some results of Bestvina, Mess and Dranishnikov about the virtual cohomological dimension of Coxeter groups. In Section 3.2, using a result of Dranishnikov, we give some properties about the virtual cohomological dimension of Coxeter groups over principal ideal domains. In Section 3.3, for a right-angled Coxeter system (W, S) with $\operatorname{vcd}_R W = n$, we construct a sequence $W_{T_0} \subset W_{T_1} \subset \cdots \subset W_{T_{n-1}}$ of parabolic subgroups with $\operatorname{vcd}_R W_{T_i} = i$.

§3.1. Results of Bestvina, Mess and Dranishnikov

We introduce some results of Bestvina, Mess and Dranishnikov. We first introduce definitions of the local cohomological dimension and the global cohomological dimension of simplicial complexes.

Definition 3.1.1 ([Dr3]). For a finite simplicial complex K and an abelian group

G, the *local cohomological dimension of K over G* is defined as

$$\operatorname{lcd}_G K = \max_{\sigma\in K}\{i \mid H^i(\mathrm{St}(\sigma, K), \mathrm{Lk}(\sigma, K); G) \neq 0\},$$

and the *global cohomological dimension of K over G* is

$$\operatorname{cd}_G K = \max\{i \mid \tilde{H}^i(K; G) \neq 0\}.$$

When $\tilde{H}^i(K; G) = 0$ for each i, then we consider $\operatorname{cd}_G K = -1$. We note that $H^i(\mathrm{St}(\sigma, K), \mathrm{Lk}(\sigma, K); G)$ is isomorphic to $\tilde{H}^{i-1}(\mathrm{Lk}(\sigma, K); G)$. Hence,

$$\operatorname{lcd}_G K = \max_{\sigma\in K}\{\operatorname{cd}_G \mathrm{Lk}(\sigma, K) + 1\}.$$

In [Dr3], Dranishnikov showed the following relation of $\operatorname{lcd}_G K$ and $\operatorname{cd}_G K$.

Theorem 3.1.2 (Dranishnikov [Dr3]**).** *For every abelian group G and every finite simplicial complex K, the inequality $\operatorname{lcd}_G K \geq \operatorname{cd}_G K$ holds.*

In [Dr3], Dranishnikov gave the following formula for the virtual cohomological dimension of Coxeter groups.

Theorem 3.1.3 (Dranishnikov [Dr3]**).** *Let (W, S) be a Coxeter system and R a principal ideal domain. Then there exists the formula*

$$\operatorname{vcd}_R W = \operatorname{lcd}_R CL = \max\{\operatorname{lcd}_R L, \operatorname{cd}_R L + 1\},$$

where $L = L(W, S)$ and CL is the simplicial cone of L.

Dranishnikov also proved the following theorem as an application of Theorem 3.1.3.

Theorem 3.1.4 (Dranishnikov [Dr3]**).** *A Coxeter group W has the following properties:*

(a) $\operatorname{vcd}_{\mathbb{Q}} W \leq \operatorname{vcd}_R W$ *for each principal ideal domain R.*

(b) $\operatorname{vcd}_{\mathbb{Z}_p} W = \operatorname{vcd}_{\mathbb{Q}} W$ *for all but finite primes p.*

(c) *There exists a prime p such that* $\operatorname{vcd}_{\mathbb{Z}_p} W = \operatorname{vcd} W$.

(d) $\operatorname{vcd} W \times W = 2 \operatorname{vcd} W$.

In Section 3.2, we extend this theorem to one over principal ideal domain coefficients.

In [BM], M. Bestvina and G. Mess proved the following theorem for hyperbolic groups and therir boundaries.

Theorem 3.1.5 (Bestvina and Mess [BM]**).** *Let Γ be a hyperbolic group and R a commutative ring with identity. Then there exists the formula*

$$\text{c-dim}_R \, \partial\Gamma = \operatorname{vcd}_R \Gamma - 1,$$

where $\partial\Gamma$ is the boundary of Γ.

Definitions of hyperbolic groups and their boundaries are found in [G], [GH] and [CP]. An analogous theorem for Coxeter groups is proved by the same argument (cf. [Dr2]).

Theorem 3.1.6 (Bestvina and Mess [BM], [B2]**).** *Let (W, S) be a Coxeter system and R a commutative ring with identity. Then there exists the formula*

$$\text{c-dim}_R \, \partial\Sigma(W, S) = \operatorname{vcd}_R W - 1.$$

§3.2. The virtual cohomological dimension of Coxeter groups over principal ideal domains

In this section, we extend Theorem 3.1.4 to an analogous theorem over principal ideal domain coefficients by using an argument similar to one in [Dr3]. We first prove the following lemma needed later.

Lemma 3.2.1. *Let R be a principal ideal domain. Let $t \geq 2$ be an integer. Then*

(i) *if the tensor product* $\mathbb{Z}_t \otimes R$ *is trivial, then the tensor product* $\mathbb{Z}_t \otimes R/I$ *and the torsion product* $\mathrm{Tor}(\mathbb{Z}_t, R/I)$ *are trivial for each ideal* I *in* R*, and*

(ii) *if* R *is not a field and the tensor product* $\mathbb{Z}_t \otimes R/I$ *is trivial for every non-trivial prime ideal* I *in* R*, then the tensor product* $\mathbb{Z}_t \otimes R$ *and the torsion product* $\mathrm{Tor}(\mathbb{Z}_t, R)$ *are trivial.*

Proof. Let $r_t \in R$ be the t sum $1_R + \cdots + 1_R$ of 1_R. Define the homomorphism $\varphi : R \to R$ by $\varphi(r) = r_t r$. Then there exists the following exact sequence:

$$0 \longrightarrow \mathrm{Tor}(\mathbb{Z}_t, R) \longrightarrow R \xrightarrow{\varphi} R \longrightarrow \mathbb{Z}_t \otimes R \longrightarrow 0.$$

Hence the kernel of φ is isomorphic to $\mathrm{Tor}(\mathbb{Z}_t, R)$ and the cokernel of φ is isomorphic to $\mathbb{Z}_t \otimes R$.

(i) Suppose that $\mathbb{Z}_t \otimes R$ is trivial. It follows from $0 = \mathbb{Z}_t \otimes R \cong R/r_t R$ and the non-triviality of φ that r_t is a non-zero unit element of R. Since R is a principal ideal domain, φ is a monomorphism. This means that $\mathrm{Tor}(\mathbb{Z}_t, R) = 0$.

Let I be a non-trivial ideal in R. Consider the following exact sequence:

$$\mathrm{Tor}(\mathbb{Z}_t, R) \to \mathrm{Tor}(\mathbb{Z}_t, R/I) \to \mathbb{Z}_t \otimes I \to \mathbb{Z}_t \otimes R \to \mathbb{Z}_t \otimes R/I \to 0,$$

which is induced by the natural short exact sequence $I \hookrightarrow R \to R/I$. Then it is clear that $\mathbb{Z}_t \otimes R/I = 0$. We also see that $\mathrm{Tor}(\mathbb{Z}_t, R/I) \cong \mathbb{Z}_t \otimes I = 0$, since r_t is a unit element of R.

(ii) We note that there exists a non-trivial prime ideal I in R, because R is not a field.

Suppose that $\mathbb{Z}_t \otimes R/I$ is trivial for every non-trivial prime ideal I in R.

First, we show that $r_t \neq 0$ in R. If $r_t = 0$ in R, then for a non-trivial prime ideal I the homomorphism $R/I \to R/I$ defined by $r + I \mapsto r_t r + I$ is trivial. Hence $\mathbb{Z}_t \otimes R/I$ is isomorphic to $R/I \neq 0$. This contradicts the assumption $\mathbb{Z}_t \otimes R/I = 0$. Therefore $r_t \neq 0$.

Then φ is a monomorphism, because R is an integral domain. Hence $\mathrm{Tor}(\mathbb{Z}_t, R)$ is trivial.

Next, we show that r_t is a unit. Suppose that r_t is not a unit. Since R is a principal ideal domain, r_t is presented as $r_t = p_1 \cdots p_k$ by some prime elements $p_1, \ldots, p_k$ of R. Then $I = p_1 R$ is a non-trivial prime ideal in R. The homomorphism $R/I \to R/I$ defined by $r + I \mapsto r_t r + I$ is trivial, because $r_t r + I = p_1(p_2 \cdots p_k r) + I = I$. Hence $\mathbb{Z}_t \otimes R/I$ is isomorphic to $R/I \neq 0$. This contradicts the assumption: $\mathbb{Z}_t \otimes R/I = 0$. Therefore r_t is a unit.

Then φ is an epimorphism. This means that $\mathbb{Z}_t \otimes R$ is trivial. □

Theorem 3.2.2. *Let W be a Coxeter group and R a principal ideal domain. Then W has the following properties:*

(a) $\mathrm{vcd}_{\mathbb{Q}} W \leq \mathrm{vcd}_{R/I} W \leq \mathrm{vcd}_R W \leq \mathrm{vcd}\, W$ *for each prime ideal I in R.*

(b) $\mathrm{vcd}_{R/I} W = \mathrm{vcd}_{\mathbb{Q}} W$ *for all but finite prime ideals I in R, if R is not a field.*

(c) *There exists a non-trivial prime ideal I in R such that* $\mathrm{vcd}_{R/I} W = \mathrm{vcd}_R W$, *if R is not a field.*

(d) $\mathrm{vcd}_R W \times W = 2\,\mathrm{vcd}_R W$.

Proof. Let (W, S) be a Coxeter system, R a principal ideal domain, and $L = L(W, S)$. We note that R/I is a field for every non-trivial prime ideal I in R, and R has the only trivial prime ideal if R is a field.

(a) For any prime ideal I in R, $\mathrm{vcd}_{\mathbb{Q}} W \leq \mathrm{vcd}_{R/I} W$ by Theorem 3.1.4 (a), and $\mathrm{vcd}_R W \leq \mathrm{vcd}\, W$. We show the inequality $\mathrm{vcd}_{R/I} W \leq \mathrm{vcd}_R W$.

If I is trivial, then it is obvious. We suppose that I is a non-trivial prime ideal in R. Let $\mathrm{vcd}_{R/I} W = n$. Then $\mathrm{lcd}_{R/I} CL = n$ by Theorem 3.1.3. Hence there exists a simplex σ of CL such that $\tilde{H}^{n-1}(\mathrm{Lk}(\sigma, CL); R/I) \neq 0$. By the universal coefficient formula, either $\tilde{H}^{n-1}(\mathrm{Lk}(\sigma, CL)) \otimes R/I$ or $\mathrm{Tor}(\tilde{H}^{n}(\mathrm{Lk}(\sigma, CL)), R/I)$

is non-trivial. Since $\tilde{H}^{n-1}(\mathrm{Lk}(\sigma, CL))$ and $\tilde{H}^{n}(\mathrm{Lk}(\sigma, CL))$ are finitely generated abelian groups, $\tilde{H}^{n-1}(\mathrm{Lk}(\sigma, CL)) \otimes R \neq 0$ or $\tilde{H}^{n}(\mathrm{Lk}(\sigma, CL)) \otimes R \neq 0$ by Lemma 3.2.1 (i). By the universal coefficient formula, $\tilde{H}^{n-1}(\mathrm{Lk}(\sigma, CL); R) \neq 0$ or $\tilde{H}^{n}(\mathrm{Lk}(\sigma, CL); R) \neq 0$. In both cases, $\mathrm{vcd}_R W = \mathrm{lcd}_R CL \geq n$ by Theorem 3.1.3.

(b) Let $\mathrm{vcd}_{\mathbb{Q}} W = n$. We define $\mathcal{A}$ as the set of non-trivial prime ideals I in R such that $\tilde{H}^{i}(\mathrm{Lk}(\sigma, CL)) \otimes R/I \neq 0$ for some simplex σ of CL and integer $i \geq n$. We show that $\mathcal{A}$ contains every non-trivial prime ideal I in R with $\mathrm{vcd}_{R/I} W \neq n$.

Suppose that I is a non-trivial prime ideal in R with $\mathrm{vcd}_{R/I} W \neq n$. Then $\mathrm{lcd}_{R/I} CL = \mathrm{vcd}_{R/I} W > n$ by Theorem 3.1.3 and Theorem 3.1.4 (a). Hence there exist a simplex σ of CL and an integer $i \geq n$ such that $\tilde{H}^{i}(\mathrm{Lk}(\sigma, CL); R/I) \neq 0$. By the universal coefficient formula, either $\tilde{H}^{i}(\mathrm{Lk}(\sigma, CL)) \otimes R/I$ or $\mathrm{Tor}(\tilde{H}^{i+1}(\mathrm{Lk}(\sigma, CL)), R/I)$ is non-trivial. Here we note that for a field F and an integer $t \geq 2$, the tensor product $\mathbb{Z}_t \otimes F$ is trivial if and only if the torsion product $\mathrm{Tor}(\mathbb{Z}_t, F)$ is trivial. Therefore $\tilde{H}^{i}(\mathrm{Lk}(\sigma, CL)) \otimes R/I \neq 0$ or $\tilde{H}^{i+1}(\mathrm{Lk}(\sigma, CL)) \otimes R/I \neq 0$ because R/I is a field. In both cases, I is an element of $\mathcal{A}$. Therefore to prove our desired property, it is sufficient to show that $\mathcal{A}$ is finite.

Let T be the set of all torsion coefficients of $\tilde{H}^{i}(\mathrm{Lk}(\sigma, CL))$ for each simplex σ of CL and integer $i \geq n$. Since CL is a finite simplicial complex and $\tilde{H}^{i}(\mathrm{Lk}(\sigma, CL))$ is a finitely generated torsion group for each simplex σ of CL and $i \geq n$, which is by $\mathrm{lcd}_{\mathbb{Q}} CL = \mathrm{vcd}_{\mathbb{Q}} W = n$, we have that T is finite. For each $t \in T$, we define $\mathcal{B}_t$ as the set of non-trivial prime ideals I such that $\mathbb{Z}_t \otimes R/I \neq 0$. Then we note that $\mathcal{A} = \bigcup_{t \in T} \mathcal{B}_t$.

We show that $\mathcal{B}_t$ is finite for each $t \in T$. Let $r_t \in R$ be the t sum $1_R + \cdots + 1_R$ of 1_R. Since R is a principal ideal domain, R is a unique factorization domain. Hence r_t is presented as $r_t = p_1 \cdots p_k$ by some prime elements $p_1, \ldots, p_k$. Let

I be a non-trivial prime ideal in R such that $\mathbb{Z}_t \otimes R/I$ is non-trivial. For the homomorphism $\bar{\varphi} : R/I \to R/I$ defined by $r + I \mapsto r_t r + I$, the cokernel of $\bar{\varphi}$ is isomorphic to $\mathbb{Z}_t \otimes R/I$. Since R/I is a field, $\bar{\varphi}$ is trivial. Hence I is a member of $\{p_1 R, \dots, p_k R\}$, because $p_1, \cdots, p_k$ are prime elements. Therefore the cardinality of $\mathcal{B}_t$ is at most k. Hence $\mathcal{A}$ is finite, because T is finite.

(c) Let $\mathrm{vcd}_R W = n$. Then there exists a simplex σ of CL such that $\tilde{H}^{n-1}(\mathrm{Lk}(\sigma, CL); R) \neq 0$ by Theorem 3.1.3. By the universal coefficient formula, either $\tilde{H}^{n-1}(\mathrm{Lk}(\sigma, CL)) \otimes R$ or $\mathrm{Tor}(\tilde{H}^n(\mathrm{Lk}(\sigma, CL)), R)$ is non-trivial.

First, we show that $\tilde{H}^{n-1}(\mathrm{Lk}(\sigma, CL)) \otimes R$ is non-trivial. To show the fact, we suppose that $\mathrm{Tor}(\tilde{H}^n(\mathrm{Lk}(\sigma, CL)), R)$ is non-trivial. Let the numbers $s_1, \dots, s_l$ be the torsion coefficients of $\tilde{H}^n(\mathrm{Lk}(\sigma, CL))$. Then there exists a number s_j such that $\mathrm{Tor}(\mathbb{Z}_{s_j}, R) \neq 0$. By Lemma 3.2.1 (ii), there exists a non-trivial prime ideal I in R such that $\mathbb{Z}_{s_j} \otimes R/I \neq 0$. Then $\tilde{H}^n(\mathrm{Lk}(\sigma, CL)) \otimes R/I$ is non-trivial. By the universal coefficient formula, $\tilde{H}^n(\mathrm{Lk}(\sigma, CL); R/I)$ is non-trivial. Hence $\mathrm{vcd}_{R/I} W = \mathrm{lcd}_{R/I} CL \geq n + 1$ by Theorem 3.1.3. On the other hand, $\mathrm{vcd}_{R/I} W \leq \mathrm{vcd}_R W = n$ by Theorem 3.2.2 (a). This is a contradiction. Thus $\mathrm{Tor}(\tilde{H}^n(\mathrm{Lk}(\sigma, CL)), R)$ is trivial. Therefore $\tilde{H}^{n-1}(\mathrm{Lk}(\sigma, CL)) \otimes R$ must be non-trivial.

Next, we show that $\tilde{H}^{n-1}(\mathrm{Lk}(\sigma, CL)) \otimes R/I$ is non-trivial for some non-trivial prime ideal I in R. Let β be the Betti number and the numbers $t_1, \dots, t_k$ the torsion coefficients of $\tilde{H}^{n-1}(\mathrm{Lk}(\sigma, CL))$. If β is non-zero, then it is clear that $\tilde{H}^{n-1}(\mathrm{Lk}(\sigma, CL)) \otimes R/I$ is non-trivial for each non-trivial prime ideals I in R. If β is zero, then there exists a number t_i such that $\mathbb{Z}_{t_i} \otimes R \neq 0$. By Lemma 3.2.1 (ii), there exists a non-trivial prime ideal I in R such that $\mathbb{Z}_{t_i} \otimes R/I \neq 0$. Then $\tilde{H}^{n-1}(\mathrm{Lk}(\sigma, CL)) \otimes R/I$ is non-trivial.

By the universal coefficient formula and Theorem 3.1.3, we have that

$\operatorname{vcd}_{R/I} W \geq n$. Hence, $\operatorname{vcd}_{R/I} W = n$ by Theorem 3.2.2 (a).

(d) In general, for groups W_1, W_2 the inequality $\operatorname{vcd}_R W_1 \times W_2 \leq \operatorname{vcd}_R W_1 + \operatorname{vcd}_R W_2$ holds, where the equality holds, if R is a field ([Bi, Theorem 4 c)]). Hence, in our case, the equality $\operatorname{vcd}_R W \times W = 2 \operatorname{vcd}_R W$ holds, if R is a field. We suppose that R is not a field. Then the inequality $\operatorname{vcd}_R W \times W \leq 2 \operatorname{vcd}_R W$ holds. We show that $\operatorname{vcd}_R W \times W \geq 2 \operatorname{vcd}_R W$. By Theorem 3.2.2 (c), there exists a non-trivial prime ideal I in R such that $\operatorname{vcd}_{R/I} W = \operatorname{vcd}_R W$. We note that R/I is a field. Then $2 \operatorname{vcd}_R W = 2 \operatorname{vcd}_{R/I} W = \operatorname{vcd}_{R/I} W \times W$. Since $W \times W$ is also a Coxeter group, $\operatorname{vcd}_{R/I} W \times W \leq \operatorname{vcd}_R W \times W$ by Theorem 3.2.2 (a). Therefore we have that $\operatorname{vcd}_R W \times W = 2 \operatorname{vcd}_R W$. $\square$

§3.3. A sequence of parabolic subgroups of a right-angled Coxeter system

In this section, we prove the following theorem.

Theorem 3.3.1. *Let (W, S) be a right-angled Coxeter system with $\operatorname{vcd}_R W = n$, where R is a principal ideal domain. Then there exists a sequence $T_0 \subset T_1 \subset \cdots \subset T_{n-1} \subset S$ such that $\operatorname{vcd}_R W_{T_i} = i$ for each $i = 0, \ldots, n-1$. In particular, we can obtain a sequence of simplexes $\tau_0 \succ \tau_1 \succ \cdots \succ \tau_{n-1}$ such that T_i is the vertex set of $\operatorname{Lk}(\tau_i, L(W,S))$ and $L(W_{T_i}, T_i) = \operatorname{Lk}(\tau_i, L(W,S))$.*

We note that Theorem 3.3.1 is not always true for general Coxeter groups. Indeed, there exists the following counter-example.

Example 3.3.2. We consider the Coxeter system (W, S) defined by $S =$

$\{v_1, v_2, v_3\}$ and

$$m(v_i, v_j) = \begin{cases} 1 & \text{if } i = j, \\ 3 & \text{if } i \neq j. \end{cases}$$

Then W is not right-angled, and $L(W, S)$ is not a flag complex. Indeed, $W_{\{v_i, v_j\}}$ is finite for each $i, j \in \{1, 2, 3\}$, but W is infinite (cf. [Bo, p.98, Proposition 8]). Since $\operatorname{cd} L(W, S) = 1$ and $\operatorname{lcd} L(W, S) = 1$, we have that $\operatorname{vcd} W = 2$ by Theorem 3.1.3. For each proper subset $T \subset S$, $\operatorname{vcd} W_T = 0$, because W_T is a finite group. Hence there does not exist a subset $T \subset S$ such that $\operatorname{vcd} W_T = 1$.

We first show some lemmas.

Lemma 3.3.3. *Let L be a simplicial complex. If τ is a simplex of L and τ' is a simplex in the link $\operatorname{Lk}(\tau, L)$, then the join $\tau * \tau'$ is a simplex of L and $\operatorname{Lk}(\tau', \operatorname{Lk}(\tau, L)) = \operatorname{Lk}(\tau * \tau', L)$.*

Proof. Let τ be a simplex of L and τ' in $\operatorname{Lk}(\tau, L)$. Since τ' is in $\operatorname{Lk}(\tau, L)$, the join $\tau * \tau'$ is a simplex of L and $\tau \cap \tau' = \emptyset$. For a simplex σ of L, σ is in $\operatorname{Lk}(\tau', \operatorname{Lk}(\tau, L))$ if and only if $\sigma * \tau'$ is in $\operatorname{Lk}(\tau, L)$ and $\sigma \cap \tau' = \emptyset$, i.e., $\sigma * \tau' * \tau$ is a simplex of L and $\sigma \cap (\tau * \tau') = \emptyset$. Hence σ is in $\operatorname{Lk}(\tau', \operatorname{Lk}(\tau, L))$ if and only if σ is in $\operatorname{Lk}(\tau * \tau', L)$. Thus we have that $\operatorname{Lk}(\tau', \operatorname{Lk}(\tau, L)) = \operatorname{Lk}(\tau * \tau', L)$. $\square$

Lemma 3.3.4. *Let L be a simplicial complex and G an abelian group. For each simplex τ of L, the inequality $\operatorname{lcd}_G \operatorname{Lk}(\tau, L) \leq \operatorname{lcd}_G L$ holds.*

Proof. Let $\operatorname{lcd}_G \operatorname{Lk}(\tau, L) = n$. Then there exists a simplex τ' in $\operatorname{Lk}(\tau, L)$ such that $\tilde{H}^{n-1}(\operatorname{Lk}(\tau', \operatorname{Lk}(\tau, L)); G) \neq 0$. By Lemma 3.3.3, $\operatorname{Lk}(\tau', \operatorname{Lk}(\tau, L)) = \operatorname{Lk}(\tau * \tau', L)$. Hence $\operatorname{lcd}_G L \geq n$. $\square$

Using Theorem 3.1.2 and the lemmas above, we show the following key lemma.

Lemma 3.3.5. *Let (W, S) be a right-angled Coxeter system with $\operatorname{vcd}_R W = n$, where S is nonempty and R is a principal ideal domain. Then there exists a* *proper subset T of S such that $\mathrm{vcd}_R W_T = n$ or $n-1$. In particular, we can obtain a simplex σ of $L(W,S)$ such that T is the vertex set of $\mathrm{Lk}(\sigma, L(W,S))$ and $L(W_T,T) = \mathrm{Lk}(\sigma, L(W,S))$.*

Proof. Since $\mathrm{vcd}_R W = n$, we have that $\mathrm{lcd}_R L(W,S) = n$ or $\mathrm{cd}_R L(W,S) = n-1$ by Theorem 3.1.3. If $\mathrm{lcd}_R L(W,S) \leq n-1$, then $\mathrm{cd}_R L(W,S) = n-1$, and $\mathrm{lcd}_R L(W,S) = n-1$ by Theorem 3.4. Hence $\mathrm{lcd}_R L(W,S) = n$ or $n-1$.

We set $m := \mathrm{lcd}_R L(W,S)$. Then there exists a simplex σ of $L(W,S)$ such that $\tilde{H}^{m-1}(\mathrm{Lk}(\sigma, L(W,S)); R) \neq 0$ and $\tilde{H}^i(\mathrm{Lk}(\sigma, L(W,S)); R) = 0$ for each $i \geq m$. Hence $\mathrm{cd}_R \mathrm{Lk}(\sigma, L(W,S)) = m-1$. Let T be the vertex set of $\mathrm{Lk}(\sigma, L(W,S))$. We note that T is a proper subset of S.

Then we show that

$$(*) \qquad L(W_T,T) = \mathrm{Lk}(\sigma, L(W,S)).$$

It is clear that the vertex set of $L(W_T,T)$ is the vertex set of $\mathrm{Lk}(\sigma, L(W,S))$. Let $\{v_0, \dots, v_k\}$ be a subset of T which spans a simplex of $L(W_T,T)$. Since $\{v_0, \dots, v_k\}$ generates a finite subgroup of $W_T \subset W$, $\{v_0, \dots, v_k\}$ spans a simplex of $L(W,S)$. It follows from $v_i \in T = \mathrm{Lk}(\sigma, L(W,S))^{(0)}$ that the join $v_i * \sigma$ forms a simplex of $L(W,S)$ and $v_i \notin \sigma$ for each $i = 0, \dots, k$. We note that $L(W,S)$ is a flag complex, since W is right-angled. Hence the join $|v_0, \dots, v_k| * \sigma$ forms a simplex of $L(W,S)$ and $|v_0, \dots, v_k| \cap \sigma = \emptyset$, i.e., $|v_0, \dots, v_k|$ is a simplex in $\mathrm{Lk}(\sigma, L(W,S))$. Conversely, let $\{v_0, \dots, v_k\}$ be a subset of T which spans a simplex in $\mathrm{Lk}(\sigma, L(W,S))$. Then $\{v_0, \dots, v_k\}$ generates a finite subgroup of W. Since $\{v_0, \dots, v_k\} \subset T$, $\{v_0, \dots, v_k\}$ generates a finite subgroup of W_T. Hence $\{v_0, \dots, v_k\}$ spans a simplex of $L(W_T,T)$. Thus $L(W_T,T) = \mathrm{Lk}(\sigma, L(W,S))$.

We note that $\mathrm{cd}_R L(W_T,T) = m-1$, and $\mathrm{lcd}_R L(W_T,T) \leq m$ by $(*)$ and Lemma 3.3.4. Hence $\mathrm{vcd}_R W_T = m$ by Theorem 3.1.3. Thus we have that

$\operatorname{vcd}_R W_T = n$ or $n-1$. $\square$

Using this lemma, we prove Theorem 3.3.1.

Proof of Theorem 3.3.1. Let (W, S) be a right-angled Coxeter system with $\operatorname{vcd}_R W = n$, where R is a principal ideal domain.

By Lemma 3.3.5, we can obtain subsets $\{S_i\}_i$ of S and simplexes $\{\sigma_i\}_i$ of $L(W, S)$ satisfying the following conditions:

(1) $S_0 = S$,

(2) S_{i+1} is a proper subset of S_i,

(3) $L(W_{S_{i+1}}, S_{i+1}) = \mathrm{Lk}(\sigma_{i+1}, L(W_{S_i}, S_i))$, and

(4) $\operatorname{vcd}_R W_{S_{i+1}} = \operatorname{vcd}_R W_{S_i}$ or $\operatorname{vcd}_R W_{S_i} - 1$.

Then we note, by the conditions (1), (3) and Lemma 3.3.3, that

$$\begin{aligned}
L(W_{S_i}, S_i) &= \mathrm{Lk}(\sigma_i, L(W_{S_{i-1}}, S_{i-1})) \\
&= \mathrm{Lk}(\sigma_i, \mathrm{Lk}(\sigma_{i-1}, L(W_{S_{i-2}}, S_{i-2}))) \\
&= \mathrm{Lk}(\sigma_{i-1} * \sigma_i, L(W_{S_{i-2}}, S_{i-2})) \\
&= \cdots \\
&= \mathrm{Lk}(\sigma_1 * \cdots * \sigma_i, L(W_{S_0}, S_0)) \\
&= \mathrm{Lk}(\sigma_1 * \cdots * \sigma_i, L(W, S)).
\end{aligned}$$

Since S is finite, there exists a number m such that S_m is the empty set by the condition (2). Then $\operatorname{vcd}_R W_{S_m} = 0$, because W_{S_m} is the trivial group. Hence we can have a subsequence $\{S_{i_j}\}_j$ of $\{S_i\}_i$ such that $\operatorname{vcd}_R W_{S_{i_j}} = n - j$ for each $j = 1, \ldots, n$ by the condition (4).

We set $T_j := S_{i_{n-j}}$ and $\tau_j := \sigma_1 * \cdots * \sigma_{i_{n-j}}$ for each $j = 0, \ldots, n-1$. Then $T_j \subset T_{j+1}$, $\tau_j \succ \tau_{j+1}$, $\operatorname{vcd}_R W_{T_j} = j$ and $L(W_{T_j}, T_j) = \mathrm{Lk}(\tau_j, L(W, S))$ for each j by our construction. $\square$

By Theorem 3.1.6, we obtain the following corollary.

Corollary 3.3.6. *For a right-angled Coxeter system (W, S) with* $\text{c-dim}_R\,\partial\Sigma = n$, *where R is a principal ideal domain, there exists a sequence $\partial\Sigma_{T_0} \subset \partial\Sigma_{T_1} \subset \cdots \subset \partial\Sigma_{T_{n-1}}$ of the boundaries of parabolic subgroups of (W, S) such that* $\text{c-dim}_R\,\partial\Sigma_{T_i} = i$ *for each $i = 0, 1, \ldots, n-1$.*

In general, for a finite dimensional compact metric spaces X, the equality $\text{c-dim}_{\mathbb{Z}}\,X = \dim X$ holds (Theorem 1.3.3). Since the boundaries of Coxeter groups are always finite dimensional, we obtain the following corollary.

Corollary 3.3.7. *For a right-angled Coxeter system (W, S) with $\dim \partial\Sigma = n$, there exists a sequence $\partial\Sigma_{T_0} \subset \partial\Sigma_{T_1} \subset \cdots \subset \partial\Sigma_{T_{n-1}}$ of the boundaries of parabolic subgroups of (W, S) such that $\dim \partial\Sigma_{T_i} = i$ for each $i = 0, 1, \ldots, n-1$.*

Chapter 4

The cohomology of Coxeter groups

In this chapter, we study the cohomology of Coxeter groups. In Section 4.1, we introduce a formula of the cohomology of Coxeter groups given by M. W. Davis. After some preliminaries in Section 4.2, we reformulate the Davis formula, and we study the problem as to when the i-th cohomology of a Coxeter group is finitely generated in Section 4.3,

§4.1. The Davis formula for the cohomology of Coxeter groups

Let (W, S) be a Coxeter system. Let K be the simplicial cone over the barycentric subdivision $\operatorname{sd} L$ of $L = L(W, S)$. For each $s \in S$, the closed star of s in $\operatorname{sd} L$ is denoted by K_s. The closed star K_s is a subcomplex of K. For each nonempty subset T of S, we set

$$K^T := \bigcup_{s \in T} K_s.$$

We note that K^T has the same homotopy type as $L_T = L(W_T, T)$.

For each $w \in W$, the set $S(w)$ is defined in Section 2.1 as follows:

$$S(w) := \{s \in S \mid \ell(ws) < \ell(w)\},$$

where $\ell(w)$ is the minimum length of word in S which represents w. For each subset T of S, we recall the following subsets of W:

$$A_T := \{w \in W \mid \ell(wt) > \ell(w), \text{ for all } t \in T\} = \{w \in W \mid T \subset S \setminus S(w)\},$$

$$C_T := \{w \in W \mid \ell(wt) < \ell(w), \text{ for all } t \in T\} = \{w \in W \mid T \subset S(w)\}, \text{ and}$$

$$W^T := \{w \in W \mid S(w) = T\} = C_T \cap A_{S \setminus T}.$$

In [D3], M. W. Davis gave the following formula.

Theorem 4.1.1 (Davis [D3]). *Let (W, S) be a Coxeter system. For a torsion-free subgroup Γ of finite index in W, there exists the following isomorphism:*

$$H^*(\Gamma; \mathbb{Z}\Gamma) \cong \bigoplus_{T \in \mathcal{S}^f} \left(\mathbb{Z}(W^T) \otimes H^*(K, K^{S \setminus T})\right),$$

where $\mathbb{Z}(W^T)$ is the free abelian group on W^T.

Since K is contractible and $K^{S\setminus T}$ has the same homotopy type as $L_{S\setminus T}$, the formula above is rewritten as

$$H^*(\Gamma; \mathbb{Z}\Gamma) \cong \bigoplus_{T \in \mathcal{S}^f} \left(\mathbb{Z}(W^T) \otimes \tilde{H}^{*-1}(L_{S\setminus T})\right),$$

where $\tilde{H}^*$ denotes the reduced cohomology.

§4.2. Lemmas on Coxeter groups

In this section, we prove some lemmas for Coxeter groups which are used later. Let (W, S) be a Coxeter system and $\mathcal{S}^f$, $S(w)$, A_T, C_T and W^T denote the sets defined in Section 2.1.

Lemma 4.2.1 (cf. [D3, Lemma 1.10]**).** *Suppose that $T \in \mathcal{S}^f$. Then W^T is a singleton if and only if W decomposes as the direct product: $W = W_{S\setminus T} \times W_T$.*

Proof. The "only if" part was proved by Davis ([D3, Lemma 1.10]). We prove the "if" part.

Suppose that W decomposes as the direct product: $W = W_{S\setminus T} \times W_T$. Let w_T be the element of longest length in W_T. By Lemma 2.1.9 (ii), $S(w_T) = T$, i.e., $w_T \in W^T$. We show that $W^T = \{w_T\}$.

Let $w \in W^T$. Then $S(w) = T$ by definition. Applying Lemma 2.1.10, we see that $w = uw_T$ for some $u \in A_T$. Suppose that $u \neq 1$. Then there exists $s \in S(u)$ by Lemma 2.1.6 (i). Since $u \in A_T$, $\ell(ut) > \ell(u)$ for each $t \in T$. On the other hand, $\ell(us) < \ell(u)$, since $s \in S(u)$. Thus $s \in S \setminus T \subset W_{S\setminus T}$. Since $W = W_{S\setminus T} \times W_T$, $w_T s = s w_T$. Hence,

$$\ell(ws) = \ell(uw_Ts) = \ell(usw_T) \leq \ell(us) + \ell(w_T) < \ell(u) + \ell(w_T) = \ell(uw_T) = \ell(w),$$

where the equality $\ell(u) + \ell(w_T) = \ell(uw_T)$ follows from Lemma 2.1.7 (iii). Thus we have that $s \in S(w)$. Hence $s \in S(w) = T$, since $w \in W^T$. This contradicts the fact $s \in S \setminus T$. Thus $u = 1$ and $w = w_T$. Therefore $W^T = \{w_T\}$, i.e., W^T is a singleton. $\square$

Lemma 4.2.2. *Suppose that $T \in \mathcal{S}^f$. Then,*

(i) $C_T = A_T w_T$.

(ii) $W^T = C_T \cap A_{S\setminus T}$.

(iii) *If W is infinite, then C_T is infinite.*

(iv) $(W_{S'})^T \subset W^T$ *for each subset S' of S containing T.*

Proof. (i) By the definition of C_T and Lemma 2.1.10, we have

$$C_T = \{w \in W \mid T \subset S(w)\} = A_T w_T.$$

(ii) By the definition of $A_{S\setminus T}$,

$$\begin{aligned} A_{S\setminus T} &= \{w \in W \mid S \setminus T \subset S \setminus S(w)\} \\ &= \{w \in W \mid S(w) \subset T\}. \end{aligned}$$

Hence,

$$\begin{aligned} W^T &= \{w \in W \mid S(w) = T\} \\ &= \{w \in W \mid T \subset S(w)\} \cap \{w \in W \mid S(w) \subset T\} \\ &= C_T \cap A_{S\setminus T}. \end{aligned}$$

(iii) We note that C_T is not empty because $w_T \in C_T$. Suppose that W is infinite. Then for each $w \in C_T$ there exists $s \in S$ such that $\ell(sw) = \ell(w) + 1$. Since $w \in C_T$, $\ell(swt) \leq 1 + \ell(wt) < 1 + \ell(w) = \ell(sw)$ for each $t \in T$. Hence $sw \in C_T$. Thus there is no element of longest length in C_T. Hence C_T is infinite.

(iv) Let $w \in (W_{S'})^T$. Then $T = S'(w) = \{s \in S' \mid \ell_{S'}(ws) < \ell_{S'}(w)\}$, where the length $\ell_{S'}(w)$ of w in $W_{S'}$ is equal to $\ell(w)$ by Lemma 2.1.6 (iii). Hence,

$$T = S'(w) = \{s \in S' \mid \ell(ws) < \ell(w)\} = S(w) \cap S'.$$

Thus we have that $T \subset S(w)$. To prove the reverse inclusion $S(w) \subset T$, we show that $S \setminus T \subset S \setminus S(w)$. Let $s \in S \setminus T$. If $s \in S'$, then $\ell(ws) > \ell(w)$ because $s \notin T = S'(w)$. If $s \notin S'$, then $\ell(ws) > \ell(w)$ by Lemma 2.1.6 (iv) because $w \in W_{S'}$ and $s \in S \setminus S'$. In either case, $\ell(ws) > \ell(w)$, i.e., $s \in S \setminus S(w)$. Hence $S \setminus T \subset S \setminus S(w)$, that is, $S(w) \subset T$. Thus we have that $S(w) = T$ and $w \in W^T$. Therefore $(W_{S'})^T \subset W^T$. $\square$

Lemma 4.2.3. *Let $t_0 \in S$. If $W^{\{t_0\}}$ and $W_{S\setminus\{t_0\}}$ are finite, then W is finite.*

Proof. By Lemma 2.1.6 (i), we have

$$\begin{aligned} W^{\{t_0\}} &= \{w \in W \mid S(w) = \{t_0\}\} \\ &= \{w \in W \mid S(w) \subset \{t_0\}\} \setminus \{w \in W \mid S(w) = \emptyset\} \\ &= A_{S\setminus\{t_0\}} \setminus \{1\}. \end{aligned}$$

Since $W^{\{t_0\}}$ is finite, $A_{S\setminus\{t_0\}}$ is finite. Hence $C_{S\setminus\{t_0\}}$ is finite because $C_{S\setminus\{t_0\}} = A_{S\setminus\{t_0\}} w_{S\setminus\{t_0\}}$ by $S \setminus \{t_0\} \in \mathcal{S}^f$ and Lemma 4.2.2 (i). Thus W is finite by Lemma 4.2.2 (iii). $\square$

Lemma 4.2.4. *Let $T \in \mathcal{S}^f$, $t_0 \in T$ and $S_0 := (S \setminus T) \cup \{t_0\}$. Then $(W_{S_0})^{\{t_0\}} t_0 w_T \subset W^T$.*

Proof. It is sufficient to show that $S(wt_0w_T) = T$ for each $w \in (W_{S_0})^{\{t_0\}}$ by the definition of W^T. Let $w \in (W_{S_0})^{\{t_0\}}$.

First we show that $T \subset S(wt_0w_T)$. We note that $T \subset S(wt_0w_T)$ if and only if $wt_0 \in A_T$ (i.e., $\ell((wt_0)t) > \ell(wt_0)$ for all $t \in T$) by Lemma 2.1.10. Let $t \in T$. If $t \neq t_0$, then $\ell((wt_0)t) = \ell(wt_0) + 1$ by Lemma 2.1.6 (iv) because $wt_0 \in W_{S_0}$ and $t \notin S_0$. If $t = t_0$, then $\ell((wt_0)t) = \ell(w) = (\ell(w) - 1) + 1 = \ell(wt_0) + 1$ because $w \in (W_{S_0})^{\{t_0\}}$. Hence $\ell((wt_0)t) > \ell(wt_0)$ for all $t \in T$, i.e., $wt_0 \in A_T$. Thus we have $T \subset S(wt_0w_T)$ by Lemma 2.1.10.

Next we show that $S(wt_0w_T) = T$. Suppose that there exists $s \in S(wt_0w_T) \setminus T$. Then $T \cup \{s\} \subset S(wt_0w_T)$. There exists $u \in A_{T\cup\{s\}}$ such that $wt_0w_T = uw_{T\cup\{s\}}$ by Lemma 2.1.10. Since t_0 and w_T are involutions, $w = uw_{T\cup\{s\}}w_Tt_0$. Here we note that $u \in A_{T\cup\{s\}}$ and $w_{T\cup\{s\}}w_T t_0 \in W_{T\cup\{s\}}$. Then,

$$\begin{aligned}
\ell(w) &= \ell(u(w_{T\cup\{s\}}w_T t_0)) \\
&= \ell(u) + \ell(w_{T\cup\{s\}}w_T t_0) && \text{by Lemma 2.1.7} \\
&= \ell(u) + \ell(w_{T\cup\{s\}}) - \ell(w_T t_0) && \text{by Lemma 2.1.9 (iii)} \\
&= \ell(u) + \ell(w_{T\cup\{s\}}) - \ell(w_T) + 1 && \text{by Lemma 2.1.9 (iii),}
\end{aligned}$$

because $w_T t_0 \in W_{T\cup\{s\}}$. By the same argument, we have

$$\begin{aligned}
\ell(ws) &= \ell(u(w_{T\cup\{s\}}w_T t_0 s)) \\
&= \ell(u) + \ell(w_{T\cup\{s\}}w_T t_0 s) && \text{by Lemma 2.1.7} \\
&= \ell(u) + \ell(w_{T\cup\{s\}}) - \ell(w_T t_0 s) && \text{by Lemma 2.1.9 (iii)} \\
&= \ell(u) + \ell(w_{T\cup\{s\}}) - (\ell(w_T t_0) + 1) && \text{by Lemma 2.1.6 (iv)} \\
&= \ell(u) + \ell(w_{T\cup\{s\}}) - (\ell(w_T) - 1 + 1) && \text{by Lemma 2.1.9 (iii)} \\
&= \ell(u) + \ell(w_{T\cup\{s\}}) - \ell(w_T),
\end{aligned}$$

because $u \in A_{T\cup\{s\}}$, $w_{T\cup\{s\}}w_T t_0 s \in W_{T\cup\{s\}}$ and $w_T t_0 s \in W_{T\cup\{s\}}$. Hence $\ell(ws) < \ell(w)$, i.e., $s \in S(w)$. Since $s \in S \setminus T \subset S_0$ and $w \in W_{S_0}$, we have that $s \in S_0(w)$. On the other hand, $S_0(w) = \{t_0\}$, since $w \in (W_{S_0})^{\{t_0\}}$. This is a contradiction, because $s \neq t_0$ by the definition of s. Hence $S(wt_0 w_T) = T$, that is, $wt_0 w_T \in W^T$. Therefore we have that $(W_{S_0})^{\{t_0\}} t_0 w_T \subset W^T$. $\square$

It is known that a Coxeter group W always has a torsion-free subgroup Γ of finite index ([D1, Corollary 5.2]). Such a torsion-free subgroup Γ is of type FL. Indeed, Γ acts freely, simplicially and cocompactly on the contractible simplicial complex Σ ([D1]). Since $H^*(W; \mathbb{Z}W) \cong H^*(\Gamma; \mathbb{Z}\Gamma) \cong \check{H}^{*-1}(\partial\Sigma)$, Theorem 1.2.9 implies the following:

(i) $H^i(W;\mathbb{Z}W)$ is finitely generated for each i if and only if W is a virtual Poincaré duality group.

(ii) $\check{H}^i(\partial\Sigma)$ is finitely generated for each i if and only if the Čech cohomology of $\partial\Sigma$ is isomorphic to the cohomology of an n-sphere for some n.

§4.3. The number of elements of W^T

We use the notation in Sections 2.1, 2.2 and 4.1, and (W, S) denotes a Coxeter system. In this section, we determine $T \in \mathcal{S}^f$ such that W^T is finite and $L_{S\setminus T}$ is not acyclic. The following lemma plays a key role in the later argument.

Lemma 4.3.1. *Suppose that $T \in \mathcal{S}^f$. If W^T is finite and not a singleton, then $L_{S\setminus T}$ is contractible.*

Proof. Suppose that W^T is finite and not a singleton. Then W does not decompose as the direct product of $W_{S\setminus T}$ and W_T by Lemma 4.2.1. Hence there exist $s_0 \in S \setminus T$ and $t_0 \in T$ such that $m(s_0, t_0) \neq 2$.

Let $S_0 := (S \setminus T) \cup \{t_0\}$. Since W^T is finite, $(W_{S_0})^{\{t_0\}}$ is finite by Lemma 4.2.4. Since $m(s_0, t_0) \neq 2$, we have that $W_{S_0} \neq W_{S_0\setminus\{t_0\}} \times W_{\{t_0\}}$. Hence $(W_{S_0})^{\{t_0\}}$ is not a singleton by Lemma 4.2.1. Suppose that this lemma has been proved when T is a singleton. Since $(W_{S_0})^{\{t_0\}}$ is finite and not a singleton, $L_{S_0\setminus\{t_0\}} = L_{S\setminus T}$ is contractible. Thus the general case follows. Hence it suffices to show the lemma in the case $T = \{t_0\}$. We note that $m(s_0, t_0) \neq 2$.

First we show the following:

$$(*) \qquad L = t_0 * L_{S\setminus\{t_0\}}.$$

Let σ be a simplex of $L_{S\setminus\{t_0\}}$ and let U be the vertex set of σ. Then W_U is finite. Since $W^{\{t_0\}}$ is finite, $(W_{U\cup\{t_0\}})^{\{t_0\}} \subset W^{\{t_0\}}$ is finite by Lemma 4.2.2 (iv). Hence

$W_{U\cup\{t_0\}}$ is finite by Lemma 4.2.3. Thus the join $t_0 * \sigma$ is in L. This proves $(*)$.

Next we show the following:

$$(**) \qquad L_{S\setminus\{t_0\}} = s_0 * L_{S\setminus\{s_0,t_0\}}.$$

Let s be a vertex of $L_{S\setminus\{s_0,t_0\}}$ (i.e., $s \in S \setminus \{s_0, t_0\}$). Suppose that $\{s_0, s\}$ does not span a 1-simplex in $L_{S\setminus\{t_0\}}$. Then $L_{\{t_0,s_0,s\}} = t_0 * \{s_0, s\}$ by $(*)$ (see Figure 1). We note that $L_{\{s_0,s\}}$ is a 0-sphere, and L_V is contractible for each

FIGURE 1. $L_{\{t_0,s_0,s\}}$

subset $V \subset \{t_0, s_0, s\}$ which is not equal to $\{s_0, s\}$. By Theorem 4.1.1, we have

$$H^i(\Gamma_{\{t_0,s_0,s\}}; \mathbb{Z}\Gamma_{\{t_0,s_0,s\}}) \cong \begin{cases} \mathbb{Z}((W_{\{t_0,s_0,s\}})^{\{t_0\}}) \otimes \tilde{H}^0(L_{\{s_0,s\}}), & i = 1, \\ 0, & i \neq 1, \end{cases}$$

where $\Gamma_{\{t_0,s_0,s\}}$ is a torsion-free subgroup of finite index in $W_{\{t_0,s_0,s\}}$ and $\tilde{H}^0(L_{\{s_0,s\}}) \cong \mathbb{Z}$. Since $W^{\{t_0\}}$ is finite, $(W_{\{t_0,s_0,s\}})^{\{t_0\}}$ is finite by Lemma 4.2.2 (iv). Hence $H^1(\Gamma_{\{t_0,s_0,s\}}; \mathbb{Z}\Gamma_{\{t_0,s_0,s\}})$ is a free abelian group of finite rank. By Theorem 1.2.9, $\Gamma_{\{t_0,s_0,s\}}$ is a 1-dimensional Poincaré duality group, i.e., $H^1(\Gamma_{\{t_0,s_0,s\}}; \mathbb{Z}\Gamma_{\{t_0,s_0,s\}}) \cong \mathbb{Z}$. Hence $(W_{\{t_0,s_0,s\}})^{\{t_0\}}$ is a singleton and $W_{\{t_0,s_0,s\}} = W_{\{t_0\}} \times W_{\{s_0,s\}}$ by Lemma 4.2.1. On the other hand, $m(s_0, t_0) \neq 2$ by the definitions of s_0 and t_0. This contradiction implies that $\{s_0, s\}$ spans a 1-simplex in $L_{S\setminus\{t_0\}}$ for every vertex s of $L_{S\setminus\{s_0,t_0\}}$.

Suppose for each $(n-1)$-simplex σ' of $L_{S\setminus\{s_0,t_0\}}$, the join $s_0 * \sigma'$ is in $L_{S\setminus\{t_0\}}$. Let σ be an n-simplex of $L_{S\setminus\{s_0,t_0\}}$ and let U be the vertex set of σ. We show that $s_0 * \sigma$ is in $L_{S\setminus\{t_0\}}$.

Suppose that $s_0 * \sigma$ is not in $L_{S\setminus\{t_0\}}$. By the inductive hypothesis, $s_0 * \sigma'$ is in $L_{S\setminus\{t_0\}}$ for every $(n-1)$-face σ' of σ. Hence $L_{U\cup\{s_0\}}$ is the boundary $\partial(s_0 * \sigma)$ of the $(n+1)$-simplex $s_0 * \sigma$, and $L_{U\cup\{t_0,s_0\}} = t_0 * \partial(s_0 * \sigma)$ by $(*)$. Here we note that $L_{U\cup\{s_0\}}$ is an n-sphere and L_V is contractible for each subset $V \subset U \cup \{t_0, s_0\}$ which is not equal to $U \cup \{s_0\}$. By Theorem 4.1.1, we have

$$H^i(\Gamma_{U\cup\{t_0,s_0\}}; \mathbb{Z}\Gamma_{U\cup\{t_0,s_0\}}) \cong \begin{cases} \mathbb{Z}((W_{U\cup\{t_0,s_0\}})^{\{t_0\}}) \otimes \tilde{H}^n(L_{U\cup\{s_0\}}), & i = n+1, \\ 0, & i \neq n+1, \end{cases}$$

where $\Gamma_{U\cup\{t_0,s_0\}}$ is a torsion-free subgroup of finite index in $W_{U\cup\{t_0,s_0\}}$ and $\tilde{H}^n(L_{U\cup\{s_0\}}) \cong \mathbb{Z}$. Since $W^{\{t_0\}}$ is finite, $(W_{U\cup\{t_0,s_0\}})^{\{t_0\}}$ is finite by Lemma 4.2.2 (iv). Hence $H^{n+1}(\Gamma_{U\cup\{t_0,s_0\}}; \mathbb{Z}\Gamma_{U\cup\{t_0,s_0\}})$ is a free abelian group of finite rank. By Theorem 1.2.9, $\Gamma_{U\cup\{t_0,s_0\}}$ is an $(n+1)$-dimensional Poincaré duality group. Thus $(W_{U\cup\{t_0,s_0\}})^{\{t_0\}}$ is a singleton and $W_{U\cup\{t_0,s_0\}} = W_{\{t_0\}} \times W_{U\cup\{s_0\}}$ by Lemma 4.2.1. On the other hand, $m(s_0, t_0) \neq 2$. This contradiction implies that the $(n+1)$-simplex $s_0 * \sigma$ is in $L_{S\setminus\{t_0\}}$.

By induction, we have the conclusion $(**)$. Therefore $L_{S\setminus\{t_0\}}$ is contractible. □

Definition 4.3.2. A Coxeter system (W, S) is said to be *irreducible* if, for any nonempty and proper subset T of S, W does not decompose into the direct product of W_T and $W_{S\setminus T}$.

Definition 4.3.3. Let (W, S) be a Coxeter system. Then there exists a unique decomposition $\{S_1, \ldots, S_r\}$ of S such that W is the direct product of the parabolic subgroups $W_{S_1}, \ldots, W_{S_r}$ and each Coxeter system (W_{S_i}, S_i) is irreducible (cf. [Bo], [Hu, p.30]). Here we enumerate $\{S_i\}$ so that $S_1, \ldots, S_q \in \mathcal{S}^f$ and $S_{q+1}, \ldots, S_r \notin \mathcal{S}^f$. Let $\tilde{T} := \bigcup_{i=1}^{q} S_i$ and $\tilde{S} := S \setminus \tilde{T}$. We say that $W_{\tilde{S}}$ is the *essential parabolic subgroup* in W. We note that $W_{\tilde{T}}$ is finite and W is the direct product of $W_{\tilde{S}}$ and $W_{\tilde{T}}$.

Remark. The essential parabolic subgroup $W_{\tilde{S}}$ has a finite index in W. Hence a torsion-free subgroup Γ of finite index in $W_{\tilde{S}}$ has a finite index in W as well, and $H^*(W;\mathbb{Z}W) \cong H^*(\Gamma;\mathbb{Z}\Gamma) \cong H^*(W_{\tilde{S}};\mathbb{Z}W_{\tilde{S}})$.

If W is finite, then $\tilde{T} = S$ and $\tilde{S}$ is empty, hence the essential parabolic subgroup is the trivial subgroup.

Lemma 4.3.4. *Let T be a subset of S. If $\tilde{T} \setminus T$ is nonempty, then $L_{S\setminus T}$ is contractible.*

Proof. Suppose that $\tilde{T} \setminus T$ is nonempty. By definition, W is the direct product of $W_{\tilde{S}}$ and $W_{\tilde{T}}$. Hence

$$W_{S\setminus T} = W_{\tilde{S}\setminus T} \times W_{\tilde{T}\setminus T} \text{ and}$$

$$L_{S\setminus T} = L_{\tilde{S}\setminus T} * L_{\tilde{T}\setminus T}.$$

Since $W_{\tilde{T}}$ is finite, $W_{\tilde{T}\setminus T}$ is finite, i.e., $L_{\tilde{T}\setminus T}$ is a simplex. Thus $L_{S\setminus T}$ is contractible. □

We obtain the following lemma by Lemma 4.3.1 and Lemma 4.3.4.

Lemma 4.3.5. *Suppose that $T \in \mathcal{S}^f$ and $L_{S\setminus T}$ is not contractible. Then W^T is finite if and only if $T = \tilde{T}$.*

Proof. Since W is the direct product of $W_{S\setminus\tilde{T}}$ and $W_{\tilde{T}}$, $W^{\tilde{T}}$ is a singleton by Lemma 4.2.1. Thus W^T is finite if $T = \tilde{T}$.

Suppose that W^T is finite and $L_{S\setminus T}$ is not contractible. Since $L_{S\setminus T}$ is not contractible, $\tilde{T} \setminus T$ is empty by Lemma 4.3.4. Hence $\tilde{T} \subset T$. Since W^T is finite and $L_{S\setminus T}$ is not contractible, W^T is a singleton by Lemma 4.3.1. Hence W is the direct product of $W_{S\setminus T}$ and W_T by Lemma 4.2.1. Then

$$W = W_{S\setminus T} \times W_T = W_{S\setminus\tilde{T}} \times W_{\tilde{T}}.$$

Since W_T is finite and $\tilde{T} \subset T$, we have $T = \tilde{T}$ by the definition of $\tilde{T}$. □

By Lemma 4.3.5, we can reformulate Theorem 4.1.1 as follows:

Theorem 4.4.1. *Let (W, S) be a Coxeter system and Γ a torsion-free subgroup of finite index in W. Then*

$$\begin{aligned} H^*(\Gamma; \mathbb{Z}\Gamma) &\cong \tilde{H}^{*-1}(L_{\tilde{S}}) \oplus \left(\bigoplus_{\substack{T \in \mathcal{S}^f \\ \tilde{T} \subsetneq T}} \bigoplus_{\mathbb{Z}} \tilde{H}^{*-1}(L_{S \setminus T}) \right) \\ &\cong \tilde{H}^{*-1}(\tilde{L}) \oplus \left(\bigoplus_{T \in \tilde{\mathcal{S}}^f \setminus \{\emptyset\}} \bigoplus_{\mathbb{Z}} \tilde{H}^{*-1}(\tilde{L}_{\tilde{S} \setminus T}) \right), \end{aligned}$$

where $\tilde{S}$ is the subset of S such that $W_{\tilde{S}}$ is the essential parabolic subgroup in W, $\tilde{T} = S \setminus \tilde{S}$, $\tilde{L} = L(W_{\tilde{S}}, \tilde{S})$ and $\tilde{\mathcal{S}}^f = \mathcal{S}^f(W_{\tilde{S}}, \tilde{S}) = \mathcal{S}^f \cap \tilde{S}$.

Proof. We note that $W^{\tilde{T}}$ is a singleton by Lemma 4.2.1. By Theorem 4.1.1 and Lemma 4.3.5, we have that

$$H^*(\Gamma; \mathbb{Z}\Gamma) \cong \tilde{H}^{*-1}(L_{S \setminus \tilde{T}}) \oplus \left(\bigoplus_{T \in \tilde{\mathcal{S}}^f \setminus \{\tilde{T}\}} \bigoplus_{\mathbb{Z}} \tilde{H}^{*-1}(L_{S \setminus T}) \right).$$

If $\tilde{T} \not\subset T$ (i.e., $\tilde{T} \setminus T$ is nonempty), then $L_{S \setminus T}$ is contractible by Lemma 4.3.4. Hence,

$$H^*(\Gamma; \mathbb{Z}\Gamma) \cong \tilde{H}^{*-1}(L_{S \setminus \tilde{T}}) \oplus \left(\bigoplus_{\substack{T \in \mathcal{S}^f \\ \tilde{T} \subsetneq T}} \bigoplus_{\mathbb{Z}} \tilde{H}^{*-1}(L_{S \setminus T}) \right).$$

The parabolic subgroup $W_{\tilde{S}}$ has a finite index in W, and $W_{\tilde{S}}$ is the essential parabolic subgroup in the Coxeter system $(W_{\tilde{S}}, \tilde{S})$. Therefore,

$$H^*(\Gamma; \mathbb{Z}\Gamma) \cong \tilde{H}^{*-1}(\tilde{L}) \oplus \left(\bigoplus_{T \in \tilde{\mathcal{S}}^f \setminus \{\emptyset\}} \bigoplus_{\mathbb{Z}} \tilde{H}^{*-1}(\tilde{L}_{\tilde{S} \setminus T}) \right)$$

by Theorem 4.1.1 and Lemma 4.3.5. $\square$

By Theorem 4.4.1, we have the following corollary.

Corollary 4.4.2. *Let (W, S) be a Coxeter system, Γ a torsion-free subgroup of finite index in W, $\tilde{S}$ the subset of S such that $W_{\tilde{S}}$ is the essential parabolic subgroup in W, and $\tilde{T} = S \setminus \tilde{S}$. Then the following statements are equivalent:*

(i) $H^i(\Gamma; \mathbb{Z}\Gamma)$ *is finitely generated;*

(ii) $H^i(\Gamma; \mathbb{Z}\Gamma)$ *is isomorphic to* $\tilde{H}^{i-1}(L_{\tilde{S}})$*;*

(iii) $\tilde{H}^{i-1}(L_{S\setminus T}) = 0$ *for each* $T \in \mathcal{S}^f$ *such that* $\tilde{T} \subsetneq T$.

Example 4.4.3. It is known that, for every finite simplicial complex M, there exists a right-angled Coxeter system (W, S) such that $L(W, S)$ is equal to the barycentric subdivision of M ([D1, Lemma 11.3]).

Let (W, S) be a Coxeter system such that $L = L(W, S)$ is the barycentric subdivision of a triangulation of the projective plane. In [Dr3], A. N. Dranishnikov showed that $\operatorname{vcd}_{\mathbb{Z}} W = 3$ and $\operatorname{vcd}_{\mathbb{Q}} W = 2$, where $\operatorname{vcd}_R W$ is the virtual cohomological dimension of W over R. Now, using Theorem 4.4.1, we calculate the cohomology of a torsion-free subgroup Γ of finite index in W.

Since L is the projective plane,

$$\tilde{H}^i(L) \cong \begin{cases} \mathbb{Z}_2, & i = 2, \\ 0, & i \neq 2. \end{cases}$$

Since $L = L_{S\setminus\emptyset}$ is not contractible and $W^{\emptyset}$ is a singleton, $\tilde{T}$ is the empty set (i.e., $W = W_S$ is the essential parabolic subgroup) by Lemma 4.3.5. For each $T \in \mathcal{S}^f \setminus \{\emptyset\}$, $L_{S\setminus T}$ has the same homotopy type as a circle. Hence,

$$\tilde{H}^i(L_{S\setminus T}) \cong \begin{cases} \mathbb{Z}, & i = 1, \\ 0, & i \neq 1. \end{cases}$$

Therefore, by Theorem 4.4.1, we have

$$H^i(\Gamma; \mathbb{Z}\Gamma) \cong \tilde{H}^{i-1}(L) \oplus \left(\bigoplus_{T \in \tilde{\mathcal{S}}^f \setminus \{\emptyset\}} \bigoplus_{\mathbb{Z}} \tilde{H}^{i-1}(L_{S\setminus T}) \right) \cong \begin{cases} \mathbb{Z}_2, & i = 3, \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots, & i = 2, \\ 0, & \text{otherwise.} \end{cases}$$

By the example above, we see that there exists a Coxeter group W such that $H^i(\Gamma; \mathbb{Z}\Gamma)$ is finitely generated and $H^j(\Gamma; \mathbb{Z}\Gamma)$ is infinitely generated for some $i \neq j$.

CHAPTER 5

Geometrically finite groups

In this chapter, we investigate limit sets of geometrically finite groups acting on Busemann spaces. Coxeter groups and parabolic subgroups are examples of geometrically finite groups acting on Busemann spaces. In Section 5.1, we recall definitions and basic properties of hyperbolic spaces and their boundaries in the sense of Gromov. In Section 5.2, we give an equivalent condition for a group acting on a hyperbolic space to be geometrically finite. In Section 5.3, we recall definitions and basic properties of Busemann spaces and their boundaries. In Section 5.4, we show a Busemann space-analogue of several results proved by A. Ranjbar-Motlagh for geometrically finite groups acting on hyperbolic spaces.

§5.1. HYPERBOLIC SPACES AND THEIR BOUNDARIES

We introduce definitions of hyperbolic spaces and their boundaries in the sense of Gromov ([Gr], [GH], [CP]).

Definition 5.1.1. Let (X, d) be a metric space and $x, y, w \in X$. The Gromov product of x and y with respect to w is defined as

$$\langle x|y\rangle_w := \frac{1}{2}(d(x,w) + d(y,w) - d(x,y)).$$

For some $\delta \geq 0$, we say that a geodesic space (X, d) is *δ-hyperbolic*, if for each $x, y, z, w \in X$

$$\langle x|y\rangle_w \geq \min\{\langle x|z\rangle_w,\ \langle y|z\rangle_w\} - \delta.$$

Also we say that X is *hyperbolic*, if X is δ-hyperbolic for some $\delta \geq 0$.

Definition 5.1.2. Let (X, d) be a hyperbolic space. A sequence $\{x_i\}$ of points of X is said to *converge to infinity*, if for some (arbitrary) basepoint $w \in X$

$$\lim_{i,j\to\infty} \langle x_i|x_j\rangle_w = \infty.$$

Let $S_\infty(X)$ denote the set of all sequences convergent to infinity, and define the equivalence relation

$$\{x_i\}\mathcal{R}\{y_i\} \iff \lim_{i\to\infty} \langle x_i|y_i\rangle_w = \infty.$$

The *boundary of* X is defined as $\partial X := S_\infty(X)/\mathcal{R}$. We say that $\{x_i\} \in S_\infty(X)$ converges to $x \in \partial X$, if the equivalent class of $\{x_i\}$ with respect to $\mathcal{R}$ is x, and we write $x_i \to x$. Now we extend the Gromov product to the boundary as follows: For each $x, y \in X \cup \partial X$, we define

$$\langle x|y\rangle_w := \inf\{\lim_{i\to\infty} \langle x_i|y_i\rangle_w\},$$

where the infimum is taken over all pairs of sequences $x_i \to x$ and $y_i \to y$. Then $X \cup \partial X$ has a natural topology, in which X is an open subspace, and a neighborhood basis for each point $x \in \partial X$ is given by the sets

$$N(x;\epsilon) := \{y \in X \cup \partial X \mid \langle x|y\rangle_w > \epsilon\},$$

where $\epsilon > 0$. It is known that this topology is not dependent on the basepoint $w \in X$. For a geodesic ray $\xi : [0,\infty) \to X$, there exists a unique point $x \in \partial X$

such that $\{\xi(t_i)\} \to x$ for each sequence $\{t_i\}$ of non-negative real numbers such that $\{t_i\} \to \infty$. Then we write $x = \xi(\infty)$.

Definition 5.1.3. Let X be a geodesic space. Let $x, y, z \in X$ and $\triangle := \triangle xyz$ a geodesic triangle in X. Then there exist unique non-negative numbers a, b, c such that

$$d(x,y) = a + b,\ d(y,z) = b + c,\ d(z,x) = c + a.$$

Indeed $a = \langle y|z\rangle_x$, $b = \langle z|x\rangle_y$ and $c = \langle x|y\rangle_z$. Then we can consider the metric tree $T_\triangle$ that has three vertexes of valence one, one vertex of valence three, and edges of length a, b and c. Let o be the vertex of valence three in $T_\triangle$ and let v_x, v_y, v_z be the vertexes of $T_\triangle$ such that $d(o, v_x) = a$, $d(o, v_y) = b$ and $d(o, v_z) = c$. Then the map $\{x, y, z\} \to \{v_x, v_y, v_z\}$ extends uniquely to a map $f : \triangle \to T_\triangle$ whose restriction to each side of $\triangle$ is an isometry. For some $\delta \geq 0$, the geodesic triangle $\triangle$ is said to be *δ-thin*, if $d(p,q) \leq \delta$ for each points $p, q \in \triangle$ with $f(p) = f(q)$.

The following lemma is well-known.

Lemma 5.1.4 ([GH], [CP, p.8–10])**.** *Let (X, d) be a proper δ-hyperbolic space.*

(i) *Every geodesic triangle in X is 4δ-thin.*

(ii) *Let $\xi, \zeta : [0, \infty) \to X$ be geodesic rays with $\xi(\infty) = \zeta(\infty)$. Then $d(\xi(t), \operatorname{Im}\zeta) \leq d(\xi(0), \zeta(0)) + 8\delta$ for each $t \geq 0$. Furthermore, there exists $T \geq 0$ such that $d(\xi(t), \operatorname{Im}\zeta) \leq 8\delta$ for each $t \geq T$.*

(iii) *For each pair of distinct points $\alpha, \beta \in \partial X$, there exists a geodesic line in X with endpoints α and β.*

§5.2. Geometrically finite groups acting on hyperbolic spaces

In this section, we give an equivalent condition for a group acting on a hyperbolic space to be geometrically finite. We first give a definition of limit sets.

Definition 5.2.1. Let (X, d) be a hyperbolic space and Γ a group which acts properly discontinuously on X. The *limit set of* Γ (*with respect to* X) is defined as

$$\partial\Gamma = \mathrm{cl}_{X\cup\partial X}(\Gamma x_0) \cap \partial X,$$

where $\mathrm{cl}_{X\cup\partial X}$ means the closure in $X \cup \partial X$, and x_0 is a point in X. The limit set $\partial\Gamma$ is independent of the choice of the point $x_0 \in X$.

A geometrically finite group acting on a hyperbolic space is defined as follows:

Definition 5.2.2 ([R]). Let (X, d) be a proper hyperbolic space and Γ a group which acts properly discontinuously on X. We say that (the action of) Γ is *geometrically finite* (*with respect to* X), if there exists a compact subset K of X such that $\mathcal{L}(\partial\Gamma) \subset \Gamma K$, where $\mathcal{L}(\partial\Gamma)$ is the union of the images of all geodesic lines in X with the endpoints in $\partial\Gamma$.

Lemma 5.2.3. *Let (X, d) be a proper δ-hyperbolic space, $\xi_0 : \mathbb{R} \to X$ a geodesic line in X, and $\xi_1, \xi_2 : [0, \infty) \to X$ geodesic rays in X such that $\xi_1(0) = \xi_2(0) = x_0$, $\xi_0(\infty) = \xi_1(\infty)$ and $\xi_0(-\infty) = \xi_2(\infty)$. Then, for each $i \in \{0, 1, 2\}$ and each point $x \in \mathrm{Im}\,\xi_i$, the inequality $d(x, Y_i) \le 12\delta$ holds, where $Y_i = \bigcup_{j\in\{0,1,2\}\setminus\{i\}} \mathrm{Im}\,\xi_j$.*

Proof. Let $i \in \{0, 1, 2\}$ and $x \in \mathrm{Im}\,\xi_i$.

By Lemma 5.1.4 (ii), there exists $T > 0$ such that

$$d(\xi_j(t), \mathrm{Im}\,\xi_0) \le 8\delta \text{ for each } t \ge T \text{ and } j = 1, 2, \text{ and}$$

$$d(\xi_0(s), \mathrm{Im}\,\xi_1 \cup \mathrm{Im}\,\xi_2) \le 8\delta \text{ for each } |s| \ge T.$$

It is clear that if $x \in \xi_0((-\infty, -T] \cup [T, \infty)) \cup \bigcup_{j=1,2} \xi_j([T, \infty))$, then $d(x, Y_i) \leq 8\delta$.

Let $S > T$ be a large number. Then geodesic triangles $\triangle x_0\xi_1(S)\xi_2(S)$, $\triangle \xi_0(S)\xi_1(S)\xi_2(S)$ and $\triangle \xi_0(S)\xi_0(-S)\xi_2(S)$ are 4δ-thin by Lemma 5.1.4 (i). Hence if $x \in \xi_0((-T, T)) \cup \bigcup_{j=1,2} \xi_j([0, T))$, then $d(x, Y_i) \leq 12\delta$. $\square$

Using Lemma 5.2.3, we show the following proposition. Here we note that there is no obvious inclusion between $\mathcal{L}(\partial\Gamma)$ and $\mathcal{L}^+_{x_0}(\partial\Gamma)$ in general.

Proposition 5.2.4. *Let (X, d) be a proper hyperbolic space and Γ a group which acts properly discontinuously on X. Suppose that the cardinality of $\partial\Gamma$ is greater than one. Then the following statements are equivalent:*

(1) *The action of Γ is geometrically finite.*

(2) *There exists a compact subset K of X such that $\mathcal{L}^+_{x_0}(\partial\Gamma) \subset \Gamma K$ for some $x_0 \in X$, where $\mathcal{L}^+_{x_0}(\partial\Gamma)$ is the union of the images of all geodesic rays ξ issuing from x_0 with $\xi(\infty) \in \partial\Gamma$.*

Proof. Let X be δ-hyperbolic.

(1)$\Rightarrow$(2): Let $\sigma : \mathbb{R} \to X$ be a geodesic line with $\sigma(-\infty), \sigma(\infty) \in \partial\Gamma$ and let $x_0 := \sigma(0)$. Suppose that (1) holds. Then $\mathcal{L}(\partial\Gamma) \subset \Gamma B(x_0, N)$ for some $N > 0$, where $B(x_0, N)$ is the metric ball of radius N about x_0. Let ξ be a geodesic ray issuing from x_0 with $\xi(\infty) \in \partial\Gamma$. There exists a geodesic line $\tau : \mathbb{R} \to X$ such that $\tau(\infty) = \sigma(\infty)$ and $\tau(-\infty) = \xi(\infty)$ by Lemma 5.1.4 (iii). For each $t \geq 0$, $d(\xi(t), \sigma([0, \infty)) \cup \operatorname{Im}\tau) \leq 12\delta$ by Lemma 5.2.3. Since $\operatorname{Im}\sigma \cup \operatorname{Im}\tau \subset \mathcal{L}(\partial\Gamma) \subset \Gamma B(x_0, N)$, $\operatorname{Im}\xi \subset \Gamma B(x_0, N + 12\delta)$. Thus $\mathcal{L}^+_{x_0}(\partial\Gamma) \subset \Gamma B(x_0, N + 12\delta)$.

(2)$\Rightarrow$(1): Suppose that (2) holds. Then $\mathcal{L}^+_{x_0}(\partial\Gamma) \subset \Gamma B(x_0, N)$ for some $N > 0$. Let $\sigma : \mathbb{R} \to X$ be a geodesic line with $\sigma(\infty), \sigma(-\infty) \in \partial\Gamma$. Let ξ and ζ be geodesic rays issuing from x_0 such that $\sigma(\infty) = \xi(\infty)$ and $\sigma(-\infty) = \zeta(\infty)$. By Lemma 5.2.3, $d(\sigma(t), \operatorname{Im}\xi \cup \operatorname{Im}\zeta) \leq 12\delta$ for each $t \in \mathbb{R}$. Since $\operatorname{Im}\xi \cup \operatorname{Im}\zeta \subset$

$\mathcal{L}^+_{x_0}(\partial\Gamma) \subset \Gamma B(x_0, N)$, $\mathrm{Im}\,\sigma \subset \Gamma B(x_0, N+12\delta)$. Thus $\mathcal{L}(\partial\Gamma) \subset \Gamma B(x_0, N+12\delta)$, i.e., Γ is geometrically finite. $\square$

Remark. By Lemma 5.1.4 (ii), the statement (2) is equivalent to the following statement:

(3) For each point $x_0 \in X$, there exists a compact subset K of X such that $\mathcal{L}^+_{x_0}(\partial\Gamma) \subset \Gamma K$.

The following results were given by A. Ranjbar-Motlagh in [R]. The aim in this chapter is to show a Busemann space analogue of these results.

Theorem 5.2.5 ([G, Theorem 3], [SS, Theorem 3.1], [R, Lemma 3.1])**.** *Let (X, d) be a proper hyperbolic space and Γ a group which acts properly discontinuously on X. If $H \subset G$ are two subgroups of Γ, and if H is a geometrically finite subgroup with $|\partial H| \geq 2$, then the following conditions are equivalent:*

(1) *$\partial G = \partial H$.*

(2) *$g(\partial H) = \partial H$ for each $g \in G$.*

(3) *$[G : H] < \infty$.*

Proposition 5.2.6 (cf. [R, Lemma 4.2])**.** *Let (X, d) be a proper hyperbolic space, Γ a group which acts properly discontinuously on X, and G a subgroup of finite index in Γ. Then Γ is geometrically finite if and only if G is geometrically finite.*

Proposition 5.2.7 ([R, Lemma 3.5])**.** *Let (X, d) be a proper hyperbolic space, Γ a group which acts properly discontinuously on X, and G a geometrically finite subgroup of Γ with $|\partial G| \geq 2$. For some element $\gamma \in \Gamma$, suppose that $\gamma G \gamma^{-1} \subset G$, then*

(1) *$\gamma(\partial G) \subset \partial G$,*

(2) *$\gamma^n \in G$ for some $n \in \mathbb{Z} \setminus \{0\}$,*

(3) $\gamma(\partial G) = \gamma^{-1}(\partial G) = \partial G$,

(4) $\gamma G \gamma^{-1} = G$,

(5) $[\langle G, \gamma \rangle : G] < \infty$.

Theorem 5.2.8 ([G, Theorem 2], [SS, Theorems 4.3, 4.4], [R, Theorems 4.4, 4.5]). *Let (X, d) be a proper hyperbolic space and Γ a group which acts properly discontinuously on X. Suppose that G_1 and G_2 are two geometrically finite subgroups of Γ. Then*

(1) *$G_1 \cap G_2$ is also geometrically finite.*

(2) *If $|\partial(G_1 \cap G_2)| \geq 2$, then $\partial(G_1 \cap G_2) = \partial G_1 \cap \partial G_2$.*

§5.3. Busemann spaces and their boundaries

In this section, we recall the definitions and some basic properties of Busemann spaces and their boundaries.

Definition 5.3.1. Let (X, d) be a geodesic space. A geodesic space X is a *Busemann space* if for each three points x_0, x_1, x_2 of X and each $t \in [0, 1]$,

$$d(\xi_1(td_1), \xi_2(td_2)) \leq td(x_1, x_2),$$

where $d_i = d(x_0, x_i)$ and $\xi_i : [0, d_i] \to X$ is a geodesic segment from x_0 to x_i for each $i = 1, 2$.

The following proposition is known (cf. [Ho]).

Proposition 5.3.2. *Let (X, d) be a proper Busemann space.*

(1) *Every CAT(0) space is a Busemann space.*

(2) *For each two points $x, y \in X$, there exists an unique geodesic segment between x and y in X.*

(3) *X is contractible.*

(4) *For each geodesic ray ξ in X and each point $x_0 \in X$, there exists a unique geodesic ray ξ' issuing from x_0 such that ξ and ξ' are asymptotic.*

Definition 5.3.3. Let (X, d) be a proper Busemann space and $x_0 \in X$. The *boundary of X with respect to x_0*, denoted by $\partial_{x_0} X$, is defined as the set of all geodesic rays issuing from x_0. Then $X \cup \partial_{x_0} X$ has a natural topology, in which X is an open subspace, and a neighborhood basis for each point $\xi \in \partial_{x_0} X$ is given by the sets

$$U(\xi; r, \epsilon) = \{x \in X \cup \partial X \mid x \notin B(x_0, r),\ d(\xi(r), \xi_x(r)) < \epsilon\},$$

where $r, \epsilon > 0$ and $\xi_x : [0, d(x_0, x)] \to X$ is the geodesic from x_0 to x ($\xi_x = x$ if $x \in \partial_{x_0} X$). This is called the *cone topology* on $X \cup \partial_{x_0} X$. It is known that $X \cup \partial_{x_0} X$ is a metrizable compactification of X (cf. [GH], [Ho]).

Let x_0 and x_1 be two points of a proper Busemann space X. By Proposition 5.3.2 (4), there exists a unique bijection $\Phi : \partial_{x_0} X \to \partial_{x_1} X$ such that ξ and $\Phi(\xi)$ are asymptotic for each $\xi \in \partial_{x_0} X$. The following theorem was proved by P. K. Hotchkiss.

Theorem 5.3.4 ([Ho]).

(1) *The above map $\Phi : \partial_{x_0} X \to \partial_{x_1} X$ is a homeomorphism.*

(2) *If X is a hyperbolic (resp. CAT(0)) space, then $\partial_{x_0} X$ is homeomorphic to the hyperbolic (resp. CAT(0)) boundary.*

Definition 5.3.5. Let X be a proper Busemann space. The asymptotic relation is an equivalence relation in the set of all geodesic rays in X. The *boundary of X*, denoted by ∂X, is defined as the set of asymptotic equivalence classes of geodesic rays. The equivalence class of a geodesic ray ξ is denoted by $\xi(\infty)$. By Proposition 5.3.2 (4), for each $x_0 \in X$ and each $\alpha \in \partial X$, there exists a unique element $\xi \in \partial_{x_0}X$ with $\xi(\infty) = \alpha$. Thus we may identify ∂X with $\partial_{x_0}X$ for each $x_0 \in X$.

Let (X, d) be a Busemann space and Γ a group which acts properly discontinuously on X. The *limit set of* Γ (*with respect to* X) is defined as

$$\partial\Gamma = \mathrm{cl}_{X \cup \partial X}(\Gamma x_0) \cap \partial X,$$

where x_0 is a point in X. The limit set $\partial\Gamma$ is independent of the choice of the point $x_0 \in X$.

Let (X, d) be a proper Busemann space and Γ a group which acts properly discontinuously on X. For each element $\gamma \in \Gamma$ and each geodesic ray $\xi : [0, \infty) \to X$, a map $\gamma\xi : [0, \infty) \to X$ defined by $(\gamma\xi)(t) := \gamma(\xi(t))$ is also a geodesic ray. If geodesic rays ξ and ξ' are asymptotic, then $\gamma\xi$ and $\gamma\xi'$ are also asymptotic. Thus γ induces a homeomorphism of ∂X and Γ acts on ∂X. We note that $\Gamma(\partial\Gamma) = \partial\Gamma$ by definition. Hence Γ also acts on $\partial\Gamma$.

§5.4. Geometrically finite groups acting on Busemann spaces

By Proposition 5.2.4, the following definition is natural.

Definition 5.4.1. Let (X, d) be a proper Busemann space and Γ a group which acts properly discontinuously on X. We say that (the action of) Γ is *geometrically finite* (*with respect to* X), if for some (arbitrary) point $x_0 \in X$ there exists a compact subset K of X such that $\mathcal{L}^+_{x_0}(\partial\Gamma) \subset \Gamma K$, where $\mathcal{L}^+_{x_0}(\partial\Gamma)$ is the union of the images of all geodesic rays ξ issuing from x_0 such that $\xi(\infty) \in \partial\Gamma$.

Example 5.4.2. Let (X, d) be a proper CAT(0) space. Let Γ be a group which acts properly discontinuously and cocompactly on X (such Γ is called a *CAT(0)* *group*). Then $\mathcal{L}^+_{x_0}(\partial\Gamma) \subset X = \Gamma K$ for some compact subset K of X by the cocompactness. Hence the action of Γ is geometrically finite.

Definition 5.4.3. A subset M of a geodesic space X is said to be *quasi-convex* if there exists $N > 0$ such that the metric N-neighborhood of M contains all geodesic segments between each two points of M. Also a subset M of a metric space X is said to be *quasi-dense* if M is N-dense for some $N > 0$, i.e., if each point of X is N-close to some point of M.

The following proposition generalizes the above observation.

Proposition 5.4.4. *Let (X, d) be a proper Busemann space and Γ a group which acts properly discontinuously on X. If Γx_0 is quasi-convex in X, then Γ is geometrically finite.*

Proof. Since Γx_0 is quasi-convex in X, there exists $N > 0$ such that the metric N-neighborhood of Γx_0 contains the geodesic from a to b for each $a, b \in \Gamma x_0$. Let $\xi : [0, \infty) \to X$ be a geodesic ray with $\xi(0) = x_0$ and $\xi(\infty) \in \partial\Gamma$. Then there exists a sequence $\{\gamma_i x_0\} \subset \Gamma x_0$ converging to $\xi(\infty)$ in $X \cup \partial X$. For each $t \geq 0$, there exists a number i such that $d(\xi(t), \xi_{\gamma_i x_0}(t)) \leq 1$, where $\xi_{\gamma_i x_0}$ is geodesic from x_0 to $\gamma_i x_0$. Hence $\xi(t) \in \Gamma B(x_0, N+1)$ because $\operatorname{Im} \xi_{\gamma_i x_0} \subset \Gamma B(x_0, N)$. Thus $\operatorname{Im} \xi \subset \Gamma B(x_0, N+1)$, and Γ is geometrically finite. $\square$

We prove a Busemann space-analogue of Theorem 5.2.5.

Theorem 5.4.5. *Let (X, d) be a proper Busemann space and Γ a group which acts properly discontinuously on X. If $H \subset G$ are two subgroups of Γ, and if H is geometrically finite, then the following conditions are equivalent:*

(1) $\partial G = \partial H$.

(2) *$H x_0$ is a quasi-dense subset of $G x_0$.*

(3) $[G : H] < \infty$.

Proof. (3) $\Rightarrow$ (2): Suppose that $m = [G : H] < \infty$. Then $\{Hg \,|\, g \in G\} = \{Hg_1, \ldots, Hg_m\}$ for some $g_1, \ldots, g_m \in G$. Let $N := \max\{d(x_0, g_i x_0) \,|\, i = 1, \ldots, m\}$. For each $g \in G$, $g = hg_i$ for some $h \in H$ and i. Then $d(hx_0, gx_0) = d(hx_0, hg_i x_0) = d(x_0, g_i x_0) \leq N$. Hence $Gx_0 \subset HB(x_0, N)$, i.e., Hx_0 is quasi-dense in Gx_0.

(2) $\Rightarrow$ (1): Suppose that Hx_0 is quasi-dense in Gx_0. Then $Gx_0 \subset HB(x_0, N)$ for some $N > 0$. For each $\alpha \in \partial G$, there exists a sequence $\{g_i x_0\} \subset Gx_0$ which converges to α in $X \cup \partial X$. Since $Gx_0 \subset HB(x_0, N)$, we can obtain a sequence $\{h_i x_0\} \subset Hx_0$ such that $d(h_i x_0, g_i x_0) \leq N$ for each i. Then $\{h_i x_0\}$ converges to α, i.e., $\alpha \in \partial H$. Hence $\partial G = \partial H$.

(1) $\Rightarrow$ (3): Suppose that $[G : H] = \infty$. Let $\{Ha \,|\, a \in G\} = \{Ha_\lambda \,|\, \lambda \in \Lambda\}$ ($Ha_\lambda \neq Ha_{\lambda'}$ if $\lambda \neq \lambda'$). Since $H \subset \Gamma$ acts properly discontinuously on X, for each $\lambda \in \Lambda$, we may choose a_λ in such a way that $d(x_0, a_\lambda x_0) = d(x_0, Ha_\lambda x_0)$. Since $X \cup \partial X$ is compact, there exists a sequence $\{g_i x_0\} \subset \{a_\lambda x_0 \,|\, \lambda \in \Lambda\}$ which converges to a point $\xi(\infty) \in \partial G$, where ξ is a geodesic ray issuing from x_0. Now we show that $\xi(\infty) \notin \partial H$.

Suppose that $\xi(\infty) \in \partial H$. Since H is geometrically finite, $\operatorname{Im} \xi \subset HB(x_0, N)$ for some $N > 0$. Let $R > N + 1$. Since $\{g_i x_0\}$ converges to $\xi(\infty)$, for large enough i, $d(x_0, g_i x_0) > R$ and $d(\xi(R), \xi_{g_i x_0}(R)) < 1$, where $\xi_{g_i x_0}$ is the geodesic segment from x_0 to $g_i x_0$. Since $\operatorname{Im} \xi \subset HB(x_0, N)$, there exists $h \in H$ such that

$d(hx_0, \xi(R)) \leq N$. Then,

$$\begin{aligned} d(x_0, h^{-1}g_ix_0) &= d(hx_0, g_ix_0) \\ &\leq d(hx_0, \xi(R)) + d(\xi(R), \xi_{g_ix_0}(R)) + d(\xi_{g_ix_0}(R), g_ix_0) \\ &< N + 1 + (d(x_0, g_ix_0) - R) \\ &= d(x_0, g_ix_0) - (R - N - 1) \\ &< d(x_0, g_ix_0). \end{aligned}$$

This contradicts the assumption $d(x_0, g_ix_0) = d(x_0, Hg_ix_0)$. Therefore $\xi(\infty) \notin \partial H$ and $\partial G \neq \partial H$. $\square$

Remark. The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) hold without the assumption of the geometrically finiteness of H.

In the case X is hyperbolic, Theorem 5.2.5 states that H is a subgroup of finite index in G if and only if $g(\partial H) = \partial H$ for each $g \in G$. On the other hand, this is not always the case if X is a Busemann space. Indeed there exists an easy counter-example.

Example 5.4.6. Let $G := \mathbb{Z} \times \mathbb{Z}$, $X := \mathbb{R} \times \mathbb{R}$ and G act on X by $(a, b) \cdot (x, y) = (x + a, y + b)$ for each $(a, b) \in G$ and $(x, y) \in X$. Then X is a Busemann space and the subgroup $H = \mathbb{Z} \times 0$ of G is geometrically finite. The limit set ∂H is the two-points set, and let denote $\partial H = \{\alpha^+, \alpha^-\}$, where $\alpha^+ = \lim_{i\to\infty}(i, 0)$ and $\alpha^- = \lim_{i\to\infty}(-i, 0)$. For each $(a, b) \in G$,

$$(a, b) \cdot \alpha^+ = \lim_{i\to\infty}(a, b) \cdot (i, 0) = \lim_{i\to\infty}(i + a, b) = \alpha^+.$$

By the same argument, $(a, b) \cdot \alpha^- = \alpha^-$. Hence $g(\partial H) = \partial H$ for each $g \in G$. On the other hand, it is clear that $[G : H] = \infty$ and $\partial G \neq \partial H$.

Corollary 5.4.7. *Let (X, d) be a proper Busemann space, Γ a group which acts properly discontinuously on X, and G a subgroup of finite index in Γ. Then Γ is geometrically finite if and only if G is geometrically finite.*

Proof. We note that $\partial\Gamma = \partial G$ by Theorem 5.4.5 and the remark.

Suppose that G is geometrically finite. Then $\mathcal{L}^+_{x_0}(\partial G) \subset GB(x_0, N)$ for some $N > 0$. Since $\partial\Gamma = \partial G$,

$$\mathcal{L}^+_{x_0}(\partial\Gamma) = \mathcal{L}^+_{x_0}(\partial G) \subset GB(x_0, N) \subset \Gamma B(x_0, N).$$

Hence Γ is also geometrically finite.

Suppose that Γ is geometrically finite. Then $\mathcal{L}^+_{x_0}(\partial\Gamma) \subset \Gamma B(x_0, N)$ for some $N > 0$. Since G is a subgroup of finite index in Γ, Gx_0 is a quasi-dense subset of Γx_0 by Theorem 5.4.5. Hence $\Gamma x_0 \subset GB(x_0, R)$ for some $R > 0$. Then

$$\mathcal{L}^+_{x_0}(\partial G) = \mathcal{L}^+_{x_0}(\partial\Gamma) \subset \Gamma B(x_0, N) \subset GB(x_0, N + R).$$

Thus G is also geometrically finite. $\square$

In Corollary 5.4.7, we can not omit the hypothesis $[\Gamma : G] < \infty$. Indeed there exists a counter-example.

Example 5.4.8. Let $G = \langle a, b\rangle$ be the rank two free group with basis $\{a, b\}$, $\Gamma := G \times \mathbb{Z}$ and $X := T \times \mathbb{R}$, where T is the Cayley graph of G with respect to $\{a, b\}$. We note that $\{(a, 0), (b, 0), (1_G, 1)\}$ is a generating set of Γ. An action of Γ on X is defined as follows:

$$(a, 0) * (t, r) = (a \cdot t, r),$$

$$(b, 0) * (t, r) = (b \cdot t, r + 2),$$

$$(1_G, 1) * (t, r) = (t, r + 1),$$

where $a \cdot t$ and $b \cdot t$ describe the natural action of G on its Cayley graph T. Then Γ acts properly discontinuously and cocompactly as isometries on the proper

CAT(0) space X. This example was given by P. L. Bowers and K. Ruane in [BR].

We show that G is not geometrically finite. Let $g_i := (a^i b^i, 0) \in G$ for $i = 1, 2, \dots$ and $x_0 := (1_G, 0) \in T \times \mathbb{R}$. Then $g_i * x_0 = (a^i b^i, 2i)$, and (a^i, i) is the midpoint of the geodesic segment from x_0 to $g_i * x_0$. Let $\xi : [0, \infty) \to X$ be the geodesic ray with $\xi(i/\sqrt{2}) = (a^i, i)$ for each i. Then $\xi(\infty) \in \partial G$, since the sequence $\{g_i * x_0\}_i$ converges to $\xi(\infty)$. In [BR, p.186 (i)], it is shown that $d((a^i, i), G * x_0) > i/3$ for each i. Hence $\operatorname{Im} \xi \not\subset G * B(x_0, N)$ for each $N > 0$. Thus G is not geometrically finite.

On the other hand, it is clear that Γ and $H := \langle a \rangle$ are geometrically finite and $H \subset G \subset \Gamma$.

By the example above, we see that a subgroup of a CAT(0) group is not always geometrically finite in general. A Coxeter group is an important example of a CAT(0) group. We show that each parabolic subgroup of every Coxeter group W is geometrically finite with respect to Σ.

Example 5.4.9. Let (W, S) be a Coxeter system and $T \subset S$. There exists a natural isometric embedding $\Sigma_T \to \Sigma$, and the W_T-action on Σ_T is the restriction of the W-action on Σ. Since W_T acts cocompactly on Σ_T, we see that each parabolic subgroup W_T is geometrically finite with respect to Σ.

The following proposition was given by A. Ranjbar-Motlagh.

Proposition 5.4.10 ([R, Proposition 4.3])**.** *Let (X, d) be a proper hyperbolic space, Γ a group which acts properly discontinuously on X, and G_1 and G_2 two subgroups of Γ. Suppose that G_1 is geometrically finite and $|\partial(G_1 \cap G_2)| \geq 2$. If $[G_i : G_1 \cap G_2] < \infty$ for each $i = 1, 2$, then $[G_1 \vee G_2 : G_1 \cap G_2] < \infty$.*

This proposition is not always true in general for Busemann spaces. We give a counter-example.

Example 5.4.11. Let $S := \{s_1, s_2, s_3, s_4\}$ and let $m : S \times S \to \mathbb{N} \cup \{\infty\}$ be the function defined by

$$m(s_i, s_j) = \begin{cases} 1 & \text{if } i = j, \\ 2 & \text{if } |i-j| = 1 \text{ or } 3, \\ \infty & \text{if } |i-j| = 2. \end{cases}$$

We define the Coxeter group $W = \langle S \mid (st)^{m(s,t)} = 1 \text{ for } s,t \in S\rangle$ and $X := \Sigma(W,S)$. Let $G_1 := W_{\{s_1,s_2,s_3\}}$ and $G_2 := W_{\{s_1,s_3,s_4\}}$. Then W acts properly discontinuously on the proper CAT(0) space X, and G_1 and G_2 are geometrically finite. We note that

$$\begin{aligned} &W \cong (\mathbb{Z}_2 * \mathbb{Z}_2) \times (\mathbb{Z}_2 * \mathbb{Z}_2), \\ &G_i \cong (\mathbb{Z}_2 * \mathbb{Z}_2) \times \mathbb{Z}_2 \text{ for each } i = 1, 2 \text{ and} \\ &G_1 \cap G_2 = W_{\{s_1,s_3\}} \cong \mathbb{Z}_2 * \mathbb{Z}_2. \end{aligned}$$

Thus $[G_i : G_1 \cap G_2] = 2$ for each $i = 1, 2$. On the other hand, $[G_1 \vee G_2 : G_1 \cap G_2] = \infty$ since $G_1 \vee G_2 = W$. We also have that $\partial(G_1 \vee G_2) \neq \partial(G_1 \cap G_2)$. In fact, $\partial(G_1 \vee G_2) = \partial W$ is a circle and $\partial(G_1 \cap G_2) = \partial W_{\{s_1,s_3\}}$ is a two-points set.

We prove the following result which corresponds to a part of Proposition 5.2.7.

Theorem 5.4.12. *Let (X, d) be a proper Busemann space and Γ a group which acts properly discontinuously on X. Suppose that G is a subgroup of Γ and $\gamma \in \Gamma$ such that $\gamma G \gamma^{-1} \subset G$. Let denote $F := \bigcup_{i\in\mathbb{Z}} \gamma^i G \gamma^{-i}$ and $H := \bigcap_{i\in\mathbb{Z}} \gamma^i G \gamma^{-i}$. Then*

(1) $\gamma(\partial G) \subset \partial G$.

Moreover, if either

(a) G is geometrically finite and $\partial F = \bigcup_{i\in\mathbb{Z}} \partial(\gamma^i G \gamma^{-i})$, or

(b) H *is geometrically finite and* $\partial H = \bigcap_{i\in\mathbb{Z}} \partial(\gamma^i G \gamma^{-i})$,

then

(2) $\gamma(\partial G) = \gamma^{-1}(\partial G) = \partial G$,

(3) $\gamma G \gamma^{-1} = G$.

We first show the following proposition.

Proposition 5.4.13. *Let* (X, d) *be a proper Busemann space,* Γ *a group which acts properly discontinuously on* X.

(i) *Let* $\{G_i \mid i = 1, 2, \cdots\}$ *be a sequence of subgroups of* Γ *such that* $G_i \subset G_{i+1}$ *for each* $i \geq 1$, *and* $F := \bigcup_{i=1}^{\infty} G_i$. *If each* G_i *is geometrically finite, then the following statements are equivalent:*

  (1) $\partial F = \bigcup_{i=1}^{\infty} \partial G_i$.

  (2) $\partial F = \partial G_n$ *for some* n.

  (3) $F = G_n$ *for some* n.

(ii) *Let* $\{G_i \mid i = 1, 2, \cdots\}$ *be a sequence of subgroups of* Γ *such that* $G_{i+1} \subset G_i$ *for each* $i \geq 1$. *If* $H := \bigcap_{i=1}^{\infty} G_i$ *is geometrically finite, then the following statements are equivalent:*

  (1) $\partial H = \bigcap_{i=1}^{\infty} \partial G_i$.

  (2) $\partial H = \partial G_n$ *for some* n.

  (3) $H = G_n$ *for some* n.

Proof. It is clear that (3) implies (2) and (2) implies (1) in each case (i) and (ii). We show that (1) implies (3) in each case.

(i) (1) $\Rightarrow$ (3): Suppose that (1) holds and $F \neq G_i$ for each i. Then there exists a subsequence $\{G_{i_j}\} \subset \{G_i\}$ such that G_{i_j} is a proper subgroup of $G_{i_{j+1}}$ for each j. Let $G'_j := G_{i_j}$. Since Γ acts properly discontinuously on X, for each j, there exists an element $g_j \in G'_j \setminus G'_{j-1}$ such that $d(x_0, g_j x_0) = d(x_0, (G'_j \setminus G'_{j-1})x_0)$.

Here $g_j \neq g_k$ for $j \neq k$ because $(G'_j \setminus G'_{j-1}) \cap (G'_k \setminus G'_{k-1}) = \emptyset$. Hence there exists a subsequence $\{g_{j_k}x_0\} \subset \{g_jx_0\}$ which converges to a point $\xi(\infty) \in \partial X$, where ξ is a geodesic ray issuing from x_0. Since $\{g_{j_k}x_0\} \subset Fx_0$, $\xi(\infty) \in \partial F = \bigcup_{i=1}^{\infty} \partial G_i$ by (1). Hence $\xi(\infty) \in \partial G'_m$ for some m. Since G'_m is geometrically finite, $\operatorname{Im}\xi \subset G'_m B(x_0, N)$ for some $N > 0$. Let $R > N + 1$. Since $\{g_{j_k}x_0\}$ converges to $\xi(\infty)$, for large enough $k > m$, $d(x_0, g_{j_k}x_0) > R$ and $d(\xi(R), \xi_{g_{j_k}x_0}(R)) < 1$, where $\xi_{g_{j_k}x_0}$ is the geodesic segment from x_0 to $g_{j_k}x_0$. Since $\operatorname{Im}\xi \subset G'_m B(x_0, N)$, there exists $g \in G'_m$ such that $d(gx_0, \xi(R)) \leq N$. Then,

$$\begin{aligned}
d(x_0, g^{-1}g_{j_k}x_0) &= d(gx_0, g_{j_k}x_0) \\
&\leq d(gx_0, \xi(R)) + d(\xi(R), \xi_{g_{j_k}x_0}(R)) + d(\xi_{g_{j_k}x_0}(R), g_{j_k}x_0) \\
&< N + 1 + (d(x_0, g_{j_k}x_0) - R) \\
&= d(x_0, g_{j_k}x_0) - (R - N - 1) \\
&< d(x_0, g_{j_k}x_0).
\end{aligned}$$

We note that $g^{-1}g_{j_k} \in G'_{j_k} \setminus G'_{j_k-1}$ because $g^{-1} \in G'_m \subset G'_{j_k-1} \subset G'_{j_k}$. This contradicts the assumption $d(x_0, g_{j_k}x_0) = d(x_0, (G'_{j_k} \setminus G'_{j_k-1})x_0)$. Therefore (1) implies (3).

(ii) (1) $\Rightarrow$ (3): Suppose that (1) holds and $H \neq G_i$ for each i. Then there exists a subsequence $\{G_{i_j}\} \subset \{G_i\}$ such that $G_{i_{j+1}}$ is a proper subgroup of G_{i_j}. Let $G'_j := G_{i_j}$. Since Γ acts properly discontinuously on X, for each j, there exists an element $g_j \in G'_j \setminus G'_{j+1}$ such that $d(x_0, g_jx_0) = d(x_0, (G'_j \setminus G'_{j+1})x_0)$. Then there exists a subsequence $\{g_{j_k}x_0\} \subset \{g_jx_0\}$ which converges to a point $\xi(\infty) \in \partial X$, where ξ is a geodesic ray issuing from x_0. For each $i \geq 1$, $\{g_{j_l}x_0\}_{l \geq k} \subset G_ix_0$ for some large number k. Hence $\xi(\infty) \in \bigcap_{i=1}^{\infty} \partial G_i = \partial H$ by (1). Since H is geometrically finite, $\operatorname{Im}\xi \subset HB(x_0, N)$ for some $N > 0$. Let $R > N + 1$. Since $\{g_{j_k}x_0\}$ converges to $\xi(\infty)$, for large enough k, $d(x_0, g_{j_k}x_0) > R$ and

$d(\xi(R), \xi_{g_{j_k}x_0}(R)) < 1$, where $\xi_{g_{j_k}x_0}$ is the geodesic segment from x_0 to $g_{j_k}x_0$. Since $\operatorname{Im}\xi \subset HB(x_0, N)$, $d(hx_0, \xi(R)) \leq N$ for some $h \in H$. Then, by the same argument as the one in (i),

$$d(x_0, h^{-1}g_{j_k}x_0) = d(hx_0, g_{j_k}x_0) < d(x_0, g_{j_k}x_0) - (R - N - 1) < d(x_0, g_{j_k}x_0).$$

We note that $h^{-1}g_{j_k} \in G'_{j_k} \setminus G'_{j_k+1}$ because $h^{-1} \in H \subset G'_{j_k+1} \subset G'_{j_k}$. This contradicts the assumption $d(x_0, g_{j_k}x_0) = d(x_0, (G'_{j_k} \setminus G'_{j_k+1})x_0)$. Therefore (1) implies (3). $\square$

Using this proposition, we prove Theorem 5.4.12.

Proof of Theorem 5.4.12. (1): By the definition of limit sets,

$$\begin{aligned}\gamma(\partial G) &= \gamma(\overline{Gx_0} \cap \partial X) = \overline{\gamma G x_0} \cap \partial X \\ &= \overline{(\gamma G\gamma^{-1})(\gamma x_0)} \cap \partial X = \partial(\gamma G \gamma^{-1}).\end{aligned}$$

Since $\gamma G\gamma^{-1} \subset G$, we have $\gamma(\partial G) = \partial(\gamma G\gamma^{-1}) \subset \partial G$.

(2) and (3): First, we show that if G is geometrically finite, then $\gamma^i G\gamma^{-i}$ is also geometrically finite for each $i \in \mathbb{Z}$. Since G is geometrically finite, $\mathcal{L}^+_{x_0}(\partial G) \subset GK$ for some compact set K. We note that $\gamma^i(\partial G) = \partial(\gamma^i G\gamma^{-i})$ by the proof of (1). For each $i \in \mathbb{Z}$,

$$\begin{aligned}\mathcal{L}^+_{\gamma^i x_0}(\partial(\gamma^i G\gamma^{-i})) = \mathcal{L}^+_{\gamma^i x_0}(\gamma^i(\partial G)) &= \gamma^i(\mathcal{L}^+_{x_0}(\partial G)) \\ &\subset \gamma^i(GK) = (\gamma^i G\gamma^{-i})(\gamma^i K).\end{aligned}$$

Since $\gamma^i K$ is compact, $\gamma^i G\gamma^{-i}$ is geometrically finite.

Now we have a sequence

$$\cdots \subset \gamma^2 G\gamma^{-2} \subset \gamma G\gamma^{-1} \subset G \subset \gamma^{-1}G\gamma \subset \gamma^{-2}G\gamma^2 \subset \cdots .$$

Applying Proposition 5.4.13 to the sequence above, if either (a) or (b) holds, then $\gamma^n G\gamma^{-n} = F$ or H for some $n \in \mathbb{Z}$. In either case, we have that $\gamma G\gamma^{-1} = G$.

Then $\gamma(\partial G) = \partial G = \gamma^{-1}(\partial G)$ by (1).

□

Proposition 5.2.7 (2) and (5) are not always the case for Busemann spaces in general.

Example 5.4.14. We consider the same situation of Example 5.4.6. Let $G := \mathbb{Z} \times \mathbb{Z}$ act on $X := \mathbb{R} \times \mathbb{R}$ by $(a, b) \cdot (x, y) = (x + a, y + b)$ for each $(a, b) \in G$ and $(x, y) \in X$, let $H := \mathbb{Z} \times 0$ and $g := (0, 1) \in G$. Then G and H are geometrically finite, and $gHg^{-1} = H$. On the other hand, $g^n = (0, n) \notin H$ for each $n \in \mathbb{Z} \setminus \{0\}$ and $[\langle H, g\rangle : H] = [\mathbb{Z} \times \mathbb{Z} : \mathbb{Z} \times 0] = \infty$.

We show a Busemann space-analogue of Theorem 5.2.8 by a similar proof to the one in [G] and [R].

Theorem 5.4.15. *Let (X, d) be a proper Busemann space and Γ a group which acts properly discontinuously on X. Suppose that G_1 and G_2 are two geometrically finite subgroups of Γ. Then*

(1) *$G_1 \cap G_2$ is also geometrically finite,*

(2) *$\partial(G_1 \cap G_2) = \partial G_1 \cap \partial G_2$.*

Proof. Since G_1 and G_2 are geometrically finite, there exists a compact subset K of X such that $\mathcal{L}^+_{x_0}(\partial G_i) \subset G_i K$ for each $i = 1, 2$.

(1) Let $H := G_1 \cap G_2$. Choose coset representatives $\{a_\lambda\}$ and $\{b_\mu\}$ so that

$$G_1 = \bigcup_{\lambda\in\Lambda} Ha_\lambda \text{ and } G_2 = \bigcup_{\mu\in M} Hb_\mu.$$

Then,

$$\mathcal{L}^+_{x_0}(\partial G_1) \subset G_1 K = H\Big(\bigcup_{\lambda\in\Lambda} a_\lambda K\Big) \text{ and}$$
$$\mathcal{L}^+_{x_0}(\partial G_2) \subset G_2 K = H\Big(\bigcup_{\mu\in M} b_\mu K\Big).$$

Hence we have that

$$\begin{aligned}\mathcal{L}^{+}_{x_0}(\partial H) &\subset \mathcal{L}^{+}_{x_0}(\partial G_1) \cap \mathcal{L}^{+}_{x_0}(\partial G_2)\\ &= H\Big(\bigcup_{\lambda\in\Lambda} a_\lambda K\Big) \cap H\Big(\bigcup_{\mu\in M} b_\mu K\Big)\\ &= H\Big(\bigcup_{h\in H}\bigcup_{\lambda\in\Lambda}\bigcup_{\mu\in M}(a_\lambda K \cap h b_\mu K)\Big).\end{aligned}$$

We show that $\tilde{K} := \bigcup_{h\in H}\bigcup_{\lambda\in\Lambda}\bigcup_{\mu\in M}(a_\lambda K \cap h b_\mu K)$ is compact.

Since Γ acts properly discontinuously on X, the set

$$\{a_\lambda^{-1} h b_\mu \mid h \in H,\ \lambda \in \Lambda,\ \mu \in M,\ K \cap (a_\lambda^{-1} h b_\mu)K \neq \emptyset\}$$

is finite. Suppose that $a_{\lambda_1}^{-1} h_1 b_{\mu_1} = a_{\lambda_2}^{-1} h_2 b_{\mu_2}$ for some $h_i \in H$, $\lambda_i \in \Lambda$ and $\mu_i \in M$ $(i = 1, 2)$. Since $a_{\lambda_2} a_{\lambda_1}^{-1} h_1 = h_2 b_{\mu_2} b_{\mu_1}^{-1} \in H$, we have that $a_{\lambda_2} a_{\lambda_1}^{-1}, b_{\mu_2} b_{\mu_1}^{-1} \in H$. Hence $\lambda_1 = \lambda_2$, $\mu_1 = \mu_2$ and $h_1 = h_2$. Thus the set

$$\{(h, \lambda, \mu) \in H \times \Lambda \times M \mid K \cap (a_\lambda^{-1} h b_\mu)K \neq \emptyset\}$$

is finite, hence $\tilde{K}$ is compact.

(2) It is clear that $\partial(G_1 \cap G_2) \subset \partial G_1 \cap \partial G_2$. We prove that $\partial G_1 \cap \partial G_2 \subset \partial(G_1 \cap G_2)$. Let ξ be a geodesic ray issuing from x_0 with $\xi(\infty) \in \partial G_1 \cap \partial G_2$. Since $\mathcal{L}^{+}_{x_0}(\partial G_j) \subset G_j K$ $(j = 1, 2)$, for each $i = 1, 2, \dots$, there exist $a_i \in G_1$ and $b_i \in G_2$ such that $\xi(i) \in a_i K \cap b_i K$. Then both the sequences $\{a_i x_0\}$ and $\{b_i x_0\}$ converge to $\xi(\infty)$. Since Γ acts properly discontinuously on X, the set $\{a_i^{-1} b_i \mid K \cap a_i^{-1} b_i K \neq \emptyset\}$ is finite. Hence there exist subsequences $\{a_{i_n} x_0\}_n$ and $\{b_{i_n} x_0\}_n$ such that $a_{i_n}^{-1} b_{i_n} = a_{i_m}^{-1} b_{i_m}$ for each n, m. Then $a_{i_m} a_{i_n}^{-1} = b_{i_m} b_{i_n}^{-1} \in G_1 \cap G_2$. Let n_0 be a fixed number and $c_j := a_{i_j} a_{i_{n_0}}^{-1}$ for each $j = 1, 2, \dots$ Then the sequence $\{c_j x_0\}$ converges to $\xi(\infty)$ because $d(a_{i_j} x_0, c_j x_0) = d(a_{i_j} x_0, a_{i_j} a_{i_{n_0}}^{-1} x_0) = d(x_0, a_{i_{n_0}}^{-1} x_0)$ is constant. Since $\{c_j\} \subset G_1 \cap G_2$, $\xi(\infty) \in \partial(G_1 \cap G_2)$. Hence we have that $\partial G_1 \cap \partial G_2 = \partial(G_1 \cap G_2)$. $\square$

Corollary 5.4.16. *Let (X, d) be a proper Busemann space and Γ a group which acts properly discontinuously on X. If G_1 and G_2 are two geometrically finite subgroups of Γ and $\partial G_1 \subset \partial G_2$, then there exists a geometrically finite subgroup $G_1' \subset G_2$ such that $\partial G_1 = \partial G_1'$.*

Proof. Let $G_1' := G_1 \cap G_2$. Then G_1' is geometrically finite by Theorem 5.4.15 (1), and $\partial G_1' = \partial G_1 \cap \partial G_2 = \partial G_1$ by Theorem 5.4.15 (2). $\square$

In view of Corollary 5.4.16, it is natural to ask whether the following statement always holds: if G_1 and G_2 are geometrically finite subgroups of Γ and $\partial G_1 \subset \partial G_2$, then there exists a geometrically finite subgroup G_2' of Γ such that $G_1 \subset G_2'$ and $\partial G_2 = \partial G_2'$. However this is not always the case. We give an easy counter-example below.

Example 5.4.17. Let $\Gamma = \langle a, b \rangle$ be the rank two free group with basis $\{a, b\}$, X the Cayley graph of Γ with respect to $\{a, b\}$. Then Γ naturally acts on its Cayley graph X. Let $G_1 := \langle a \rangle$ and $G_2 := \langle a^2, b \rangle$. Then G_1 and G_2 are geometrically finite and $\partial G_1 \subset \partial G_2$. We show that there does not exist a subgroup G_2' of Γ such that $G_1 \subset G_2'$ and $\partial G_2 = \partial G_2'$. Let G_2' be a subgroup of Γ such that $G_1 \subset G_2'$ and $\partial G_2 \subset \partial G_2'$. Then $a \in G_1 \subset G_2'$ and $b^\infty \in \partial G_2 \subset \partial G_2'$. Hence $ab^\infty \in \partial G_2'$. On the other hand, it is clear that $ab^\infty \notin \partial G_2$. Thus $\partial G_2 \neq \partial G_2'$.

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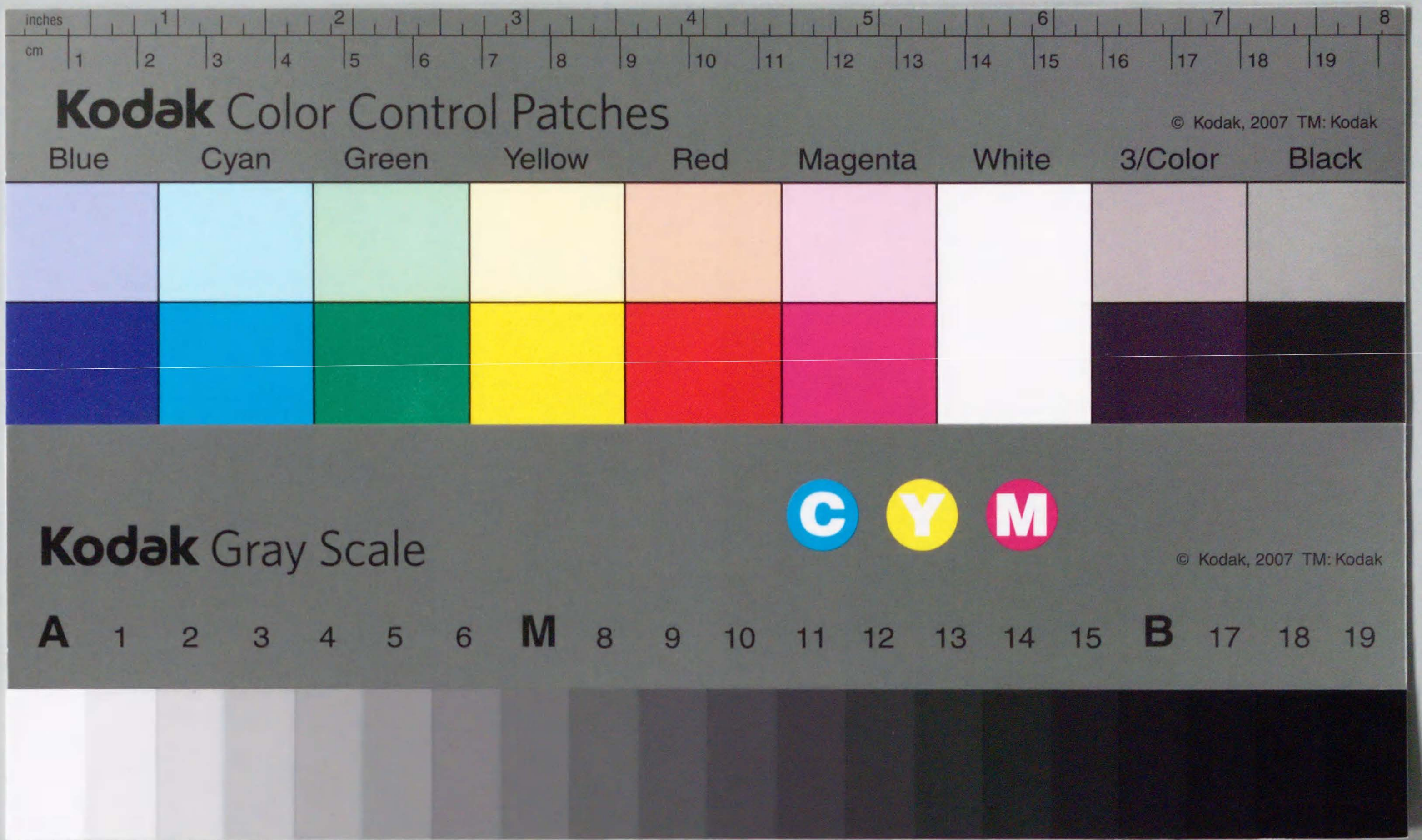
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