Extensions of Mappings on Product Spaces

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*) I would like to dedicate this work to the memory of my father.

Contents

Introduction

Throughout this paper all spaces are assumed to be T_1-spaces, and in Chapter 4 all spaces are assumed to be regular spaces. The letter γ denotes an infinite cardinal, and κ and λ denote cardinals. This paper consists of mainly three subjects. The first one is to study an extension property called P(locally-finite)-embedding. The second one is to study P(locally-finite)-embedding from the viewpoint of products. The third one is to study collectionwise normality of products from the approach by an extension property called weak z_γ-embedding.

Among properties of extending continuous mappings on a subspace over the whole space, the notions of C^*-, C- and P^γ-embeddings are most fundamental. A subspace A of a space X is said to be C^*- (respectively, C-) embedded in X if every bounded real-valued (respectively, real-valued) continuous function on A can be extended to a continuous one over X. A subspace A is said to be P^γ-embedded in X if every γ-separable continuous pseudo-metric on A can be extended to a continuous one over X. The C^*- and C-embeddings evidently come from the well-known Tietze-Urysohn's extension theorem ([61], [62]), and the notion of P^γ-embedding has its origin from a theorem of F. Hausdorff [20] on homeomorphic extensions of metric functions. These extension properties have played so far important roles in general topology and have been utilized in various areas such as dimension theory or shape theory, where normal open covers of spaces are basically used. In fact, the theorem below shows that normal open covers describe these extension properties. These covers are often easier to handle than continuous functions or pseudo-metrics.

Theorem 1 (Shapiro [57], Gantner [16]). *Let X be a space and A a subspace of X. Then, A is C^*- (respectively, C- or P^γ-) embedded in X if and only if for every normal open cover $\mathcal{U}$ of A with $|\mathcal{U}| < \omega$ (respectively, $\leq \omega$ or $\leq \gamma$), there exists a normal open cover $\mathcal{V}$ of X such that $\mathcal{V} \wedge A (= \{V \cap A : V \in \mathcal{V}\})$ refines $\mathcal{U}$.*

Recently, in consideration of the interface between set-theoretic and algebraic topology, Dydak [12] investigated an extension theory of continuous functions which take their values in metric simplicial complexes or CW-complexes. He proved some theorems characterizing several notions defined in terms of extensions of partitions of unity, and showed that these notions are closely related to P^γ-embedding. As one of such notions, it is defined in [12] that a subspace A of a space X is P^γ(locally-finite)-embedded in X if for every locally finite partition $\{f_\alpha : \alpha \in \Omega\}$ of unity on A with $|\Omega| \leq \gamma$, there exists a locally finite partition $\{g_\alpha : \alpha \in \Omega\}$ of unity on X such that $g_\alpha|A = f_\alpha$ for every $\alpha \in \Omega$. A subspace A of a space X is said to be P(locally-finite)-embedded in X if A is P^γ(locally-finite)-embedded in X for every γ. The P^γ(locally-finite)-embedding is strictly stronger than P^γ-embedding.

We remind that the notion of P(locally-finite)-embedding originally relates to Katětov [30] and Przymusiński-Wage [54]. The main purpose of Chapter 2 is to give a characterization of P^γ(locally-finite)-embedding by locally finite covers of cozero-sets as the following:

Theorem 2 (Theorem 2.1.6). *Let X be a space and A a subspace of X. Then, A is P^γ(locally-finite)-embedded in X if and only if every locally finite cover, with Card $\leq \gamma$, of cozero-sets of A can be extended to a locally finite cover of cozero-sets of X.*

Theorem 2 was proved by Przymusiński-Wage [54] assuming that X is normal and A is closed in X; the assumption is essential in their proof. Theorem 2 shows that P(locally-finite)-embedding defined by Dydak in connection with an algebraic viewpoint is precisely equal to the notion which was discussed by Katětov [30] or Przymusiński-Wage [54] in a set-theoretic topology. With the aid of Theorem 2, we can prove the following result related to products with a compact factor.

Theorem 3 (Theorem 2.2.3 (3)). *Let X be a space, A a subspace of X and γ an infinite cardinal. If A is P^γ(locally-finite)-embedded in X, then $A \times C$ is P^γ(locally-finite)-embedded in $X \times C$ for every compact Hausdorff space C with $w(C) \leq \gamma$, where $w(C)$ denotes the weight of C.*

When we put $C = I$ in Theorem 3, it is an affirmative answer to a problem posed by Dydak in [12].

In Chapter 3, the motivation of our results is from the following two theorems.

Theorem 4 (Alò-Sennott [2], Morita-Hoshina [39], Przymusiński [49]). *Let X be a space, A a subspace of X and γ an infinite cardinal. Then, the following statements are equivalent:*

(1) *A is P^γ-embedded in X;*

(2) $A \times Y$ *is* C^**-embedded in* $X \times Y$ *for every compact Hausdorff space* Y *with* $w(Y) \leq \gamma$;

(3) $A \times A(\gamma)$ *is* C^**-embedded in* $X \times A(\gamma)$, *where* $A(\gamma)$ *denotes the one-point compactification of the discrete space of cardinality* γ.

Theorem 5 (Przymusiński [52]). *Let* κ *be a cardinal,* X *a normal space and* A *a closed subspace of* X. *Then, the following statements are equivalent*:

(1) *Every countable locally finite cover of open* F_κ*-sets of* A *can be extended to a locally finite open cover of* X;

(2) $A \times J(\kappa)$ *is* C^**-embedded in* $X \times J(\kappa)$;

(3) $A \times J_0(\kappa)$ *is* C^**-embedded in* $X \times J_0(\kappa)$,

where $J(\kappa)$ *denotes the hedgehog with* κ *spines and* $J_0(\kappa)$ *denotes the zero-dimensional hedgehog with* κ *spines.*

By introducing a new space $J_\gamma(\kappa)$ and a new class of spaces of type $t(\gamma, \kappa, \lambda)$, we characterize P^γ(locally-finite)-embedding as follows:

Theorem 6 (Theorem 3.3.1). *Let* X *be a space,* A *a subspace of* X *and* γ *an infinite cardinal. Then, the following statements are equivalent*:

(1) A *is* P^γ*(locally-finite)-embedded in* X;

(2) $A \times Y$ *is* C^**-embedded in* $X \times Y$ *for every space* Y *of type* $t(\gamma, \omega, \gamma)$;

(3) $A \times J_\gamma(\omega)$ *is* C^**-embedded in* $X \times J_\gamma(\omega)$.

A product space $X \times Y$ of spaces X and Y is called rectangularly normal if $A \times B$ is C-embedded in $X \times Y$ for any closed subspace A and B of X and Y, respectively [52]. A natural and interesting question is that when $X \times Y$ is rectangularly normal. Only a few results which give rectangular normality have been known. Przymusiński proved in [52] that a space X is a countably functionally Katětov space (respectively, a countably Katětov space) if and only if $X \times J(\omega)$ (respectively, $X \times J(\kappa)$ for every κ) is rectangularly normal. Extending his result, we characterize (γ, κ)-Katětov spaces by rectangular normality of products with $J_\gamma(\kappa)$ and spaces of type $t(\gamma, \kappa, \gamma)$ as follows.

Theorem 7 (Theorem 3.4.2). *Let* X *be a space,* A *a subspace of* X *and* γ, κ *infinite cardinals. Then, the following statements are equivalent*:

(1) X *is* (γ, κ)*-Katětov*;

(2) $X \times Y$ *is rectangularly normal for every space* Y *of type* $t(\gamma, \kappa, \gamma)$;

(3) $X \times J_\gamma(\kappa)$ *is rectangularly normal.*

In Chapter 4, we apply weak z_γ-embedding, which is defined in Chapter 2 as one of extension properties, to consider the classical problem that:

> Under what conditions, is the product $X \times Y$ collectionwise normal if $X \times Y$ is normal?

Concerning this problem, in [40] Nagami showed the following:

Theorem 8 (Nagami [40]). *The following statements hold.*

(1) *For a paracompact σ-space X and a paracompact P-space Y, the product $X \times Y$ is paracompact.*

(2) *For a paracompact σ-space X and a collectionwise normal P-space Y, the product $X \times Y$ is normal if and only if $X \times Y$ is collectionwise normal.*

In (1) of Theorem 8, the case replacing "σ-space" by "M-space" was proved by Morita [34]. In another paper [41], extending σ-spaces as well as M-spaces, Nagami defined new spaces called Σ-spaces, and improved (1) of Theorem 8 as well as the Morita's result as the following:

Theorem 9 (Nagami [41]). *For a paracompact Σ-space X and a paracompact P-space Y, the product $X \times Y$ is paracompact.*

This theorem is one of well-known results asserting that the product of two paracompact spaces is paracompact. After his paper [41], taking products of P-spaces and Σ-spaces, some analogous results were obtained. In view of Theorems 8 and 9, it is natural to ask whether "σ-space" in (2) of Theorem 8 can be generalized to "Σ-space". Indeed, Yang asked it in [73] as follows:

> Let X be a paracompact Σ-space and Y a collectionwise normal P-space. Suppose that $X \times Y$ is normal. Then, is $X \times Y$ collectionwise normal?

The case (1) of the following theorem is an affirmative answer to the above problem, that is, an improvement of (2) of Theorem 8. Moreover, the cases (3) and (4) are improvements of the K. Chiba's results in [10].

Theorem 10 (Theorem 4.2.1). *Suppose that X and Y satisfy one of the following conditions. Then, $X \times Y$ is normal if and only if $X \times Y$ is collectionwise normal.*

(1) *X is a paracompact Σ-space and Y is a collectionwise normal P-space;*

(2) *X is a collectionwise normal Σ-space and Y is a collectionwise normal first countable P-space;*

(3) *X is the closed continuous image of a paracompact M-space and Y is a collectionwise normal P-space;*

(4) *X is the closed continuous image of a normal M-space and Y is a collectionwise normal first countable P-space.*

Let us note that the cases (2) and (4) seem first positive ones with no assuming the paracompactness of either X or Y.

The results in this paper are mainly quoted from [67], [68], [70] and [71]. The detailed citation will be denoted on the last part of each chapter.

Chapter 1.

Preliminaries

Throughout this paper, all spaces are assumed to be T_1-spaces, the letter γ denotes an infinite cardinal, and κ and λ denote cardinals. The letter I stands for the closed unit interval $[0, 1]$. In this chapter, we review definitions of some extension properties and their fundamental facts. As for basic references, see Alò-Shapiro [3], Engelking [13], Gillman-Jerison [18] and Hoshina [24].

1. Definitions of basic extension properties

First, we state the most basic notion in our research.

Definition 1.1.1 (cf. [18]). A subspace A of a space X is said to be C^*-*embedded* (respectively, C-*embedded*) in X if every bounded real-valued (respectively, real-valued) continuous function on A can be continuously extended over X.

A subspace A of a space X is said to be a *zero-set* of X if $A = f^{-1}(\{0\})$ for some continuous function $f : X \to I$. The complement of a zero-set is called a *cozero-set*.

Definition 1.1.2 (cf. [3]). A subspace A of a space X is said to be z-*embedded* in X if every zero-set in A is the intersection of A with a zero-set of X.

Let A_1- and A_2-embeddings be extension properties. We mean by "A_1-embedding *implies* A_2-embedding" that every A_1-embedded subspace of any space X is A_2-embedded in X and write "$A_1 \to A_2$". Likewise, when A_1-embedding is equivalent to A_2-embedding, we write "$A_1 = A_2$". We denote by "$A_1 + A_2$" A_1-embedding and A_2-embedding. By Definitions 1.1.1 and 1.1.2, it is clear that C-embedding implies C^*-embedding, and the latter implies z-embedding.

For a cover $\mathcal{U}$ of a space X, put $\mathcal{U}^* = \{\mathrm{St}(U,\mathcal{U}) : U \in \mathcal{U}\}$. A sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of open covers of a space X is said to be *normal* if $\mathcal{U}^*_{n+1} < (=$ refines) $\mathcal{U}_n$ for each $n \in \mathbb{N}$. An open cover $\mathcal{U}$ of X is said to be *normal* if there exists a normal sequence $\{\mathcal{U}_n : n \in \mathbb{N}\}$ of X such that $\mathcal{U}_1 < \mathcal{U}$.

We adopt the definition of P^γ-embedding as follows instead of the original one which was stated in the introduction (cf. Thorem 1.2.5).

Definition 1.1.3 (cf. [3]). A subspace A of a space X is said to be *P^γ-embedded* in X if for every normal open cover $\mathcal{U}$ of A with $|\mathcal{U}| \leq \gamma$, there exists a normal open cover $\mathcal{V}$ of X such that $\mathcal{V} \wedge A (= \{V \cap A : V \in \mathcal{V}\}) < \mathcal{U}$. A subspace A of a space X is said to be *P-embedded* in X if A is P^γ-embedded in X for every γ.

Definition 1.1.4 (Blair [6]). A subspace A of a space X is said to be *z_γ-embedded* in X if every normal open cover $\mathcal{U}$ of A with $|\mathcal{U}| \leq \gamma$, there exists a cozero-set G of X containing A and a normal open cover $\mathcal{V}$ of G such that $\mathcal{V} \wedge A < \mathcal{U}$. A subspace A of a space X is said to be *z_∞-embedded* in X if A is z_γ-embedded in X for every γ.

Note that z_γ-embedding was defined as a cardinal generalization of z-embedding, i.e., "$z_\omega = z$" holds.

Let X be a space and $\mathcal{A} = \{A_\alpha : \alpha \in \Omega\}$ a collection of subsets of X. Then, $\mathcal{A}$ is said to be *uniformly locally finite* (respectively, *uniformly discrete*) in X if there exist a locally finite (respectively, discrete) collection $\{G_\alpha : \alpha \in \Omega\}$ of cozero-sets of X and a collection $\{Z_\alpha : \alpha \in \Omega\}$ of zero-sets of X such that $A_\alpha \subset Z_\alpha \subset G_\alpha$ for every $\alpha \in \Omega$ (Morita [37], Ohta [43] and Blair [6]).

Definition 1.1.5 (Hoshina [21]). A subspace A of a space X is said to be *U^γ-embedded* in X if every uniformly locally finite collection $\mathcal{U}$ of subsets of A with $|\mathcal{U}| \leq \gamma$ is uniformly locally finite in X. A subspace A of a space X is said to be *U-embedded* in X if A is U^γ-embedded in X for every γ.

Note that P^γ-embedding implies U^γ-embedding ([21]).

Some of the properties like the above are often called "embedding", "weak-embedding" or "weak extension properties". In this paper, we call properties defined like the above simply "extension properties".

2. Review of characterizations of basic extension properties

A collection $\{f_\alpha : \alpha \in \Omega\}$ of continuous functions from a space X into I is said to be a *partition of unity* on X if $\sum_{\alpha\in\Omega} f_\alpha(x) = 1$ for every $x \in X$, where $\sum_{\alpha \in \Omega} f_\alpha(x)$ means the least upper bound of all sums of finitely many $f_\alpha(x)$' s. A partition $\{f_\alpha : \alpha \in \Omega\}$ of unity on X is said to be *subordinated* to a cover $\{U_\alpha : \alpha \in \Omega\}$ of X if $f_\alpha^{-1}((0,1]) \subset U_\alpha$ for every $\alpha \in \Omega$. Disjoint subsets A_1 and A_2 of a space X are said to be *completely separated* in X if there exists a continuous function $f : X \to I$ such that $f(A_1) = 0$ and $f(A_2) = 1$. Clearly, A_1 and A_2 are completely separated in X if and only if there exist disjoint zero-sets Z_1 and Z_2 of X such that $A_i \subset Z_i$ $(i = 1, 2)$. Normal open covers can be represented by various forms as the following:

Theorem 1.2.1 ([3], [33], [34]). *For an open cover $\mathcal{U}$ of a space X, the following statements are equivalent:*

(1) *$\mathcal{U}$ is normal;*

(2) *$\mathcal{U}$ is refined by a locally finite cover of cozero-sets of X;*

(3) *$\mathcal{U}$ is refined by a σ-locally finite (or σ-discrete) cover of cozero-sets of X;*

(4) *$\mathcal{U}$ has a partition of unity subordinated to it;*

(5) *$\mathcal{U}$ is refined by a locally finite cover $\{V_U : U \in \mathcal{U}\}$ of cozero-sets of X such that $\overline{V_U}$ and $X - U$ are completely separated in X for each $U \in \mathcal{U}$.*

Here, we review characterizations of C^*-, C- and z-embedding as the following. In the next theroem, (1) $\Leftrightarrow$ (2) is due to Gillman-Jerison [18], (1) $\Leftrightarrow$ (3) is due to Morita-Hoshina [38], and (1) $\Leftrightarrow$ (4) is due to Morita [35]. The abbreviated word AR means the absolute retract for the class of metrizable spaces.

Theorem 1.2.2 ([18], [35], [38]). *Let X be a space and A a subspace of X. Then, the following statements are equivalent:*

(1) *A is C^*-embedded in X;*

(2) *Every disjoint zero-sets Z_1 and Z_2 in A are completely separated in X;*

(3) *For every finite normal open cover $\mathcal{U}$ of A, there exists a normal open cover $\mathcal{V}$ of X such that $\mathcal{V} \wedge A < \mathcal{U}$;*

(4) *Every continuous map $f : A \to Y$ into a compact AR is continuously extended over X.*

A subspace A of a space X is called *well-embedded* in X if every zero-set disjoint from A and A are completely separated in X.

In the following theorem, (1) $\Leftrightarrow$ (2) is due to Gillman-Jerison [18] and Blair-Hager [7], (1) $\Leftrightarrow$ (3) is due to Gantner [16], and (1) $\Leftrightarrow$ (4) is due to Morita [35].

Theorem 1.2.3 ([7], [16], [18], [35]). *Let X be a space and A a subspace of X. Then, the following statements are equivalent:*

(1) *A is C-embedded in X*;

(2) *A is z- (or C^*-) embedded and well-embedded in X*;

(3) *For every countable normal open cover $\mathcal{U}$ of A, there exists a normal open cover $\mathcal{V}$ of X such that $\mathcal{V} \wedge A < \mathcal{U}$*;

(4) *Every continuous map $f : A \to Y$ into a Čech-complete separable AR is continuously extended over X.*

Especially, we have the following:

Corollary 1.2.4 (Gantner [16]). *The P^{ω}-embedding equals C-embedding.*

Let κ be a cardinal. Let $I_\beta = I \times \{\beta\}$ for every $\beta < \kappa$. Define the equivalence relation E on $\bigcup_{\beta<\kappa} I_\beta$ as $(x,\beta_1)E(y,\beta_2)$ whenever $x = y = 0$ or ($x = y$ and $\beta_1 = \beta_2$). Denote by $J(\kappa)$ the set of all equivalence classes with respect to E and define a metric on $J(\kappa)$ as follows:

$$\rho((x,\beta_1),(y,\beta_2)) = \begin{cases} |x-y| & \text{if } \beta_1 = \beta_2, \\ x+y & \text{if } \beta_1 \neq \beta_2 \end{cases}$$

for every (x,β_1), $(y,\beta_2) \in J(\kappa)$. The set $J(\kappa)$ with this metric is called the *hedgehog with κ spines.* The p stands for the class of $J(\kappa)$ consisting of $(0,\beta), \beta < \kappa$. The letter $\langle x,\beta\rangle$ denotes the equivalence class of (x,β).

The letter $J_0(\kappa)$ denotes the zero-dimensional hedgehog with κ spines, i.e., the subspace $\{p\} \cup \{\langle 1/n,\beta\rangle : n \in \mathbb{N}, \beta < \kappa\}$ of $J(\kappa)$. The hedgehog is usually defined for infinite cardinals (cf. [13]). Notice that our definition admits the case that κ is finite.

For a space Y, $w(Y)$ denotes the weight of Y. In the following theorem, (1) $\Leftrightarrow$ (2) is due to Hoshina [21] or Morita-Hoshina [39], (1) $\Leftrightarrow$ (3) is due to Blair [6], (4) is the original definition of P^{γ}-embedding by Shapiro [57] (see [3]), and (1) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) are due to Morita [35] or Przymusiński [49].

Theorem 1.2.5 ([6], [21], [35], [49], [57]). *Let X be a space and A a subspace of X. Then, the following statements are equivalent:*

(1) *A is P^{γ}-embedded in X*;

(2) *A is U^{γ}-embedded and z- (or C^*-, C-) embedded in X*;

(3) *A is z_γ-embedded and well- (or C-) embedded in X*;

(4) *Every γ-separable continuous pseudo-metric on A can be extended to a continuous pseudo-metric on X*;

(5) *Every continuous map $f : A \to J(\gamma)$ is continuously extended over X*;

(6) *Every continuous map $f : A \to Y$ into a Čech-complete AR space with $w(Y) \leq \gamma$ is continuously extended over X.*

Theorem 1.2.6 (Hoshina [23]). *Let X be a space and A a subspace of X. Then the following statements are equivalent:*

(1) *A is z_γ-embedded in X;*

(2) *For every locally finite cover $\mathcal{U}$ of cozero-sets of A with $|\mathcal{U}| \leq \gamma$, there exists a σ-locally finite collection $\mathcal{V}$ of cozero-sets of X such that $\mathcal{V}$ covers A and $\mathcal{V} \wedge A < \mathcal{U}$.*

3. Review of basic facts

In this section, we mention useful facts for our later discussion. First we state the most basic result in our research. The implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are well-known as Tietze-Urysohn's extension theorem ([61], [62]) (cf. [3]).

Theorem 1.3.1 (Tietze-Urysohn's extension theorem [61], [62]). *For a space X, the following statements are equivalent:*

(1) *X is normal;*

(2) *Every closed subspace of X is C-embedded in X;*

(3) *Every closed subspace of X is C^*-embedded in X;*

(4) *Every closed subspace of X is z-embedded in X.*

A space X is said to be *γ-collectionwise normal* if every discrete closed collection $\mathcal{F}$ of X with $|\mathcal{F}| \leq \gamma$ can be separated by a disjoint open collection of X. A space X is said to be collectionwise normal if X is γ-collectionwise normal for every γ. It is well-known that X is ω-collectionwise normal if and only if X is normal (cf. [3] or [13]).

In the following theorem, (1) $\Leftrightarrow$ (2) is due to Dowker [11], and (1) $\Leftrightarrow$ (3) is due to Blair [6].

Theorem 1.3.2 ([6], [11]). *For a space X, the following statements are equivalent:*

(1) *X is γ-collectionwise normal;*

(2) *Every closed subspace of X is P^γ-embedded in X;*

(3) *Every closed subspace of X is z_γ-embedded in X.*

It follows from Theorem 1.3.2 that a space X is collectionwise normal if and only if every closed subspace of X is P- (or z_∞-) embedded in X.

The union of a locally finite collection of closed subsets is closed. On the other hand, the union of a locally finite collection of zero-sets is not necessarily a zero-set (see [24]). For calculation of some collection of zero-sets, the theorem below is useful.

Theorem 1.3.3 (Morita-Hoshina [39]). *Let X be a space and $\{A_\alpha : \alpha \in \Omega\}$ a uniformly locally finite collection of zero-sets of X. Then, $\bigcup\{A_\alpha : \alpha \in \Omega\}$ is a zero-set of X.*

Theorem 1.3.4 (Morita [37]). *Let X be a space and $\{A_\alpha : \alpha \in \Omega\}$ be a uniformly locally finite collection of C^*- (respectively, C- or P^γ-) embedded subspaces of X. If $A_\alpha \cup A_\beta$ is C^*-embedded in X for every $\alpha, \beta \in \Omega$, then $\bigcup\{A_\alpha : \alpha \in \Omega\}$ is C^*- (respectively, C- or P^γ-) embedded in X.*

Next we review basic results of extensions of mappings on products with a compact or a metric factor.

For the following result concerned with a compact factor, (1) $\Leftrightarrow$ (2) is due to Alò-Sennott [2], and (1) $\Leftrightarrow$ (3) is due to Morita-Hoshina [39] or Przymusiński [49].

Theorem 1.3.5 ([2], [39], [49]). *Let X be a space and A a subspace of X. Then, the following statements are equivalent:*

(1) A is P^γ-embedded in X;

(2) $A \times Y$ is C^*- (or P^γ-) embedded in $X \times Y$ for every compact Hausdorff space Y with $w(Y) \leq \gamma$;

(3) There exists a compact Hausdorff space Y with $w(Y) = \gamma$ such that $A \times Y$ is C^*- (or P^γ-) embedded in $X \times Y$.

By combining Theorems 1.3.4, 1.3.5 and 1.3.11, we have the following result; for a space Y, $\ell w(Y)$ denotes the *local weight* of Y, i.e., $\ell w(Y) = \sup\{w(y, Y) : y \in Y\}$ where $w(y, Y) = \min\{w(U) : U$ is a neighborhood of $y\}$.

Corollary 1.3.6. *Let X be a space and A a subspace of X. Then, A is P^γ-embedded in X if and only if $A \times Y$ is C^*- (or P^γ-) embedded in $X \times Y$ for every locally compact paracompact Hausdorff space Y with $\ell w(Y) \leq \gamma$.*

An application of Theorem 1.3.5 is the following result related to the homotopy extension property:

Theorem 1.3.7 (Morita-Hoshina [38]). *Let X be a space and A a subspace of X. Then, the following statements are equivalent:*

(1) A is P^γ-embedded in X;

(2) $(X \times B) \cup (A \times Y)$ is P^γ-embedded in $X \times Y$ for every compact Hausdorff space Y with $w(Y) \leq \gamma$ and every closed subspace B of Y;

(3) $(X \times \{0\}) \cup (A \times I)$ is P^γ-embedded in $X \times I$.

Related to a metric factor, we introduce the following result; it was recently proved by Gutev-Ohta answering to Przymusiński's problem in [51].

Theorem 1.3.8 (Gutev-Ohta [19]). *Let X be a space, A a P^γ-embedded subspace of X and Y a metric space. Then, the following statements are equivalent:*

(1) *$A \times Y$ is C^*-embedded in $X \times Y$;*

(2) *$A \times Y$ is U^ω-embedded in $X \times Y$;*

(3) *$A \times Y$ is P^γ-embedded in $X \times Y$.*

For Przymusiński's problem [51], see also [19], [24], [25] and [45]. In comparison with Theorem 1.3.8, note that C^*-embedding on (2) and (3) of Theorem 1.3.5 can not be changed into U^ω-embedding in general (see [21]).

Next, we review some basic results concerning with π-embedding.

Definition 1.3.9 (Przymusiński [50]). A subspace A of a space X is said to be *π-embedded* in X if $A \times Y$ is C^*-embedded in $X \times Y$ for every space Y.

Notice that A is π-embedded in X if and only if $A \times Y$ is P-embedded in $X \times Y$ for every space Y. The following theorem is known (see [24, Lemma 4.4]).

Theorem 1.3.10. *Every compact subspace of a Tychonoff space X is π-embedded in X.*

Theorem 1.3.11 (Morita [36]). *Every closed subspace of a locally compact paracompact Hausdorff space X is π-embedded in X.*

Theorem 1.3.12 (Michael, cf. [59]). *Every closed subspace of a metric space X is π-embedded in X.*

Finally, we list other important results:

Theorem 1.3.13 (Morita [33]). *For a normal γ-paracompact space X and a compact Hausdorff space Y with $w(Y) \leq \gamma$, the product $X \times Y$ is normal.*

We denote by $A(\gamma)$ the one-point compactification of the discrete space with cardinality γ.

Theorem 1.3.14 (Alas [1]). *A space X is γ-collectionwise normal and countably paracompact if and only if $X \times A(\gamma)$ is normal.*

Chapter 2.

Extensions of locally finite partitions of unity

In this chapter, we give a set-theoretical characterization of P(locally-finite)-embedding. By using it, we show that P(locally-finite)-embedding is preserved by the product with a compact factor. Our basic idea is to "exactly" extend locally finite covers of cozero-sets. We will make it clear how such extensions differ from so called extensions of refinements.

1. P(locally-finite)-embedding and its characterization

Let X be a space and A a subspace of X. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Omega\}$ be an indexed collection of subsets of A and $\mathcal{V} = \{V_\alpha : \alpha \in \Omega\}$ an indexed collection of subsets of X. If $V_\alpha \cap A = U_\alpha$ for every $\alpha \in \Omega$, we say $\mathcal{U}$ is *extended* to $\mathcal{V}$ or $\mathcal{V}$ is an *extention* of $\mathcal{U}$. If $U_\alpha \subset V_\alpha$ for every $\alpha \in \Omega$, we say $\mathcal{V}$ *expands* $\mathcal{U}$ or $\mathcal{V}$ is an *expansion* of $\mathcal{U}$. A partition $\{f_\alpha : \alpha \in \Omega\}$ of unity on X is said to be *locally finite* if $\{f_\alpha^{-1}((0,1]) : \alpha \in \Omega\}$ is locally finite in X.

Dydak defined in [12] the following notion.

Definition 2.1.1(Dydak [12]). Let X be a space and A a subspace of X. Then, A is *P^γ(locally-finite)-embedded* in X if for every locally finite partition $\{f_\alpha : \alpha \in \Omega\}$ of unity on A with $|\Omega| \leq \gamma$, there exists a locally finite partition $\{g_\alpha : \alpha \in \Omega\}$ of unity on X such that $g_\alpha | A = f_\alpha$ for every $\alpha \in \Omega$. If A is P^γ(locally-finite)-embedded in X for every γ, A is said to be *P(locally-finite)-embedded* in X.

In [12], it is stated that P^γ(locally-finite)-embedding implies P^γ-embedding, and the inverse implication need not hold (see Example 2.1.7 below).

From a set-theoretic viewpoint, we remind that the notion of P(locally-finite)-embedding originally relates to Katětov [30] and Przymusiński-Wage [54]. Katětov introduced in [30] some extension properties which are generalizations of collectionwise normal countably paracompactness; these properties was named later by Przymusiński-Wage [54].

Definition 2.1.2 ([30], [54]). A space X is said to be *Katětov* (respectively, *countably Katětov*) if X is normal and for every closed subspace A of X, every locally finite (respectively, countable locally finite) open cover of A can be extended to a locally finite open cover of X. A space X is said to be *functionally Katětov* (respectively, *countably functionally Katětov*) if X is normal and for every closed subspace A of X, every locally finite (respectively, countable locally finite) cover of cozero-sets of A can be extended to a locally finite open cover of X.

Katětov showed in [30] that every collectionwise normal and countably paracompact space is Katětov and that every functionally Katětov space is collectionwise normal. Similarily, he also stated in [30] that every normal and countably paracompact space is countably Katětov. So these extension properties are concluded as the following:

Diagram 2.1.3. The following implications hold, where CN means "collectionwise normal", CP means "countably paracompact", C. means "countably" and F. means "functionally".

$$\begin{array}{ccccccc}
\text{CN+CP} & \longrightarrow & \text{Katětov} & \longrightarrow & \text{F. Katětov} & \longrightarrow & \text{CN} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{N+CP} & \longrightarrow & \text{C. Katětov} & \longrightarrow & \text{C. F. Katětov} & \longrightarrow & \text{N}
\end{array}$$

Przymusiński-Wage showed by examples in [54] any of implications is not reversed. They also showed in [54] the following result; in their proof of (1) of the "only if" part, the normality of X and the closedness of A are essentially used.

Theorem 2.1.4 (Przymusiński-Wage [54]). *The following statements hold.*

(1) *A space X is functionally Katětov if and only if every locally finite partition of unity on any closed subspace A of X can be extended to a locally finite partition of unity on X.*

(2) *A space X is collectionwise normal if and only if every locally finite partition of unity on any closed subspace A of X can be extended to a (not necessarily locally finite) partition of unity on X.*

They also comment that (2) in the above can be generalized like (1) $\Leftrightarrow$ (2) of the following theorem; Dydak showed in [12] all of the conditions below are equivalent.

Theorem 2.1.5 ([12], [54]). *For a space X and a subspace A of X, the following statements are equivalent:*

(1) *A is P^γ-embedded in X;*

(2) *Every locally finite partition, with Card $\leq \gamma$, of unity on A can be extended to a (not necessarily locally finite) partition of unity on X;*

(3) *Every partition, with Card $\leq \gamma$, of unity on A can be extended to a partition of unity on X.*

Thus, it is natural to ask whether a subspace A of a space X is P(locally-finite)-embedded in X if and only if every locally finite cover of cozero-sets of A can be extended to a locally finite cover of cozero-sets of X. From these points of view, we prove this equivalence as follows:

Theorem 2.1.6 (Main). *Let X be a space and A a subspace of X. Then, A is P^γ(locally-finite)-embedded in X if and only if every locally finite cover, with Card $\leq \gamma$, of cozero-sets of A can be extended to a locally finite cover of cozero-sets of X.*

Theorem 2.1.6 shows that P(locally-finite)-embedding, which was defined by Dydak in connection with algebraic viewpoints, is precisely equal to the notion which had discussed by Katětov or Przymusiński-Wage in a set-theoretic topology. From the points of Theorems 2.1.4 and 2.1.6, one may ask the following:

> Is it true that A is P^γ-embedded in X if and only if locally finite cover, with Card $\leq \gamma$, of cozero-sets of A can be extended to a cover of cozero-sets of X ?

On the case of $\gamma = \omega$, this is affirmatively answered easily. However on the case of $\gamma > \omega$, this is negative. Indeed, in Bing's space H ([5, Example H], see also [48]), there exists a closed subset A which is not P^γ-embedded in H. We have that $U \cup (H - A)$ is a cozero-set of H for every cozero-set U of A. Therefore every locally finite cover of cozero-sets of A can be extended to a cover of cozero-sets of H.

From now on, we use P(locally-finite)-embedding under the meaning of the charactrization in Theorem 2.1.6 without reference. Here, we hold an example which directly follows from Theorems 1.3.2 and 2.1.4.

Example 2.1.7 (Przymusiński-Wage [54, Example 3]). The P-embedding need not imply P^ω(locally-finite)-embedding.

We say that a subspace A of a space X is *L^γ-embedded* in X if every locally finite collection $\mathcal{U}$ of cozero-sets of A with $|\mathcal{U}| \leq \gamma$ has a locally finite expansion of cozero-sets of X. A subspace A of a space X is said to be L-embedded in X if A is L^γ-embedded in X for every γ.

Proposition 2.1.8. *Let X be a space, A a subspace of X and γ an infinite cardinal. Then, A is P^γ(locally-finite)-embedded in X if and only if A is C- and L^γ-embedded in X.*

From Proposition 2.1.8, it also follows that "P^γ(locally-finite)$=P^\gamma + L^\gamma$".

For the proof of Theorem 2.1.6, the following characterization of C-embedding is essential (cf. Remark 2.1.10).

Lemma 2.1.9 . *Let X be a space and A a subspace of X. Then, A is C-embedded in X if and only if for every continuous function $f : A \to I$ and disjoint zero-sets Z_0, Z_1 of X with $Z_i \cap A = f^{-1}(\{i\})$ $(i = 0, 1)$, there exists a continuous extension $g : X \to I$ of f such that $Z_i = g^{-1}(\{i\})$ $(i = 0, 1)$.*

By Ishii-Ohta [27], a subspace A of a space X is said to be *C_1-embedded* in X if any zero-set Z_1 of X and any zero-set Z_2 of A disjoint from Z_1 are completely separated in X. In [27] it is proved that C_1-embedding implies well-embedding; and by [21] U^ω-embedding implies C_1-embedding. Hence it follows from Theorem 1.2.3 that A is C-embedded in X if and only if A is C^*- and C_1-embedded in X ([27]).

Proof of Lemma 2.1.9. To prove the "if" part, assume that for every continuous function $f : A \to I$ and disjoint zero-sets Z_0, Z_1 of X with $Z_i \cap A = f^{-1}(\{i\})$ $(i = 0, 1)$, there exists a continuous extension $g : X \to I$ of f such that $Z_i = g^{-1}(\{i\})$ $(i = 0, 1)$. To prove C-embeddability of A in X, it suffices to show that any continuous function $f : A \to (0, 1)$ can be extended to a continuous function $g : X \to (0, 1)$. Regard f as $f : A \to I$ and apply the condition to $Z_0 = Z_1 = \emptyset$. Then the extension g of f satisfying the condition maps X into $(0, 1)$. Hence g is the desired extension.

To prove the "only if" part, suppose A is C-embedded in X. Let $f : A \to I$ be a continuous function and Z_0, Z_1 disjoint zero-sets of X with $Z_i \cap A = f^{-1}(\{i\})$ $(i = 0, 1)$. Let $\ell : X \to I$ be a continuous function satisfying that $\ell^{-1}(\{i\}) = Z_i$ $(i = 0, 1)$. At first, we prove the following claim.

Claim. There exists a continuous extension $h : X \to I$ of f such that $Z_i \subset h^{-1}(\{i\})$ $(i = 0, 1)$.

Proof of Claim. By induction, we shall construct continuous functions h_n :

$X \to [-1/2^{n-1}, 1/2^{n-1}]$ ($n \in \mathbb{N}$) which satisfy the following conditions:

(1) $h_1^{-1}(\{i\}) \supset Z_i$ ($i = 0, 1$) and $h_n^{-1}(\{0\}) \supset Z_0 \cup Z_1$ ($n \geq 2$); and

(2) $\left|f - \sum_{i=1}^{n} (h_i|A)\right| < 1/2^n$ ($n \in \mathbb{N}$).

Let $k_1 = f - \ell|A$ and put $F_1 = k_1^{-1}\big([-1, -1/2] \cup [1/2, 1]\big)$. Then, F_1 is a zero-set of A disjoint from $Z_0 \cup Z_1$. Since A is C_1-embedded in X, there exists a continuous function $j_1 : X \to I$ such that

$$j_1^{-1}(\{1\}) \supset F_1 \text{ and } j_1^{-1}(\{0\}) = Z_0 \cup Z_1.$$

Since A is C^*-embedded in X, there exists a continuous function $f_1 : X \to I$ such that $f_1|A = f$. Define a continuous function $h_1 : X \to I$ by

$$h_1(x) = j_1(x) \cdot f_1(x) + (1 - j_1(x)) \cdot \ell(x)$$

for every $x \in X$. Then, h_1 trivially satisfies the conditions (1) and (2).

Next assume that the continuous functions $h_1, \ldots, h_n$ are defined with the properties (1) and (2) for $i = 1, \ldots, n$. Put $k_{n+1} = f - \sum_{i=1}^{n}(h_i|A)$. Then, by the assumption (2), k_{n+1} takes its value in $[-1/2^n, 1/2^n]$. Put

$$F_{n+1} = k_{n+1}^{-1}\Big(\big[-\frac{1}{2^n}, -\frac{1}{2^{n+1}}\big] \cup \big[\frac{1}{2^{n+1}}, \frac{1}{2^n}\big]\Big).$$

Then, F_{n+1} is a zero-set of A disjoint from $Z_0 \cup Z_1$. Since A is C_1-embedded in X, there exists a continuous function $j_{n+1} : X \to I$ such that

$$j_{n+1}^{-1}(\{1\}) \supset F_{n+1} \text{ and } j_{n+1}^{-1}(\{0\}) = Z_0 \cup Z_1.$$

Since A is C^*-embedded in X, there exists a continuous function $f_{n+1} : X \to [-1/2^n, 1/2^n]$ such that $f_{n+1}|A = k_{n+1}$. Define a continuous function h_{n+1} by

$$h_{n+1}(x) = f_{n+1}(x) \cdot j_{n+1}(x)$$

for every $x \in X$. Then $h_{n+1} : X \to [-1/2^n, 1/2^n]$ is a continuous function satisfying (1) and (2). Hence the induction completes.

Put $h = ((\sum_{i \in \mathbb{N}} h_i) \wedge 1) \vee 0$. It is not hard to see that h is continuous, $h|A = f$ and $Z_i \subset h^{-1}(\{i\})$ ($i = 0, 1$). It completes the proof of Claim. □

Here, put $D = h^{-1}(\{0\}) \cup h^{-1}(\{1\}) - Z_0 \cup Z_1$. Notice that D can be represented as $D = \bigcup_{i \in \mathbb{N}} D_i$, where each D_i is a zero-set of X. Since $A \cap h^{-1}(\{i\}) = f^{-1}(\{i\}) = A \cap Z_i$ ($i = 0, 1$), we have $A \cap D = \emptyset$ and hence $A \cap D_i = \emptyset$ ($i \in \mathbb{N}$). Since A is well-embedded in X, there exists zero-set F_i of X such that $F_i \cap D_i = \emptyset$ and $A \subset F_i$. Since $\bigcap_{i \in \mathbb{N}} F_i$ is a zero-set of X, there exists a continuous function $\varphi : A \to I$ such that $\bigcap_{i\in\mathbb{N}} F_i = \varphi^{-1}(\{1\})$. Then it follows that

$$A \subset \varphi^{-1}(\{1\}) \quad \text{and} \quad \varphi^{-1}(\{1\}) \cap \big(h^{-1}(\{0\}) \cup h^{-1}(\{1\}) - Z_0 \cup Z_1\big) = \emptyset.$$

Define a continuous function $g : X \to I$ by

$$g(x) = \varphi(x) \cdot h(x) + (1 - \varphi(x)) \cdot \ell(x)$$

for every $x \in X$. Then, g is an extension of f. Finally we shall show that $Z_i = g^{-1}(\{i\})$ $(i = 0, 1)$. Since $Z_i = \ell^{-1}(\{i\}) \subset h^{-1}(\{i\})$ $(i = 0, 1)$, we have $Z_i \subset g^{-1}(\{i\})$ $(i = 0, 1)$. Suppose $x \notin Z_0 \cup Z_1$. Then $0 < \ell(x) < 1$. If $\varphi(x) = 1$, then $0 < h(x) < 1$ because of the definition of φ; it follows that $0 < g(x) < 1$. If $\varphi(x) < 1$, then $g(x) \geq (1 - \varphi(x)) \cdot \ell(x) > 0$ and $g(x) < \varphi(x) \cdot 1 + (1 - \varphi(x)) \cdot 1 = 1$; it follows that $0 < g(x) < 1$. These show that $X - Z_0 \cup Z_1 \subset g^{-1}\big((0, 1)\big)$. Thus we have $Z_i = g^{-1}(\{i\})$ $(i = 0, 1)$. The proof of Lemma 2.1.9 is completed. □

Proof of Theorem 2.1.6. The "only if" part is easy to see. Assume that every locally finite cover, with Card $\leq \gamma$, of cozero-sets of A can be extended to a locally finite cover of cozero-sets of X. By Theorems 1.2.1 and 1.2.3, we first note that A is C-embedded in X. Let $\{f_\alpha : \alpha < \gamma\}$ be a locally finite partition of unity on A. From the assumption, there exists a locally finite cover $\{U_\alpha : \alpha < \gamma\}$ of cozero-sets of X such that $U_\alpha \cap A = f_\alpha^{-1}((0, 1])$ for every $\alpha < \gamma$. By Lemma 2.1.9, there exists a continuous extension $g_\alpha : X \to I$ of f_α such that $g_\alpha^{-1}((0, 1]) = U_\alpha$ for every $\alpha < \gamma$. It is easy to see that $\sum_{\beta<\gamma} g_\beta$ is continuous and positive-valued. Hence $\{g_\alpha / \sum_{\beta<\gamma} g_\beta : \alpha < \gamma\}$ is the required locally finite partition of unity on X. □

Remark 2.1.10. Frantz proved in [14] Lemma 2.1.9 assuming the normality of X and the closedness of A. According to [14], Frantz's result shows that Tietze-Urysohn's extension theorem admits controlling the extended function so as to take on certain specified values. Lemma 2.1.9 shows the controlling extension itself equals C-embedding.

2. P(locally-finite)-embedding on products with a compact factor

In this section, we discuss the extension of P(locally-finite)-embedding with a compact factor. Our motivation of this section is the following problem posed by Dydak in [12].

Problem 2.2.1 (Dydak [12]). Let A be a P^γ(locally-finite)-embedded subspace of a space X. Then, is $A \times I$ P^γ(locally-finite)-embedded in $X \times I$?

He posed Problem 2.2.1 investigating the homotopy extensions of some extension properties containing P(locally-finite)-embedding. From this point of view, he proved the following result.

Theorem 2.2.2 (Dydak [12]). *Let X be a space and A a subspace of X. If $A \times I$ is P^γ(locally-finite)-embedded in $X \times I$, then $(X \times \{0\}) \cup (A \times I)$ is P^γ(locally-finite)-embedded in $X \times I$.*

For the products with a compact factor, we have the following conclusion. The (1) and (3) were known for the case of P-embedding (cf. Theorem 1.3.10 and Corollary 1.3.6). An affirmative answer to Problem 2.2.1 follows from (3) immediately.

Theorem 2.2.3. *Let X be a space and A a subspace of X. Then, the following statements hold.*

(1) *Let X be Tychonoff and A compact. Then for any space Y, $A \times Y$ is P(locally-finite)-embedded in $X \times Y$.*

(2) *Let A_α, $\alpha \in \Omega$, be L^γ-embedded subspace of X. If $\{A_\alpha : \alpha \in \Omega\}$ has a locally finite expansion of cozero-sets of X, then $\bigcup_{\alpha\in\Omega} A_\alpha$ is L^γ-embedded in X.*

(3) *Let A be P^γ(locally-finite)-embedded in X and Y a locally compact paracompact Hausdorff space with $\ell w(Y) \leq \gamma$. Then, $A \times Y$ is P^γ(locally-finite)-embedded in $X \times Y$.*

As an application of Theorem 2.2.3, we show the following result; the case of P^γ-embedding was known (cf. Theorem 1.3.7). The implication "(1) $\Rightarrow$ (3)" is also shown by Theorem 2.2.2 and (3) of Theorem 2.2.3.

Corollary 2.2.4. *Let X be a space and A a subspace of X. Then, the following statements are equivalent:*

(1) *A is P^γ(locally-finite)-embedded in X;*

(2) *$(X \times B) \cup (A \times Y)$ is P^γ(locally-finite)-embedded in $X \times Y$ for every compact Hausdorff space Y with $w(Y) \leq \gamma$ and every closed subspace B of Y;*

(3) *$(X \times \{0\}) \cup (A \times I)$ is P^γ(locally-finite)-embedded in $X \times I$.*

Let us prove Theorem 2.2.3.

Proof of Theorem 2.2.3. To prove (1), let X be Tychonoff, A compact and Y a space. By Theorem 1.3.10, $A \times Y$ is P-embedded in $X \times Y$. By Proposition 2.1.8, it suffices to show $A \times Y$ is L-embedded in $X \times Y$. To prove this, let $\{U_\alpha : \alpha \in \Omega\}$ be a locally finite cover of cozero-sets of $A \times Y$. Let $p_Y : A \times Y \to Y$ be the projection. Since the image of a cozero-set under an open perfect map is a cozero-set (see [15, Lemma 3.4] or [13, 1.5.L]), $p_Y(U_\alpha)$ is a cozero-set for every $\alpha \in \Omega$. Hence, $\{X \times p_Y(U_\alpha) : \alpha \in \Omega\}$ is a locally finite cover of cozero-sets of $X \times Y$ and expands $\{U_\alpha : \alpha \in \Omega\}$. It follows that $A \times Y$ is L-embedded in $X \times Y$. So, (1) holds.

To prove (2), let $\{U_\beta : \beta < \gamma\}$ be a locally finite collection of cozero-sets of $\bigcup_{\alpha\in\Omega} A_\alpha$. Let $\{G_\alpha : \alpha \in \Omega\}$ be a locally finite expansion of $\{A_\alpha : \alpha \in \Omega\}$ of cozero-sets of X. For every $\alpha \in \Omega$, there exists a locally finite collection $\{B^\alpha_\beta : \beta < \gamma\}$ of cozero-sets of X such that $U_\beta \cap A_\alpha \subset B^\alpha_\beta$ for every $\beta < \gamma$. Then, it is easily shown that $\{\bigcup_{\alpha\in\Omega}(B^\alpha_\beta \cap G_\alpha) : \beta < \gamma\}$ is a locally finite collection of cozero-sets of X such that $U_\beta \subset \bigcup_{\alpha\in\Omega}(B^\alpha_\beta \cap G_\alpha)$ for every $\beta < \gamma$. It follows that $\bigcup_{\alpha\in\Omega} A_\alpha$ is L^γ-embedded in X.

To prove (3), let A be a P^γ(locally-finite)-embedded subset of X. Let C be a compact Hausdorff space with $w(C) \leq \gamma$. To prove $A \times C$ is L^γ-embedded in $X \times C$, let $\mathcal{U} = \{U_\alpha : \alpha < \gamma\}$ be a locally finite collection of cozero-sets of $A \times Y$. Let $p_A : A \times C \to A$ be the projection. Since $\{p_A(U_\alpha) : \alpha < \gamma\}$ is a locally finite collection of cozero-sets of A, there exists a locally finite expansion $\{V_\alpha : \alpha \in \Omega\}$ of $\{p_A(U_\alpha) : \alpha < \gamma\}$ of cozero-sets of X. Clearly, $\{V_\alpha \times C : \alpha < \gamma\}$ is locally finite in $X \times C$ and $U_\alpha \subset V_\alpha \times C$ for each $\alpha < \gamma$. It shows that $A \times C$ is L^γ-embedded in $X \times C$.

Since Y has a uniformly locally finite cover of compact subsets with weight $\leq \gamma$, from (1) and (2) of this proposition and the fact shown above, $A \times Y$ is L^γ-embedded in $X \times Y$.

On the other hand, by Corollary 1.3.6, $A \times Y$ is C-embedded in $X \times Y$. Hence by Proposition 2.1.8, $A \times Y$ is P^γ(locally-finite)-embedded in $X \times Y$. It completes the proof. □

Proof of Corollary 2.2.4. (1) ⇒ (2): Let Y be a compact Hausdorff space with $w(Y) \leq \gamma$ and B a closed subspace of Y. Assume A is P^γ(locally-finite)-embedded in X. By Theorem 1.3.7, $(X \times B) \cup (A \times Y)$ is C-embedded in $X \times Y$. By Proposition 2.2.3, $(X \times B) \cup (A \times Y)$ is L^γ-embedded in $X \times Y$. Hence it follows from Proposition 2.1.8 that $(X \times B) \cup (A \times Y)$ is P^γ(locally-finite)-embedded in $X \times Y$.

(2) ⇒ (3): Obvious.

(3) ⇒ (1): Assume (3). Let $\{U_\alpha : \alpha < \gamma\}$ be a locally finite cover of cozero-sets of A. For a locally finite cover $\{U_\alpha \times (1/3, 1] : \alpha < \gamma\} \cup \{(A \times [0, 2/3)) \cup (X \times \{0\})\}$ of cozero-sets of $(X \times \{0\}) \cup (A \times I)$, there exists a locally finite cover $\{V_\alpha : \alpha < \gamma\} \cup \{W\}$ of cozero-sets of $X \times I$ such that $V_\alpha \cap ((X \times \{0\}) \cup (A \times I)) = U_\alpha \times (1/3, 1]$ for every $\alpha < \gamma$ and $W \cap ((X \times \{0\}) \cup (A \times I)) = (A \times [0, 2/3)) \cup (X \times \{0\})$. Let $V^*_0 =$

$(V_0 \cup W) \cap (X \times \{1\})$. Then $\{V_0^*\} \cup \{V_\alpha \cap (X \times \{1\}) : 1 \le \alpha < \gamma\}$ can be regarded as a locally finite cover of X extending $\{U_\alpha : \alpha < \gamma\}$. Hence A is P^γ(locally-finite)-embedded in X. The proof is completed. □

Remark 2.2.5. In (3) of Proposition 2.2.3, local-compactness of Y can not be replaced by Čech-completeness. Indeed, for Michael line X (see [13, 5.5.3]) and the irrationals Y, $\mathbb{Q}$ (= the rationals in X) is P(locally-finite)-embedded in X and $\mathbb{Q} \times Y$ is not C^*-embedded in $X \times Y$.

Finally, we introduce another problem posed by Dydak in [12]. In parallel with Problem 2.2.1, he posed the following:

Problem 2.2.6 (Dydak [12]). Let A be a P^γ(point-finite)-embedded* subspace of a space X. Then, is $A \times I$ P^γ(point-finite)-embedded in $X \times I$?

3. P(locally-finite)-embedding and functionally Katětov spaces

In this section, we give basic facts of P(locally-finite)-embedding and functionally Katětov spaces. For every $n \in \mathbb{N}$, the symbol $[\gamma]^n$ stands for $\{\delta \subset \gamma : |\delta| = n\}$. The $[\gamma]^{<\omega}$ stands for $\{\delta \subset \gamma : |\delta| < \omega\}$.

Our motivation is the following result by Smith-Krajewski [58]:

A space X is γ-expandable (i.e. every locally finite collection $\mathcal{U}$ of closed subsets of X with $|\mathcal{U}| \le \gamma$ can be expanded to a locally finite open collection of X) if and only if X is γ-boundedly expandable (i.e. every locally finite collection $\mathcal{U}$, with finite order, of closed subsets of X with $|\mathcal{U}| \le \gamma$ can be expanded to a locally finite open collection of X) and ω-expandable.

If X is assumed to be normal, the above fact is precisely the well-known Katětov's characterization of collectionwise normal and countably paracompactness ([30], see also [3, Theorems 12.4, 21,25 and 21,26] and [13, 5.5.17]). Motivated by the above result, we characterize P(locally-finite)-embedding and functionally Katětov spaces by the statements composed by the countable cardinal case and the finite order case.

Theorem 2.3.1. *Let X be a space and A a subspace of X. Then, A is P^γ(locally-finite)-embedded in X if and only if A is P^ω(locally-finite)-embedded in X and for every locally finite collection $\{U_\alpha : \alpha < \gamma\}$ of cozero-sets of A with finite*

A subspace A of a space X is said to be P^γ(point-finite)-embedded* in X if for every point-finite partition $\{f_\alpha : \alpha \in \Omega\}$ of unity on A with $|\Omega| \le \gamma$, there exists a point-finite partition $\{g_\alpha : \alpha \in \Omega\}$ of unity on X such that $g_\alpha|A = f_\alpha$ for every $\alpha \in \Omega$.

order, there exists a locally finite collection $\{V_\alpha : \alpha < \gamma\}$ of cozero-sets of X such that $U_\alpha \subset V_\alpha$ for every $\alpha < \gamma$.

Corollary 2.3.2. *A space X is functionally Katětov if and only if X is countably functionally Katětov and for every closed subspace A of X and every locally finite collection $\{U_\alpha : \alpha < \gamma\}$ of cozero-sets of A with finite order, there exists a locally finite collection $\{V_\alpha : \alpha < \gamma\}$ of cozero-sets of X such that $U_\alpha \subset V_\alpha$ for every $\alpha < \gamma$.*

To prove Theorem 2.3.1, we need a lemma.

Lemma 2.3.3. *Let $\mathcal{U} = \{U_\alpha : \alpha < \gamma\}$ be a locally finite collection of cozero-sets of a space X. For every $\delta \in \bigcup_{n\in\mathbb{N}}[\gamma]^n$, let $B_\delta = \bigcap_{\alpha\in\delta} U_\alpha - \bigcup_{\beta\notin\delta} U_\beta$. Then, for every $n \in \mathbb{N}$, there exists a locally finite disjoint collection $\{B^*_\delta : \delta \in [\gamma]^n\}$ of cozero-sets of X such that $B_\delta \subset B^*_\delta \subset \bigcap_{\alpha\in\delta} U_\alpha$ for every $\delta \in [\gamma]^n$.*

Proof. Fix $n \in \mathbb{N}$. We can express $B_\delta = \bigcup_{j\in\mathbb{N}} Z^j_\delta$, where Z^j_δ is a zero-set of X. Notice that $\{Z^j_\delta : \delta \in [\gamma]^n\}$ is uniformly locally finite in X for every $j \in \mathbb{N}$. Hence there exist a uniformly locally finite collection $\{F^j_\delta : \delta \in [\gamma]^n\}$ of zero-sets and a collection $\{G^j_\delta : \delta \in [\gamma]^n\}$ of cozero-sets of X such that $Z^j_\delta \subset G^j_\delta \subset F^j_\delta \subset \bigcap_{\alpha\in\delta} U_\alpha$ for every $\delta \in [\gamma]^n$. Put

$$B^*_\delta = \bigcup_{i\in\mathbb{N}}\Big(G^i_\delta - \bigcup\{F^j_\mu : j \leq i, \mu \in [\gamma]^n \text{ and } \mu \neq \delta\}\Big)$$

for every $\delta \in [\gamma]^n$. Since $\{F^j_\mu : \mu \in [\gamma]^n\}$ is uniformly locally finite, by Theorem 1.3.3, every B^*_δ is a cozero-set of X. It is easily shown that $\{B^*_\delta : \delta \in [\gamma]^n\}$ is the required collection. This completes the proof. □

Proof of Theorem 2.3.1. It suffices to show the "if" part. Assume that A is P^ω(locally-finite)-embedded in X and for every locally finite collection $\{U_\alpha : \alpha < \gamma\}$ of cozero-sets of A with finite order, there exists a locally finite collection $\{V_\alpha : \alpha < \gamma\}$ of cozero-sets of X such that $U_\alpha \subset V_\alpha$ for every $\alpha < \gamma$. Since A is C-embedded in X, by Proposition 2.1.8, it suffices to show that A is L^γ-embedded in X. Let $\mathcal{U} = \{U_\alpha : \alpha < \gamma\}$ be a locally finite cover of cozero-sets of A. Let $A_n = \{x \in A : \mathrm{ord}(x, \mathcal{U}) \geq n\}$ $(n \in \mathbb{N})$. Then A_n, $n \in \mathbb{N}$, is a cozero-set of A, because $A_n = \bigcup\{\bigcap_{\alpha\in\delta} U_\alpha : \delta \in [\gamma]^n\}$. Since $\{A_n : n \in \mathbb{N}\}$ is a locally finite cover of A, by the assumption, there exists a locally finite cover $\{A^*_n : n \in \mathbb{N}\}$ of cozero-sets of X such that $A^*_n \cap A = A_n$ for every $n \in \mathbb{N}$. For every $\delta \in \bigcup_{n\in\mathbb{N}}[\gamma]^n$, let $B_\delta = \bigcap_{\alpha\in\delta} U_\alpha - \bigcup_{\beta\notin\delta} U_\beta$. Then by Lemma 2.3.3, for every $n \in \mathbb{N}$, there exists a locally finite disjoint collection $\{B^*_\delta : \delta \in [\gamma]^n\}$ of cozero-sets of A such that $B_\delta \subset B^*_\delta$ for every

$\delta \in [\gamma]^n$. From the assumption, for every $n \in \mathbb{N}$, there exists a locally finite collection $\{V_\delta : \delta \in [\gamma]^n\}$ of cozero-sets of X such that $B_\delta^* \subset V_\delta$ for every $\delta \in [\gamma]^n$. Then, $\{V_\delta \cap A_n^* : \delta \in [\gamma]^n, n \in \mathbb{N}\}$ is a locally finite collection of cozero-sets of X. Put

$$U_\alpha^* = \bigcup \Big\{ V_\delta \cap A_n^* : \delta \in [\gamma]^n \text{ and } \alpha \in \delta, n \in \mathbb{N} \Big\}$$

for every $\alpha < \gamma$. Then we have that $\{U_\alpha^* : \alpha < \gamma\}$ is a locally finite collection of cozero-sets of X and $U_\alpha \subset U_\alpha^*$ for every $\alpha < \gamma$. Hence A is L^γ-embedded in X. It follows that A is P^γ(locally-finite)-embedded in X. It completes the proof. □

In view of Proposition 2.1.8, it is natural to ask the following two problems.

Problem 2.3.4. Suppose that a subspace A of a space X is L^ω- and P^γ-embedded (or equivalently, P^ω(locally-finite)- and P^γ-embedded) in X. Then, is A P^γ(locally-finite)-embedded in X?

Problem 2.3.5. Suppose that a space X is countably functionally Katětov and collectionwise normal. Then, is X functionally Katětov?

Problem 2.3.5 is compared with Przymusiński-Wage's question in [54, Question 3] that:

"Is a countably Katětov and collectionwise normal space a Katětov space?"

If Problem 2.3.4 is affirmative, then Problem 2.3.5 is also affirmative. Theorem 2.3.1 or Cororally 2.3.2 may be regarded as a partial answer to each of these problems.

4. Exact extensions versus extensions of refinements

The purpose of this section is to compare extensions of covers (we often call them exact extensions) and extensions of refinements of covers, and to show that they essentially differ.

Obviously we have that:

A subspace A of a space X is P^γ-embedded in X if and only if every normal open cover, with Card $\le \gamma$, of A can be extended to a normal open cover of X.

This shows that extensions of normal open covers are the same as extensions of some refinements of normal open covers.

Next, with the indexed forms, we express the facts already given.

(1) A is P^{γ}(locally-finite)-embedded in X if and only if for every locally finite cover $\{U_\alpha : \alpha < \gamma\}$ of cozero-sets of A, there exists a locally finite cover $\{H_\alpha : \alpha < \gamma\}$ of cozero-sets of X such that $H_\alpha \cap A = U_\alpha$ for each $\alpha < \gamma$.
(2) A is P^{γ}-embedded in X if and only if for every locally finite cover $\{U_\alpha : \alpha < \gamma\}$ of cozero-sets of A, there exists a locally finite cover $\{H_\alpha : \alpha < \gamma\}$ of cozero-sets of X such that $H_\alpha \cap A \subset U_\alpha$ for each $\alpha < \gamma$.

Example 2.1.7 shows, on the statement (2) of the above, "$\subset$" can not be changed into "$=$"(cf. (A) and (D) in the picture below). It shows that, on the case of locally finite covers of cozero-sets, exact extensions and extenions of some refinements are different.

Here, recall (3) of Theorem 1.2.1, a characterization of normal open covers. The P^{γ}-embedding is expressed as the following indexed form.

A subspace A of a space X is P^{γ}-embedded in X if and only if for every locally finite cover $\{U_\alpha : \alpha < \gamma\}$ of cozero-sets of A, there exists a cover $\{H_{i\alpha} : i < \omega, \alpha < \gamma\}$ of cozero-sets of X such that $\{H_{i\alpha} : \alpha < \gamma\}$ is locally finite in X $(i < \omega)$ and $(\bigcup_{i<\omega} H_{i\alpha}) \cap A \subset U_\alpha$ for every $\alpha < \gamma$.

From our viewpoint, we ask the following:

In the above result, can "$\subset$" be changed into "$=$"?

To answer this question, first, we give a definition of weak z_γ-embedding[†], which will be a key notion in our discussion.

Definition 2.4.1. A subspace A of a space X is *weakly z_γ-embedded* in X if for any uniformly discrete collection $\{F_\alpha : \alpha < \gamma\}$ of zero-sets of A, there exist locally finite collections $\mathcal{H}_i = \{H_{i\alpha} : \alpha < \gamma\}$ $(i < \omega)$ of cozero-sets of X such that $F_\alpha \subset \bigcup_{i<\omega} H_{i\alpha}$ for each $\alpha < \gamma$. If A is weakly z_γ-embedded in X for every γ, A is said to be *weakly z_∞-embedded* in X.

Notice that any subspace A of any space X is weakly z_ω-embedded in X. The following characterizations show that for weak z_γ-embedding, local-finiteness and uniformly local-finiteness play similar roles.

[†]This notion is introduced in [69] to discuss a cardinal generalization of C^*-embedding concerned with Ohta's problem in [46]. In this paper, we only give natural characterizations of weak z_γ-embedding and their proofs (cf. [69, Added in proof (2)]). For the details about this topic, see [46] and [69].

Theorem 2.4.2. *Let X be a space and A a subspace of X. Then, the following statements are equivalent:*

(1) *A is weakly z_γ-embedded in X;*

(2) *For every locally finite collection $\{U_\alpha : \alpha < \gamma\}$ of cozero-sets of A, there exist locally finite collections $\mathcal{H}_i = \{H_{i\alpha} : \alpha < \gamma\}$ $(i < \omega)$ of cozero-sets of X such that $U_\alpha \subset \bigcup_{i<\omega} H_{i\alpha}$ for each $\alpha < \gamma$;*

(3) *For every uniformly locally finite collection $\{U_\alpha : \alpha < \gamma\}$ of cozero-sets of A, there exist uniformly locally finite collections $\mathcal{H}_i = \{H_{i\alpha} : \alpha < \gamma\}$ $(i < \omega)$ of cozero-sets of X such that $U_\alpha \subset \bigcup_{i<\omega} H_{i\alpha}$ for each $\alpha < \gamma$.*

From Theorem 2.4.2, it follows that:

> *A subspace A of a space X is weakly z_γ-embedded in X if and only if for any collection $\{U_\alpha : \alpha < \gamma\}$ of A with locally finite expansion of cozero-sets of A, there exist locally finite collections $\mathcal{H}_i = \{H_{i\alpha} : \alpha < \gamma\}$ $(i < \omega)$ of cozero-sets of X such that $U_\alpha \subset \bigcup_{i<\omega} H_{i\alpha}$ for each $\alpha < \gamma$.*

From Thorem 2.4.2, we have the following corollaries. Corollary 2.4.5 answers the question mentioned above affirmatively.

Corollary 2.4.3. *A subspace A of a space X is z_γ-embedded in X if and only if A is weakly z_γ-embedded and z-embedded in X. Moreover, a subspace A of a space X is P^γ-embedded in X if and only if A is weakly z_γ-embedded and C-embedded in X.*

Corollary 2.4.4. *A subspace A of a space X is z_γ-embedded in X if and only if for every locally finite collection $\{U_\alpha : \alpha < \gamma\}$ of cozero-sets of A, there exist locally finite collections $\mathcal{H}_i = \{H_{i\alpha} : \alpha < \gamma\}$ $(i < \omega)$ of cozero-sets of X such that $U_\alpha = (\bigcup_{i<\omega} H_{i\alpha}) \cap A$ for each $\alpha < \gamma$.*

Corollary 2.4.5. *A subspace A of a space X is P^γ-embedded in X if and only if for every locally finite cover $\{U_\alpha : \alpha < \gamma\}$ of cozero-sets of A, there exist locally finite collections $\mathcal{H}_i = \{H_{i\alpha} : \alpha < \gamma\}$ $(i < \omega)$ of cozero-sets of X such that $U_\alpha = (\bigcup_{i<\omega} H_{i\alpha}) \cap A$ for each $\alpha < \gamma$ and $X = \bigcup_{i<\omega, \alpha<\gamma} H_{i\alpha}$.*

Let us proceed to the proofs. To prove Theorem 2.4.2, we need a lemma.

Lemma 2.4.6. *Any discrete collection of zero-sets of a space X with a locally finite expansion of cozero-sets is uniformly discrete in X.*

Proof. Let $\{F_\alpha : \alpha \in \Omega\}$ be a discrete collection of zero-sets of a space X and $\{G_\alpha : \alpha \in \Omega\}$ a locally finite expansion of $\{F_\alpha : \alpha \in \Omega\}$ of cozero-sets of X. By Theorem 1.3.3, $G_\alpha - \bigcup_{\beta \neq \alpha} F_\beta$ is a cozero-set for each $\alpha \in \Omega$.

Then, there exist a cozero-set W_α and a zero-set Z_α such that $F_\alpha \subset W_\alpha \subset Z_\alpha \subset G_\alpha - \bigcup_{\beta\neq\alpha} F_\beta$ for every $\alpha \in \Omega$. Take a cozero-set U_α such that $F_\alpha \subset U_\alpha \subset \overline{U_\alpha} \subset W_\alpha - \bigcup_{\beta\neq\alpha} Z_\beta$ for every $\alpha \in \Omega$. Then, we can see that $\{U_\alpha : \alpha \in \Omega\}$ is discrete. □

Proof of Theorem 2.4.2. (1) ⇒ (2): Let $\{U_\alpha : \alpha < \gamma\}$ be a locally finite collection of cozero-sets of A. For every $\delta \in [\gamma]^{<\omega}$, put $V_\delta = \bigcap_{\alpha\in\delta} U_\alpha - \bigcup_{\beta\notin\delta} U_\beta$. Fix $n < \omega$ arbitrarily. For every $\delta \in [\gamma]^n$, we can express $V_\delta = \bigcup_{k<\omega} Z_\delta^k$, where every Z_δ^k is a zero-set of A. Since $\{\bigcap_{\alpha\in\delta} U_\alpha : \delta \in [\gamma]^n\}$ is a locally finite collection of cozero-sets and $\{V_\delta : \delta \in [\gamma]^n\}$ is disjoint, $\{Z_\delta^k : \delta \in [\gamma]^n\}$ is a discrete collection of zero-sets of A with a locally finite expansion of cozero-sets for every $k < \omega$. Then, by Lemma 2.4.6, $\{Z_\delta^k : \delta \in [\gamma]^n\}$ is a uniformly discrete collection of A for every $k < \omega$. Fix $k < \omega$ arbitrarily. From the assumption, there exists a locally finite collection $\{W_\delta^{k,m} : \delta \in [\gamma]^n\}$ of cozero-sets of X for every $m < \omega$ such that $Z_\delta^k \subset \bigcup_{m<\omega} W_\delta^{k,m}$ for every $\delta \in [\gamma]^n$. Let us define now, for every $k, m, n < \omega$ and $\alpha < \gamma$, $H_{k,m,n,\alpha} = \bigcup\{W_\delta^{k,m} : \delta \in [\gamma]^n \text{ and } \alpha \in \delta\}$. Then, $\{H_{k,m,n,\alpha} : \alpha < \gamma\}$ is a locally finite collection of cozero-sets of X for every $k, m, n < \omega$. Moreover, we have that $U_\alpha \subset \bigcup_{k,m,n<\omega} H_{k,m,n,\alpha}$ for every $\alpha < \gamma$.

(2) ⇒ (3): Since for any cozero-set H there exist cozero-sets H_n, zero-sets Z_n and cozero-sets W_n $(n < \omega)$ such that $H = \bigcup_{n<\omega} H_n$ and $H_n \subset Z_n \subset W_n \subset H$ for every $n \in \mathbb{N}$, (3) follows.

(3) ⇒ (1): Since any uniformly discrete collection of zero-sets has a uniformly discrete expansion of cozero-sets, (1) obviously follows. It completes the proof. □

Proof of Corollary 2.4.3. The "if" part follows from (1) ⇔ (2) of Theorem 1.2.6 and (2) of Theorem 2.4.2 immediately.

To prove the "only if" part, assume that A is z_γ-embedded in X. Clearly A is z-embedded in X. To prove A is weakly z_γ-embedded in X, let $\{F_\alpha : \alpha < \gamma\}$ be a uniformly discrete collection of zero-sets of A. Let $\{U_\alpha : \alpha < \gamma\}$ be a discrete expansion of $\{F_\alpha : \alpha < \gamma\}$. By Theorem 1.2.6, there exists a σ-locally finite collection $\mathcal{H}$ of cozero-sets of X such that $\mathcal{H} \wedge A < \{U_\alpha : \alpha < \gamma\} \cup \{A - \bigcup_{\alpha<\gamma} F_\alpha\}$. So, we can easily construct sequences of locally finite collection of cozero-sets of X as the definition of weak z_γ-embedding.

Moreover, another statement also holds from Theorems 1.2.3 and 1.2.5 immediately. It completes the proof. □

Proof of Corollary 2.4.4. It follows from (1) ⇔ (2) of Theorem 2.4.2 and Corollary 2.4.3. □

Proof of Corollary 2.4.5. The "if" part follows from the definition of P-

embedding and Theorem 1.2.1 immediately. The "only if" part follows from Corollary 2.4.4 and (1) $\Leftrightarrow$ (3) of Theorem 1.2.5. □

For convenience, we prepare some extension properties by the indexed forms:

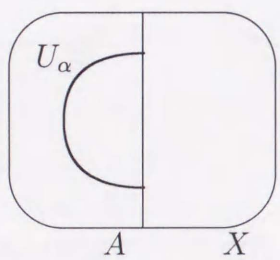
For a space X and a subspace A of X, some extension properties are expressed as follows:

(1) A is z_γ-embedded in X if and only if for every locally finite cover $\{U_\alpha : \alpha < \gamma\}$ of cozero-sets of A, there exists a locally finite cover $\{H_\alpha : \alpha < \gamma\}$ of cozero-sets of some cozero-set G of X containing A such that $H_\alpha \cap A \subset U_\alpha$ for each $\alpha < \gamma$.

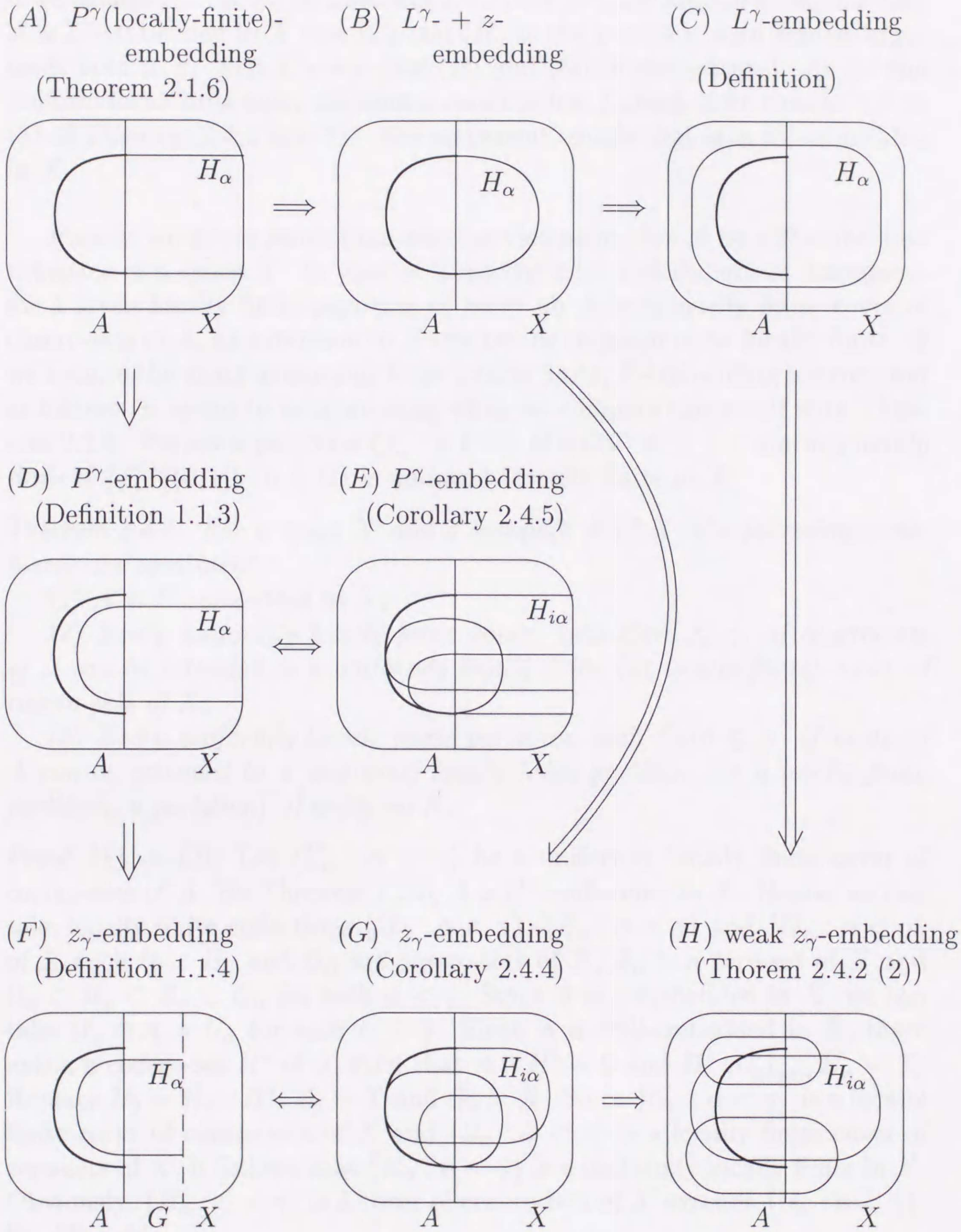
(2) A is L^γ-embedded in X if and only if for every locally finite cover $\{U_\alpha : \alpha < \gamma\}$ of cozero-sets of A, there exists a locally finite collection $\{H_\alpha : \alpha < \gamma\}$ of cozero-sets of X such that $U_\alpha \subset H_\alpha$ for each $\alpha < \gamma$.

(3) A is L^γ- and z-embedded in X if and only if for every locally finite cover $\{U_\alpha : \alpha < \gamma\}$ of cozero-sets of A, there exists a locally finite collection $\{H_\alpha : \alpha < \gamma\}$ of cozero-sets of X such that $H_\alpha \cap A = U_\alpha$ for each $\alpha < \gamma$.

Let X be a space and A a subspace of X. Let $\{U_\alpha : \alpha < \gamma\}$ be a locally finite cover of cozero-sets of A. Fix a U_α. We illustrate this situation as follows:



The following picture illustrate the relation of U_α and H_α (or $H_{i\alpha}$) on some extension properties.



If we change "ω" in Corollary 2.4.5 into "1", the condition equals that A is P^γ(locally-finite)-embedded in X (see (A) and (E) in the picture). Similarily if we change "ω" in (2) of Theorem 2.4.2 into "1", the statement equals that A is L^γ-embedded in X (see (C) and (H) in the picture). Also similar argument hold in z_γ- and L^γ- $+z$- (see (B) and (G) in the picture). As for the uniform local-finite case, the similar result holds. Indeed, if we change "ω" in (3) of Theorem 2.4.2 into "1", the statement equals that A is U^γ-embedded in X.

Finally, we give a result from another viewpoint. Let A be a P-embedded subspace of a space X. In view of Theorem 2.1.5 and the above discussion, for a given locally finite partition of unity on A or a locally finite cover of cozero-sets of A, its extension to X can not be required to be locally finite. If we require the exact extensions to be locally finite, P-embedding is expressed as follows. It seems to be interesting when we compare this result with Theorem 2.1.6. We say a partition $\{f_\alpha : \alpha \in \Omega\}$ of unity on X is *uniformly locally finite* if $\{f_\alpha^{-1}((0,1]) : \alpha \in \Omega\}$ is uniformly locally finite in X.

Theorem 2.4.7. *For a space X and a subspace A of X, the following statements are equivalent:*

(1) *A is P^γ-embedded in X;*

(2) *Every uniformly locally finite cover, with Card $\leq \gamma$, of cozero-sets of A can be extended to a uniformly locally finite (or locally finite) cover of cozero-sets of X;*

(3) *Every uniformly locally finite partition, with Card $\leq \gamma$, of unity of A can be extended to a uniformly locally finite partition (or a locally finite partition, a partition) of unity on X.*

Proof. (1) $\Rightarrow$ (2): Let $\{U_\alpha : \alpha < \gamma\}$ be a uniformly locally finite cover of cozero-sets of A. By Theorem 1.2.5, A is U^γ-embedded in X. Hence, we can take locally finite collections $\{H_\alpha : \alpha < \gamma\}$, $\{Z_\alpha : \alpha < \gamma\}$ and $\{G_\alpha : \alpha < \gamma\}$ of X such that H_α and G_α are cozero-sets of X, Z_α is a zero-set of X and $U_\alpha \subset H_\alpha \subset Z_\alpha \subset G_\alpha$ for each $\alpha < \gamma$. Since A is z-embedded in X, we can take $H_\alpha \cap A = U_\alpha$ for each $\alpha < \gamma$. Since A is well-embedded in X, there exists a cozero-set H^* of X such that $A \cap H^* = \emptyset$ and $H^* \cup \bigcup_{\alpha<\gamma} H_\alpha = X$. Replace $H_0 = H_0 \cup H^*$, $Z_0 = X$ and $G_0 = X$. Since $\{G_\alpha : \alpha < \gamma\}$ is a locally finite cover of cozero-sets of X and $\{Z_\alpha : \alpha < \gamma\}$ is a locally finite cover of zero-sets of X, it follows that $\{H_\alpha : \alpha < \gamma\}$ is a uniformly locally finite in X. Obviously, $\{H_\alpha : \alpha < \gamma\}$ is a cover of cozero-sets of X extends $\{U_\alpha : \alpha < \gamma\}$. So, (2) holds.

(2) $\Rightarrow$ (1): Let $\mathcal{U}$ be a normal open cover of A with $|\mathcal{U}| \leq \gamma$. From (1) $\Leftrightarrow$ (5) of Theorem 1.2.1, $\mathcal{U}$ has a uniformly locally finite refinement $\mathcal{V}$ of cozero-sets of A with $|\mathcal{V}| \leq \gamma$. Hence, by (3), $\mathcal{V}$ can be extended to a locally finite cover of cozero-sets of X. It shows that A is P^γ-embedded in X.

(2) $\Rightarrow$ (3): By the quite same way as the proof of Theorem 2.1.6, we can show them.

(3) $\Rightarrow$ (1): Assume that every uniformly locally finite partition, with Card $\leq \gamma$, of unity of A can be extended to a partition of unity on X. To complete the proof, it suffices to show A is P^γ-embedded in X. Let $\{U_\alpha : \alpha < \gamma\}$ be a normal open cover of A. By Theorem 1.2.1, $\{U_\alpha : \alpha < \gamma\}$ has a uniformly locally finite partition $\{f_\alpha : \alpha < \gamma\}$ of unity on A subordinated to $\{U_\alpha : \alpha < \gamma\}$. Hence, by the assumption, $\{f_\alpha : \alpha < \gamma\}$ can be extended to a partition of unity on X. It shows that A is P^γ-embedded in X. It completes the proof. □

Corollary 2.4.8. *Let X be a space and A a subspace of X. Assume that every locally finite collection, with Card $\leq \gamma$, of cozero-sets of A is uniformly locally finite in A. Then, A is P^γ-embedded in X if and only if A is P^γ(locally-finite)-embedded in X.*

It follows from Corollary 2.4.8 that every collectionwise normal P-space (=every G_δ-set is open) is functionally Katětov. On the other hand, Rudin's Dowker space [55] is a collectionwise normal P-space but not countably Katětov ([54, Example 2]).

Remark 2.4.9. Theorem 2.1.6 is proved in [70], the proof in this paper is essentially the same to the original but Lemma 2.1.9 is added here. On (3) of Theorem 2.2.3, the case that Y is compact Hausdorff is proved in [70]. The definition of weak z_γ-embedding and Corollary 2.4.3 are stated in [69], and other results are added here. For detailed results related to weak z_γ-embedding, see [69].

Chapter 3.

Rectangular normality of product spaces

In this chapter, introducing a space $J_\gamma(\kappa)$ and spaces of type $t(\gamma, \kappa, \gamma)$, we first characterize P^γ(locally-finite)-embedding by products with these spaces. Next, extending Przymusiński's result in [52], we also characterize Katětov spaces and functionally Katětov spaces by rectangular normality of products with these spaces. Moreover, we give characterizations of γ-collectionwise normal spaces, and γ-collectionwise normal λ-paracompact spaces by products with these spaces.

1. The space $J_\gamma(\kappa)$ and spaces of type $t(\gamma, \kappa, \lambda)$

Let κ be a cardinal. A subspace A of a space X is called an *F_κ-set* if it is the union of κ many closed sets in X. In [52] Przymusiński proved the following result, which is a motivation of our research.

Theorem 3.1.1 (Przymusiński [52, Proposition 2.2]). *Let κ be a cardinal, X a normal space and A a closed subspace of X. Then, the following statements are equivalent:**

(1) *Every countable locally finite cover of open F_κ-sets of A can be extended to a locally finite open cover of X;*

(2) *$A \times J(\kappa)$ is C^*-embedded in $X \times J(\kappa)$;*

(3) *$A \times J_0(\kappa)$ is C^*-embedded in $X \times J_0(\kappa)$.*

Here, we give detailed definitions of Katětov spaces. Let κ be a cardinal. In [45], a subspace A of a space X is called a *κ-open set* if it is the union of

*In [52, Proposition 2.2], "C^*-embedding" in (2) and (3) is written as "C-embedding". However his proof actually shows C^*-embedding of them, and only comments about C-embedding. If we use Theorem 1.3.8 or [66, Theorem 1.1], C-embedding of (2) or (3) is implied by C^*-embedding.

less than κ many cozero-sets of X. The complement of a κ-open set is called a *κ-closed set*. The letter κ^+ denotes the smallest cardinal larger than κ. In particular, *ω_1-open sets* mean cozero-sets. A *κ-open cover* (respectively, a *κ-open collection*) of a space X means a cover (respectively, a collection) consisting of κ-open sets of X.

Let γ and κ be infinite cardinals. We say that a space X is *(γ, κ)-Katětov* if X is normal and every locally finite κ^+-open cover, with Card $\leq \gamma$, of any closed subspace of X is extended to a locally finite κ^+-open cover of X. A space X is said to be *(∞, κ)-Katětov* (respectively, *(γ, ∞)-Katětov* or *(∞, ∞)-Katětov*) if X is (γ, κ)-Katětov for every γ (respectively, for every κ or for every γ and κ). Notice that (∞, ∞)-Katětov, (ω, ∞)-Katětov, (∞, ω)-Katětov and (ω, ω)-Katětov mean Katětov, countably Katětov, functionally Katětov and countably functionally Katětov, respectively, in Definition 2.1.2. Likewise Diagram 2.1.3, they follow that any γ-collectionwise normal and countably paracompact space is (γ, ∞)-Katětov, that any (γ, κ)-Katětov space is (γ', κ')-Katětov if $\gamma' \leq \gamma$ and $\kappa' \leq \kappa$, and that any (γ, κ)-Katětov space is γ-collectionwise normal.

Next we define a space $J_\gamma(\kappa)$ and spaces of type $t(\gamma, \kappa, \lambda)$. Let γ be an infinite cardinal and κ a cardinal. Let $J_\gamma(\kappa) = \{p\} \cup \{(\alpha, \beta) : \alpha < \gamma, \beta < \kappa\}$ be a space satisfying that the point p has basic neighborhoods of the form

$$\{p\} \cup \big\{(\alpha, \beta) : \alpha \in \gamma - \delta,\ \beta < \kappa\big\}; \quad \delta \in [\gamma]^{<\omega}$$

and other points (α, β) are isolated. From now on, we denote points (α, β) by $\langle \alpha, \beta \rangle$ as the definition of $J(\kappa)$. Notice that, for each $\beta < \kappa$, $\{p\} \cup \{\langle \alpha, \beta \rangle : \alpha < \gamma\}$ can be regarded as $A(\gamma)$, where $A(\gamma)$ is the one-point compactification of the discrete space with cardinality γ. Note that $J_\gamma(1) = A(\gamma)$ and $J_\omega(\kappa)$ can be regarded as $J_0(\kappa)$.

Let γ be an infinite cardinal and κ and λ cardinals. A Tychonoff space Y is said to be a space *of type $t(\gamma, \kappa, \lambda)$* if Y satisfies the following conditions:

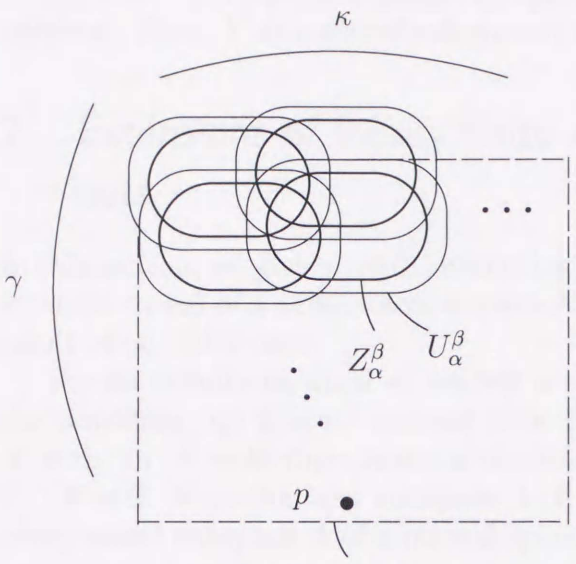
(1) Y can be represented as $Y' \cup \{p\}$, where $p \notin Y'$;

(2) there exist a locally finite open cover $\{U_\alpha^\beta : \alpha < \gamma, \beta < \kappa\}$ of Y' with $w(U_\alpha^\beta) \leq \lambda$ for every $\alpha < \gamma$ and $\beta < \kappa$ and a cover $\{Z_\alpha^\beta : \alpha < \gamma, \beta < \kappa\}$ of compact sets of Y' such that $Z_\alpha^\beta \subset U_\alpha^\beta$ for every $\alpha < \gamma$ and $\beta < \kappa$;

(3) $\{U_\delta(p) : \delta \in [\gamma]^{<\omega}\}$ is a neighborhood base of p, where $U_\delta(p)$ denotes $\{p\} \cup \bigcup\{U_\alpha^\beta : \alpha \in \gamma - \delta, \beta < \kappa\}$ for every $\delta \in [\gamma]^{<\omega}$;

(4) for every $\delta \in [\gamma]^{<\omega}$, there exists $\delta' \in [\gamma]^{<\omega}$ with $\delta \subset \delta'$ such that $\big(\bigcup\{U_\alpha^\beta : \alpha \in \delta, \beta < \kappa\}\big) \cap U_{\delta'}(p) = \emptyset$.

The condition (3) are illustrated as follows:



Notice that the existence of $\{Z_\alpha^\beta : \alpha < \gamma, \beta < \kappa\}$ of (2) implies local compactness and paracompactness of Y'. Hence we may assume each U_α^β is a cozero-set of Y' and each Z_α^β is a compact zero-set of Y'. This will be frequently used without reference.

The class of spaces of type $t(\gamma, \kappa, \lambda)$ includes spaces listed below from (a) to (d) as special cases.

(a) $A(\gamma)$ is of type $t(\gamma, 1, 1)$,

(b) $J(\kappa)$ is of type $t(\omega, \kappa, \omega)$,

(c) $J_\gamma(\kappa)$ is of type $t(\gamma, \kappa, 1)$,

(d) $A(\oplus_{\alpha<\gamma}(\lambda+1)_\alpha)$ is of type $t(\gamma, 1, \lambda)$,

where $A(\oplus_{\alpha<\gamma}(\lambda+1)_\alpha)$ is the one-point compactification of the topological sum of γ-many $\lambda + 1$' s (with the usual topologies).

Basic facts of spaces of type $t(\gamma, \kappa, \lambda)$ are the following; the proofs are easy and omitted.

Proposition 3.1.2. *The following statements hold.*

(1) *Every space of type $t(\gamma, \kappa, \lambda)$ is paracompact.*

(2) *A space of type $t(\gamma, \kappa, \lambda)$ is of type $t(\gamma', \kappa', \lambda')$ if $\gamma \leq \gamma'$, $\kappa \leq \kappa'$ and $\lambda \leq \lambda'$.*

(3) *Let Y be a closed subspace of a space of type $t(\gamma, \kappa, \lambda)$. If $p \in Y$, then Y is also a space of type $t(\gamma, \kappa, \lambda)$. If $p \notin Y$, then Y is locally compact paracompact, $w(Y) \leq \kappa \cdot \lambda$ and $\ell w(Y) \leq \lambda$.*

(4) *Let Y be a compact Hausdorff space with $w(Y) \leq \lambda$ and γ an infinite cardinal. Then, Y is a closed subspace of some space of type $t(\gamma, 1, \lambda)$.*

2. Extensions of locally finite κ^+-open covers and products

In this section, we give a result concerning with extendability of locally finite κ^+-open covers of a subspace of a space X to those of X. This will be a key result of our later ones.

For an infinite cardinal κ, we say a subspace A of a space X satisfies the *condition* $(\star_\kappa)$ if every zero-set Z of A and every κ^+-closed subset F of X with $Z \cap F = \emptyset$, there exists a cozero-set U of X such that $Z \subset U$ and $U \cap F = \emptyset$. Note that any subspace A of a space X satisfies $(\star_\omega)$, and that every closed subspace A of a normal space X satisfies $(\star_\kappa)$ for every κ.

Theorem 3.2.1. *Let γ and κ be infinite cardinals. For a space X and a subspace A of X, consider the following conditions.*

(1) *Every locally finite κ^+-open cover $\mathcal{U}$ of A with $|\mathcal{U}| \leq \gamma$ can be extended to a locally finite κ^+-open cover of X, and A is P^γ-embedded in X.*

(2) *Every locally finite κ^+-open collection $\mathcal{U}$ of A with $|\mathcal{U}| \leq \gamma$ can be extended to a locally finite κ^+-open collection of X, and A is P^γ-embedded in X.*

(3) *$A \times Y$ is C^*-embedded in $X \times Y$ for every space Y of type $t(\gamma, \kappa, \gamma)$.*

(4) *$A \times J_\gamma(\kappa)$ is C^*-embedded in $X \times J_\gamma(\kappa)$.*

Then, the implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) $\Rightarrow$ (2) always hold. If A satisfies $(\star_\kappa)$, then all conditions are equivalent.

Before the proof, we give a lemma.

Lemma 3.2.2. *Let γ be an infinite cardinal, and κ and λ cardinals. Let X be a space and Y a space of type $t(\gamma, \kappa, \lambda)$. Let $Y = Y' \cup \{p\}$ and $\{U_\alpha^\beta : \alpha < \gamma, \beta < \kappa\}$ be as the definition of $t(\gamma, \kappa, \lambda)$. Let $g_0 : X \to I$ and $g : X \times Y' \to I$ be continuous functions and $\{H_\alpha : \alpha < \gamma\}$ a locally finite open collection of X. Let $\psi : X \times Y' \to I$ be a continuous function which satisfies that $\psi^{-1}((0,1]) \subset \bigcup_{\alpha<\gamma} \bigcup_{\beta<\kappa} (H_\alpha \times U_\alpha^\beta)$. Define a function $h : X \times Y \to I$ by*

$$h(x,y) = \begin{cases} g_0(x) & \text{if } \ y = p, \\ \psi(x,y) \cdot g(x,y) + (1 - \psi(x,y)) \cdot g_0(x) & \text{otherwise} \end{cases}$$

for each $(x,y) \in X \times Y$. Then, h is continuous.

Remark 3.2.3. In Lemma 3.2.2, assume further that A is a subspace of $X \times Y$ satisfying that $p_X(A) \times \{p\} \subset A$, where $p_X : X \times Y \to X$ is the projection. If a function $f : A \to I$ is given so that $g|(A \cap (X \times Y')) = f|(A \cap (X \times Y'))$, $\{(x, y) \in A : |f(x, y) - f(p, y)| > 1/3\} \subset \psi^{-1}(\{1\})$, $\{a \in X : f(a, p) \leq 1/3\} \subset g_0^{-1}(\{0\})$ and $\{a \in X : f(a, p) \geq 2/3\} \subset g_0^{-1}(\{1\})$, then the function h defined like Lemma 3.2.2 satisfies $f^{-1}(\{i\}) \subset h^{-1}(\{i\})$ $(i = 0, 1)$. This fact was essentially proved by Przymusiński [52].

Proof of Theorem 3.2.1. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4): Obvious.

(4) $\Rightarrow$ (2): First we prove that A is P^γ-embedded in X. Since $A(\gamma)$ is homeomorphic to $\{p\} \cup \{\langle \alpha, 0 \rangle : \alpha < \gamma\}$, we may say that $A \times A(\gamma)$ is C^*-embedded in $A \times J_\gamma(\omega)$. From the assumption, $A \times A(\gamma)$ is C^*-embedded in $X \times J_\gamma(\omega)$, hence in $X \times A(\gamma)$. By Theorem 1.3.5, A is P^γ-embedded in X.

Let $\{U_\alpha : \alpha < \gamma\}$ be a locally finite κ^+-open collection of A. Since κ is infinite, U_α can be expressed as $U_\alpha = \bigcup_{\beta<\kappa} F_\alpha^\beta = \bigcup_{\beta<\kappa} W_\alpha^\beta$, where F_α^β is a zero-set of A, W_α^β is a cozero-set of A and $F_\alpha^\beta \subset W_\alpha^\beta$ for each $\beta < \kappa$. For every $\alpha < \gamma$ and $\beta < \kappa$, take a continuous function $f_\alpha^\beta : A \to I$ satisfying that $(f_\alpha^\beta)^{-1}(\{1\}) = F_\alpha^\beta$ and $(f_\alpha^\beta)^{-1}(\{0\}) = A - W_\alpha^\beta$. Define a continuous function $f : A \times J_\gamma(\kappa) \to I$ as follows: $f(x, y) = 0$ if $y = p$; $f(x, y) = f_\alpha^\beta(x)$ if $y = \langle \alpha, \beta \rangle$. By (4), there exists a continuous extension $g : X \times J_\gamma(\kappa) \to I$ of f. Put

$$V_\alpha^\beta = \left\{x \in X : |g(x, p) - g(x, \langle \alpha, \beta \rangle)| > 1/2\right\}$$

for each $\alpha < \gamma$ and $\beta < \kappa$. And, for every $\alpha < \gamma$, put $V_\alpha = \bigcup_{\beta<\kappa} V_\alpha^\beta$. Then, $\{V_\alpha : \alpha < \gamma\}$ is a locally finite κ^+-open collection of X such that $V_\alpha \cap A = U_\alpha$ for every $\alpha < \gamma$. Hence we have (2).

Next we prove (2) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3) assuming that A satisfies $(\star_\kappa)$.

(2) $\Rightarrow$ (1): Assume that A satisfies $(\star_\kappa)$. Let $\{U_\alpha : \alpha < \gamma\}$ be a locally finite κ^+-open cover of A. By (2), there exists a locally finite κ^+-open collection $\{V_\alpha : \alpha < \gamma\}$ of X such that $V_\alpha \cap A = U_\alpha$ for every $\alpha < \gamma$. Here V_α can be expressed as $\bigcup_{\beta<\kappa} W_\alpha^\beta$, where W_α^β is a cozero-set of X for every $\beta < \kappa$. Put $V = \bigcup_{\alpha<\gamma} V_\alpha$. Since $V = \bigcup_{\beta<\kappa}(\bigcup_{\alpha<\gamma} W_\alpha^\beta)$ and $\{W_\alpha^\beta : \alpha < \gamma\}$ is a locally finite collection of cozero-sets of X, V is a κ^+-open set of X. By $(\star_\kappa)$, there exists a cozero-set U of X such that $A \subset U$ and $U \subset V$. Since A is well-embedded in X, there exists a cozero-set V^* of X such that $V^* \cup U = X$ and $A \cap V^* = \emptyset$. Replace V_0 by $V_0 \cup V^*$. Then, $\{V_\alpha : \alpha < \gamma\}$ is the required locally finite κ^+-open cover of X extending $\{U_\alpha : \alpha < \gamma\}$. Hence we have (1).

(2) $\Rightarrow$ (3): Assume that A satisfies $(\star_\kappa)$. Let $Y = Y' \cup \{p\}$ be a space of type $t(\gamma, \kappa, \gamma)$, where $Y' = \{\langle \alpha, \beta \rangle : \alpha < \gamma, \beta < \kappa\}$. Let $\{U_\alpha^\beta : \alpha < \gamma, \beta < \kappa\}$ and $\{Z_\alpha^\beta : \alpha < \gamma, \beta < \kappa\}$ be the same as in the condition (2) of the definition

of Y. Let $f : A \times Y \to I$ be a continuous function. Notice that $A \times Y'$ is C-embedded in $X \times Y'$ by Corollary 1.3.6. So, there exists a continuous function $g : X \times Y' \to I$ such that $g|(A \times Y') = f|(A \times Y')$. Since A is C^*-embedded in X, there exists a continuous function $g_0 : X \to I$ such that

$$\{x \in A : f(x,p) \leq 1/3\} \subset g_0^{-1}(\{0\}) \text{ and } \{x \in A : f(x,p) \geq 2/3\} \subset g_0^{-1}(\{1\}).$$

For every $\alpha < \gamma$ and $\beta < \kappa$, define

$$G_\alpha^\beta = \left\{x \in A : |f(x,p) - f(x,y)| > 1/6 \text{ for some } y \in Z_\alpha^\beta\right\} \text{ and}$$

$$K_\alpha^\beta = \left\{x \in A : |f(x,p) - f(x,y)| \geq 1/3 \text{ for some } y \in Z_\alpha^\beta\right\}.$$

Since Z_α^β is compact, each G_α^β is a cozero-set of A and K_α^β is a zero-set of A. For every $\alpha < \gamma$, put $G_\alpha = \bigcup_{\beta<\kappa} G_\alpha^\beta$; it is a κ^+-open set of A. Notice that $\{G_\alpha : \alpha < \gamma\}$ is locally finite in A. By (2), there exists a locally finite κ^+-open collection $\{H_\alpha : \alpha < \gamma\}$ of X such that $G_\alpha = H_\alpha \cap A$ for every $\alpha < \gamma$. By $(\star_\kappa)$, for every $\alpha < \gamma$ and $\beta < \kappa$, there exists a cozero-set H_α^β of X such that $K_\alpha^\beta \subset H_\alpha^\beta \subset H_\alpha$. Define

$$G = \bigcup_{\alpha<\gamma} \bigcup_{\beta<\kappa} (H_\alpha^\beta \times U_\alpha^\beta) \text{ and } F = \bigcup_{\alpha<\gamma} \bigcup_{\beta<\kappa} (K_\alpha^\beta \times Z_\alpha^\beta).$$

Then, $F \subset G$ and G is a cozero-set of $X \times Y'$. Since $K_\alpha^\beta \times Z_\alpha^\beta \subset A \times U_\alpha^\beta$ and $\{A \times U_\alpha^\beta : \alpha < \gamma, \beta < \kappa\}$ is a locally finite collection of cozero-sets of $A \times Y'$, $\{K_\alpha^\beta \times Z_\alpha^\beta : \alpha < \gamma, \beta < \kappa\}$ is a uniformly locally finite collection of zero-sets of $A \times Y'$. Hence, by Theorem 1.3.3, F is a zero-set of $A \times Y'$. Since $A \times Y'$ is C_1-embedded in $X \times Y'$, there exists a continuous function $\psi : X \times Y' \to I$ such that $F \subset \psi^{-1}(\{1\})$ and $(X \times Y') - G \subset \psi^{-1}(\{0\})$. Then, the function $h : X \times Y \to I$ constructed as in Lemma 3.2.2 is continuous and $f^{-1}(\{i\}) \subset h^{-1}(\{i\})$ $(i = 0, 1)$ (see Remark 3.2.3). Hence $A \times Y$ is C^*-embedded in $X \times Y$; (3) is satisfied. It completes the proof. □

In Theorem 3.2.1, the infiniteness of κ is essential. Because, if κ is finite, the conditions (1) and (2) are equal to P^γ(locally-finite)-embeddability of A, and the conditions (3) and (4) are equal to P^γ-embeddability of A (cf. the proof of Theorem 3.5.1).

On the other hand, we remark that in Theorem 3.1.1, κ is not assumed to be infinite. Notice that, in Thorem 3.1.1, if κ is finite, then each of the conditions from (1) to (3) always holds for a normal space X and a closed subspace A of X.

Remark 3.2.4. The author does not know all of the conditions in Theorem 3.2.1 are equivalent without $(\star_\kappa)$. At least, either (1) or (4) in Theorem 3.2.1 need not imply $(\star_\kappa)$. For, consider the Tychonoff plank $T = ((\omega_1+1)\times(\omega+1)) - \{(\omega_1,\omega)\}$. Let $X = T$ and $A = \omega_1 \times \{\omega\}$. Then, for $\kappa = \omega_1$ and any infinite cardinal γ, A satisfies conditions (1) and (4), but not $(\star_{\omega_1})$. That A satisfies (1) and that A does not satisfy $(\star_{\omega_1})$ are easy to see (cf. Ohta [45, footnote p.6, English translation]). To prove that A satisfies (4), notice that $J_\gamma(\omega_1)$ is a Frechét space. Hence, by [13, Theorem 3.10.7], the projection $p_{J_\gamma(\omega_1)} : \omega_1 \times J_\gamma(\omega_1) \to J_\gamma(\omega_1)$ is the closed map. So, by [13, Theorem 3.12.21(a)], $\omega_1 \times J_\gamma(\omega_1)$ is C^*-embedded in $\beta\omega_1 \times J_\gamma(\omega_1)$. By the same way as [45], we have that $A \times J_\gamma(\omega_1)$ is C^*-embedded in $X \times J_\gamma(\omega_1)$. Indeed, $A \times J_\gamma(\omega_1)(= \omega_1 \times \{\omega\} \times J_\gamma(\omega_1))$ is C^*-embedded in $\beta\omega_1 \times \{\omega\} \times J_\gamma(\omega_1)$ $(= (\omega_1+1) \times \{\omega\} \times J_\gamma(\omega_1))$. Moreover, $(\omega_1+1) \times \{\omega\} \times J_\gamma(\omega_1)$ is C^*-embedded in $(\omega_1+1)\times(\omega+1)\times J_\gamma(\omega_1)$. Hence $A \times J_\gamma(\omega_1)$ is C^*-embedded in $(\omega_1+1)\times(\omega+1)\times J_\gamma(\omega_1)$; hence in $X \times J_\gamma(\omega_1)$.

3. Applications to P(locally-finite)-embedding

In this section, we characterize P(locally-finite)-embedding by products.

Theorem 3.3.1 (Main). *Let X be a space, A a subspace of X and γ an infinite cardinal. Then, the following statements are equivalent:*

(1) *A is P^γ(locally-finite)-embedded in X;*
(2) *$A \times Y$ is C^*-embedded in $X \times Y$ for every space Y of type $t(\gamma,\omega,\gamma)$;*
(3) *$A \times J_\gamma(\omega)$ is C^*-embedded in $X \times J_\gamma(\omega)$.*

In combination with (4) of Proposition 3.1.2, we have the following result. It may be natural if we compare the following result with Theorem 1.3.5.

Corollary 3.3.2. *Let X be a space, A a subspace of X and γ an infinite cardinal. Then, A is P^γ(locally-finite)-embedded in X if and only if $A \times Y$ is C^*-embedded in $X \times Y$ for every closed subspace Y of a space of type $t(\gamma,\omega,\gamma)$.*

On the case of $\gamma = \omega$, Theorem 3.3.1 can also be stated as follows (cf. Przymusiński [52]):

Corollary 3.3.3. *Let X be a space and A a subspace of X. Then, the following statements are equivalent:*

(1) *A is P^ω(locally-finite)-embedded in X;*
(2) *$A \times Y$ is C^*-embedded in $X \times Y$ for every space Y of type $t(\omega,\omega,\omega)$;*
(3) *$A \times J_0(\omega)$ is C^*-embedded in $X \times J_0(\omega)$;*

(4) *For some non-locally compact metric space* Y, $A \times Y$ *is* C^**-embedded in* $X \times Y$;

(5) $A \times Y$ *is* C^**-embedded in* $X \times Y$ *for every separable metric space* Y *such that* $Y - Y_1$ *is locally compact for some closed discrete subspace* Y_1 *of* Y.

Let us proceed to the proofs.

Proof of Theorem 3.3.1. Since A satisfies $(\star_\omega)$ and ω^+-open sets mean cozero-sets, Theorem 3.3.1 directly follows from Theorem 3.2.1. □

Proof of Corollary 3.3.2. The "if" part is contained in Theorem 3.3.1. The only "if" part follows from Corollary 1.3.6, (3) of Proposition 3.1.2 and Theorem 3.3.1. □

To prove Corollary 3.3.3, we need a lemma.

Lemma 3.3.4. *Let* γ *and* κ *be infinite cardinals. Let* X *be a space and* A *a subspace of* X. *Assume that* $A \times Y$ *is* C^**-embedded in* $X \times Y$ *for every space* Y *of type* $t(\omega, \kappa, \omega)$. *Then,* $A \times Y$ *is* C^**-embedded in* $X \times Y$ *for every metric space* Y *with* $w(Y) \leq \kappa$ *such that* $Y - Y_1$ *is locally compact for some closed discrete subspace* Y_1 *of* Y.

Proof. First, let Z be a metric space with $w(Z) \leq \kappa$ such that $Z' = Z - \{y_0\}$ is locally compact for some point $y_0 \in Z$. We shall prove Z is of type $t(\omega, \kappa, \omega)$. Take a local base $\{U_n : n < \omega\}$ of y_0 such that $Z = U_0$ and $\overline{U_{n+1}} \subset U_n$ for every $n < \omega$. There exists a countable locally finite open cover $\mathcal{V} = \bigcup_{n<\omega} \mathcal{V}_n$ of Z' such that each $\mathcal{V}_n = \{V_n^\beta : \beta < \kappa\}$ satisfies $\overline{V_n^\beta} \subset U_n - \overline{U_{n+2}}$ for every $n < \omega$ and $\beta < \kappa$, and that $\overline{V}$ is compact for every $V \in \mathcal{V}$. Hence it follows that Z is a space of type $t(\omega, \kappa, \omega)$.

To complete the proof, let Y be a metric space with weight $\leq \kappa$ and Y_1 a closed discrete subspace of Y satisfying that $Y - Y_1$ is locally compact. Then, there exists a uniformly locally finite closed cover $\mathcal{Z}$ of Y such that $\mathcal{Z} = \mathcal{Z}_1 \cup \mathcal{Z}_2$, where $\mathcal{Z}_1$ consists of compact subsets of Y and $\mathcal{Z}_2$ is a disjoint collection of subsets of Y satisfying that, for every $Z \in \mathcal{Z}_2$, $Z - \{y_0\}$ is locally compact for some point $y_0 \in Z$. Hence, by the fact shown above, every member of $\mathcal{Z}_2$ is a space of type $t(\omega, \kappa, \omega)$. Then, $A \times Z$ is C^*-embedded in $X \times Z$ for every $Z \in \mathcal{Z}$. To show this, first notice that A is C-embedded in X. Hence, if $Z \in \mathcal{Z}_1$, it follows from Thorem 1.3.5 that $A \times Z$ is C^*-embedded in $X \times Z$. If $Z \in \mathcal{Z}_2$, it follows from the assumption. Next, by Theorem 1.3.12, $A \times Z$ is C^*-embedded in $X \times Y$ for every $Z \in \mathcal{Z}$. Moreover, $A \times (Z_1 \cup Z_2)$ is C^*-embedded in $X \times Y$ for every $Z_1, Z_2 \in \mathcal{Z}$. Indeed, if $Z_1, Z_2 \in \mathcal{Z}_1$ then

$Z_1 \cup Z_2$ is compact, if $Z_1, Z_2 \in \mathcal{Z}_2$ then $Z_1 \cap Z_2 = \emptyset$, and if $Z_1 \in \mathcal{Z}_1$ and $Z_2 \in \mathcal{Z}_2$ then $Z_1 \cup Z_2$ is a space of type $t(\omega, \kappa, \omega)$. Therefore by the similar argument to the above, $A \times (Z_1 \cup Z_2)$ is C^*-embedded in $X \times Y$. It follows from Theorem 1.3.4 that $A \times Y$ is C^*-embedded in $X \times Y$. It completes the proof. □

Proof of Corollary 3.3.3. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3): It follows from Theorem 3.3.1.

(3) $\Rightarrow$ (4) and (5) $\Rightarrow$ (3): Obvious.

(4) $\Rightarrow$ (3): It can be shown similarily to [52, Proposition 2.2].

(2) $\Rightarrow$ (5): It follows from Lemma 3.3.4. □

4. Applications to (γ, κ)-Katětov spaces

Our main purpose in this section is to characterize (γ, κ)-Katětov spaces by rectangular normality of products.

First, we extend Theorem 3.1.1 to general cardinals as follows:

Theorem 3.4.1. *Let X be a normal space, A a closed subspace of X and γ and κ infinite cardinals. Then, the following statements are equivalent:*

(1) *Every locally finite κ^+-open cover $\mathcal{U}$ of A with $|\mathcal{U}| \leq \gamma$ can be extended to a locally finite κ^+-open cover of X;*

(2) *$A \times Y$ is C^*-embedded in $X \times Y$ for every space Y of type $t(\gamma, \kappa, \gamma)$;*

(3) *$A \times J_\gamma(\kappa)$ is C^*-embedded in $X \times J_\gamma(\kappa)$.*

Theorem 3.4.1 need not hold without the assumption of the normality of X if $\kappa > \omega$. Because for a non-normal countably compact Tychonoff space X with $w(X) \leq \kappa$, there exists a closed subspace A of X which satisfies (1) but does not satisfy (3).

As applications of Theorem 3.4.1, we describe (γ, κ)-Katětov spaces by rectangular normality of products with $J_\gamma(\kappa)$ and spaces of type $t(\gamma, \kappa, \gamma)$. A product space $X \times Y$ is said to be *rectangularly normal* if for every closed subspace A of X and closed subspace B of Y, $A \times B$ is C-embedded in $X \times Y$ [52]. Clearly, if $X \times Y$ is normal then $X \times Y$ is rectangularly normal, and it is not necessarily reversed (for, let X be any Dowker space and let $Y = I$). Przymusiński proved in [52, Theorems 2.3 and 2.4] that:

X is (ω, κ)-Katětov if and only if $X \times J_0(\kappa)$ (or $X \times J(\kappa)$) is rectangularly normal.

We also extend this result as follows:

Theorem 3.4.2 (Main). *Let X be a space and γ and κ infinite cardinals. Then, the following statements are equivalent:*

(1) *X is (γ, κ)-Katětov;*
(2) *$X \times Y$ is rectangularly normal for every space Y of type $t(\gamma, \kappa, \gamma)$;*
(3) *$X \times J_\gamma(\kappa)$ is rectangularly normal.*

On (2) in Theorem 3.4.2, $t(\gamma, \kappa, \gamma)$ can not be replaced by $t(\gamma, \kappa, \lambda)$ in general. Let X be a normal countably paracompact space which is not ω_1-collectionwise normal (e.g. [48]). Then, X has a closed subspace A which is not P^{ω_1}-embedded in X. Since every normal countably paracompact space is countably functionally Katětov, A is P^ω(locally-finite)-embedded in X. Let $Y = A(\oplus_{n<\omega}(\omega_1 + 1)_n)$. Then, $A \times Y$ need not be C^*-embedded in $X \times Y$ for Y. Indeed, on the contrary, $A \times Y$ is C^*-embedded in $X \times Y$. By Theorem 1.3.10 and the assumption, we have $A \times (\omega_1 + 1)$ is C^*-embedded in $X \times (\omega_1 + 1)$. It follows from Theorem 1.3.5 that A is P^{ω_1}-embedded in X, it is a contradiction.

It is known that A is C-embedded in X if and only if $A \times Y$ is C^*- (or equivalently, C-) embedded in $X \times Y$ for every locally compact metric space Y even if either Y is separable or not (see [25], also see Corollary 1.3.6). That is, this fact does not depend on the weight of Y. On the other hand, the condition on the weight of Y is essential in the following proposition.

Corollary 3.4.3. *A space X is (ω, κ)-Katětov if and only if for every metric space Y with $w(Y) \leq \kappa$ having a closed discrete subspace Y_1 with locally compact $Y - Y_1$, the product $X \times Y$ is rectangularly normal.*

Related to Corollary 3.4.3, Przymusiński states in [51, Theorem 4] that X is countably Katětov if and only if for every closed subset A of X and every σ-locally compact metric space Y, $A \times Y$ is C^*-embedded in $X \times Y$. However he gives its proof only for the case of $\dim Y = 0$, and comments "I have a very complicated proof that eliminates the assumption of $\dim Y = 0$" and asks the reasonable simple way of eliminating $\dim Y = 0$. The author does not know whether if the general case is true.

Proof of Theorem 3.4.1. Since X is normal and A is closed in X, A satisfies $(\star_\kappa)$. Hence all conditions of Theorem 3.2.1 are equivalent. To prove Theorem 3.4.1, it suffices to show that (1) of Theorem 3.4.1 implies that A is P^γ(locally-finite)-embedded in X. To prove this, let $\mathcal{U} = \{U_\alpha : \alpha < \gamma\}$ be a locally finite cover of cozero-sets of A. By (1) of Theorem 3.4.1, there exists a locally finite κ^+-open cover $\{V_\alpha : \alpha < \gamma\}$ of X such that $V_\alpha \cap A = U_\alpha$ for every $\alpha < \gamma$. Since X is normal and A is closed in X, there exists a cozero-set W_α of X such that $W_\alpha \cap A = U_\alpha$ and $W_\alpha \subset U_\alpha$ for each $\alpha < \gamma$. Take a cozero-set W' of X satisfying that $W' \cap A = \emptyset$ and $W' \cup \bigcup_{\alpha<\gamma} W_\alpha = X$. Replace W_0 by $W_0 \cup W'$. Then, $\{W_\alpha : \alpha < \gamma\}$ is a locally finite cover of cozero-sets of X extending $\mathcal{U}$. It completes the proof. □

To prove Theorem 3.4.2, some preliminary results are needed. Lemma 3.4.5 below gives a class of spaces in which every closed subspace is π-embedded.

Lemma 3.4.4. *Let X be a Tychonoff space, A a compact subspace of X and Y a space. Let $h : Y \to I$ and $f : A \times Y \to I$ be continuous functions. Then, there exists a continuous extension $g : X \times Y \to I$ of f such that $|g(x, y) - h(y)| \leq \varepsilon_y$ for every $(x, y) \in X \times Y$, where $\varepsilon_y = \sup\{|h(y) - f(a, y)| : a \in A\}$.*

Proof. By Theorem 1.3.10, A is π-embedded in X. Hence, there exists a continuous extension $\hat{f} : X \times Y \to I$ of f. Define a function $g : X \times Y \to I$ by $g(x, y) = \hat{f}(x, y) \wedge (h(y) + \varepsilon_y) \vee (h(y) - \varepsilon_y)$ for each $(x, y) \in X \times Y$. Then, g is the required continuous extension of f. □

Lemma 3.4.5. *Let γ be an infinite cardinal, and κ and λ cardinals. Then, every closed subspace of a space X of type $t(\gamma, \kappa, \lambda)$ is π-embedded in X.*

Proof. Let A be a closed subspace of a space X of type $t(\gamma, \kappa, \lambda)$ and Y a space. Let $f : A \times Y \to I$ be a continuous function. Let $X' = X - \{p\}$ be a subspace as in the condition (1) of type $t(\gamma, \kappa, \lambda)$. Let $\{U_\alpha^\beta : \alpha < \gamma, \beta < \kappa\}$ and $\{Z_\alpha^\beta : \alpha < \gamma, \beta < \kappa\}$ be the same as in the condition (2) of the definition. Take a locally finite cover $\{V_\alpha^\beta : \alpha < \gamma, \beta < \kappa\}$ of cozero-sets of X' such that $\overline{V_\alpha^\beta}$ is compact and $Z_\alpha^\beta \subset V_\alpha^\beta \subset \overline{V_\alpha^\beta} \subset U_\alpha^\beta$ for each $\alpha < \gamma$ and $\beta < \kappa$.

Case 1. Assume $p \in A$. For every $\alpha < \gamma$ and $\beta < \kappa$, we shall define a continuous function $g_\alpha^\beta : U_\alpha^\beta \times Y \to I$ as follows.

In case $A \cap \overline{V_\alpha^\beta} \neq \emptyset$, by Lemma 3.4.4, take a continuous extension $g_\alpha^\beta : U_\alpha^\beta \times Y \to I$ of $f|(A \cap \overline{V_\alpha^\beta}) \times Y$ such that $|g_\alpha^\beta(x, y) - f(p, y)| \leq \sup\{|f(a, y) - f(p, y)| : a \in A \cap \overline{V_\alpha^\beta}\}$ for every $(x, y) \in U_\alpha^\beta \times Y$.

In case $A \cap \overline{V_\alpha^\beta} = \emptyset$, define a continuous function $g_\alpha^\beta : U_\alpha^\beta \times Y \to I$ by $g_\alpha^\beta(x, y) = f(p, y)$ for each $(x, y) \in U_\alpha^\beta \times Y$.

Let $\{p_\alpha^\beta : \alpha < \gamma, \beta < \kappa\}$ be a partition of unity on X' such that $\overline{(p_\alpha^\beta)^{-1}((0, 1])} \subset V_\alpha^\beta$ for every $\alpha < \gamma$ and $\beta < \kappa$. Define a function $g : X \times Y \to I$ by

$$g(x, y) = \begin{cases} \sum\{p_\alpha^\beta(x) \cdot g_\alpha^\beta(x, y) : \alpha < \gamma, \beta < \kappa\} & \text{if } x \neq p, \\ f(p, y) & \text{if } x = p. \end{cases}$$

Clearly g is an extension of f. It is easy to see that g is continuous at (x, y) if $x \neq p$. We shall show that g is continuous at (p, y). Let $y \in Y$ and $\varepsilon > 0$. Since f is continuous, there exist $\delta \in [\gamma]^{<\omega}$ and a neighborhood O of y in Y

such that $|f(x',y')-f(p,y)|<\varepsilon/4$ for every $(x',y')\in(U_\delta(p)\cap A)\times O$. Take $\delta'\in[\gamma]^{<\omega}$ with $\delta\subset\delta'$ such that $(\bigcup\{U_\alpha^\beta:\alpha\in\delta,\beta<\kappa\})\cap U_{\delta'}(p)=\emptyset$. Let $(x',y')\in U_{\delta'}(p)\times O$. We shall show that $|g(x',y')-g(p,y)|<\varepsilon$. We may assume $x'\neq p$, because the case of $x'=p$ is easily shown.

Let $\alpha\in\gamma-\delta$ and $\beta<\kappa$ satisfy that $x'\in U_\alpha^\beta$ and $A\cap\overline{V_\alpha^\beta}\neq\emptyset$. Then,

$$\begin{aligned}|g_\alpha^\beta(x',y')-f(p,y')| &\leq \sup\Big\{|f(a,y')-f(p,y')|:a\in A\cap\overline{V_\alpha^\beta}\Big\}\\ &\leq \varepsilon/2.\end{aligned}$$

It follows that

$$\begin{aligned}|g_\alpha^\beta(x',y')-g(p,y)| &\leq |g_\alpha^\beta(x',y')-f(p,y')|+|f(p,y')-f(p,y)|\\ &\leq \varepsilon/2+\varepsilon/4=(3/4)\varepsilon.\end{aligned}$$

Let $\alpha\in\gamma-\delta$ and $\beta<\kappa$ satisfy that $x'\in U_\alpha^\beta$ and $A\cap\overline{V_\alpha^\beta}=\emptyset$. Then,

$$\begin{aligned}|g_\alpha^\beta(x',y')-g(p,y)| &\leq |g_\alpha^\beta(x',y')-f(p,y')|+|f(p,y')-f(p,y)|\\ &\leq 0+\varepsilon/4=\varepsilon/4.\end{aligned}$$

Let $\alpha<\gamma$ and $\beta<\kappa$. If $p_\alpha^\beta(x')>0$, then $x'\in U_\alpha^\beta$ and $\alpha\notin\delta$. Hence, it follows from the facts shown above that

$$|g(x',y')-g(p,y)|\leq\sum_{\alpha<\gamma,\beta<\kappa}\Big(p_\alpha^\beta(x)\cdot|g_\alpha^\beta(x',y')-g(p,y)|\Big)<\varepsilon.$$

Case 2. Assume $p\notin A$. Define a continuous function $f':(A\cup\{p\})\times Y\to I$ as follows: $f'|(A\times Y)=f$ and $f'(p,y)=0$ for each $y\in Y$. So it comes back to Case 1. It completes the proof. □

Lemma 3.4.6. *Let γ and λ be infinite cardinals and κ a cardinal. Let A be a P^λ-embedded subspace of a space X and Y a space of type $t(\gamma,\kappa,\lambda)$. Then, $A\times Y$ is well-embedded in $X\times Y$.*

Proof. Let $f:X\times Y\to I$ be a continuous function satisfying that $f^{-1}(\{0\})\cap(A\times Y)=\emptyset$. From the definition of Y, for every $\alpha<\gamma$ and $\beta<\kappa$, we can take a cozero-set V_α^β of Y' such that $\overline{V_\alpha^\beta}$ is compact and $Z_\alpha^\beta\subset V_\alpha^\beta\subset\overline{V_\alpha^\beta}\subset U_\alpha^\beta$. By Corollary 1.3.6, $A\times Y'$ is C-embedded in $X\times Y'$. There exists a continuous function $g:X\times Y'\to I$ such that $A\times Y'\subset g^{-1}(\{0\})$ and $f^{-1}(\{0\})\cap(X\times Y')\subset g^{-1}(\{1\})$. Moreover, let $g_0:X\to I$ be a continuous function such that $A\subset g_0^{-1}(\{0\})$ and $\{x\in X:f(x,p)=0\}\subset g_0^{-1}(\{1\})$. For every $\alpha<\gamma,\beta<\kappa$ and $n\in\mathbb{N}$, put

$$H_{\alpha\beta}^n=\Big\{x\in X:|f(x,y)-f(x,p)|>1/(n+1)\text{ for some }y\in\overline{V_\alpha^\beta}\Big\}.$$

Note that $\{\bigcup_{\beta<\kappa} H^n_{\alpha\beta} : \alpha < \gamma\}$ is locally finite in X for each $n \in \mathbb{N}$. Since $\overline{V^\beta_\alpha}$ is compact, we have $H^n_{\alpha\beta}$ is a cozero-set of X. For every $n \in \mathbb{N}$, put

$$H_n = \bigcup_{\alpha<\gamma} \bigcup_{\beta<\kappa} \Big(H^n_{\alpha\beta} \times V^\beta_\alpha \Big).$$

Then, H_n is a cozero-set of $X \times Y'$. For every $n \in \mathbb{N}$, let $\psi_n : X \times Y' \to I$ be a continuous function such that $H_n = \psi_n^{-1}((0,1])$. For every $n \in \mathbb{N}$, define a continuous function $h_n : X \times Y \to I$ as in Lemma 3.2.2. Put

$$W = \Big(\bigcup_{n\in\mathbb{N}} {h_n}^{-1}((0,1]) \Big) \cup \Big(g_0^{-1}((0,1]) \times Y \Big),$$

which is a cozero-set of $X \times Y$. Then one can show that $(A \times Y) \cap W = \emptyset$ and $f^{-1}(\{0\}) \subset W$. Hence $A \times Y$ is well-embedded in $X \times Y$. The proof is completed. □

Proof of Theorem 3.4.2. (1) $\Rightarrow$ (2): Let Y be a space of type $t(\gamma, \kappa, \gamma)$. Let A be a closed subspace of X and B a closed subspace of Y. By Lemma 3.4.5, $A \times B$ is C-embedded in $A \times Y$. It follows from Theorem 3.4.1 and Lemma 3.4.6 that $A \times Y$ is C-embedded in $X \times Y$. Hence, $A \times B$ is C-embedded in $X \times Y$. Thus, $X \times Y$ is rectangularly normal.

(2) $\Rightarrow$ (3): Obvious.

(3) $\Rightarrow$ (1): The normality of X easily follows. It follows from Thorem 3.4.1 that X is (γ, κ)-Katětov. It completes the proof. □

Proof of Corollary 3.4.3. It suffices to show the "only if" part. Let X be an (ω, κ)-Katětov space, Y a space as in the proposition, A a closed subspace of X and B a closed subspace of Y. By Theorem 1.3.12, $A \times B$ is C-embedded in $A \times Y$. By Lemma 3.3.4 and Theorem 3.4.2, $A \times Y$ is C^*-embedded in $X \times Y$. By Theorem 1.3.8, $A \times Y$ is C-embedded in $X \times Y$. Hence it follows that $X \times Y$ is rectangularly normal.□

Remark 3.4.7. (a) On Theorems 3.4.1 and 3.4.2, Corollary 3.4.3 and Lemmas 3.4.5 and 3.4.6, all of C^*-embedding can be replaced by C-embedding (use Theorem 1.3.8 or Lemma 3.4.6).

(b) On Theorems 3.4.1 and 3.4.2 and Lemma 3.4.5 and 3.4.6, the part of "spaces of type $t(\gamma, \kappa, \lambda)$" can be changed into "closed subspaces of a space of type $t(\gamma, \kappa, \lambda)$". Indeed, by (3) of Proposition 3.1.2, if $p \in Y$ then the assertion is obvious because Y is of type $t(\gamma, \kappa, \lambda)$. If $p \notin Y$ then Y is locally compact paracompact Hausdorff and $\ell w(Y) \leq \lambda$, hence the assertion is contained in the known results (cf. Corollary 1.3.6).

5. Applications to γ-collectionwise normal λ-paracompact spaces

The aims of this section are to state γ-collectionwise normal spaces, and γ-collectionwise normal λ-paracompact spaces along the same line of Theorem 3.4.2, and to explain Diagram 2.1.3 from the viewpoint of products. We have the following results.

Proposition 3.5.1. *Let X be a space, γ an infinite cardinal and $n \in \mathbb{N}$. Then, the following conditions are equivalent:*
(1) *X is γ-collectionwise normal;*
(2) *$X \times Y$ is rectangularly normal for every space Y of type $t(\gamma, n, \gamma)$;*
(3) *$X \times J_\gamma(n)$ is rectangularly normal.*

Proposition 3.5.2. *Let X be a space, γ and λ infinite cardinals and κ a cardinal. Then, the following statements are equivalent:*
(1) *X is γ-collectionwise normal and λ-paracompact;*
(2) *$X \times Y$ is normal for every space Y of type $t(\gamma, \kappa, \lambda)$;*
(3) *$X \times A(\oplus_{\alpha<\gamma}(\lambda + 1)_\alpha)$ is normal.*

Corollary 3.5.3. *Let X be a space, γ an infinite cardinal and κ a cardinal. Then, the following statements are equivalent:*
(1) *X is γ-collectionwise normal and countably paracompact;*
(2) *$X \times Y$ is normal for every space Y of type $t(\gamma, \kappa, \omega)$;*
(3) *$X \times J_\gamma(\kappa)$ is normal.*
Namely, $X \times J_\gamma(\kappa)$ is normal if and only if $X \times J_\gamma(1)$ is normal.

On Proposition 3.5.2, an equivalent condition similar to (3) was also obtained by Katuta [29, Theorem 1.2]. Moreover, he proved in [29, Theorem 1.2] the equivalence of (1) of Proposition 3.5.2 and the normality of $X \times Y$ for arbitrarily compact space Y with $w(Y) \leq \gamma$ and $v(Y) \leq \kappa$, and Ohta showed in [44] the condition $v(Y)$ can be replaced by a smaller cardinal $u(Y)$ (see [29] and [44] for the definitions of v and u). On Corollary 3.5.3, the equivalence of (1) and the normality of $X \times J_\gamma(1)$ was proved by Alas (Theorem 1.3.14). An equivalent condition similar to (3) was also obtained by Katuta [29, Proposition 3.6]. On Corollary 3.5.3, (1) $\Rightarrow$ (3) also follows from by Katuta's result in [28].

Let us proceed to the proofs.

Proof of Proposition 3.5.1. (1) $\Rightarrow$ (2): Let Y a space of type $t(\gamma, n, \gamma)$. Notice that Y is compact Hausdorff and $w(Y) \leq \gamma$. By Theorems 1.3.2, 1.3.5 and 1.3.10, $X \times Y$ is rectangularly normal.

(2) $\Rightarrow$ (3): Obvious.

(3) $\Rightarrow$ (1): Let A be a closed subspace of X. Then, $A \times J_\gamma(n)$ is C^*-embedded in $X \times J_\gamma(n)$. By Theorem 1.3.5, A is P^γ-embedded in X. Hence X is γ-collectionwise normal; this completes the proof. □

Next we prove Proposition 3.5.2. The proof is given along the same line to our previous discussion using Lemma 3.2.2.

Proof of Proposition 3.5.2. (1) $\Rightarrow$ (2): Let Y be a space of type $t(\gamma, \kappa, \lambda)$ and F_0 and F_1 be disjoint closed subspaces of $X \times Y$. Put $A = F_0 \cup F_1 \cup (X \times \{p\})$. Since X is normal, we can take a continuous function $f : A \to I$ satisfying that $f(F_i) = i$ $(i = 0, 1)$. Define a function $g_0 : X \to I$ by $g_0(x) = 0 \vee (3 \cdot f(x,p) - 1) \wedge 1$. Let $F^* = \big(F_0 \cap (\{x \in X : f(x,p) \geq 1/3\} \times Y)\big) \cup \big(F_1 \cap (\{x \in X : f(x,p) \leq 2/3\} \times Y)\big)$. Then, F^* is a closed subspace in $X \times Y$ disjoint from $X \times \{p\}$. Let $p : X \times Y \to X$ be the projection. For every $\alpha < \gamma$, put $A_\alpha = p\big(F^* \cap (X \times \bigcup_{\beta<\kappa} Z_\alpha^\beta)\big)$. Then, $\{A_\alpha : \alpha < \gamma\}$ is locally finite in X. Since X is γ-collectionwise normal and countably paracompact, there exists a locally finite open collection $\{H_\alpha : \alpha < \gamma\}$ of X such that $\overline{A_\alpha} \subset H_\alpha$ for each $\alpha < \gamma$. Let

$$F = \bigcup_{\alpha<\gamma} \bigcup_{\beta<\kappa} \Big(\overline{A_\alpha} \times Z_\alpha^\beta\Big) \quad \text{and} \quad G = \bigcup_{\alpha<\gamma} \bigcup_{\beta<\kappa} \Big(H_\alpha \times U_\alpha^\beta\Big).$$

Then, it follows that F is closed in $X \times Y'$, G is open in $X \times Y'$ and $F \subset G$. Notice that $X \times Y'$ is normal, since it has a locally finite closed cover of normal subspaces $X \times Z_\alpha^\beta$ $(\alpha < \gamma, \beta < \kappa)$. Hence, there exist continuous functions ψ, $g : X \times Y' \to I$ such that $F \subset \psi^{-1}(\{1\})$, $X \times Y' - G \subset \psi^{-1}(\{0\})$ and $g|(A \cap (X \times Y')) = f|(A \cap (X \times Y'))$. The continuous function $h : X \times Y \to I$ defined as in Lemma 3.2.2 satisfies $F_i \subset h^{-1}(\{i\})$ $(i = 0, 1)$ (see Remark 3.2.3). It follows that $X \times Y$ is normal.

(2) $\Rightarrow$ (3): Obvious.

(3) $\Rightarrow$ (1): Since $A(\gamma)$ and $\lambda + 1$ can be seen as closed subspaces of $A(\oplus_{\alpha<\gamma}(\lambda + 1)_\alpha)$, by Theorem 1.3.14 and [52, Corollary 3.7] it follows. It completes the proof. □

Proof of Corollary 3.5.3. By Proposition 3.5.2, (1) $\Rightarrow$ (2) follows. The implication (2) $\Rightarrow$ (3) are obvious, and (3) $\Rightarrow$ (1) follows by Theorem 1.3.14 immediately. □

6. Locations of extension properties

In this section, let us comment where P(locally-finite)-embedding locates among extension properties. Let $\mathcal{C}$ be a class of spaces. A subspace A of a space X is said to be *$\pi_{\mathcal{C}}$-embedded* in X if $A \times Y$ is C^*-embedded in $X \times Y$ for every space Y belonging to $\mathcal{C}$ [50]. Let $\mathcal{M}_\kappa$ be the class of all metrizable spaces with weight $\leq \kappa$ and $\mathcal{M}$ the class of all metrizable spaces.

Answering to Przymusiński's problem in [50], Gutev-Ohta characterized in [19] $\pi_{\mathcal{M}_\kappa}$-embedding introducing the following notions. In [19] a map $\mathcal{G} : \kappa^{<\omega} \to Coz(A)^\kappa$ is said to be *monotone decreasing* if $\{\mathcal{G}[\sigma^\wedge\alpha](\beta) : \beta < \kappa\}$ refines $\{\mathcal{G}[\sigma](\beta) : \beta < \kappa\}$ for every $\sigma \in \kappa^{<\omega}$ and $\alpha \in \kappa$, where $Coz(X)$ denotes the collection of all cozero-sets of X.

Theorem 3.6.1(Gutev-Ohta [19]). *Let X be a space and A a subspace of X. Then, A is $\pi_{\mathcal{M}_\kappa}$-embedded in X if and only if A is C-embedded in X and A has the following property $(\sharp)_\kappa$;*
$(\sharp)_\kappa$ *Every monotone decreasing map $\mathcal{G} : \kappa^{<\omega} \to Coz(A)^\kappa$ has an expansion $\mathcal{H} : \kappa^{<\omega} \to Coz(X)^\kappa$ (i.e., $\mathcal{G}[\sigma](\alpha) \subset \mathcal{H}[\sigma](\alpha)$ for every $\sigma \in \kappa^{<\omega}$ and $\alpha \in \kappa$) such that $\bigcap_{n<\omega} \overline{\bigcup_{\alpha\in\kappa} \mathcal{H}[t|n](\alpha)}^X = \emptyset$ for every $t \in \kappa^\omega$ with $\bigcap_{n<\omega} \overline{\bigcup_{\alpha\in\kappa} \mathcal{G}[t|n](\alpha)}^A = \emptyset$.*

Related to this result, we have the following:

Proposition 3.6.2. *Let X be a space and A a subspace of X with the property $(\sharp)_\kappa$. Then, every countable locally finite κ^+-open collection of A can be expanded to a locally finite κ^+-open collection of X.*

Proof. Let $\{U_n : n < \omega\}$ be a locally finite collection of κ^+-open sets of A. For every $n < \omega$, let $U_n = \bigcup_{\alpha\in\kappa} U_n^\alpha$, where each U_n^α is a cozero-set of A. Define a monotone decreasing map $\mathcal{G} : \kappa^{<\omega} \to Coz(A)^\kappa$ by

$$\mathcal{G}[\sigma](\alpha) = \begin{cases} \bigcup_{i\geqslant n} U_i^\alpha & \text{if } \sigma = (0,\ldots,0)_n,\ \alpha \in \kappa, \\ \emptyset & \text{otherwise,} \end{cases}$$

where $(0,\ldots,0)_n \in \kappa^n$. Then, $\mathcal{G}$ is monotone decreasing. By the assumption, there exists an expansion $\mathcal{H} : \kappa^{<\omega} \to Coz(X)^\kappa$ such that $\bigcap_{n<\omega} \overline{\bigcup_{\alpha\in\kappa} \mathcal{H}[t|n](\alpha)}^X = \emptyset$ for every $t \in \kappa^\omega$ with $\bigcap_{n<\omega} \overline{\bigcup_{\alpha\in\kappa} \mathcal{G}[t|n](\alpha)}^A = \emptyset$. Notice that $\bigcap_{n<\omega} \overline{\bigcup_{\alpha\in\kappa} \mathcal{G}((0,\ldots,0)_n)(\alpha)}^A = \emptyset$. Hence $\bigcap_{n<\omega} \overline{\bigcup_{\alpha\in\kappa} \mathcal{H}((0,\ldots,0)_n)(\alpha)}^X = \emptyset$. For every $n < \omega$, define V_n as follows: $V_1 = \bigcup_{\alpha\in\kappa} \mathcal{H}[(0)](\alpha)$ and $V_n = (\bigcup_{\alpha\in\kappa} \mathcal{H}[(0,\ldots,0)_n](\alpha)) \cap V_{n-1}$ for $n > 1$. Then, $\{V_n : n < \omega\}$ is a locally finite collection of κ^+-open sets of X and $U_n \subset V_n$ for each $n < \omega$. The proof is completed. □

By Theorem 3.3.3 or Proposition 3.6.2, we have the following result. It clarifies the location of P^ω(locally-finite)-embedding in the realm of extension properties.

Corollary 3.6.3. *The $\pi_{\mathcal{M}_\omega}$-embedding implies P^ω(locally-finite)-embedding.*

Remark 3.6.4. (1) Michael's example ([13, 5.1.32]) shows that Corollary 3.6.3 can not be reversed (see [36], see also [24, Example 4.13]). Related to it, we comment about the products with $J(\kappa)^\omega$. Let X be a space, A a subspace of X and κ an infinite cardinal. The letter $\mathcal{C}_\kappa$ means the class of all Čech-complete metric spaces with weight $\leq \kappa$. Then, we can easily show that: *A is $\pi_{\mathcal{C}_\kappa}$-embedded in X if and only if $A \times J(\kappa)^\omega$ is C^*-embedded in $X \times J(\kappa)^\omega$*, and that: *For Michael line X and a subspace $\mathbb{Q}$ of X, $\mathbb{Q} \times J(\omega)$ is C^*-embedded in $X \times J(\omega)$ and $\mathbb{Q} \times J(\omega)^\omega$ is not C^*-embedded in $X \times J(\omega)^\omega$.*

(2) The $\pi_{\mathcal{M}}$-embedding need not imply P^γ-embedding in the case $\gamma > \omega$ (see [13, 5.5.3] or [48]). Namely, Corollary 3.6.3 need not hold for the general cardinality.

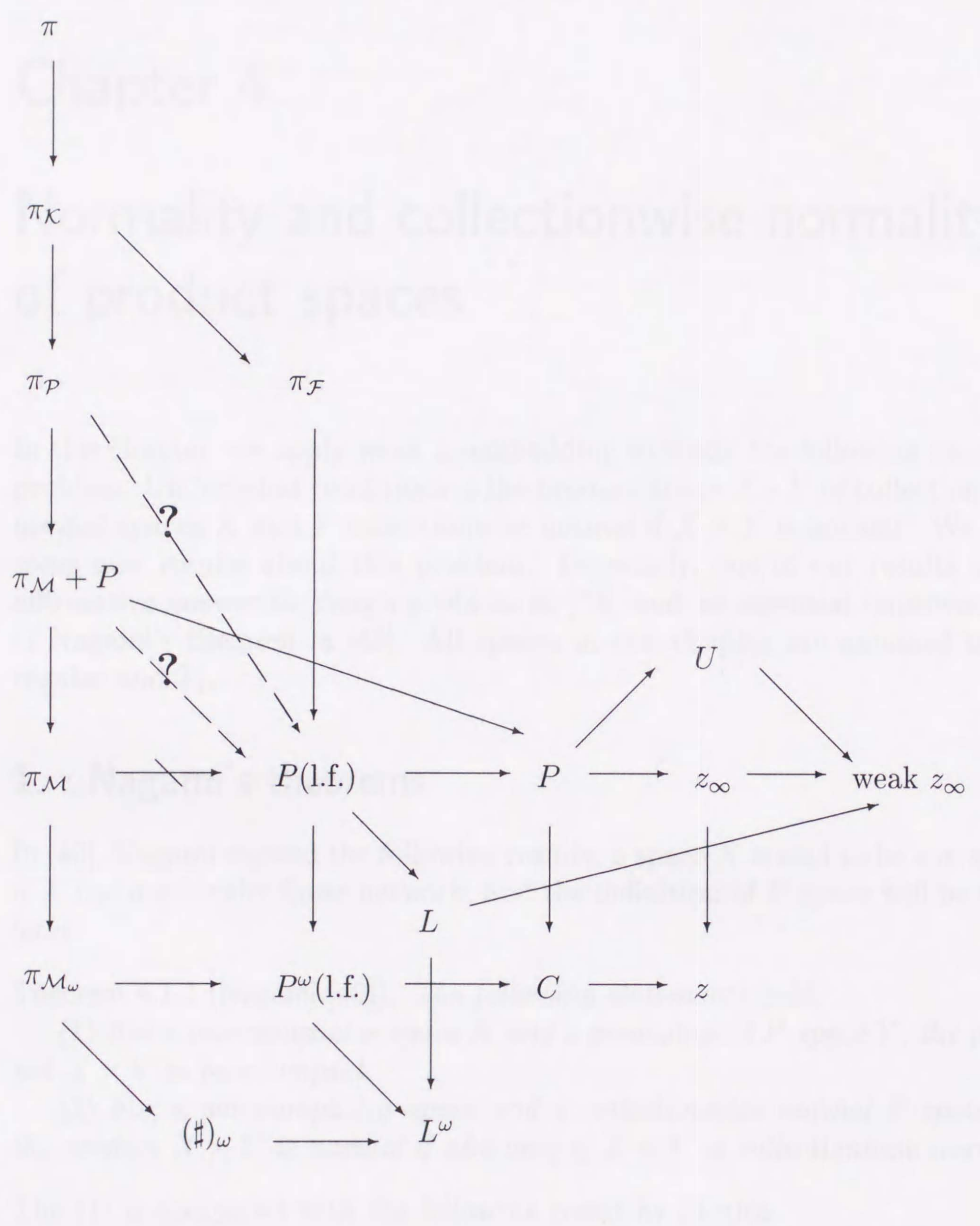
Waśko gave in [63] and [64] locations of extension properties like $\pi_{\mathcal{C}}$-embedding. She showed that "$\pi_{\mathcal{M}} + P = \pi_{\mathcal{M}\times\mathcal{C}}$", where $\mathcal{M} \times \mathcal{C}$ is the class of spaces consisting of all of the product spaces of metric spaces and compact spaces [64]. Since $J_\gamma(\omega)$ is Fréchet σ-compact, by Theorem 3.3.1, $\pi_{\mathcal{F}}$-embedding implies P(locally-finite)-embedding, where $\mathcal{F}$ means the class of all Fréchet σ-compact spaces. From the above argument, it seems to be natural to ask the followings:

Problem 3.6.5. Does $\pi_{\mathcal{M}} + P$-embedding imply P(locally-finite)-embedding?

Problem 3.6.6. Let $\mathcal{P}$ be a class of all paracompact M-spaces. Does $\pi_{\mathcal{P}}$-embedding imply P(locally-finite)-embedding?

Related to Problem 3.6.5, note that in Example 2.1.7 the subspace A is not $\pi_{\mathcal{M}}$-embedded in X.

Diagram 3.6.7. Some extension properties are located as follows: the symbol $\mathcal{K}$ means a class of all paracompact k-spaces, and P(l.f.) stands for P(locally-finite).



Remark 3.6.8. The definitions of $J_\gamma(\kappa)$ and spaces of type $t(\gamma, \kappa, \lambda)$ are given in [71]. All of the results in Sections 2, 3, 4 and 5 and Corollary 3.6.3 are proved in [71]. Other results are added here.

Chapter 4.

Normality and collectionwise normality of product spaces

In this chapter, we apply weak z_γ-embedding to study the following classical problem: Under what conditions is the product space $X \times Y$ of collectionwise normal spaces X and Y collectionwise normal if $X \times Y$ is normal? We give some new results about this problem. Especially, one of our results is an affirmative answer to Yang's problem in [73], and an essential improvement of Nagami's theorem in [40]. All spaces in this chapter are assumed to be regular and T_1.

1. Nagami's theorems

In [40], Nagami showed the following results; a space X is said to be a *σ-space* if X has a σ-locally finite network, and the definition of P-space will be seen later.

Theorem 4.1.1 (Nagami [40]). *The following statements hold.*

(1) *For a paracompact σ-space X and a paracompact P-space Y, the product $X \times Y$ is paracompact.*

(2) *For a paracompact σ-space and a collectionwise normal P-space Y, the product $X \times Y$ is normal if and only if $X \times Y$ is collectionwise normal.*

The (1) is compared with the following result by Morita.

Theorem 4.1.2 (Morita [34]). *For a paracompact M-space X and a paracompact P-space Y, the product $X \times Y$ is paracompact.*

In another paper [41], extending the notions of σ-spaces and M-spaces, Nagami defined a new class of spaces called Σ-spaces, and gave the following results. Especially (1) is an essential improvement of the result (1) of Theorem 4.1.1 as well as Theorem 4.1.2.

Theorem 4.1.3 (Nagami [41]). *The following statements hold.*

(1) *For a paracompact Σ-space X and a paracompact P-space Y, the product $X \times Y$ is paracompact.*

(2) *For a paracompact Σ-space X and a normal P-space Y, if $X \times Y$ is normal then $X \times Y$ is countably paracompact.*

The (1) of Theorem 4.1.3 is famous as one of those that give a paracompact product of two spaces, and was suggestive to subsequent studies. In fact, some analogous results hold as follows (see Nagami [41], Mizokami [32], Lutzer [31] and Burke [8]):

Theorem 4.1.4 ([8], [31], [32], [41]). *Let X and Y be spaces. If X is a Lindelöf (respectively, metacompact, subparacompact or submetacompact) Σ-space and Y is a Lindelöf (respectively, metacompact, subparacompact or submetacompact) P-space, then $X \times Y$ is Lindelöf (respectively, metacompact, subparacompact or submetacompact).*

In comparison with Theorems 4.1.1 and 4.1.3, it is natural to ask whether "σ-space" in (2) of Theorem 4.1.1 can be generalized to "Σ-space". Indeed, Yang posed it as a problem in [73] as follows:

Problem 4.1.5 (Yang [73]). Let X ba a paracompact Σ-space and Y a collectionwise normal P-space and $X \times Y$ normal. Then, is $X \times Y$ collectionwise normal?

In [73], Yang proved Problem 4.1.5 affirmatively assuming that Y is countably compact; and his assumption was improved to that Y is a Σ-space in [26] by Hoshina and the author. However this problem has been unknown even in the case that X is a perfect space. We shall give an affirmative answer to Problem 4.1.5 in the next section.

In this section, let us comment about differences of the behavior between σ-spaces and Σ-spaces. By Hoshina [22], the inverse implication of (2) of Theorem 4.1.3 is also true. Hence it follows that:

> *For a paracompact σ-space X and a normal P-space Y, the product $X \times Y$ is normal if and only if $X \times Y$ is countably paracompact.*

As is [22], (2) of Theorem 4.1.1 can also be proved by the above result and Theorem 1.3.14 indirectly. We shall demonstrate his proof from [22]. Let X be a paracompact σ-space and Y a γ-collectionwise normal P-space and

$X \times Y$ normal. By the above result, the normal space $X \times Y$ is countably paracompact. Hence, $(X \times Y) \times A(\gamma)$ is countably paracompact. Since $Y \times A(\gamma)$ is a normal P-space, by the above result, the countably paracompact space $(X \times Y) \times A(\gamma) = X \times (Y \times A(\gamma))$ is normal. Hence, by Theorem 1.3.14, $X \times Y$ is γ-collectionwise normal. This proof shows, under some kinds of conditions, "the equivalence of normality and collectionwise normality" indirectly follows from "the equivalence of normality and countable paracompactness".

On the other hand, the inverse implication of (2) of Theorem 4.1.3 itself does not hold in general (consider $(\omega_1 + 1) \times \omega_1$). It means that the indirect method demonstrated above can not be used in proving Problem 4.1.5 affirmatively.

For two collectionwise normal spaces X and Y, the result which asserts normality of $X \times Y$ implies its collectionwise normality has been proved in some cases. These are mainly as the following: (1) is due to Okuyama [47] and improved as (3) or (4); (2) is due to Starbird [59] and improved as (4); (3) is due to Hoshina [22]; (4) is due to Rudin-Starbird [56]; (5) is due to K. Chiba [10]; (6) is due to Nagami [40]; and (7) and (8) are due to K. Chiba [10].

Theorem 4.1.6 ([10], [22], [40], [47], [56], [59]). *Let X and Y be collectionwise normal spaces. If X and Y satisfy one of the following conditions, then $X \times Y$ is normal if and only if $X \times Y$ is collectionwise normal*:

(1) *Y is a metrizable space*;

(2) *Y is a compact space*;

(3) *Y is a Lašnev (= the closed image* of a metrizable) space*;

(4) *Y is a paracompact M-space*;

(5) *Y is a σ-locally compact paracompact Hausdorff space*;

(6) *X is a paracompact σ-space and Y is a collectionwise normal P-space*;

(7) *X is the closed image of a normal M-space and Y is a paracompact first countable P-space*;

(8) *X is the closed image of a paracompact first countable M-space and Y is a collectionwise normal Σ-space.*

A space Y is a *P-space* [34] if for any index set Ω and for any collection $\{G(\alpha_1, \ldots, \alpha_n) : \alpha_1, \ldots, \alpha_n \in \Omega, n \in \mathbb{N}\}$ of open subsets of Y such that $G(\alpha_1, \ldots, \alpha_n) \subset G(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$ for $\alpha_1, \ldots, \alpha_n, \alpha_{n+1} \in \Omega$, there exists a collection $\{F(\alpha_1, \ldots, \alpha_n) : \alpha_1, \ldots, \alpha_n \in \Omega,\ n \in \mathbb{N}\}$ of closed subsets of Y such that the conditions (a) and (b) below are satisfied:

(a) $F(\alpha_1, \ldots, \alpha_n) \subset G(\alpha_1, \ldots, \alpha_n)$ for $\alpha_1, \ldots, \alpha_n \in \Omega$,

*The closed image means the image of some continuous closed map.

(b) $Y = \bigcup\{G(\alpha_1, \dots, \alpha_n) : n \in \mathbb{N}\} \Rightarrow Y = \bigcup\{F(\alpha_1, \dots, \alpha_n) : n \in \mathbb{N}\}$.

A Σ-space [41] is a space X having a sequence, called a *Σ-net*, $\{\mathcal{E}_n : n \in \mathbb{N}\}$ of locally finite closed covers of X which satisfies the following conditions:

(c) $\mathcal{E}_n$ is written as $\{E(\alpha_1, \dots, \alpha_n) : \alpha_1, \dots, \alpha_n \in \Omega\}$ with an index set Ω,
(d) $E(\alpha_1, \dots, \alpha_n) = \bigcup\{E(\alpha_1, \dots, \alpha_n, \alpha_{n+1}) : \alpha_{n+1} \in \Omega\}$ for $\alpha_1, \dots, \alpha_n \in \Omega$,
(e) For every $x \in X$, $C(x)$ is countably compact, and there exists a sequence $\alpha_1, \alpha_2, \dots \in \Omega$ such that $C(x) \subset V$ with V open implies $C(x) \subset E(\alpha_1, \dots, \alpha_n) \subset V$ for some n, where $C(x) = \bigcap\{E : y \in E \in \mathcal{E}_n,\ n \in \mathbb{N}\}$.
We call $\{E(\alpha_1, \dots, \alpha_n) : n \in \mathbb{N}\}$ a *local net* of $C(x)$.

2. Results

In this section, we obtain some new results related to the problem stated in the previous section. In the following theorem, (1) is an affirmative answer to Problem 4.1.5, i.e. an improvement of (2) of Theorem 4.1.1 (i.e. (6) of Theorem 4.1.6), and (3) and (4) in the following result are also improvements of (7) and (8) of Theorem 4.1.6. Notice that $(\omega_1 + 1) \times \omega_1$ shows, under each of the conditions from (1) to (4) of the following theorem, $X \times Y$ is not necessarily normal.

Theorem 4.2.1(Main). *If X and Y satisfy one of the following conditions, then $X \times Y$ is normal if and only if $X \times Y$ is collectionwise normal:*

(1) *X is a paracompact Σ-space and Y is a collectionwise normal P-space;*

(2) *X is a collectionwise normal Σ-space and Y is a collectionwise normal first countable P-space;*

(3) *X is the closed image of a paracompact M-space and Y is a collectionwise normal P-space;*

(4) *X is the closed image of a normal M-space and Y is a collectionwise normal first countable P-space.*

It should be noted that (2) and (4) in the above theorem seem to be first cases that these conclusion are implied under the conditions that neither X nor Y is paracompact.

Our motivation of (2) of Theorem 4.2.1 is K. Chiba's result in [9] as follows:

If X is a collectionwise normal Σ-space and Y is a paracompact first countable P-space, then $X \times Y$ is collectionwise normal.

3. Key lemmas for the proof

In this section, for the proof of Theorem 4.2.1, we prepare key lemmas.

Lemma 4.3.1. *Let X be a paracompact Σ-space and Y a γ-collectionwise normal P-space. Then, every closed subspace of $X \times Y$ is weakly z_γ-embedded in $X \times Y$.*

Lemma 4.3.2. *Let X be a collectionwise normal Σ-space and Y a γ-collectionwise normal first countable P-space. Then, every closed subspace of $X \times Y$ is weakly z_γ-embedded in $X \times Y$.*

Since proofs of Lemmas 4.3.1 and 4.3.2 are similar, we only prove Lemma 4.3.1.

Proof of Lemma 4.3.1. Let A be a closed subspace of $X \times Y$ and $\{F_\beta : \beta < \gamma\}$ be a uniformly discrete collection of zero-sets of $X \times Y$. Let $\{\mathcal{E}_n : n \in \mathbb{N}\}$, where $\mathcal{E}_n = \{E(\alpha_1, \dots, \alpha_n) : \alpha_1, \dots, \alpha_n \in \Omega\}$ $(n \in \mathbb{N})$, be a Σ-net of X. Since X is collectionwise normal and countably paracompact, for each $n \in \mathbb{N}$, $\mathcal{E}_n$ has a locally finite expansion $\{L(\alpha_1, \dots, \alpha_n) : \alpha_1, \dots, \alpha_n \in \Omega\}$ of cozero-sets of X. For each $\alpha_1, \dots, \alpha_n \in \Omega, n \in \mathbb{N}$ and $\delta \in [\gamma]^{<\omega}$, put

$$(*)\qquad G_\delta(\alpha_1, \dots, \alpha_n) = \bigcup\{O : O \text{ is open in } Y \text{ and } (E(\alpha_1, \dots, \alpha_n) \times O) \cap (\bigcup\{F_\beta : \beta \notin \delta\}) = \emptyset\},$$

and $G(\alpha_1, \dots, \alpha_n) = \bigcup\{G_\delta(\alpha_1, \dots, \alpha_n) : \delta \in [\gamma]^{<\omega}\}$. Then, we have $G(\alpha_1, \dots, \alpha_n) \subset G(\alpha_1, \dots, \alpha_n, \alpha_{n+1})$ for each $\alpha_1, \dots, \alpha_n, \alpha_{n+1} \in \Omega$. Since Y is a P-space, there exists a closed collection $\{M(\alpha_1, \dots, \alpha_n) : \alpha_1, \dots, \alpha_n \in \Omega, n \in \mathbb{N}\}$ of Y such that $M(\alpha_1, \dots, \alpha_n) \subset G(\alpha_1, \dots, \alpha_n)$ for each $\alpha_1, \dots, \alpha_n \in \Omega$ and $n \in \mathbb{N}$, and

$$Y = \bigcup\{G(\alpha_1, \dots, \alpha_n) : n \in \mathbb{N}\} \Longrightarrow Y = \bigcup\{M(\alpha_1, \dots, \alpha_n) : n \in \mathbb{N}\}.$$

Here we may assume that $M(\alpha_1, \dots, \alpha_n) \subset M(\alpha_1, \dots, \alpha_n, \alpha_{n+1})$ for each $\alpha_1, \dots, \alpha_n, \alpha_{n+1} \in \Omega$. Define

$$(**)\qquad P_\delta(\alpha_1, \dots, \alpha_n) = \{y \in Y : (E(\alpha_1, \dots, \alpha_n) \times \{y\}) \cap F_\beta \neq \emptyset \text{ if and only if } \beta \in \delta\}$$

for each $\alpha_1, \dots, \alpha_n \in \Omega$, $n \in \mathbb{N}$ and $\delta \in [\gamma]^{<\omega}$.

Fix $\alpha_1, \dots, \alpha_n \in \Omega$ and $n \in \mathbb{N}$ arbitrarily.

Claim 1. The collection $\{M(\alpha_1, \dots, \alpha_n) \cap P_\delta(\alpha_1, \dots, \alpha_n) : \delta \in [\gamma]^{<\omega}\}$ is locally finite in Y.

Proof of Claim 1. Let $y \in Y$. Since $M(\alpha_1, \dots, \alpha_n)$ is closed in Y, to prove

Claim 1 we may assume that $y \in M(\alpha_1, \ldots, \alpha_n)$. Since $y \in G(\alpha_1, \ldots, \alpha_n)$, there exists $\delta_y \in [\gamma]^{<\omega}$ such that $y \in G_{\delta_y}(\alpha_1, \ldots, \alpha_n)$. Suppose that $G_{\delta_y}(\alpha_1, \ldots, \alpha_n) \cap P_\delta(\alpha_1, \ldots, \alpha_n) \neq \emptyset$. Then we shall show $\delta \subset \delta_y$. To show this, let $\beta \in \delta$. Select a point $z \in G_{\delta_y}(\alpha_1, \ldots, \alpha_n) \cap P_\delta(\alpha_1, \ldots, \alpha_n)$. Since $(E(\alpha_1, \ldots, \alpha_n) \times \{z\}) \cap F_\beta \neq \emptyset$, we have $(E(\alpha_1, \ldots, \alpha_n) \times G_{\delta_y}(\alpha_1, \ldots, \alpha_n)) \cap F_\beta \neq \emptyset$. By the definition of $G_{\delta_y}(\alpha_1, \ldots, \alpha_n)$, we have

$$(E(\alpha_1, \ldots, \alpha_n) \times G_{\delta_y}(\alpha_1, \ldots, \alpha_n)) \cap (\bigcup\{F_\mu : \mu \notin \delta_y\}) = \emptyset.$$

It shows that $\beta \in \delta_y$. Hence $\delta \subset \delta_y$; it completes the proof of Claim 1. □

Since Y is γ-collectionwise normal and countably paracompact, $\{M(\alpha_1, \ldots, \alpha_n) \cap P_\delta(\alpha_1, \ldots, \alpha_n) : \delta \in [\gamma]^{<\omega}\}$ has a locally finite expansion $\{H_\delta(\alpha_1, \ldots, \alpha_n) : \delta \in [\gamma]^{<\omega}\}$ of cozero-sets of X. Define

$$H_{n\beta} = \bigcup\{L(\alpha_1, \ldots, \alpha_n) \times H_\delta(\alpha_1, \ldots, \alpha_n) : \delta \in [\gamma]^{<\omega} \text{ and } \beta \in \delta;\ \alpha_1, \ldots, \alpha_n \in \Omega\}$$

for each $n \in \mathbb{N}$ and $\beta < \gamma$. Then, it follows that $\{H_{n\beta} : \beta < \gamma\}$ is a locally finite collection of cozero-sets of $X \times Y$ for each $n \in \mathbb{N}$.

Claim 2. $F_\beta \subset \bigcup\{H_{n\beta} : n \in \mathbb{N}\}$ for every $\beta < \gamma$.

Proof of Claim 2. Let $(x, y) \in F_\beta$. Choose $\alpha_1, \alpha_2, \cdots \in \Omega$ such that $\{E(\alpha_1, \ldots, \alpha_n) : n \in \mathbb{N}\}$ is a local net of $C(x)$. Before everything, we show that $Y = \bigcup\{G(\alpha_1, \ldots, \alpha_n) : n \in \mathbb{N}\}$. Let $z \in Y$ and $\delta_{xz} = \{\mu < \gamma : (C(x) \times \{z\}) \cap F_\mu \neq \emptyset\}$. Then, δ_{xz} is finite. Moreover, since $C(x) \times \{z\}$ is compact (on the case of Lemma 4.3.2, since $C(x) \times \{z\}$ is countably compact and Y is first countable), there exist open subsets O and O' of X and Y, respectively, such that $C(x) \times \{z\} \subset O \times O' \subset X \times Y - \bigcup\{F_\mu : \mu \notin \delta_{xz}\}$. From the property of the local net, there exists an $n \in \mathbb{N}$ such that $C(x) \subset E(\alpha_1, \ldots, \alpha_n) \subset O$. Therefore $(E(\alpha_1, \ldots, \alpha_n) \times O') \cap (\bigcup\{F_\mu : \mu \notin \delta_{xz}\}) = \emptyset$. Hence, $z \in O' \subset G_{\delta_{xz}}(\alpha_1, \ldots, \alpha_n) \subset G(\alpha_1, \ldots, \alpha_n)$; it shows that $Y = \bigcup\{G(\alpha_1, \ldots, \alpha_n) : n \in \mathbb{N}\}$.

So we have $Y = \bigcup\{M(\alpha_1, \ldots, \alpha_n) : n \in \mathbb{N}\}$. There exists an $n \in \mathbb{N}$ such that $y \in M(\alpha_1, \ldots, \alpha_n)$. Let $\delta_{xy} = \{\mu < \gamma : (C(x) \times \{y\}) \cap F_\mu \neq \emptyset\}$. Likewise the matter shown above, we have $\delta_{xy} \in [\gamma]^{<\omega}$, and there exist open subsets O_x and O_y of X and Y, respectively, such that

$$C(x) \times \{y\} \subset O_x \times O_y \subset X \times Y - \bigcup\{F_\mu : \mu \notin \delta_{xy}\}.$$

From the property of the local net, there exists an $m < \omega$ such that $C(x) \times \{y\} \subset E(\alpha_1, \ldots, \alpha_m) \times \{y\} \subset O_x \times O_y$, where we can select $m \geq n$. Hence,

$$\left(E(\alpha_1, \ldots, \alpha_m) \times \{y\}\right) \cap \left(\bigcup\{F_\mu : \mu \notin \delta_{xy}\}\right) = \emptyset.$$

Moreover, by the definition of δ_{xy} and the fact $C(x) \subset E(\alpha_1, \dots, \alpha_m)$, we have $(E(\alpha_1, \dots, \alpha_m) \times \{y\}) \cap F_\mu \neq \emptyset$ for every $\mu \in \delta_{xy}$. It follows that $y \in P_{\delta_{xy}}(\alpha_1, \dots, \alpha_m)$ and $y \in M(\alpha_1, \dots, \alpha_n) \subset M(\alpha_1, \dots, \alpha_m)$. So we have

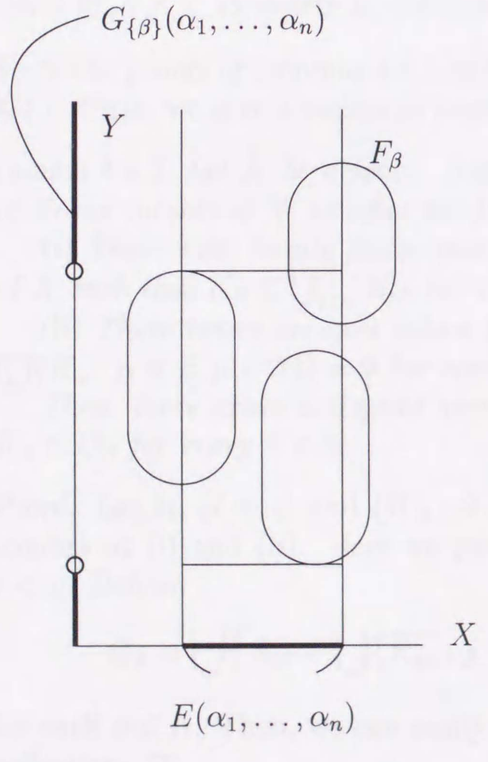
$$y \in M(\alpha_1, \dots, \alpha_m) \cap P_{\delta_{xy}}(\alpha_1, \dots, \alpha_m) \subset H_{\delta_{xy}}(\alpha_1, \dots, \alpha_m).$$

Thus, $(x, y) \in \bar{L}(\alpha_1, \dots, \alpha_m) \times H_{\delta_{xy}}(\alpha_1, \dots, \alpha_m)$. Since $(x, y) \in F_\beta$, we have that $\beta \in \delta_{xy}$. It follows that $(x, y) \in H_{m\beta}$, which proves that $F_\beta \subset \bigcup\{H_{n\beta} : n \in \mathbb{N}\}$. This completes the proof of Claim 2. □

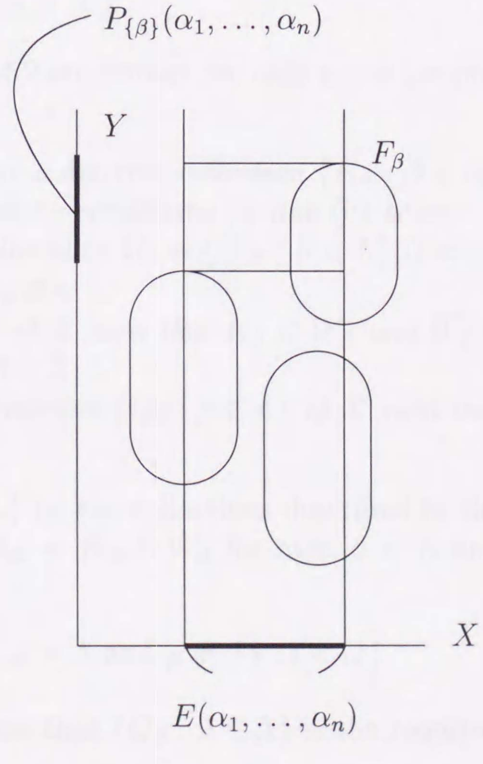
Hence, it follows that A is weakly z_γ-embedded in $X \times Y$. It completes the proof of Lemma 4.3.1. □

Defining as (**) is our essential idea of the proof. We compare (**) with (*); defining like (*) is often used in proving results of products. In general, $G_\delta(\alpha_1, \dots, \alpha_n)$ and $P_\delta(\alpha_1, \dots, \alpha_n)$ are not included in each other.

(*)



(**)



4. Proofs

First we prove the cases (1) and (2) of Theorem 4.2.1.

Proofs of the cases (1) *and* (2) *of Theorem 4.2.1.* Since the proofs of the cases of (1) and (2) of the theorem are similar, we only prove the case (1). Let X and Y be as in the conditions in (1). Assume $X \times Y$ is normal and A a closed subspace of X. By Theorem 1.3.1, A is z-embedded (or equivalently, C-embedded) in X. By Lemma 4.3.1, A is weakly z_γ-embedded in $X \times Y$. By Proposition 2.4.3, A is z_γ-embedded (or equivalently, P^γ-embedded) in $X \times Y$. It follows from Theorem 1.3.2 that X is γ-collectionwise normal. It completes the proof. □

As the proofs of the cases (1) and (2), in order to prove the cases (3) and (4), it suffices to show the following two lemmas.

Lemma 4.4.1. *Let X be the closed image of a paracompact M-space, Y a γ-collectionwise normal P-space and $X \times Y$ normal. Then, every closed subspace of $X \times Y$ is weakly z_γ-embedded in $X \times Y$.*

Lemma 4.4.2. *Let X be the closed image of a normal M-space, Y a γ-collectionwise normal first countable P-space and $X \times Y$ is normal. Then, every closed subspace of $X \times Y$ is weakly z_γ-embedded in $X \times Y$.*

Since the proofs of Lemmas 4.4.1 and 4.4.2 are similar, we only prove Lemma 4.4.1. First, we give a technical lemma.

Lemma 4.4.3. *Let X be a space. Suppose a discrete collection $\{K_\beta : \beta \in \Lambda\}$ of closed subsets of X satisfies the following conditions* (i) *and* (ii) *below:*

(i) *There exist locally finite open collections $\mathcal{H}_i = \{H_{i\beta} : \beta \in \Lambda\}$ $(i < \omega)$ of X such that $K_\beta \subset \bigcup_{i<\omega} H_{i\beta}$ for every $\beta \in \Lambda$;*

(ii) *There exists an open subset W_β of X such that $K_\beta \subset W_\beta$ and $\overline{W_\beta} \cap (\bigcup\{K_\mu : \mu \neq \beta, \mu \in \Lambda\}) = \emptyset$ for every $\beta \in \Lambda$.*

Then, there exists a disjoint open collection $\{Q_\beta : \beta \in \Lambda\}$ of X such that $K_\beta \subset Q_\beta$ for every $\beta \in \Lambda$.

Proof. Let $\mathcal{H}_i$ $(i < \omega)$ and $\{W_\beta : \beta \in \Lambda\}$ be the collections described in the conditions (i) and (ii). Here we put $R_{i\beta} = H_{i\beta} \cap W_\beta$ for each $\beta \in \Lambda$ and $i < \omega$. Define

$$Q_\beta = \bigcup\Big\{R_{i\beta} - \bigcup\{\overline{R_{j\mu}} : j \leq i, \mu \in \Lambda \text{ and } \mu \neq \beta\} : i < \omega\Big\}$$

for each $\beta \in \Lambda$. Then, we can easily show that $\{Q_\beta : \beta \in \Lambda\}$ is the required collection. □

Proof of Lemma 4.4.1. Let $X \times Y$ be normal, A a closed subspace of $X \times Y$ and $\{D_\beta : \beta < \gamma\}$ a uniformly discrete collection of zero-sets of $X \times Y$. Let Z be a paracompact M-space and f a closed continuous map from Z onto X. By Nagata [42, Theorem VII,4], we can express that $X = \bigcup\{X_i : i \geq 0\}$, where X_i is closed discrete for every $i \geq 1$ and $f^{-1}(x)$ is compact for each $x \in X_0$

First we remark that for a subset A of $Z \times Y$ the following equality holds:

$$(f \times 1_Y)(\overline{A}) \cap (X_0 \times Y) = \overline{(f \times 1_Y)(A)} \cap (X_0 \times Y).$$

For each $i \geq 1$, X and Y are collectionwise normal, we can take a discrete collection $\{H_{i\beta} : \beta < \gamma\}$ of cozero-sets of $X \times Y$ such that $D_\beta \cap (X_i \times Y) \subset H_{i\beta}$ for every $\beta < \gamma$.

Let $F_\beta = D_\beta - \bigcup\{H_{i\beta} : i \geq 1\}$ for each $\beta < \gamma$. Then, it follows that $\{(f \times 1_Y)^{-1}(F_\beta) : \beta < \gamma\}$ is a discrete closed collection of $Z \times Y$.

Claim. The $\{(f \times 1_Y)^{-1}(F_\beta) : \beta < \gamma\}$ has a disjoint open expansion of $Z \times Y$.

Proof of Claim. Let $A = \bigcup_{\beta<\gamma}(f \times 1_Y)^{-1}(F_\beta)$. Then A is closed subspace of $Z \times Y$. Since Z is a collectionwise normal Σ-space, by Lemma 4.3.1, A is weakly z_γ-embedded in $Z \times Y$. Hence, there exists a locally finite collection $\{H_{i\beta} : \beta < \gamma\}$ of cozero-sets of $Z \times Y$ for each $i < \omega$ and $(f \times 1_Y)^{-1}(F_\beta) \subset \bigcup_{i<\omega} H_{i\beta}$ for every $\beta < \gamma$. Since $X \times Y$ is normal, for each $\beta < \gamma$, there exists an open subset W_β of $X \times Y$ such that $F_\beta \subset W_\beta \subset \overline{W_\beta} \subset X \times Y - \bigcup_{\mu\neq\beta} F_\mu$. Then we have $(f \times 1_Y)^{-1}(F_\beta) \subset (f \times 1_Y)^{-1}(W_\beta)$ and $\overline{(f \times 1_Y)^{-1}(W_\beta)} \cap \left(\bigcup_{\mu\neq\beta}(f \times 1_Y)^{-1}(F_\mu)\right) = \emptyset$ for every $\beta < \gamma$. By Lemma 4.4.3, Claim follows. $\square$

Define $V_\beta = X \times Y - \overline{(f \times 1_Y)(Z \times Y - Q_\beta)}$ for each $\beta < \gamma$. Then, $\{V_\beta : \beta < \gamma\}$ is a disjoint open collection of $X \times Y$. Since $F_\beta \subset X_0 \times Y$, we can show that $F_\beta \subset V_\beta$ for each $\beta < \gamma$. By the normality of $X \times Y$, there exists a discrete collection $\{H_{0\beta} : \beta < \gamma\}$ of cozero-sets of $X \times Y$ such that $F_\beta \subset H_{0\beta} \subset V_\beta$ for each $\beta < \gamma$. The collection $\{H_{i\beta} : \beta < \gamma, i \geq 0\}$ has the properties that $D_\beta \subset \bigcup\{H_{i\beta} : i \geq 0\}$ for each $\beta < \gamma$ and that $\{H_{i\beta} : \beta < \gamma\}$ is discrete for each $i \geq 0$. It follows that A is weakly z_γ-embedded in $X \times Y$. It completes the proof. $\square$

5. Related problems and results

If we replace the paracompactness of X and Y by the collectionwise normality in (1) of Theorem 4.1.1, then even the normality of $X \times Y$ need not be implied in general. Thus the following problem naturally arises. The cases of (1) and (2) in Theorem 4.2.1 can be regarded as partial answers to this problem.

Problem 4.5.1. Let X be a collectionwise normal Σ-space and Y a collectionwise normal (or a paracompact) P-space. Is $X \times Y$ collectionwise normal if it is normal?

Corresponding to Nagami's result above, we have the following theorem.

Theorem 4.5.2. *Let X be the closed image of a paracompact M-space and Y a paracompact P-space. Then, $X \times Y$ is normal if and only if $X \times Y$ is paracompact.*

Proof. First we note the fact that for spaces X and Y given in the theorem, $X \times Y$ is normal if and only if $X \times Y$ is countably paracompact; the proof is similar to Bešlagić-Chiba [4, Section 5]. Assume that $X \times Y$ is normal and K is a compact space. Then, $Y \times K$ is a paracompact P-space, and by the fact above the countably paracompact space $(X \times Y) \times K = X \times (Y \times K)$ is normal. It follows from Tamano's theorem [60, Theorem 2] that $X \times Y$ is paracompact. □

In view of Theorem 4.5.2, under the similar consideration to Problem 4.5.1, the following problem also arises. The cases of (3) and (4) of Theorem 4.2.1 can be regarded as partial answers.

Problem 4.5.3. Let X be the closed image of a normal M-space and Y a collectionwise normal (or a paracompact) P-space. Is $X \times Y$ collectionwise normal if it is normal?

Remark 4.5.4. Defining like $(**)$ is first introduced in [72] for the other purpose. The (1) of Theorem 4.2.1 is proved in [67], and (2), (3) and (4) of Theorem 4.2.1 are proved in [68] by the direct way, i.e. proving collectionwise normality of $X \times Y$ under the assumption of normality of $X \times Y$. The proof of Lemma 4.3.1 in this paper is essentially the same of [68, Lemma 2.2]. All of the results in Sections 4 and 5 are stated in [68].

The proof of Lemma 4.3.1 actually shows that $X \times Y$ has the following property: *For every locally finite closed collection $\{F_\alpha : \alpha < \gamma\}$, there exist locally finite open collections $\{H_{n\alpha} : \alpha < \gamma\}$ $(n \in \mathbb{N})$ such that $F_\alpha \subset \bigcup_{n \in \mathbb{N}} H_{n\alpha}$ for each $\alpha < \gamma$* (see also [68, Lemma 2.2]). This property is called as σ-expandable by Zhong [74]. Yajima discussed in [65] some properties of products by the notion of special refinements; see [65] for some related results to (1) of Theorem 4.2.1. For problems concerning extensions of mappings on products of Σ-spaces and P-spaces, see Ohta [46]. Other results on extension properties on products, see [66] or [72].

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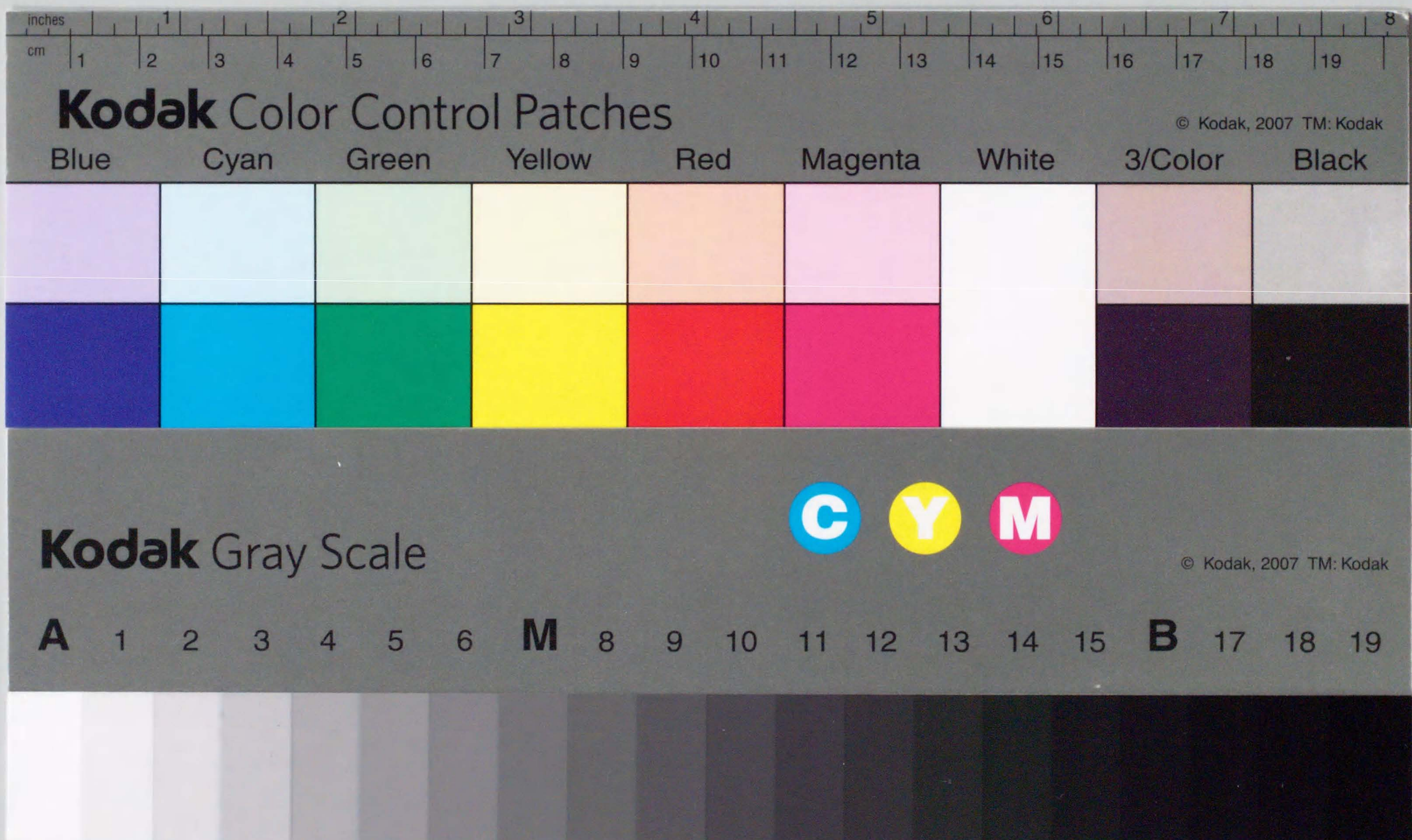
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