Localization of Derived Categories and Ring Homomorphisms

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CONTENTS

Introduction

The notion of quotient and localization of abelian categories by dense subcategories (i.e. Serre classes) was introduced by Gabriel, and plays an important role in ring theory [10], [33]. The notion of triangulated categories was introduced by Grothendieck and developed by Verdier [14], [39], and is useful in representation theory [6], [12], [33]. The quotient of triangulated categories by épaisse subcategories was constructed in [39]. These two quotients were indicated by Grothendieck, and they resemble each other. We define localization of triangulated categories, and study a relation between localizations and épaisse subcategories. Beilinson, Bernstein and Deligne introduced the notion of t-structure which is similar to torsion theory in abelian categories [4]. We consider a stable t-structure, which consists of épaisse subcategories, and show that there is a correspondence between localizations of triangulated categories and stable t-structures. Moreover, we show that if $\mathcal{A}/\mathcal{C}$ is the quotient of an abelian category $\mathcal{A}$ by a Serre subcategory $\mathcal{C}$, then the natural functor from the derived category $D^*(\mathcal{A})$ of $\mathcal{A}$ to the derived category $D^*(\mathcal{A}/\mathcal{C})$ of $\mathcal{A}/\mathcal{C}$ is a quotient functor, where $* = -$, $+$ or b (Theorem 3.2).

For a ring A, we denote by Mod A (resp., mod A) the category of right A-modules (resp., finitely presented right A-modules). Let T be a finitely generated right A-module, B an endomorphism ring $\mathrm{End}_A(T)$ of T, and I a trace ideal of T. If T is a projective A-module, then we have an exact sequence of abelian categories: $0 \to \mathrm{Mod}\,A/I \xrightarrow{\mathrm{I}} \mathrm{Mod}\,A \xrightarrow{\mathrm{Q}} \mathrm{Mod}\,B \to 0$, where Q and I are the natural quotient and the natural inclusion functors, respectively. According to Theorem 3.2, we get an exact sequence of derived categories: $0 \to D^*_{\mathrm{Mod}A/I}(\mathrm{Mod}\,A) \xrightarrow{\mathrm{I}^*} D^*(\mathrm{Mod}\,A) \xrightarrow{\mathrm{Q}^*} D^*(\mathrm{Mod}\,B) \to 0$, where $* = +, -$ or b. Moreover, Q^+ (resp., Q^-) is a localization (resp., a colocalization) functor (cf. [8, Proposition 2.1]). This result indicates us three problems. First, what condition is $0 \to D^b(\text{Mod } A/I) \overset{I^b}{\to} D^b(\text{Mod } A) \overset{Q^b}{\to} D^b(\text{Mod } B) \to 0$ to be an exact sequence of derived categories? Secondly, when does a right A-module T induce an exact sequence of derived categories: $0 \to \text{Ker } R^b\text{F} \to D^b(\text{Mod } A) \overset{R^b\text{F}}{\to} D^b(\text{Mod } B) \to 0$, where $\text{F} = \text{Hom}_A(T, -)$? Furthermore, what condition is $D^b(\text{Mod } A) \overset{R^b\text{F}}{\to} D^b(\text{Mod } B)$ to be a colocalization functor? Thirdly, when does a ring morphism $A \to C$ induce an exact sequence of derived categories: $0 \to D^b(\text{Mod } C) \overset{I^b}{\to} D^b(\text{Mod } A) \to D^b(\text{Mod } A)/D^b(\text{Mod } C) \to 0$? Moreover, what condition is $D^b(\text{Mod } A) \to D^b(\text{Mod } A)/D^b(\text{Mod } C)$ to be a localization or colocalization functor? Considering the second problem, we are given a hint by results on equivalences for derived categories. Miyashita introduced the notion of tilting modules of finite projective dimension [27]. Afterward, Happel, Cline, Parshall and Scott showed that a tilting module T_A induces a derived equivalence between $D^b(\text{Mod}A)$ and $D^b(\text{Mod}B)$ [6], [12], [13]. Rickard introduced the notion of a tilting complex $T_A^\bullet$ and showed that $D^*(\text{Mod}A)$ is derived equivalent to $D^*(\text{Mod}B)$, where B is an endomorphism ring $\text{End}_{D^*(\text{Mod}A)}(T_A^\bullet)$ and $* = -$ or b [33], [34]. Thus, a tilting module or a tilting complex plays same role in a derived category as a finitely generated projective generator in a module category. Then, we consider a partial tilting module T in a derived category as a finitely generated projective module in a module category. We show that projective dimension of a partial tilting module T as an $\text{End}_A(T)$-module is finite if and only if $D^b(\text{mod } A) \overset{R^b\text{F}}{\to} D^b(\text{mod } B)$ is a colocalization functor in case of A being a finite dimensional algebra over a field k. Furthermore, we give the upper bound of the number of isomorphism classes of indecomposable direct summands of such a partial tilting module by calculating the Grothendieck groups of derived categories. Concerning to the third problem, we give a necessary and sufficient condition for the existence of a localization functor E : $D^b(\text{Mod } A) \to D^b(\text{Mod } C)$ which is the left adjoint of $D^b(\text{Mod } C) \xrightarrow{i^b} D^b(\text{Mod } A)$. Moreover, we study a relation between the second problem and the third problem. Concerning to the first problem, the notion of recollement, which was introduced by Beilinson, Bernstein and Deligne, is useful [4]. Cline, Parshall and Scott studied algebraic stratification which induces recollement of derived categories of module categories. In particular, they introduced heredity ideals which are idempotent ideals inducing recollement of derived categories of module categories, and studied quasi-hereditary algebras which have a sequence of heredity ideals [8], [30], [31]. We give a necessary and sufficient condition for an idempotent ideal AeA to induce an exact sequence of derived categories: $0 \to D^b(\text{mod } A/AeA) \xrightarrow{i^b} D^b(\text{mod } A) \xrightarrow{Q^b} D^b(\text{mod } eAe) \to 0$ in case of A being a finite dimensional algebra over a field k. Furthermore, we give a necessary and sufficient condition for an idempotent ideal AeA to induce recollement of derived categories in case of A being a left Noetherian or semiprimary ring. In particular, we show that a minimal idempotent ideal satisfies recollement conditions if and only if it is projective as both right and left modules.

Various results on extensions of algebras and extensions of tilting functors were given in representation theory. Let $0 \to A \to \Lambda$ be an extension of a ring A. What condition is $T \otimes_A \Lambda$ to be a tilting Λ-module? What is the relation between $0 \to A \to \Lambda$ and $\text{End}_A(T) \to \text{End}_\Lambda(T \otimes_A \Lambda)$? Tachikawa and Wakamatsu showed that $T \otimes_A \Lambda$ is a classical tilting module under the condition that T is a classical tilting A-module and that Λ is a trivial extension algebra $A \ltimes \text{D}(\text{t}A)$ of A by $\text{D}(\text{t}A)$, where $\text{t}A$ is a trace ideal of T [38]. In case of $\Lambda = A \ltimes M$, we had a necessary and sufficient condition for $T \otimes_A \Lambda$ to be a classical tilting module [22]. Assem and Marmaridis gave a necessary and sufficient condition for $T \otimes_A \Lambda$ to be a classical tilting module, in case of split-by-nilpotent extensions of rings [1]. Miyashita considered a condition for $T \otimes_A \Lambda$ to be a tilting module [27]. Hoshino showed a necessary and sufficient condition for $T \otimes_A \Lambda$ to be a tilting module, in case of split extensions of rings [5]. Rickard gave a sufficient condition for $T^{\bullet} \otimes^{L}_{A} \Lambda$ to be a tilting complex, in case of $\Lambda = A \ltimes M$ [35], [36]. Also, Rickard showed that a finite dimensional algebra which is derived equivalent to a symmetric algebra is itself symmetric, and that if A and B are derived equivalent algebras, then a trivial extension algebra $A \ltimes DA$ and a trivial extension algebra $B \ltimes DB$ are also derived equivalent (see [36] for details). In these two cases, there is the structure similar to Frobenius extensions. In case that T is a finitely generated projective generator, Miyashita showed that if $0 \to A \to \Lambda$ is a Frobenius extension, then $0 \to \mathrm{End}_A(T) \to \mathrm{End}_\Lambda(T \otimes_A \Lambda)$ is also a Frobenius extension [26]. We give conditions for extensions of rings inducing tilting complexes. Moreover we show that Frobenius extensions are invariant under derived equivalences which are induced by these tilting complexes.

In Chapter I, we study quotient and localization of abelian categories and of triangulated categories, and the relation of them. In Section 1, we recall standard notations and terminologies of quotient and localization of abelian categories, and torsion theories. In Section 2, we define localization of triangulated categories, and consider a relation between localizations and stable t-structures (Theorem 2.6). In Section 3, we show that if $\mathcal{A} \to \mathcal{A}/\mathcal{C}$ is a quotient of abelian categories, then $D^*(\mathcal{A}) \to D^*(\mathcal{A}/\mathcal{C})$ is a quotient of triangulated categories, where $* = +, -$ or b (Theorem 3.2). Moreover, in case of $\mathcal{A}/\mathcal{C}$ having enough injectives, if $\mathcal{A} \to \mathcal{A}/\mathcal{C}$ is a localization of abelian categories, then $D^+(\mathcal{A}) \to D^+(\mathcal{A}/\mathcal{C})$ is a localization of triangulated categories (Corollary 3.3).

In Chapter II, we apply the results of Chapter I to derived categories of module categories. In section 4, we study quotient and localization of derived categories of modules by using partial tilting modules of finite projective dimension [6], [27] (Propositions 4.2, 4.3, Corollaries 4.4). Moreover, in case of a finite dimensional algebra A over a field k, we show that if projective dimension of a partial tilting module T as an $\mathrm{End}_A(T)$-module is finite, then the number of isomorphism classes of indecomposable direct summands of T is at most the rank of the Grothendieck group of A (Propositions 4.5, 4.6, Corollaries 4.7, 4.8). In Section 5, we consider relations between ring epimorphisms and localizations (Proposition 5.1, Theorem 5.2). Moreover, we consider that partial tilting modules induce ring epimorphisms and their localization of derived categories (Proposition 5.3, Corollary 5.4). In Section 6, we give necessary and sufficient conditions for $\{D^b(\mathrm{Mod}\ A/AeA)\ ,\ D^b(\mathrm{Mod}\ A),\ D^b(\mathrm{Mod}\ eAe)\}$ to be recollement in case of A being left Noetherian or semiprimary (Theorems 6.2, 6.3). In particular, we study when a minimal idempotent ideal satisfies recollement conditions (Propositions 6.6, 6.7).

In Chapter III, we study extensions of rings and extensions of tilting complexes. In Section 7, in case of split-extensions of rings, we give a necessary and sufficient condition for $T^{\bullet}\otimes^{L}_{A}\Lambda$ to be a tilting complex (Theorem 7.7). In Section 8, in case of arbitrary extensions of rings, we give a necessary and sufficient condition for $T^{\bullet}\otimes^{L}_{A}\Lambda$ to be a tilting complex and $\mathrm{End}_{D(\mathrm{Mod} A)}(T^{\bullet}) \to \mathrm{End}_{D(\mathrm{Mod} \Lambda)}(T^{\bullet}\otimes^{L}_{A}\Lambda)$ is ring extension (Theorem 8.1). Furthermore, we show that Frobenius extensions are invariant under derived equivalences which are induced by tilting complexes satisfying the condition of Theorem 8.1 (Theorem 8.3).

In this chapter, we assume that all categories are skeletally small.

§1. Preliminaries.

Let $\mathcal{A}$ be an abelian category. A collection $\mathcal{S}$ of arrows of $\mathcal{A}$ is called a multiplicative system if it satisfies the following conditions:

(FR-1) If $f, g \in \mathcal{S}$, and $f \circ g$ exists, then $f \circ g \in \mathcal{S}$. For any $X \in \mathcal{A}$, id_X belongs to $\mathcal{S}$.

(FR-2) In $\mathcal{A}$, any diagram:

$$\begin{array}{ccc} & & Y \\ & & \downarrow s \\ Z & \xrightarrow{f} & X \end{array},$$

with $s \in \mathcal{S}$, can be completed to a commutative diagram:

$$\begin{array}{ccc} Z & \xrightarrow{f'} & Y \\ t \downarrow & & \downarrow s \\ Z & \xrightarrow{f} & X \end{array},$$

with $t \in \mathcal{S}$. Ditto for the opposed statement.

(FR-3) If f and g are morphisms in $\mathcal{A}$, the following properties are equivalent:

(i) there exists $s \in \mathcal{S}$ such that $s \circ f = s \circ g$;

(ii) there exists $t \in \mathcal{S}$ such that $f \circ t = g \circ t$.

A full subcategory $\mathcal{C}$ of $\mathcal{A}$ is called dense provided that for every exact sequence in $\mathcal{A}$:

$$0 \to X \to Y \to Z \to 0,$$

X and Z belong to $\mathcal{C}$ if and only if Y belongs to $\mathcal{C}$.

We denote by $\phi(\mathcal{C})$ the system of all morphisms f such that Ker f and Coker f are in $\mathcal{C}$. Then $\phi(\mathcal{C})$ is a multiplicative system, and the quotient category $\mathcal{A}/\mathcal{C}$ can be defined. Moreover, $\mathcal{C}$ and $\mathcal{A}/\mathcal{C}$ are also abelian categories, and the natural quotient functor $\mathcal{A} \overset{Q}{\to} \mathcal{A}/\mathcal{C}$ is an exact functor. In this case, we will say that $0 \to \mathcal{C} \to \mathcal{A} \overset{Q}{\to} \mathcal{A}/\mathcal{C} \to 0$ is an exact sequence of abelian categories.

The right adjoint of Q (if it exists) is called a section functor. If there exists a section functor S, then $\{\mathcal{A}/\mathcal{C}; Q, S\}$ is called a localization of $\mathcal{A}$. In this case, $\mathcal{C}$ is called a localizing subcategory of $\mathcal{A}$. If $\{\mathcal{A}/\mathcal{C}; Q, S\}$ is a localization of $\mathcal{A}$, then S is fully faithful. On the other hand, if $T : \mathcal{A} \to \mathcal{B}$ is an exact functor between abelian categories which has a fully faithful right adjoint $S: \mathcal{B} \to \mathcal{A}$, then Ker T is a localizing subcategory of $\mathcal{A}$, and T induces an equivalence between $\mathcal{A}$/KerT and $\mathcal{B}$. Colocalization of $\mathcal{A}$ is also defined, and similar results hold (see [10] and [32] for details).

A torsion theory for $\mathcal{A}$ consists of a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories satisfying the following conditions:

(a) $\mathrm{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0$,

(b) for every object $X \in \mathcal{A}$, there exists an exact sequence:

$$0 \to X' \to X \to X'' \to 0,$$

with $X' \in \mathcal{T}$ and $X'' \in \mathcal{F}$.

A torsion theory $(\mathcal{T}, \mathcal{F})$ is called hereditary if $\mathcal{T}$ is closed under subobjects. If $(\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory for $\mathcal{A}$, then $\mathcal{T}$ is a dense subcategory of $\mathcal{A}$, and then $0 \to \mathcal{T} \to \mathcal{A} \xrightarrow{Q} \mathcal{A}/\mathcal{T} \to 0$ is exact. For an object $X \in \mathcal{F}$, X is called $\mathcal{T}$-closed if $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{T}, X) = 0$. Let $\mathcal{V}$ be the full subcategory of $\mathcal{A}$ consisting of all $\mathcal{T}$-closed objects. For a hereditary torsion theory $(\mathcal{T}, \mathcal{F})$ for $\mathcal{A}$, if $\mathcal{T}$ is a localizing subcategory of $\mathcal{A}$, then $\{\mathcal{A}/\mathcal{C}; Q, S\}$ is localization of $\mathcal{A}$, and then Q and S induce an equivalence between $\mathcal{V}$ and $\mathcal{A}/\mathcal{T}$. We apply these ideas to triangulated categories in the next section.

§2. Localization of Triangulated Categories.

Definitions. A triangulated category $\mathcal{T}$ is an additive category endued with an autofunctor $\mathrm{T} : \mathcal{T} \to \mathcal{T}$ (we often denote $\mathrm{T}^i X$ by $X[i]$ for $X \in \mathcal{T}$) and a collection S of triangles $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \mathrm{T}X$, called the distinguished triangles of $\mathcal{T}$, which satisfies the following:

(TR-1) Every triangle which is isomorphic to some distinguished triangle is distinguished. Every morphism $X \to Y$ can be imbedded in a distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \mathrm{T}X$. The triangle $X \xrightarrow{1} X \to 0 \to \mathrm{T}X$ is distinguished for every object $X \in \mathcal{T}$.

(TR-2) $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \mathrm{T}X$ is distinguished if and only if $Y \xrightarrow{v} Z \xrightarrow{w} \mathrm{T}X \xrightarrow{-\mathrm{T}(u)} \mathrm{T}Y$ is distinguished.

(TR-3) Given two distinguished triangles $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \mathrm{T}X$ and $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \mathrm{T}X'$, and morphisms $f: X \to X'$ and $g: Y \to Y'$ which satisfy $g \circ u = u' \circ f$, there exists a morphism $h: Z \to Z'$ such that $h \circ v = v' \circ g$, $\mathrm{T}(f) \circ w = w' \circ h$.

(TR-4) Given distinguished triangles $X \xrightarrow{u} Y \to Z' \to \mathrm{T}X$, $X \xrightarrow{v \circ u} Z \to Y' \to \mathrm{T}X$ and $Y \xrightarrow{v} Z \to X' \to \mathrm{T}Y$, there exists a distinguished triangle $Z' \to Y' \to X' \to \mathrm{T}Z'$ which satisfies the following commutative diagram:

$$\begin{array}{ccccccccc}
\mathrm{T}^{-1}Y' & \to & X & = & X & & & & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
\mathrm{T}^{-1}X' & \to & Y & \to & Z & \to & X' & \to & \mathrm{T}Y \\
 & & \downarrow & & \downarrow & & \| & & \downarrow \\
 & & Z' & \to & Y' & \to & X' & \to & \mathrm{T}Z' \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathrm{T}X & = & \mathrm{T}X & & & &
\end{array}$$

.

Given two triangulated categories $\mathcal{D}$ and $\mathcal{D}'$, a grade functor from $\mathcal{D}$ to $\mathcal{D}'$ is a pair of an additive functor $F: \mathcal{D} \to \mathcal{D}'$ and an isomorphism Φ : $FT \to T'F$, where T and T' are the translation functors of $\mathcal{D}$ and $\mathcal{D}'$, respectively. A grade functor (F, Φ) is called a ∂- functor if for every distinguished triangle (X, Y, Z, u, v, w) in $\mathcal{D}$, $(X, Y, Z, Fu, Fv, \Phi_X \circ Fw)$ is distinguished in $\mathcal{D}'$ (we often simply write F unless it confounds us) [14, Chapter I, §1], [39, I, §1, n° 1]. Let $F : \mathcal{D} \to \mathcal{D}'$ be a ∂-functor. If F has a right or left adjoint G, then G is also a ∂- functor [18, 1.6 Proposition].

A subcategory $\mathcal{U}$ of $\mathcal{D}$ is called épaisse if $\mathcal{U}$ is a full triangulated subcategory and $\mathcal{U}$ satisfies the following condition: For any $f : X \to Y$, which factors through an object in $\mathcal{U}$ and which has a mapping cone in $\mathcal{U}$, X and Y are objects in $\mathcal{U}$. We denote by $\phi(\mathcal{U})$ the set of morphisms f which is contained in a distinguished triangle (X, Y, Z, f, g, h) where Z is an object of $\mathcal{U}$. Then $\phi(\mathcal{U})$ is a multiplicative system which satisfies the following conditions:

(FR-4) $s \in \phi(\mathcal{U})$ if and only if $Ts \in \phi(\mathcal{U})$, where T is the translation functor.

(FR-5) Given distinguished triangles (X, Y, Z, u, v, w), (X', Y', Z', u', v', w'), if f and g are morphisms in $\phi(\mathcal{U})$ such that $u' \circ f = g \circ u$, then there exists a morphism h in $\phi(\mathcal{U})$ such that (f, g, h) is a morphism of distinguished triangles (see [39, I, §2, n° 1] for details).

In this case, the quotient category $\mathcal{D}/\mathcal{U}$ is defined, and we will say that $0 \to \mathcal{U} \xrightarrow{K} \mathcal{D} \xrightarrow{Q} \mathcal{D}/\mathcal{U} \to 0$ is an exact sequence of triangulated categories (see [4, 1.4.4], [39, I, §2, n° 3] for details).

LEMMA 2.1. *Let $\mathcal{D}$ be a triangulated category, $\mathcal{U}$ an épaisse subcategory* *of $\mathcal{D}$, and $Q : \mathcal{D} \to \mathcal{D}/\mathcal{U}$ a natural quotient. For $M \in \mathcal{D}$, the following are equivalent.*

(a) *For every $f: X \to Y \in \phi(\mathcal{U})$, $\mathrm{Hom}(f, M) : \mathrm{Hom}_{\mathcal{D}}(Y, M) \to \mathrm{Hom}_{\mathcal{D}}(X, M)$ is bijective.*

(b) $\mathrm{Hom}_{\mathcal{D}}(\mathcal{U}, M) = 0$.

(c) *For every $X \in \mathcal{D}$, $Q(X, M) : \mathrm{Hom}_{\mathcal{D}}(X, M) \to \mathrm{Hom}_{\mathcal{D}/\mathcal{U}}(QX, QM)$ is bijective.*

Proof. (a) $\Rightarrow$ (b): For every object $U \in \mathcal{U}$, $0 \to U \xrightarrow{1} U \to$ is a distinguished triangle. Then $0 = \mathrm{Hom}_{\mathcal{D}}(0, M) \cong \mathrm{Hom}_{\mathcal{D}}(U, M)$.

(b) $\Rightarrow$ (c): Every morphism of $\mathrm{Hom}_{\mathcal{D}/\mathcal{U}}(QX, QM)$ is represented by a diagram:

$$\begin{array}{ccc} & X' & \\ s \swarrow & & \searrow f \\ X & & M \end{array},$$

where s is contained in a distinguished triangle $U \to X' \to X \to$ with $U \in \mathcal{U}$. Then there exists $f' : X \to M$ in $\mathcal{D}$ such that $f = f' \circ s$, because $\mathrm{Hom}_{\mathcal{D}}(\mathcal{U}, M) = 0$. Hence $Q(X, M)$ is surjective. Let $U \xrightarrow{r} X' \xrightarrow{s} X \xrightarrow{t}$ be a distinguished triangle with $U \in \mathcal{U}$. If a morphism $g : X \to M$ satisfies $g \circ s = 0$, then there exist $u : U[1] \to M$ such that $g = u \circ t$. Therefore $g = 0$, because $u \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{U}, M) = 0$. Hence $Q(X, M)$ is injective.

(c) $\Rightarrow$ (a): Let $f: X \to Y$ be a morphism in $\phi(\mathcal{U})$. Then we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(Y, M) & \xrightarrow{\mathrm{Hom}(f,M)} & \mathrm{Hom}_{\mathcal{D}}(X, M) \\ Q(Y, M) \downarrow & & \downarrow Q(X, M) \\ \mathrm{Hom}_{\mathcal{D}/\mathcal{U}}(QY, QM) & \xrightarrow{\mathrm{Hom}(Qf,QM)} & \mathrm{Hom}_{\mathcal{D}/\mathcal{U}}(QX, QM) \end{array}$$

According to (c), $Q(X, M)$ and $Q(Y, M)$ are bijective. Since $QU = 0$, $\mathrm{Hom}(Qf, QM)$ is bijective. Hence $\mathrm{Hom}(f, M)$ is bijective.

An object M is called $\mathcal{U}$-closed if it satisfies the equivalent conditions of Lemma 2.1. Let $0 \to \mathcal{U} \to \mathcal{D} \to \mathcal{D}/\mathcal{U} \to 0$ be an exact sequence of triangulated categories. The right adjoint of Q is called a section functor. If there exists a section functor S, then $\{\mathcal{D}/\mathcal{U}; Q, S\}$ is called a localization of $\mathcal{D}$, and $0 \to \mathcal{U} \xrightarrow{K} \mathcal{D} \xrightarrow{Q} \mathcal{D}/\mathcal{U} \to 0$ is called localization exact.

LEMMA 2.2. *For every object* $V \in \mathcal{D}/\mathcal{U}$, SV *is* $\mathcal{U}$*-closed.*

Proof. For every $f: X \to Y \in \phi(\mathcal{U})$, we have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(Y, SV) & \xrightarrow{\mathrm{Hom}(f,SV)} & \mathrm{Hom}_{\mathcal{D}}(X, SV) \\ \wr\wr & & \wr\wr \\ \mathrm{Hom}_{\mathcal{D}/\mathcal{U}}(QY, V) & \underset{\cong}{\xrightarrow{\mathrm{Hom}(Qf,V)}} & \mathrm{Hom}_{\mathcal{D}/\mathcal{U}}(QX, V) \end{array}$$

Therefore $\mathrm{Hom}(f, SV)$ is an isomorphism. By Lemma 2.1, SV is $\mathcal{U}$-closed.

Let $\Phi: QS \to 1_{\mathcal{D}/\mathcal{U}}$ and $\Psi: 1_{\mathcal{D}} \to SQ$ be adjunction arrows.

PROPOSITION 2.3. *Let* $\{\mathcal{D}/\mathcal{U}; Q, S\}$ *be a localization of* $\mathcal{D}$.

(a) Φ *is an isomorphism (i.e.* S *is fully faithful).*

(b) *For every object* $X \in \mathcal{D}$, *the distinguished triangle* $U \to X \xrightarrow{\Psi_X} SQX \to$ *satisfies that* U *is in* $\mathcal{U}$.

Proof. (a) For every $X \in \mathcal{D}$ and $Y \in \mathcal{D}/\mathcal{U}$, we have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(X, SY) & = & \mathrm{Hom}_{\mathcal{D}}(X, SY) \\ Q(X, SY) \downarrow & & \wr\wr \\ \mathrm{Hom}_{\mathcal{D}/\mathcal{U}}(QX, QSY) & \overset{\mathrm{Hom}(QX, \Phi_Y)}{\rightarrow} & \mathrm{Hom}_{\mathcal{D}/\mathcal{U}}(QX, Y). \end{array}$$

By Lemma 2.1 and 2.2, $Q(X, SY)$ is an isomorphism. Then $\mathrm{Hom}(QX, \Phi_Y)$ is an isomorphism. For any $Z \in \mathcal{D}/\mathcal{U}$, there exists $X \in \mathcal{D}$ such that $Z \cong QX$. Hence Φ is an isomorphism.

(b) It suffices to show that for any $X \in \mathcal{D}$, $Q\Psi_X$ is an isomorphism. By the property of adjunction arrows, we have $QX \overset{Q\Psi_X}{\rightarrow} QSQX \overset{\Phi_{QX}}{\cong} QX = 1_{QX}$, and hence $Q\Psi_X$ is an isomorphism.

Corollary 2.4. *Let $M \in \mathcal{D}$. Then M is $\mathcal{U}$-closed if and only if $M \cong SQM$.*

Proposition 2.5. *Let $\mathcal{D}$ and $\mathcal{E}$ be triangulated categories, $F : \mathcal{D} \to \mathcal{E}$ a ∂-functor which has a fully faithful right adjoint $S : \mathcal{E} \to \mathcal{D}$. Then F induces an equivalence between $\mathcal{D}/\mathrm{Ker}F$ and $\mathcal{E}$.*

Proof. Consider $Q : \mathcal{D} \to \mathcal{D}/\mathrm{Ker}F$. Then by the universal property of Q we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{D} & \overset{F}{\rightarrow} & \mathcal{E} \\ Q\searrow & & \nearrow R \\ & \mathcal{D}/\mathrm{Ker}F & \end{array} .$$

If $f : X \to Y$ is a morphism in $\mathcal{D}$, then Ff is an isomorphism if and only if Qf is an isomorphism. For every object $M \in \mathcal{D}$, $FM \to FSFM$ is an isomorphism, and then $QM \to QSFM$ is an isomorphism. Therefore $Q \to QSF$ is an isomorphism. By the universal property of Q and $QSF = QSRQ$, we have $1_{\mathcal{D}/\mathrm{Ker}F} \cong QSR$. Since, $RQS = FS \cong 1_{\mathcal{E}}$, R is an equivalence.

Let $\mathcal{U}$ and $\mathcal{V}$ be full subcategories of $\mathcal{D}$ such that: a) $\mathcal{U}$ and $\mathcal{V}$ are stable for translations; b) $\mathrm{Hom}_{\mathcal{D}}(\mathcal{U}, \mathcal{V}) = 0$; c) For every $X \in \mathcal{D}$, there exists a distinguished triangle $U \to X \to V \to$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$. Then $\mathcal{U}$ and $\mathcal{V}$ are épaisse subcategories of $\mathcal{D}$, and $(\mathcal{U}, \mathcal{V})$ is t-structure in the sense of Beilinson–Bernstein–Deligne [4, 1.3]. We will call $(\mathcal{U}, \mathcal{V})$ a stable t-structure. Moreover, there exist exact sequences $0 \to \mathcal{U} \overset{K}{\to} \mathcal{D} \overset{Q}{\to} \mathcal{V} \to 0$ and $0 \to \mathcal{V} \overset{R}{\to} \mathcal{D} \overset{Q'}{\to} \mathcal{U} \to 0$ such that Q is the left adjoint of R and that Q' is the right adjoint of K, where K and R are natural inclusions (see [4, 1.4.4] for details). Namely, $\{\mathcal{V}; Q, R\}$ is a localization of $\mathcal{D}$, and $\{\mathcal{U}; K, Q'\}$ is a colocalization of $\mathcal{D}$. By Proposition 2.5 and [39, 6-6 Proposition], and their duals, $\mathcal{D}/\mathcal{U}$ is a localization of $\mathcal{D}$ if and only if $\mathcal{U}$ is a colocalization of $\mathcal{D}$, and $\mathcal{D}/\mathcal{U}$ is a colocalization of $\mathcal{D}$ if and only if $\mathcal{U}$ is a localization of $\mathcal{D}$. We later see that recollement, in the sense of [4, 1.4.3], is equivalent to bilocalization.

Proposition 2.6. *Let $\mathcal{D}$ be a triangulated category. If $\{\mathcal{V}; Q, R\}$ is a localization of $\mathcal{D}$, then R is fully faithful, and $(K\mathcal{U}, R\mathcal{V})$ is a stable t-structure, where $\mathcal{U}$ = Ker Q and K is a natural inclusion. Conversely, if $(\mathcal{U}, \mathcal{V})$ is a stable t-structure in $\mathcal{D}$, then a natural inclusion $R : \mathcal{V} \to \mathcal{D}$ has a left adjoint Q such that $\{\mathcal{V}; Q, R\}$ is a localization.*

Proof. Let $\{\mathcal{V}; Q, R\}$ be a localization of $\mathcal{D}$. Then, by $\mathrm{Hom}_{\mathcal{D}}(K\mathcal{U}, R\mathcal{V}) \cong \mathrm{Hom}_{\mathcal{D}}(QK\mathcal{U}, \mathcal{V}) = 0$ and Proposition 2.3, it is clear that R is fully faithful and $(K\mathcal{U}, R\mathcal{V})$ is a stable t-structure. The converse is true by the above.

We have the same result of Cline–Parshall–Scott [8, §1, Theorem 1.1] under the weak conditions.

Proposition 2.7. *Let $F : \mathcal{D} \to \mathcal{E}$ be a ∂-functor of triangulated categories.*

Assume that F *has a fully faithful right (resp., left) adjoint* $G : \mathcal{E} \to \mathcal{D}$. *If* F *has a left (resp., right) adjoint* $H : \mathcal{E} \to \mathcal{D}$, *then* H *is a fully faithful* ∂-*functor. In this case,* (KerF, $\mathcal{D}$, $\mathcal{E}$) *is a recollement.*

Proof. According to Proposition 2.5, Proposition 2.3, Theorem 2.6, and their duals, it is clear.

§3. Localization of Derived Categories.

Let $\mathcal{A}$ be an additive category, $K(\mathcal{A})$ a homotopy category of $\mathcal{A}$, and $K^+(\mathcal{A})$, $K^-(\mathcal{A})$ and $K^b(\mathcal{A})$ full subcategories of $K(\mathcal{A})$ generated by the bounded below complexes, the bounded above complexes and the bounded complexes, respectively. For an abelian category $\mathcal{A}$, a derived category $D(\mathcal{A})$ (resp., $D^+(\mathcal{A})$, $D^-(\mathcal{A})$ and $D^b(\mathcal{A})$) of $\mathcal{A}$ is a quotient of $K(\mathcal{A})$ (resp., $K^+(\mathcal{A})$, $K^-(\mathcal{A})$ and $K^b(\mathcal{A})$) by a multiplicative set of quasi-isomorphisms. Then $K^*(\mathcal{A})$ and $D^*(\mathcal{A})$ are triangulated categories, where $*$ = nothing, +, - or b [14], [39]. In general, we denote by $K^*(\mathcal{A})$ a localizing subcategory of $K(\mathcal{A})$ (i.e. $K^*(\mathcal{A})$ is a full triangulated subcategory of $K(\mathcal{A})$ and $D^*(\mathcal{A}) \to D(\mathcal{A})$ is a fully faithful ∂- functor, where $D^*(\mathcal{A})$ is a quotient of $K^*(\mathcal{A})$ by a multiplicative set of quasi-isomorphisms) [14, I, §5], [39, II,§1, n°1]. For a thick abelian subcategory $\mathcal{C}$ of $\mathcal{A}$ (i.e. $\mathcal{C}$ is extension closed in $\mathcal{A}$), we denote by $D^*_{\mathcal{C}}(\mathcal{A})$ a full subcategory of $D^*(\mathcal{A})$ generated by complexes of which all homologies are in $\mathcal{C}$ [14, I, §4]. Let $\partial(D^*(\mathcal{A}), D(\mathcal{B}))$ be a category of ∂- functors from $D^*(\mathcal{A})$ to $D(\mathcal{B})$ and $\mathrm{Hom}_\partial(\mathrm{F},\mathrm{G})$ the set of morphisms from F to G for F, G $\in \partial(D^*(\mathcal{A}), D(\mathcal{B}))$. Given a ∂-functor $\mathrm{F} : K^*(\mathcal{A}) \to K(\mathcal{B})$, we obtain a right derived functor $R^*\mathrm{F}$: $D^*(\mathcal{A}) \to D(\mathcal{B})$ if there exists an object $R^*\mathrm{F}$ in $\partial(D^*(\mathcal{A}), D(\mathcal{B}))$ such that $\mathrm{Hom}_\partial(R^*\mathrm{F}, -)$ is isomorphic to $\mathrm{Hom}_\partial(\mathrm{Q}_{\mathcal{A}}{}^* \circ \mathrm{F}, - \circ \mathrm{Q}_{\mathcal{B}})$ in $\partial(D^*(\mathcal{A}), D(\mathcal{B}))$, where $\mathrm{Q}_{\mathcal{A}}{}^* : K^*(\mathcal{A}) \to D^*(\mathcal{A})$, $\mathrm{Q}_{\mathcal{B}}: K(\mathcal{B}) \to D(\mathcal{B})$ are natural quotients, [14, I, §5], [39, I, §2]. When $R^+\mathrm{F} : D^+(\mathcal{A}) \to D(\mathcal{B})$ exists, we say F has right homological dimension $\leq n$ on $\mathcal{A}$ if $R^i\mathrm{F}(X) = 0$ for all $X \in \mathcal{A}$ and for all $i > n$ [14, I, §5], [39, I, §2, n°2]. And an object X in $\mathcal{A}$ is called a right F-acyclic object if $R^i\mathrm{F}(X) = 0$ for all $i > 0$. We also denote by $R^*\mathrm{F}$ a right derived functor of an induced ∂- functor from F: $\mathcal{A} \to \mathcal{B}$. Let F : $\mathcal{A} \to \mathcal{B}$ be a left exact additive functor between abelian categories. If $\mathcal{A}$ has enough injectives, and F has finite right homological dimension on

$\mathcal{A}$, then $R^{+}\mathrm{F}$, $R^{-}\mathrm{F}$ and $R^{b}\mathrm{F}$ exist, and $R\mathrm{F}|_{D^{*}(\mathcal{A})} \cong R^{*}\mathrm{F}$, and moreover, $R^{*}\mathrm{F}$ has image in $D^{*}(\mathcal{B})$, where $* = +, -$ or b [14, I, §5]. We often denote $R^{*}\mathrm{F}|_{D^{\#}(\mathcal{A})}$ by $R^{*,\#}\mathrm{F}$, where $D^{\#}(\mathcal{A})$ is a full subcategory of $D^{*}(\mathcal{A})$. On the other hand, if $\mathcal{A}$ and $\mathcal{B}$ have enough injectives and projectives, respectively, and if the derived functor $R^{+,b}\mathrm{F} : D^{b}(\mathcal{A}) \to D(\mathcal{B})$ has image in $D^{b}(\mathcal{B})$ and $R^{+,b}\mathrm{F} : D^{b}(\mathcal{A}) \to D^{b}(\mathcal{B})$ has a left adjoint, then F has a left adjoint $\mathrm{G} : \mathcal{B} \to \mathcal{A}$ and the derived functor $L^{-,b}\mathrm{G} : D^{b}(\mathcal{B}) \to D(\mathcal{A})$ has image in $D^{b}(\mathcal{A})$ which is the left adjoint of $R^{+,b}\mathrm{F}$ [6, (3.1) Lemma].

LEMMA 3.1. *Let $\mathcal{D}$ and $\mathcal{E}$ be triangulated categories and $\mathrm{F} : \mathcal{D} \to \mathcal{E}$ a ∂-functor. Consider the following commutative diagram :*

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\mathrm{F}} & \mathcal{E} \\ & \mathrm{Q}\searrow \nearrow \mathrm{F'} & \\ & \mathcal{D}/\mathrm{Ker}\mathrm{F} & \end{array} .$$

If F' *is full dense, then* F' *is an equivalence.*

Proof. It suffices to show that F' is faithful. Let f be a morphism in $\mathcal{D}/\mathrm{Ker}\mathrm{F}$ such that $\mathrm{F}'f = 0$. Then f is represented by a diagram in $\mathcal{D}$:

$$\begin{array}{ccc} & Y' & \\ f'\nearrow & & \nwarrow s \\ X & & Y \end{array} ,$$

with $s : Y \to Y' \in \phi(\mathrm{Ker\ F})$ and $\mathrm{F}f = 0$. Let $X \xrightarrow{f'} Y' \xrightarrow{g} Z \xrightarrow{h} X[1]$ be the distinguished triangle which contains f' in $\mathcal{D}$. Then we have a morphism of distinguished triangles in $\mathcal{D}$:

$$\begin{array}{ccccccc}
FX & \overset{Ff'}{\to} & FY' & \overset{Fg}{\to} & FZ & \overset{Fh}{\to} & FX[1] \\
\| & & \| & & \downarrow \begin{pmatrix} x \\ Fg \end{pmatrix} & & \| \\
FX & \overset{0}{\to} & FY' & \overset{\binom{1}{0}}{\to} & FY' \oplus FX[1] & \overset{(0\ 1)}{\to} & FX[1].
\end{array}$$

Since F' is full, there exists a morphism $s' : Y' \to Y'' \in \phi(\mathrm{Ker}\ F)$ and $y : Z \to Y''$ in $\mathcal{D}$ such that $Fs' \circ x = Fy$. Then $Fs' = Fs' \circ x \circ Fg = Fy \circ Fg = F(y \circ g)$ is an isomorphism. Therefore we have $y \circ g \in \phi(\mathrm{Ker}\ F)$, and $y \circ g \circ s \in \phi(\mathrm{Ker}\ F)$. Then f is represented by a diagram in $\mathcal{D}$:

$$\begin{array}{ccc}
 & Y'' & \\
y \circ g \circ f' \nearrow & & \nwarrow y \circ g \circ s \\
X & & Y,
\end{array}$$

and $y \circ g \circ f' = 0$. Hence f is zero morphism in $\mathcal{D}/\mathrm{Ker}\ F$.

THEOREM 3.2. *Let* $0 \to \mathcal{C} \to \mathcal{A} \to \mathcal{A}/\mathcal{C} \to 0$ *be an exact sequence of abelian categories. Then* $0 \to D^*_{\mathcal{C}}(\mathcal{A}) \to D^*(\mathcal{A}) \overset{Q^*}{\to} D^*(\mathcal{A}/\mathcal{C}) \to 0$ *is an exact sequence of triangulated categories, where* $* = +, -$ *or b.*

Proof. According to Ker $Q^* = D^*_{\mathcal{C}}(\mathcal{A})$ and Lemma 3.1, it suffices to show that the induced ∂- functor $Q'^* : D^*(\mathcal{A})/D^*_{\mathcal{C}}(\mathcal{A}) \to D^*(\mathcal{A}/\mathcal{C})$ is full dense.

(I) *The case of* $* = b$. (i) Q'^b *is dense.* Let $X^\bullet : \ldots \to 0 \to X^{-n} \overset{\partial_{-n}}{\to} X^{-n+1} \overset{\partial_{-n+1}}{\to} \ldots \overset{\partial_{-1}}{\to} X^0 \to 0 \to \ldots$ be a complex in $D^b(\mathcal{A}/\mathcal{C})$. Then $X^\bullet$ is represented by a diagram in $\mathcal{A}$:

$$\begin{array}{ccccccc} & X'^{-n} & & X'^{-n+1} & & X'^{-1} & \\ & s_{-n} \swarrow \searrow d_{-n} & s_{-n+1} & \swarrow \searrow d_{-n+1} & \ldots s_{-1} & \swarrow \searrow d_{-1} & \\ \ldots \to 0 \to X^{-n} & & X^{-n+1} & & & X^{0} \to 0 \to \ldots, & \end{array}$$

where $s_i \in \phi(\mathcal{C})$ for all i. By induction on i, we have the following commutative diagram in $\mathcal{A}$:

$$\begin{array}{ccc} X''^{-i} & \xrightarrow{d'_{-i}} & X''^{-i+1} \\ t_{-i} \downarrow & & \downarrow s'_{-i+1} \\ X'^{-i} & \xrightarrow{d_{-i}} & X^{-i+1} \end{array},$$

and we have $s'_{-i} = s_{-i} \circ t_{-i} \in \phi(\mathcal{C})$ and $d'_{-i+1} \circ d'_{-i} = 0$ for all i. Indeed, it is clear in case of $i \geq 1$ by taking $X''_{-i+1} = X'_{-i+1}$, $X''_{-i} = X_{-i}$, $s_{-i+1} = 1_{X^{-i+1}}$, $t_{-i} = 1_{X^{-i}}$ and $d'_{-i} = d_{-i}$. Next, by the property of a multiplicative system, we have the following commutative diagram in $\mathcal{A}$:

$$\begin{array}{ccc} X'''^{-i-1} & \xrightarrow{d''_{-i-1}} & X''^{-i} \\ s''_{-i-1} \downarrow & & \downarrow s'_{-i} \\ X'^{-i-1} & \xrightarrow{d_{-i-1}} & X^{-i} \end{array},$$

where $s''_{-i-1} \in \phi(\mathcal{C})$. Since $\partial_{-i} \circ \partial_{-i-1} = 0$, there exists $t'_{-i-1} : X'''^{-i-1} \to X''''^{-i-1}$ such that $s_{-i-1} \circ s''_{-i-1} \circ t'_{-i-1} \in \phi(\mathcal{C})$ and $s'_{-i+1} \circ d'_{-i} \circ d''_{-i-1} \circ t'_{-i-1} = 0$. Then there exists $t''_{-i-1} : X''^{-i-1} \to X'''^{-i-1} \in \phi(\mathcal{C})$ such that $d'_{-i} \circ d'_{-i-1} \circ t'_{-i-1} \circ t''_{-i-1} = 0$. Let $t_{-i-1} = s''_{-i-1} \circ t'_{-i-1} \circ t''_{-i-1}$, $d'_{-i-1} = d''_{-i-1} \circ t'_{-i-1} \circ t''_{-i-1}$, $s'_{-i-1} = s_{-i-1} \circ t_{-i-1}$. Then we have the following commutative diagram in $\mathcal{A}$:

$$\begin{array}{ccc} X''^{-i-1} & \xrightarrow{d'_{-i-1}} & X''^{-i} \\ t_{-i-1} \downarrow & & \downarrow s'_{-i} \\ X'^{-i-1} & \xrightarrow{d_{-i-1}} & X^{-i} \end{array},$$

and we have $s'_{-i+1} = s_{-i+1} \circ t_{-i+1} \in \phi(\mathcal{C})$ and $d'_{-i} \circ d'_{-i+1} = 0$. It is easy to see that $X'^{\bullet} = (X''^{i}, d'_{i})$ is a complex in $D^{b}(\mathcal{A})$ such that $QX'^{\bullet} \cong X^{\bullet}$, and $Q'^{b}X'^{\bullet} \cong X^{\bullet}$.

(ii) Q'^{b} *is full.* (a) We first show that for every morphism $f : X^{\bullet} \to Y^{\bullet}$ of complexes in $K^{b}(\mathcal{A}/\mathcal{C})$, there exist a complex $X'^{\bullet}$ and morphisms $s' : X'^{\bullet} \to Y^{\bullet}$ and $f' : X'^{\bullet} \to Y^{\bullet}$ of complexes in $K^{b}(\mathcal{A})$ such that $f \circ Qs' = Qf'$ and Qs' is an isomorphism in $K^{b}(\mathcal{A}/\mathcal{C})$. By (i), $f = (f_{i}) : X^{\bullet} \to Y^{\bullet}$ is represented by the following diagram in $\mathcal{A}$:

$$\begin{array}{ccccccccccc}
\dots \to 0 \to & X^{-n} & \xrightarrow{\partial_{-n}} & X^{-n+1} & \xrightarrow{\partial_{-n+1}} & \dots & \xrightarrow{\partial_{-1}} & X^{0} & \to 0 \\
& \uparrow s_{-n} & & \uparrow s_{-n+1} & & & & \uparrow s_{0} & \\
& X'^{-n} & & X'^{-n+1} & & \dots & & X'^{0} & \\
& \downarrow f'_{-n} & & \downarrow f'_{-n+1} & & & & \downarrow f'_{0} & \\
\dots \to 0 \to & Y^{-n} & \xrightarrow{\partial_{-n}} & Y^{-n+1} & \xrightarrow{\partial_{-n+1}} & \dots & \xrightarrow{\partial_{-1}} & Y^{0} & \to 0
\end{array},$$

where $s_{i} \in \phi(\mathcal{C})$ for all i. By induction on i, we have the following commutative diagram in $\mathcal{A}$:

$$\begin{array}{ccc}
X^{-i} & \xrightarrow{\partial_{-i}} & X^{-i+1} \\
s'_{-i} \uparrow & & \uparrow s'_{-i+1} \\
X''^{-i} & \xrightarrow{\partial'_{-i}} & X''^{-i+1} \\
f''_{-i} \downarrow & & \downarrow f''_{-i+1} \\
Y^{-i} & \xrightarrow{d_{-i}} & Y^{-i+1}
\end{array},$$

where $s'_{-i} \in \phi(\mathcal{C})$, and we have $f_{-i} \circ Qs''_{-i} = Qf''_{-i}$ and $\partial'_{-i+1} \circ \partial'_{-i} = 0$ for all i. Indeed, it is clear in case of $i \geq 0$ by taking $s'_{-i} = s_{-i}$, $f''_{-i} = f'_{-i}$ and $\partial'_{-i} = \partial_{-i}$. Next, by the property of a multiplicative system, we have the following commutative diagram in $\mathcal{A}$:

$$\begin{array}{ccc} X^{-i-1} & \overset{\partial_{-i-1}}{\rightarrow} & X^{-i} \\ s'''_{-i-1} \uparrow & & \uparrow s'_{-i} \\ X''''^{-i-1} & \overset{\partial'''_{-i-1}}{\rightarrow} & X''^{-i} \end{array} \quad ,$$

where $s'''_{-i-1} \in \phi(\mathcal{C})$. Since f is a morphism of complexes in $C^b(\mathcal{A}/\mathcal{C})$, there exist $t'_{-i-1} : X'''^{-i-1} \rightarrow X''''^{-i-1}$ and $t_{-i-1} : X'''^{-i-1} \rightarrow X'^{-i-1}$ such that $s'''_{-i-1} \circ t'_{-i-1} = s_{-i-1} \circ t_{-i-1} \in \phi(\mathcal{C})$ and such that $d_{-i-1} \circ f'_{-i-1} \circ t_{-i-1} = f''_{-i} \circ \partial'''_{-i-1} \circ t'_{-i-1}$. Let $\partial''_{-i-1} = \partial'''_{-i-1} \circ t'_{-i-1}$, $f'''_{-i-1} = f'_{-i-1} \circ t_{-i-1}$, $s''_{-i-1} = s'''_{-i-1} \circ t'_{-i-1} = s_{-i-1} \circ t_{-i-1}$. Then we have the following commutative diagram in $\mathcal{A}$:

$$\begin{array}{ccc} X^{-i-1} & \overset{\partial_{-i-1}}{\rightarrow} & X^{-i} \\ s''_{-i-1} \uparrow & & \uparrow s'_{-i} \\ X'''^{-i-1} & \overset{\partial''_{-i-1}}{\rightarrow} & X''^{-i} \\ f'''_{-i-1} \downarrow & & \downarrow f''_{-i} \\ Y^{-i-1} & \overset{d_{-i-1}}{\rightarrow} & Y^{-i} \end{array} \quad ,$$

where $s''_{-i-1} \in \phi(\mathcal{C})$ and we have $f_{-i-1} \circ Qs''_{-i-1} = Qf'''_{-i-1}$. Since $s'_{-i+1} \circ \partial'_{-i} \circ \partial''_{-i-1} = \partial_{-i} \circ \partial_{-i-1} \circ s''_{-i-1} = 0$, there exists $t''_{-i-1} : X''^{-i-1} \rightarrow X'''^{-i-1} \in \phi(\mathcal{C})$ such that $\partial'_{-i} \circ \partial''_{-i-1} \circ t''_{-i-1} = 0$. Let $\partial'_{-i-1} = \partial''_{-i-1} \circ t''_{-i-1}$, $f''_{-i-1} = f'''_{-i-1} \circ t''_{-i-1}$, $s'_{-i-1} = s''_{-i-1} \circ t''_{-i-1}$. Then we have the following commutative diagram in $\mathcal{A}$:

$$\begin{array}{ccc} X^{-i-1} & \overset{\partial_{-i-1}}{\rightarrow} & X^{-i} \\ s'_{-i-1} \uparrow & & \uparrow s'_{-i} \\ X''^{-i-1} & \overset{\partial'_{-i-1}}{\rightarrow} & X''^{-i} \\ f''_{-i-1} \downarrow & & \downarrow f''_{-i} \\ Y^{-i-1} & \overset{d_{-i-1}}{\rightarrow} & Y^{-i} \end{array} \quad ,$$

where $s'_{-i-1} \in \phi(\mathcal{C})$, and we have $f_{-i-1} \circ Qs'_{-i-1} = Qf''_{-i-1}$ and $\partial'_{-i} \circ \partial'_{-i-1} = 0$. It is easy to see that $X''^{\bullet} = (X''^{i}, \partial'_{i})$ is a complex in $K^b(\mathcal{A})$ and $s' = (s'_i)$, $f'' =$

(f''_i) are morphisms in $K^b(\mathcal{A})$ such that $f\circ Qs' = Qf''$. (b) Any morphism f: $X^\bullet \to Y^\bullet$ in $D^b(\mathcal{A}/\mathcal{C})$ is represented by the following diagram in $K^b(\mathcal{A}/\mathcal{C})$:

$$\begin{array}{ccc} & X_1^\bullet & \\ t \swarrow & & \searrow f' \\ X^\bullet & & Y^\bullet \end{array},$$

where t is a quasi-isomorphism. According to (a), it is easy to see that there exist morphisms $t' : X_2^\bullet \to X^\bullet$ and $s : X_2^\bullet \to X_1^\bullet$ in $K^b(\mathcal{A})$ such that $t \circ Qs = Qt'$, Qs is an isomorphism in $K^b(\mathcal{A}/\mathcal{C})$ and Qt' is a quasi-isomorphism, and that there exist morphisms $f'' : X_3^\bullet \to Y^\bullet$ and $s' : X_3^\bullet \to X_1^\bullet$ in $K^b(\mathcal{A})$ such that $f' \circ Qs' = Qf''$ and Qs' is an isomorphism in $K^b(\mathcal{A}/\mathcal{C})$. We have the following morphism of distinguished triangles in $K^b(\mathcal{A})$:

$$\begin{array}{ccccc} X_4^\bullet & \xrightarrow{r} & X_3^\bullet & \to & Z^\bullet \to \\ t'' \downarrow & & \downarrow s' & & \| \\ X_2^\bullet & \xrightarrow{s} & X_1^\bullet & \to & Z^\bullet \to \end{array}.$$

By $QZ^\bullet = 0$ in $K^b(\mathcal{A}/\mathcal{C})$, r and t'' are isomorphisms in $K^b(\mathcal{A}/\mathcal{C})$. Then $Q(t' \circ t'')$ is a quasi-isomorphism in $K^b(\mathcal{A}/\mathcal{C})$. Denoting by the same symbols the induced morphisms in $D^b(\mathcal{A})$, $f \circ r$ and $t' \circ t''$ are morphisms in $D^b(\mathcal{A})$ such that $t' \circ t'' \in \phi(D^*_{\mathcal{C}}(\mathcal{A}))$ and $f \circ Q'^b(t' \circ t'') = f \circ Rs \circ Q'^b(s \circ t'') = Rf' \circ Q'^b(s' \circ r) = Q'^b(f'' \circ r)$, where R: $K^b(\mathcal{A}/\mathcal{C}) \to D^b(\mathcal{A}/\mathcal{C})$ is a natural quotient. Hence Q'^b is full.

(II) *The case of* $* = -$. Let $X^\bullet : \ldots \to X^{-n} \to X^{-n+1} \to \ldots \to X^0 \to 0 \to \ldots$ be a complex in $D^-(\mathcal{A}/\mathcal{C})$, and $X_i^\bullet : \ldots \to 0 \to X^{-i} \to X^{-i+1} \to \ldots \to X^0 \to 0 \to \ldots$ a truncated complex in $D^b(\mathcal{A}/\mathcal{C})$. Then, by (I), there exists a complex $X_i'^\bullet$ in $D^b(\mathcal{A})$ such that $s_i : Q'^b X_i'^\bullet \cong X_i^\bullet$.

Moreover, for a natural inclusion $X_i^\bullet \to X_{i+1}{}^\bullet$, we have a commutative diagram:

$$\begin{array}{ccc} Q'^b X_i'^\bullet & \to & Q'^b X_{i+1}'^\bullet \\ s_i \wr\!\wr & & \wr\!\wr\, s_{i+1} \\ X_i^\bullet & \to & X_{i+1}{}^\bullet . \end{array}$$

Hence $Q'^- X'^\bullet \cong \varinjlim_{i\to\infty} Q'^b X_i'^\bullet \cong X^\bullet$, where $X'^\bullet = \varinjlim_{i\to\infty} X_i'^\bullet$. For any morphism $f: X^\bullet \to Y^\bullet$ in $K^-(\mathcal{A}/\mathcal{C})$, we have a commutative diagram:

$$\begin{array}{ccc} X_i^\bullet & \xrightarrow{f_i} & Y_i^\bullet \\ \downarrow & & \downarrow \\ X^\bullet & \xrightarrow{f} & Y^\bullet \end{array} .$$

According to (I), there exist a complex $Z_i^\bullet$ and morphisms $s_i: Z_i^\bullet \to X_i^\bullet$ and $f_i' : X'^\bullet \to Y'^\bullet$ of complexes in $K^b(\mathcal{A})$ such that $f_i \circ Qs_i' = Qf_i'$. Moreover, for all i, we have the following commutative diagram:

$$\begin{array}{ccccc} X_i^\bullet & \xleftarrow{s'_i} & Z_i^\bullet & \xrightarrow{f'_i} & Y_i^\bullet \\ \downarrow & & \downarrow & & \downarrow \\ X_{i+1}{}^\bullet & \xleftarrow{s'_{i+1}} & Z_{i+1}{}^\bullet & \xrightarrow{f'_{i+1}} & Y_{i+1}{}^\bullet \end{array} .$$

Then we have $f \circ Qs' = Qf'$, where $Qs' = \varinjlim_{i\to\infty} Qs_i'$ is an isomorphism in $K^b(\mathcal{A}/\mathcal{C})$ and $Qf' = \varinjlim_{i\to\infty} Qf_i'$. In the same way as (I), Q'^- is also full.

(III) *The case of* $* = +$. By (I) with the arrows reversed and the dual of (II), it is trivial.

Remark. By the proof of Theorem 3.2, $K^*(\mathcal{A}) \overset{Q^*}{\to} K^*(\mathcal{A}/\mathcal{C})$ is also a quotient functor, where $* = +, -$ or b.

COROLLARY 3.3. *Let* $0 \to \mathcal{C} \to \mathcal{A} \to \mathcal{A}/\mathcal{C} \to 0$ *be a localization* $\{\mathcal{A}/\mathcal{C}; Q, S\}$ *of* $\mathcal{A}$. *Assume that* $\mathcal{A}/\mathcal{C}$ *has enough injectives. Then* $0 \to D^+_{\mathcal{C}}(\mathcal{A}) \to D^+(\mathcal{A}) \to D^+(\mathcal{A}/\mathcal{C}) \to 0$ *is localization exact, that is,* $\{D^+(\mathcal{A}/\mathcal{C}); Q^+, R^+S\}$ *is a localization of* $D^+(\mathcal{A})$.

Proof. For any $Y^\bullet \in D^+(\mathcal{A}/\mathcal{C})$, there exists a complex $I^\bullet = (I_i, d_i) \in K^+(\mathcal{A}/\mathcal{C})$ where all I_i are injective such that $Y^\bullet \cong I^\bullet$ in $D^+(\mathcal{A}/\mathcal{C})$. Then, given any $X^\bullet \in D^+(\mathcal{A})$ and $Y^\bullet \in D^+(\mathcal{A}/\mathcal{C})$, we have $\mathrm{Hom}_{D^+(\mathcal{A}/\mathcal{C})}(Q^+X^\bullet, Y^\bullet) \cong \mathrm{Hom}_{K^+(\mathcal{A}/\mathcal{C})}(Q^+X^\bullet, I^\bullet) \cong \mathrm{Hom}_{K^+(\mathcal{A})}(X^\bullet, SI^\bullet)$. Since Q is exact and S is the right adjoint of Q, $SI^\bullet = (SI^i, Sd_i)$ is a complex in $K^+(\mathcal{A})$, where all SI^i are injective. Then we have $\mathrm{Hom}_{K^+(\mathcal{A})}(X^\bullet, SI^\bullet) \cong \mathrm{Hom}_{D^+(\mathcal{A})}(X^\bullet, R^+SY^\bullet)$. Therefore R^+S is the right adjoint of Q^+. According to Theorem 3.2 and Theorem 2.6, we are done.

Chapter II. Partial Tilting Modules and Ring Epimorphisms

Let A be a ring. We denote by Free$A$ (resp., free$A$) the category of free right A-modules (resp., finitely generated free A-modules). For a right A-module T, we denote by Sum$T$ the category of direct sums of copies of T.

§4. Localization of Derived Categories of Modules.

Equivalences of derived categories of modules were considered in [12], [6], [33]. For a ring A, we call a right A-module T a partial tilting right A-module provided that it satisfies the conditions: a) $0 \to P_n \to \dots \to P_1 \to P_0 \to T \to 0$ is exact, where all P_i are finitely generated projective; b) $\mathrm{Ext}^i_A(T, T) = 0$ for all $i > 0$. Cline, Parshall and Scott showed that if T is a partial tilting right A-module, then $\{D^-(\mathrm{Mod}\, B); -\otimes^{L^-}_B T, R^-\mathrm{Hom}_A(T, -)\}$ is a colocalization of $D^-(\mathrm{Mod}\, A)$, where $B = \mathrm{End}_A(T)$. Moreover, if pdim ${}_B T < \infty$, then $\{D^b(\mathrm{Mod}\, B); -\otimes^{L^b}_B T, R^b\mathrm{Hom}_A(T, -)\}$ is a colocalization of $D^b(\mathrm{Mod}\, A)$ (see [6, (4.2)]). In this section, we consider quotient and localization of derived categories of modules categories for rings.

For a complex $X^\bullet = (X_i, d_i)$, we define the following truncations [14, I, §7]:

$$\sigma_{>n}(X^\bullet): \dots \to 0 \to \mathrm{Im}\, d_n \to X_{n+1} \to X_{n+2} \to \dots,$$
$$\sigma_{\le n}(X^\bullet): \dots \to X_{n-2} \to X_{n-1} \to \mathrm{Ker}\, d_n \to 0 \to \dots.$$

For $m \le n$, we denote by $D^{[m,n]}(\mathrm{Mod}\, A)$ a full subcategory of $D^b(\mathrm{Mod}\, A)$ generated by complexes of which homology $\mathrm{H}^i = 0$ ($i < m$ or $n < i$).

LEMMA 4.1. *Let* $\mathrm{F} : \mathcal{C} \to \mathcal{D}$ *be a* ∂*- functor between triangulated categories. Suppose there exists a family* $\mathcal{T}$ *of objects in* $\mathcal{C}$ *satisfying the following conditions:*

(a) *For every* $X \in \mathcal{C}$, *there exists an object* $T_X \in \mathcal{T}$ *and a morphism* s_X : $T_X \to X$ *such that* Z *belongs to* Ker F, *where* $T_X \to X \to Z \to$ *is a distinguished triangle.*

(b) *For* X *and* $Y \in \mathcal{C}$, *there exists a morphism* f' : $T_X \to Y$ *in* $\mathcal{C}$ *such that* $f \circ \mathrm{F}s_Y = \mathrm{F}f'$ *for any* $f \in \mathrm{Hom}_{\mathcal{D}}(\mathrm{F}X, \mathrm{F}Y)$.

(c) *For every* $Y \in \mathcal{D}$, *there exists an object* $X \in \mathcal{T}$ *such that* $Y \cong \mathrm{F}X$.

Then $0 \to \mathrm{Ker\,F} \to \mathcal{C} \to \mathcal{D} \to 0$ *is exact.*

Proof. It is clear by Lemma 3.1.

PROPOSITION 4.2. *Let* T *be a partial tilting right A-module,* $B = \mathrm{End}_A(T)$ *and* $\mathrm{F} = \mathrm{Hom}_A(T, -) : \mathrm{Mod}\,A \to \mathrm{Mod}\,B$. *If projective dimension of* T_A *is at most one, then* $0 \to \mathrm{Ker}\,R^b\mathrm{F} \to D^b(\mathrm{Mod}\,A) \overset{R^b\mathrm{F}}{\to} D^b(\mathrm{Mod}\,B) \to 0$ *is exact.*

Proof. Let F-rac A be the full subcategory of Mod$A$ generated by the modules M such that $\mathrm{Ext}^i_A(T, M) = 0$ for all $i > 0$, and let $\mathcal{T}$ be a family of complexes $X^\bullet : \ldots \to 0 \to X^m \to \ldots \to X^{n-1} \to X^n \to 0 \to \ldots$ (for all $m \leq n$) $\in K^b(\mathrm{Mod}\,A)$, where $X^m \in$ F-rac A and X^i is direct sums of T ($m < i \leq n$). It suffices to show that $\mathcal{T}$ satisfies the conditions of Lemma 4.1. Since $\mathrm{Ext}^i_A(T, T) = 0$ $(i > 0)$ and pdim $T_A \leq 1$, if X is generated by T, then $\mathrm{Ext}^i_A(T, X) = 0$ $(i > 0)$. The condition (b) implies the existence of $R^b\mathrm{F}$. Since $D^b(\mathrm{Mod}\,B) \cong K^{-,b}(\mathrm{Free\ B})$, for $Y^\bullet \in D^{[m,n]}(\mathrm{Mod}\,B)$, there exists $T^\bullet \in K^-(\mathrm{Sum}\,T)$ such that $\mathrm{F}T^\bullet \cong Y^\bullet$ in $D^b(\mathrm{Mod}\,B)$. Furthermore, $\mathrm{F}(\sigma_{>t}T^\bullet) \cong \sigma_{>t}(\mathrm{F}T^\bullet) \cong \mathrm{F}T^\bullet$ $(t < m-1)$. Since Im d_t is generated by T, $\sigma_{>t}(T^\bullet)$ is in $\mathcal{T}$. Then (c) of Lemma 4.1 is satisfied. Given $X^\bullet \in D^{[m,n]}(\mathrm{Mod}\,A)$, there exists $I^\bullet \in K^{[m,n+1]}(\text{F-rac } A)$ such that $X^\bullet \cong I^\bullet$ in $D^b(\text{Mod } A)$. For $\text{F}I^\bullet$, there exist $T^\bullet \in K^-(\text{Sum } T)$ and $f: T^\bullet \to I^\bullet$ in $K^-(\text{Mod}A)$ such that $\text{F}f$ is an isomorphism in $D^-(\text{Mod } B)$. Similarly, $f = T^\bullet \to \sigma_{>t}(T^\bullet) \xrightarrow{g} I^\bullet$ and $\text{F}g$ is an isomorphism in $D^b(\text{Mod } B)$ ($t < m-1$). Then (a) of Lemma 4.1 is satisfied. Moreover, for every $X^\bullet$ and $Y^\bullet \in D^{[m,n]}(\text{Mod } A)$, there exist $T^\bullet$ and $T'^\bullet \in K^-(\text{Sum } T)$ such that $\text{F}T^\bullet \cong X^\bullet$ and $\text{F}T'^\bullet \cong Y^\bullet$, and then $\text{Hom}_{D^b(\text{Mod}B)}(R^b\text{F}(X^\bullet), R^b\text{F}(Y^\bullet)) \cong \text{Hom}_{D^b(\text{Mod}B)}(\text{F}T^\bullet, \text{F}T'^\bullet)$. Since $\text{F}T^\bullet$ and $\text{F}T'^\bullet \in D^{[m,n+1]}(\text{Mod } B)$, for $t < m-1$, we have $\text{Hom}_{D^-(\text{Mod}B)}(\text{F}T^\bullet, \sigma_t(\text{F}T'^\bullet)[i]) = 0$ for all i. Then we get

$$\begin{aligned}\text{Hom}_{D^-(\text{Mod}B)}(\text{F}T^\bullet, \text{F}T'^\bullet) &\cong \text{Hom}_{D^-(\text{Mod}B)}(\text{F}T^\bullet, \sigma_{>t}(\text{F}T'^\bullet)) \\ &\cong \text{Hom}_{K^-(\text{Mod}B)}(\text{F}T^\bullet, \sigma_{>t}(\text{F}T'^\bullet)).\end{aligned}$$

Hence we have

$$\begin{aligned}\text{Hom}_{K^-(\text{Mod}B)}(\text{F}T^\bullet, \sigma_{>t}(\text{F}T'^\bullet)) &\cong \text{Hom}_{K^b(\text{Mod}B)}(\sigma_{>t-1}(\text{F}T^\bullet), \sigma_{>t}(\text{F}T'^\bullet)) \\ &\cong \text{Hom}_{K^b(\text{Mod}A)}(\sigma_{>t-1}T^\bullet, \sigma_{>t}T'^\bullet).\end{aligned}$$

The condition (b) of Lemma 4.1 is satisfied.

PROPOSITION 4.3. *Let A and B be semiprimary rings and* $\text{F}: \text{Mod } A \to \text{Mod } B$ *a left exact additive functor. Assume that* $R^{+,b}\text{F}$ *has image in* $D^b(\text{Mod } B)$ *and that* $R^{+,b}\text{F}: D^b(\text{Mod } A) \to D^b(\text{Mod } B)$ *is a colocalization. Then there exists a right B-A-bimodule T such that*: a) $\text{F} \cong \text{Hom}_A(T,-)$; b) $B \cong \text{End}_A(T)$; c) $\text{Ext}^i_A(T, T) = 0$ ($i \geq 1$); d) $\text{pdim } T_A$, $\text{pdim }_B T < \infty$. *Furthermore,* $R^{+,b}\text{F} \cong R^b\text{F}$ and $L^b\text{G}$ *is the left adjoint of* $R^b\text{F}$, where $\text{G} = -\otimes_B T$.

Proof. There exists a left adjoint G of F such that $L^{-,b}\text{G}: D^b(\text{Mod } B)$

$\to D^b(\mathrm{Mod}\,A)$ is the left adjoint of $R^{+,b}\mathrm{F}$, by [6, (3,1) Lemma]. Let $T = \mathrm{G}B$, and then T is a B-A-bimodule such that $\mathrm{F} \cong \mathrm{Hom}_A(T, -)$ and $\mathrm{G} \cong -\otimes_B T$. Let J_A and J_B be Jacobson radicals of A and B, respectively. Since $R^{+,b}\mathrm{F}$ has image in $D^b(\mathrm{Mod}\,B)$, $R^{+,b}\mathrm{F}(A/J_A)$ is in $D^b(\mathrm{Mod}\,A)$, and then there exists an integer n such that $R^i\mathrm{F}(A/J_A) \cong \mathrm{Ext}^i_A(T, A/J_A) = 0$ $(i > n)$. By [2, Proposition 7], pdim $T_A < \infty$. Similarly, pdim $_BT < \infty$. Then $R^{+,b}\mathrm{F} \cong R^b\mathrm{F}$ and $L^{-,b}\mathrm{G} \cong L^b\mathrm{G}$. Next, since $R^b\mathrm{F}$ is a colocalization and $T \cong L^b\mathrm{G}(B)$, $B \cong R^b\mathrm{F}\circ L^b\mathrm{G}(B) \cong R^b\mathrm{F}(T)$ and T is Ker $R^b\mathrm{F}$-coclosed, by the dual of Lemma 2.2. Hence $R^i\mathrm{F}(\mathrm{T}) \cong \mathrm{Ext}^i_A(T, T) = 0$ $(i \neq 0)$, and $B \cong \mathrm{End}_B(B) \cong \mathrm{End}_A(T)$ as rings, by the dual of Lemma 2.1.

Corollary 4.4. *Under the condition of Proposition* 4.5, *we have* gl dim B $\leqq$ gl dim A + pdim $_BT$.

Proof. Since $L^b\mathrm{G} : D^b(\mathrm{Mod}\,B) \to D^b(\mathrm{Mod}\,A)$ is fully faithful, for all B-modules M, N, we have

$$\begin{aligned}\mathrm{Ext}^i_B(M, N) &\cong \mathrm{Hom}_{D^b(\mathrm{Mod}\,B)}(M, N[i]) \\ &\cong \mathrm{Hom}_{D^b(\mathrm{Mod}\,A)}(L^b\mathrm{G}(M), L^b\mathrm{G}(N)[i]).\end{aligned}$$

Let pdim $_BT = n$, then $L^b\mathrm{G}(\mathrm{M})$ and $L^b\mathrm{G}(\mathrm{N})$ are in $D^{[-n,0]}(\mathrm{Mod}\,B)$. Hence we have $\mathrm{Hom}_{D^b(\mathrm{Mod}A)}(L^b\mathrm{G}(M), L^b\mathrm{G}(N)[i]) = 0$ for $i >$ gl dim $A + n$.

Let A be a finite dimensional algebra over a fixed field k. Then $\mathrm{D} = \mathrm{Hom}_k(-,k)$ induces a duality between mod A and A-mod, where A-mod is the category of finitely generated left A-modules. Therefore, D induces the duality, which we use the same symbol D, between $D^*(\mathrm{mod}\,A)$ and $D^\#(A\text{-mod})$, where $(*,\#) = (+,-), (-,+)$ or (b,b), by $(\mathrm{D}X^\bullet)^i = \mathrm{D}X^{-i}$, where $X^\bullet = (X^i, d_i)$. For a finite dimensional algebra A, we know that the

Grothendieck group of mod A is isomorphic to a free abelian group which has the complete set of non-isomorphic indecomposable projective A-modules as a basis. We denote by Grot($\mathcal{A}$) the Grothendieck group of $\mathcal{A}$, where $\mathcal{A}$ is an abelian category or a triangulated category. Here, we use Grot(mod A) $\cong$ Grot(D^b(mod A)) and Proposition of Grothendieck (see [11] for details).

Proposition 4.5. *Let A be a finite dimensional k-algebra and T a partial tilting right A-module. If projective dimension of T_A is at most one, then* $0 \to \mathrm{Ker}\, R^b\mathrm{F} \to D^b(\mathrm{mod}\, A) \overset{R^b\mathrm{F}}{\to} D^b(\mathrm{mod}\, B) \to 0$ *is exact.*

Proof. It is trivial by Proposition 4.3.

Remark. According to Bongartz's lemma [5, 2.1 Lemma] and an equivalence of derived categories ([6, (2.1) Theorem] or [33, Theorem 3.1.2]), we get another proof of Proposition 5.1 by Theorem 3.2.

Proposition 4.6. *Let A and B be finite dimensional algebras,* F : mod A $\to$ mod B *a left exact additive functor. Then $R^{+,b}\mathrm{F}$ has image in* $D^b(\mathrm{mod} B)$ *and* $R^{+,b}\mathrm{F} : D^b(\mathrm{mod}\, A) \to D^b(\mathrm{mod}\, B)$ *is a colocalization if and only if there exists a finitely generated B-A-bimodule T such that:*

(a) T_A *is a partial tilting right A-module,*

(b) $\mathrm{F} \cong \mathrm{Hom}_A(T, -)$,

(c) $\mathrm{B} \cong \mathrm{End}_A(T)$,

(d) $\mathrm{pdim}\, _BT < \infty$.

Proof. By Proposition 4.3 and [6, (4.2)], it is clear.

Corollary 4.7. *Under the condition of Proposition* 4.6, *we have* gl dim

$B \leq$ gl dim A + pdim $_B T$.

Proof. It is trivial by Corollary 4.4.

For a finitely generated A-module M, Let $n(M)$ be the number of isomorphism classes of indecomposable direct summands of M.

COROLLARY 4.8. *Let T be a finitely generated right A-module such that:* a) $\mathrm{Ext}^i_A(T,T) = 0$ ($i \geq 1$); b) pdim T_A, pdim $_B T < \infty$, *where* $B = \mathrm{End}_A(T)$. *Then we have* $n(T) \leq n(A)$.

Proof. According to Proposition 4.6, $0 \to \mathrm{Ker}\, R^b\mathrm{F} \to D^b(\mathrm{mod}\, A) \to D^b(\mathrm{mod}\, B) \to 0$ is a colocalization. Then, by [11, §3], we have the following split exact sequence:

$$0 \to \mathrm{Grot}(\mathrm{Ker}\, R^b\mathrm{F}) \to \mathrm{Grot}(D^b(\mathrm{mod}\, A)) \to \mathrm{Grot}(D^b(\mathrm{mod}\, B)) \to 0.$$

Since $\mathrm{Grot}(D^b(\mathrm{mod}\, A)) \cong \mathrm{Grot}(\mathrm{mod}\, A) \cong Z^{n(A)}$ and $\mathrm{Grot}(D^b(\mathrm{mod}\, B)) \cong \mathrm{Grot}(\mathrm{mod}\, B) \cong Z^{n(T)}$, we have $n(T) \leq n(A)$.

Remark. Under the conditions of Proposition 4.6, in case that $\mathrm{Ker}\, R^b\mathrm{F}$ being not zero implies $\mathrm{Grot}(\mathrm{Ker}\, R^b\mathrm{F})$ being not zero (for example, A is hereditary), $D^b(\mathrm{mod}\, A)$ is equivalent to $D^b(\mathrm{mod}\, B)$ if and only if $n(T) = n(A)$.

§5. Ring Epimorphisms and Derived Categories.

In this section, we consider conditions that ring homomorphisms induce localization and colocalization of derived categories. Moreover, we consider the case that a partial tilting module induces a ring morphism which induces a colocalization of derived categories.

For a ring A, we denote by $\mathcal{P}_A$ the category of finitely generated projective right A-modules.

PROPOSITION 5.1. *Let* $\pi : A \to C$ *be a ring homomorphism between finite dimensional k-algebras,* $\mathrm{E} = - \otimes_C C_A : \mathrm{mod}\ C \to \mathrm{mod}\ A$, *and* $\mathrm{E}' = {}_A C \otimes_C - : C\text{-mod} \to A\text{-mod}$. *Then the following are equivalent.*

(a) $\mathrm{E} : D^-(\mathrm{mod}\ C) \to D^-(\mathrm{mod}\ A)$ *has a left adjoint* $\hat{\mathrm{G}}$ *such that* $\{D^-(\mathrm{mod}\ C); \hat{\mathrm{G}}, \mathrm{E}\}$ *is a localization of* $D^-(\mathrm{mod}\ A)$.

(b) $\mathrm{E} : D^+(\mathrm{mod}\ C) \to D^+(\mathrm{mod}\ A)$ *has a right adjoint* $\hat{\mathrm{F}}$ *such that* $\{D^+(\mathrm{mod}\ C); \mathrm{E}, \hat{\mathrm{F}}\}$ *is a colocalization of* $D^+(\mathrm{mod}\ A)$.

(c) $\mathrm{E}' : D^-(C\text{-mod}) \to D^-(A\text{-mod})$ *has a left adjoint* $\hat{\mathrm{G}}'$ *such that* $\{D^-(C\text{-mod}); \hat{\mathrm{G}}', \mathrm{E}'\}$ *is a localization of* $D^-(A\text{-mod})$.

(d) $\mathrm{E}' : D^+(C\text{-mod}) \to D^+(A\text{-mod})$ *has a right adjoint* $\hat{\mathrm{F}}'$ *such that* $\{D^+(C\text{-mod}); \mathrm{E}', \hat{\mathrm{F}}'\}$ *is a colocalization of* $D^+(A\text{-mod})$.

(e) π *is a ring epimorphism, and* $\mathrm{Tor}_i^A(C, C) = 0$ *for all* $i > 0$.

Moreover, in this case, $\mathrm{Ext}_A^i(C_A, C_A) = \mathrm{Ext}_A^i({}_A C, {}_A C) = 0$ *for all* $i > 0$.

Proof. It is well known that π is a ring epimorphism if and only if the natural morphism $C \otimes_A C \to C$ is an isomorphism as an C-C-bimodule morphism. If π is a ring epimorphism, then the natural ring morphism $C \to \mathrm{End}(C_A)$ is an isomorphism (see [37]).

(e) $\Rightarrow$ (a): Let $\mathrm{G} = - \otimes_A C$, and then G is the left adjoint of E. For $X^\bullet \in D^-(\mathrm{mod}\ A)$, there exists a complex $P^\bullet \in K^-(\mathrm{free}\ A)$ such that $X^\bullet \cong P^\bullet$ in

$D^-(\text{mod } A)$. Given $Y^\bullet \in D^-(\text{mod } C)$, we have

$$\begin{aligned}\mathrm{Hom}_{D^-(\mathrm{mod} A)}(X^\bullet, \mathrm{E}Y^\bullet) &\cong \mathrm{Hom}_{K^-(\mathrm{mod} A)}(P^\bullet, \mathrm{E}Y^\bullet) \\ &\cong \mathrm{Hom}_{K^-(\mathrm{mod} C)}(\mathrm{G}P^\bullet, Y^\bullet).\end{aligned}$$

Since E is exact and G is the left adjoint of E, $\mathrm{G}P^\bullet$ is in $K^-(\mathcal{P}_C)$. Then we have

$$\mathrm{Hom}_{D^-(\mathrm{mod} A)}(X^\bullet, \mathrm{E}Y^\bullet) \cong \mathrm{Hom}_{D^-(\mathrm{mod} C)}(L^-\mathrm{G}(X^\bullet), Y^\bullet).$$

And for $Y^\bullet \in D^-(\text{mod } C)$, there exists a complex $Q^\bullet \in K^-(\text{free C})$ such that $Y^\bullet \cong Q^\bullet$ in $D^-(\text{mod } C)$. Since $C \otimes_A C \cong C$ and $\mathrm{Tor}_i^A(C, C) = 0$ for all $i > 0$, we have $L^-\mathrm{G} \circ \mathrm{E}(Y^\bullet) \cong \mathrm{G} \circ \mathrm{E}(Q^\bullet) \cong Q^\bullet$ by [21, Chapter XII, Theorem 12.1, 12,2]. Hence we have $L^-\mathrm{G} \circ \mathrm{E} \cong \mathrm{id}_{D^-(\mathrm{mod} C)}$.

(a) $\Rightarrow$ (e): By the above, we have $\hat{\mathrm{G}} \cong L^-\mathrm{G}$, where $\mathrm{G} = - \otimes_A C_C$. Then $L^-\mathrm{G} \circ \mathrm{E}(C) \cong C$ in $D^-(\text{mod } C)$, and hence the natural morphism $C \otimes_A C \to C$ is an isomorphism and $\mathrm{Tor}_i^A(C, C) = 0$ for all $i > 0$.

(c) $\Leftrightarrow$ (e): It is similar to (a) $\Leftrightarrow$ (e).

(a) $\Leftrightarrow$ (d) and (b) $\Leftrightarrow$ (c): Since $\mathrm{DED} \cong \mathrm{E}'$ and $\mathrm{DE}'\mathrm{D} \cong \mathrm{E}$, they are trivial by the duality.

The conditions (b) and (d) imply $C \cong R^+\mathrm{Hom}_A(C_A, -) \circ \mathrm{E}(C)$ in $D^+(\text{mod } C)$ and $C \cong R^+\mathrm{Hom}_A({}_AC, -) \circ \mathrm{E}'(C)$ in $D^+(C\text{-mod})$, respectively. Hence we have $\mathrm{Ext}_A^i(C_A, C_A) = \mathrm{Ext}_A^i({}_AC, {}_AC) = 0$ for all $i > 0$.

Remark. Replacing mod$A$ and mod$C$ by Mod$A$ and Mod$C$ in Proposition 5.1, respectively, the proof of Proposition 5.1 implies that the assertions (a), (c) and (e) are equivalent for arbitrary rings A and C.

Theorem 5.2. Let $\pi : A \to C$ *be a ring homomorphism between finite* *dimensional k-algebras,* $\mathrm{E} = - \otimes_A C_C$: mod C $\rightarrow$ mod A *, and* $\mathrm{E}' = {}_A C \otimes_C -$: C-mod $\rightarrow$ A-mod. *Then the following are equivalent.*

(a) $\mathrm{E} : D^b(\mathrm{mod}\ C) \rightarrow D^b(\mathrm{mod}\ A)$ *has a right adjoint* $\hat{\mathrm{F}}$ *such that* $\{D^b(\mathrm{mod}\ C); \mathrm{E}, \hat{\mathrm{F}}\}$ *is a colocalization of* $D^b(\mathrm{mod}\ A)$.

(b) $\mathrm{E}' : D^b(C\text{-mod}) \rightarrow D^b(A\text{-mod})$ *has a left adjoint* $\hat{\mathrm{G}}'$ *such that* $\{D^b(C\text{-mod}); \hat{\mathrm{G}}', \mathrm{E}'\}$ *is a localization of* $D^b(A\text{-mod})$.

(c) i) The natural morphism $C \rightarrow \mathrm{End}(C_A)$ is an isomorphism as a ring, ii) pdim $C_A < \infty$, and iii) $\mathrm{Ext}^i_A(C_A, C_A) = 0$ for all $i > 0$.

(d) i) p is a ring epimorphism, ii) pdim $C_A < \infty$, and iii) $\mathrm{Tor}^A_i(C,C) = 0$ for all $i > 0$.

Proof. (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (d) and (a) $\Rightarrow$ (c): They are trivial by Proposition 5.2.

(c) $\Rightarrow$ (a): It is trivial by Proposition 4.6.

For a finitely generated right A-module T_A, let add T_A be the full subcategory of mod A generated by direct summands of finite direct sums of T_A.

Proposition 5.3. *Let A be a finite dimensional k-algebra, T a finitely generated right A-module,* $B = \mathrm{End}(T_A)$ *and* $C = \mathrm{End}({}_B T)^{\mathrm{op}}$. *Assume that T satisfies the following conditions:*

(a) $\mathrm{Ext}^i_A(T,T) = 0$ $(i \geq 1)$,

(b) pdim $T_A < \infty$,

(c) *there exists an exact sequence* $0 \rightarrow C \rightarrow T_0 \rightarrow T_1 \rightarrow \ldots \rightarrow T_n \rightarrow 0$ in mod A, *where all* T_i *are in* add T_A.

Then $R^{+,b}\mathrm{F}$ *has image in* $D^b(\mathrm{mod}\ B)$ *and* $R^{+,b}\mathrm{F} : D^b(\mathrm{mod}\ A) \rightarrow D^b(\mathrm{mod}\ B)$ *is a colocalization. In this case,* ${}_B T$ *is a left tilting B-module with finite projective dimension, in the sense of* [27], *and C satisfies the conditions* *of Theorem* 5.2.

Proof. First, it suffices to show that pdim ${}_BT < \infty$. By the conditions (a) and (c), we have the following exact sequence in B-mod: $0 \to \mathrm{Hom}_A(T_n, T) \to \dots \to \mathrm{Hom}_A(T_0, T) \to \mathrm{Hom}_A(C,T) \to 0$. It easy to see that ${}_BT$ is a direct summand of $\mathrm{Hom}_A(C,T)$, and that all $\mathrm{Hom}_A(T_i,T)$ are left projective B-modules. Then pdim ${}_BT \leq n$. Next, applying $\mathrm{Hom}_B(-,T)$ to the above sequence, we get the following commutative diagram:

$$\begin{array}{ccccccc} \mathrm{Hom}_B(\mathrm{Hom}_A(C,T),T) & \to & \mathrm{Hom}_B(\mathrm{Hom}_A(T_0,T),T) & \to \dots \to & \mathrm{Hom}_B(\mathrm{Hom}_A(T_n,T),T) & & \\ \wr\| & & \wr\| & & \wr\| & & \\ 0 \to C & \to & T_0 & \to \dots \to & T_n & \to & 0. \end{array}$$

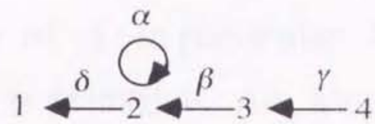
Then $\mathrm{Ext}^i_B(\mathrm{Hom}_A(C,T),T) = 0$, and $\mathrm{Ext}^i_B(T,T) = 0$ for all $i > 0$. By the condition (b), we have a projective resolution of T in mod A : $0 \to P_m \to \dots \to P_1 \to P_0 \to T \to 0$. Applying $\mathrm{Hom}_A(-,T)$ to it, we get an exact sequence in B-mod:

$$0 \to \mathrm{Hom}_A(T,T) \to \mathrm{Hom}_A(P_0,T) \to \mathrm{Hom}_A(P_1,T) \to \dots \to \mathrm{Hom}_A(P_m,T) \to 0,$$

where $B = \mathrm{Hom}_A(T,T)$ and all $\mathrm{Hom}_A(P_i,T)$ are in add T_A. Hence ${}_BT$ is a left tilting B-module with finite projective dimension. Then it is easy to see that $\{D^b(\mathrm{mod}\ C); - \otimes_A C_C, R^b\mathrm{Hom}_A(C_A,-)\}$ is a colocalization of $D^b(\mathrm{mod}A)$.

Corollary 5.4. *Let T be a finitely generated right A-module such that*: a) $\mathrm{Ext}^i_A(T,T) = 0$ (i $\geq$ 1); b) pdim $T_A < \infty$; c) *there exists an exact sequence* $0 \to C \to T_0 \to T_1 \to \dots \to T_n \to 0$ *in* mod A , *with* $T_i \in$ add T *for all i , where* C = Biend(T_A). *Then we have* $n(C) = n({}_BT) = n(T_A) \leq n(A)$.

Example. Let A be a finite dimensional algebra over a field k with the following quiver with relations:



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with $\delta\alpha = \alpha^2 = \delta\beta = \beta\gamma = 0$. Then gl dim $A = \infty$. Let $T = I(3)\oplus(I(3)/S(3))$, where $S(3)$ is a simple right A-module corresponding to a vertex 3 , and $I(3)$ is an injective hull of $S(3)$. Then pdim $T_A = 2$ and $\mathrm{Ext}^i_A(T,T) = 0$ for all $i > 0$. Moreover, T satisfies the conditions of Proposition 5.3. Next, $B = \mathrm{End}_A(T)$ have a quiver with a relation:

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with $\zeta^2 = 0$. Then we have gl dim $B = \infty$ and pdim ${}_BT = 1$. Hence $R^b\mathrm{Hom}_A(T, -) : D^b(\mathrm{mod}\ A) \to D^b(\mathrm{mod}\ B)$ is a colocalization functor which has $- \otimes^{L^b}_B T$ as a cosection functor.

§6. Idempotent Ideals and Derived Categories.

Recall that an ideal I of a ring A is called idempotent if $I = AeA$ for some idempotent e of A; in particular, I is a minimal idempotent ideal provided that e is primitive. An ideal J of A is said to be a heredity ideal of A if $J^2 = J$, $J(Rad\ A)J = 0$, and J_A is projective. Then, in case of A being a semiprimary ring, J is a heredity ideal if and only if there exists an idempotent e of A such that: (1) $J = AeA$; (2) $Ae\otimes_{eAe}eA \cong AeA$; (3) eAe is a semisimple ring [9], [31]. In this case, Cline, Parshall and Scott showed that $\{D^b(\mathrm{Mod}\ A/AeA)\ ,\ D^b(\mathrm{Mod}\ A)\ ,\ D^b(\mathrm{Mod}\ eAe)\}$ is recollement [31]. Moreover, they studied idempotent ideals which induce recollement of derived categories of modules [31, Theorem 2.7], [8, §1 and 2]. Auslander, Platzeck and Todorov considered homological properties of idempotent ideals [3]. We give a necessary and sufficient condition for idempotent ideals to induce recollement of derived categories of modules.

PROPOSITION 6.1. *Let A be a finite dimensional k-algebra, e an idempotent of A,* $\mathrm{I} : \mathrm{mod}\ A/AeA \to \mathrm{mod}\ A$ *the natural inclusion and* $\mathrm{Q} : \mathrm{mod}\ A \to \mathrm{mod}\ eAe$ *the natural quotient. Then the following are equivalent.*

(a) $0 \to D^b(\mathrm{mod}\ A/AeA) \overset{\mathrm{I}^b}{\to} D^b(\mathrm{mod}\ A) \overset{\mathrm{Q}^b}{\to} D^b(\mathrm{mod}\ eAe) \to 0$ *is exact.*

(b) $\mathrm{Tor}_i^A(A/AeA, A/AeA) = 0$ *for all* $i > 0$.

(c) (i) $\mathrm{Tor}_i^A(AeA, AeA) = 0$ for all $i > 0$, (ii) $AeA\otimes_A AeA \cong AeA$.

Proof. (a) ⇒ (b): According to (a), I^b is fully faithful. Then we have

$$\begin{aligned}\mathrm{Tor}_i^A(A/AeA, A/AeA) &\cong \mathrm{DExt}_A^i(A/AeA_A, \mathrm{D}(A/AeA)_A) \\ &\cong \mathrm{Hom}_{D^b(\mathrm{mod}\ A/AeA)}(A/AeA, \mathrm{D}(A/AeA)[i]) \\ &= 0 \text{ for all } i > 0.\end{aligned}$$

(b) $\Rightarrow$ (a): By Proposition 5.1, I^b is fully faithful. According to [8, (1.3)], $D^b(\text{mod } A/AeA) \cong D^b_{\text{mod}A/AeA}(\text{mod } A)$. Then we are done by Theorem 3.2.

(b) $\Leftrightarrow$ (c): It is easy.

Remark. In the same way as [31, Theorem 2.7], it is easy to see that $0 \to D^-(\text{mod } A/AeA) \xrightarrow{I^-} D^-(\text{mod } A) \xrightarrow{Q^-} D^-(\text{mod } eAe) \to 0$ is colocalization exact if and only if (i) $\text{Tor}_i^{eAe}(Ae, eA) = 0$ for all $i > 0$, and (ii) $Ae \otimes_{eAe} eA \cong AeA$. In this case, we have $\text{Tor}_i^A(A/AeA, A/AeA) = 0$ for all $i > 0$.

COROLLARY 6.2. *The following are equivalent.*

(a) $0 \to D^b(\text{mod } A/AeA) \xrightarrow{I^b} D^b(\text{mod } A) \xrightarrow{Q^b} D^b(\text{mod } eAe) \to 0$ *is colocalization exact.*

(b) (i) $\text{Tor}_i^A(A/AeA, A/AeA) = 0$ *for all* $i > 0$, (ii) $\text{pdim}_A AeA < \infty$.

(c) (i) $\text{Ext}_A^i(A/AeA_A, A/AeA_A) = 0$ *for all* $i > 0$, (ii) $\text{pdim}_A AeA < \infty$.

(d) (i) $\text{Tor}_i^A(AeA, AeA) = 0$ *for all* $i > 0$, (ii) $AeA \otimes_A AeA \cong AeA$, (iii) $\text{pdim}_A AeA < \infty$.

(e) (i) $\text{Tor}_i^{eAe}(Ae, eA) = 0$ for all $i > 0$, (ii) $Ae \otimes_{eAe} eA \cong AeA$, (iii) $\text{pdim}_{eAe} eA < \infty$.

Proof. (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c): According to section 2, this is trivial by Theorem 5.2 and Proposition 6.1.

(b) $\Leftrightarrow$ (d): This is easy.

(a) $\Leftrightarrow$ (e): See [31, Theorem 2.7].

Remark. Replacing modA/AeA, mod$A$ and mod eAe by ModA/AeA, Mod$A$ and Mod eAe in Corollary 6.2, respectively, the assertions (a), (b) and (e) are equivalent for an arbitrary ring A, by the same reason of the remark after Proposition 5.1 (see also [30, (2,1) Theorem]).

THEOREM 6.3. *Suppose A is a left Noetherian or semiprimary ring. Let e be an idempotent of A. The following assertions are equivalent:*

(a) $\{D^b(\mathrm{Mod}\ A/AeA)\ ,\ D^b(\mathrm{Mod}\ A)\ ,\ D^b(\mathrm{Mod}\ eAe)\}$ *is recollement,*

(b) (i) $\mathrm{Tor}_i^A\ (A/AeA,A/AeA)=0$ *for all* $i>0$; (ii) $\{(1)$ *or* $(3)\}$ *and* $\{(2)$ *or* $(4)\}$,

(c) (i) $\mathrm{Ext}_A^i\ (A/AeA_A,A/AeA_A)=0$ *for all* $i>0$; (ii) (1) *and* $\{(2)$ *or* $(4)\}$,

(d) (i) $\mathrm{Ext}_A^i\ ({}_AA/AeA,{}_AA/AeA)=0$ *for all* $i>0$; (ii) (2) *and* $\{(1)$ *or* $(3)\}$,

(e) (i) $Ae\otimes_{eAe}eA\cong AeA$ and $\mathrm{Tor}_i^{eAe}\ (Ae,eA)=0$ *for all* $i>0$; (ii) $\{(1)$ *or* $(3)\}$ *and* $\{(2)$ *or* $(4)\}$,

where (1) $\mathrm{pdim}\ A/AeA_A<\infty$, (2) $\mathrm{pdim}\ {}_AA/AeA<\infty$, (3) $\mathrm{pdim}\ Ae_{eAe}<\infty$, *and* (4) $\mathrm{pdim}_{eAe}eA<\infty$.

Proof. First, we show that if A is left Noetherian or semiprimary, then we have $\mathrm{wdim}_AA/AeA=\mathrm{pdim}_AA/AeA$ and $\mathrm{wdim}_{eAe}eA=\mathrm{pdim}_{eAe}eA$. If A is left Noetherian, then ${}_AAeA$ is a finitely generated left A-module. Therefore we have an epimorphism ${}_AAe^{(n)}\to{}_AAeA$ for some integer n. This implies that eA is a finitely generated left eAe-module. By [2, Theorem 4], we have $\mathrm{wdim}_AA/AeA=\mathrm{pdim}_AA/AeA$ and $\mathrm{wdim}_{eAe}eA=\mathrm{pdim}_{eAe}eA$. If A is semiprimary, then we have also same results by [2, Proposition 7]. According to section 2 and 3, it suffices to show that the condition (i) in (a) - (e) hold, in order to show that (a) implies the other assertions. Conversely, if the functor $D^b(\mathrm{Mod}\ A/AeA)\to D^b(\mathrm{Mod}\ A)$ is fully faithful, then $0\to D^b(\mathrm{Mod}\ A/AeA)\to D^b(\mathrm{Mod}\ A)\to D^b(\mathrm{Mod}\ eAe)\to 0$ is exact in the sense of [4]. According to section 2, (1) and (2) are equivalent to (3) and (4) , respectively. And (ii) of the other assertions imply that $\{D^b(\mathrm{Mod}\ A/AeA)\ ,\ D^b(\mathrm{Mod}\ A)\ ,\ D^b(\mathrm{Mod}\ eAe)\}$ is recollement (see sections 2 and 3 for details).

(a) $\Rightarrow$ (b): $D^b(\mathrm{Mod}\ A/AeA)\to D^b(\mathrm{Mod}\ A)$ has a left adjoint, say G . Then $\mathrm{G}\cong L^{-,b}(-\otimes_AA/AeA)$ (see the remark after Corollary 6.2). Therefore we have the following isomorphism in $D^b(\mathrm{Mod}\ A/AeA)$:

$$A/AeA \cong L^{-,b}(-\otimes_A A/AeA)(A/AeA) .$$

In particular, we have

$$\mathrm{Tor}_i^A\ (A/AeA, A/AeA) = 0 \ \text{ for all } i > 0 .$$

(b) ⇒ (a): According to the remark after Corollary 6.2, we have a fully faithful functor $D^b(\mathrm{Mod}\ A/AeA) \to D^b(\mathrm{Mod}\ A)$.

(a) ⇔ (e): See [30, (2,1) Theorem] and [31, Theorem 2.7].

(a) ⇒ (c): This is trivial by the following isomorphisms:

$$\begin{aligned}\mathrm{Ext}_A^i\ (A/AeA_A, A/AeA_A) &\cong \mathrm{Hom}_{D^b(\mathrm{Mod}\,A)}(A/AeA_A, A/AeA_A[i]) \\ &\cong \mathrm{Hom}_{D^b(\mathrm{Mod}\,A/AeA)}(A/AeA_A, A/AeA_A[i]) \\ &= 0 \ \text{ for all } i > 0 .\end{aligned}$$

(c) ⇒ (a): By Rickard's results there exists a fully faithful functor $D^-(\mathrm{Mod}\ A/AeA) \to D^-(\mathrm{Mod}\ A)$, in particular, a fully faithful functor $D^b(\mathrm{Mod}\ A/AeA) \to D^b(\mathrm{Mod}\ A)$ (see [19], [33] and [34]).

(d) ⇒ (b): Considering (c) ⇒ (a) in case of the left module categories (we need not assume that A is right Noetherian), $\{D^b(A/AeA\text{-Mod})$, $D^b(A\text{-Mod})$, $D^b(eAe\text{-Mod})\}$ is recollement. As well as (a) ⇒ (b), we get $\mathrm{Tor}_i^A\ (A/AeA, A/AeA) = 0$ for all $i > 0$.

(b) ⇒ (d): Since the condition (b) is right and left symmetric, $\{D^b(A/AeA\text{-Mod})$, $D^b(A\text{-Mod})$, $D^b(eAe\text{-Mod})\}$ is recollement (we need not assume that A is right Noetherian). As well as (a) ⇒ (c), we get $\mathrm{Ext}_A^i\ ({}_AA/AeA, {}_AA/AeA) = 0$ for all $i > 0$.

Remark. (b) - (e) in the above theorem are also equivalent for right Noetherian rings.

Recall that a ring A is called an artin algebra if its center $Z(A)$ is an Artinian ring, and A is a finitely generated $Z(A)$-module.

PROPOSITION 6.4. *Let A be an artin algebra, and e an idempotent. The following assertions are equivalent:*
(a) $\{D^b(\text{mod } A/AeA)\,,\ D^b(\text{mod } A)\,,\ D^b(\text{mod } eAe)\}$ *is recollement,*
(b) $\{D^b(\text{Mod } A/AeA)\,,\ D^b(\text{Mod } A)\,,\ D^b(\text{Mod } eAe)\}$ *is recollement.*

Proof. In general, if R is a right coherent ring, then we have $D^b_{\text{mod } R}(\text{Mod } R) \cong D^b(\text{mod } R)$. Let J_A be the Jacobson radical of A. For a given $X \in \text{mod } R$, if $\text{Ext}^i_R(X, A/J_A) = 0$ or $\text{Tor}^A_i(A/J_A, Y) = 0$ for all $i > n$, then $\text{pdim}\, X_R \leq n$ (see [2] for details).

(a) $\Rightarrow$ (b): Let F and G be right and left adjoint functors of $D^b(\text{mod } A/AeA) \to D^b(\text{mod } A)$, respectively. Since A is Artinian and A/AeA is a finitely generated A-module, we have $G \cong L^{-,b}(- \otimes_A A/AeA)$, and $\text{Tor}^A_i(A/AeA, A/AeA) = 0$ for all $i > 0$ as in the proof (a) $\Rightarrow$ (b) of Theorem 6.3. Also we have $\text{Tor}^A_i(A/J_A, A/AeA) \cong H^i(G(A/J_A))$ for all i. Since $F(A/J_A)$ is contained in $D^b(\text{mod } A)$, $\text{pdim}\,_A A/AeA < \infty$. We have the following isomorphisms:

$$\begin{aligned}\text{Ext}^i_A(A/AeA, A/J_A) &\cong \text{Hom}_{D^b(\text{mod } A)}(G(A/AeA), A/J_A[i])\\ &\cong \text{Hom}_{D^b(\text{mod } A/AeA)}(A/AeA, F(A/J_A)[i])\\ &\cong H^i(F(A/J_A)) \text{ for all } i.\end{aligned}$$

Since $F(A/J_A)$ is contained in $D^b(\text{mod } A)$, we get $\text{pdim}\, A/AeA_A < \infty$. Hence $\{D^b(\text{Mod } A/AeA)\,,\ D^b(\text{Mod } A)\,,\ D^b(\text{Mod } eAe)\}$ is recollement by

Theorem 6.3.

(b) $\Rightarrow$ (a): Let E and H be right and left adjoint functors of D^b(Mod A/AeA) $\to D^b$(Mod A), respectively. It is clear that D^b(mod A/AeA) $\to$ D^b(mod A) has a left adjoint. Since A is an artin algebra, and A/AeA is finitely generated, Ext^i_A $(A/AeA, X)$ is a finitely generated A/AeA-module for all $X \in$ mod A. Then it is easy to see that $\mathrm{Im}\, \mathrm{H}|_{D^b(\mathrm{mod}\, A)}$ is contained in $D^b{}_{\mathrm{mod}\, A}$(Mod A). By the above equivalence, D^b(mod A/AeA) $\to D^b$(mod A) has a right adjoint. We are done by Theorem 6.3.

Let A be a left (or right) Noetherian or semiprimary ring. An ideal I of A is called a recollement ideal of A if $I = AeA$ with some idempotent e of A which satisfies the equivalent conditions (b) - (e) of Theorem 6.3. The next proposition is useful to exhibiting examples of recollement ideals.

Proposition 6.5. *Let R be a commutative ring, and A and B R-algebras. Suppose A is a left or right Noetherian ring and B is a finitely generated projective R-module. If I is a recollement ideal of A, then $I \otimes_R B$ is a recollement ideal of $A \otimes_R B$.*

Proof. First, $A \otimes_R B$ is a left or right Noetherian ring, because B is a finitely generated R-module. Since B is R-projective, we have pdim I_A $\geq$ pdim $I \otimes_R B_{A\otimes_R B}$ and pdim ${}_AI \geq$ pdim ${}_{A\otimes_R B}I \otimes_R B$. And let $P^\cdot$ be a projective resolution of A/I. Then we have

$$\begin{aligned}\mathrm{Tor}_i^{A\otimes_R B}(A/I \otimes_R B, A/I \otimes_R B) &\cong \mathrm{H}_i(P^\cdot \otimes_R B \otimes_{A\otimes_R B} A/I \otimes_R B)\\ &\cong \mathrm{H}_i(P^\cdot \otimes_A A/I) \otimes_R B\\ &\cong \mathrm{Tor}_i^A(A/I, A/I) \otimes_R B\\ &= 0 \text{ for all } i > 0.\end{aligned}$$

LEMMA 6.5. *If A is a local semiprimary ring, then* pdim M *is* 0 *or* ∞, *for all modules M.*

PROPOSITION 6.6. *Suppose A is a semiprimary ring. Let I be a minimal idempotent ideal of A. Then I is a recollement ideal of A if and only if I is projective as both a left and right A-module.*

Proof. If $I = AeA$ is projective as both a left and right A-module, then it is easy to see that A/AeA satisfies the condition (b) of Theorem 6.3.. Conversely, If $I = AeA$ is a recollement ideal, then AeA has finite projective dimension. Let $P^\bullet$ be a projective resolution of Ae as right eAe-modules. Then given any left A-module X, we get

$$\begin{aligned}\mathrm{Tor}_i^A(AeA,X) &\cong \mathrm{Tor}_i^A(Ae\otimes_{eAe}eA,X)\\ &\cong \mathrm{H}_i(P^\bullet\otimes_{eAe}eA\otimes_A X)\\ &\cong \mathrm{Tor}_i^{eAe}(Ae,eX).\end{aligned}$$

For every left eAe-module Y, there exists a left A-module X such that Y is isomorphic to eX. Then Ae has finite projective dimension in Mod eAe. Since I is a minimal idempotent ideal of A, eAe is a local semiprimary ring. Therefore Ae is a projective right eAe-module by Lemma 6.5. Hence AeA is a projective right A-module by the above isomorphisms. Similarly, AeA is also a projective left A-module.

According to the above proposition, in order to find minimal recollement ideals, it suffices to find idempotent ideals which are two-sided projective. But the following proposition implies that heredity ideals are best possible in case of rings of finite global dimension.

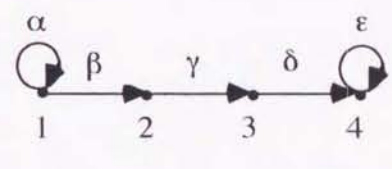
PROPOSITION 6.7. *Suppose A is a semiprimary ring of finite global dimension. Let I be a minimal idempotent ideal. Then I is a recollement ideal if and only if I is a heredity ideal.*

Proof. Let I be AeA with some idempotent e of A, and $P^{\bullet}$ a projective resolution of eAe/eJe as right eAe-modules. Then $P^{\bullet}\otimes_{eAe}eA$ is a projective resolution of $eA/eJeA$ as right A-modules, where J is the radical of A. Therefore, we get

$$\begin{aligned}\mathrm{Tor}_i^{eAe}(eAe/eJe,eX) &\cong H_i(P^{\bullet}\otimes_{eAe}eA\otimes_A X)\\ &\cong \mathrm{Tor}_i^{A}(eA/eJeA,X)\,.\end{aligned}$$

According to assumption, pdim $eA/eJeA < \infty$, and pdim $eAe/eJe < \infty$. Since eAe is a local semiprimary ring, eAe/eJe is a projective eAe-module by Lemma 6.5. Hence $eJe = 0$.

Examples. (a) Let A be a finite dimensional algebra over a field k which has a quiver with relations:



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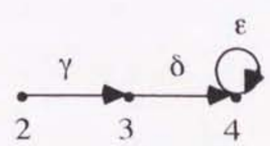
with $\alpha^2 = \varepsilon^2 = \gamma\beta = 0$. Then Ae_1A is projective as both sides. Moreover, e_1Ae_1 is isomorphic to $k[x]/(x^2)$ as a ring, and A/Ae_1A has the following quiver with relations:



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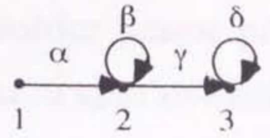
with $\varepsilon^2 = 0$. Hence we have gldim A = gldim $e_1 A e_1$ = gldim $A/Ae_1A = \infty$.

(b) Let A be a finite dimensional algebra over a field k which has a quiver with relations:



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with $\beta\alpha = \delta\gamma = \beta^2 = \delta^2 = 0$. Then $A(e_1+e_2)A$ is a recollement ideal. But Ae_2A is not a recollement ideal because of pdim $Ae_2A_A = \infty$.

Chapter III. Ring Extensions and Tilting Complexes

§7. Ring Homomorphisms and Tilting Complexes.

In this section, we consider tensor products which induced by ring morphisms. In particular, a split extension of a ring yields a necessary and sufficient condition for tensor product of a complex to be a tilting complex.

Let A be a ring. We denote by Proj-A the category of all projective right A-modules. Rickard defined a tilting complex $T^{\bullet}$ for A as follows,

(i) $T^{\bullet} \in K^{b}(\mathcal{P}_A)$,

(ii) $\mathrm{Hom}_{K(\mathrm{Mod}A)}(T^{\bullet}, T^{\bullet}[i]) = 0$ for all $i \neq 0$,

(iii) add$T^{\bullet}$, the additive category of direct summands of finite direct sums of copies of $T^{\bullet}$, generates $K^{b}(\mathcal{P}_A)$ as a triangulated category.

Rickard also showed that (iii) can be replaced by

(iii)' For each non-zero object $X^{\bullet}$ of K^{-}(Proj-A), there is a some i such that $\mathrm{Hom}_{K(\mathrm{Mod}A)}(T^{\bullet}, X^{\bullet}[i]) \neq 0$.

Then there is a derived equivalent functor $D^{-}(\mathrm{Mod}B) \to D^{-}(\mathrm{Mod}A)$ which sends B to $T^{\bullet}$, where $B = \mathrm{End}_{K(\mathrm{Mod}A)}(T^{\bullet})$ (see [34] for details).

For a tilting complex $T^{\bullet}$ for A, we call $\mathrm{H}^0T^{\bullet}$ a tilting A-module provided that $\mathrm{H}^iT^{\bullet} = 0$ for all $i \neq 0$ ([13] and [27]). In this case, we have $T^{\bullet} \cong \mathrm{H}^0T^{\bullet}$ in $D^b(\mathrm{Mod}A)$. Furthermore, we call a tilting module T a classical tilting module if projective dimension of T is less than or equal to 1.

In case that A is a finite dimensional algebra over a field k, there exist two-sided tilting complexes $\Delta^{\bullet}$ in $D^{b}(\mathrm{Mod}(B^{\mathrm{op}} \otimes_k A))$ and $\nabla^{\bullet}$ in

$D^b(\mathrm{Mod}(A^{op}\otimes_k B))$ such that $\Delta^\bullet \otimes^L_A V^\bullet \cong {}_B B_B$ and $V^\bullet \otimes^L_B \Delta^\bullet \cong {}_A A_A$ (see [36] for details).

LEMMA 7.1. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be additive categories,* $\mathrm{H}, \mathrm{L} : \mathcal{A} \to \mathcal{C}$ *additive functors,* $\eta : \mathrm{H} \to \mathrm{L}$ *a morphism of functors and* $\mathrm{G} : \mathcal{A} \to \mathcal{B}$ *an additive functor which has the right adjoint* $\mathrm{F} : \mathcal{B} \to \mathcal{A}$. *Given* $X^\bullet, Z^\bullet \in \mathrm{C}(\mathcal{A})$, $Y^\bullet \in \mathrm{C}(\mathcal{C})$ *and* $U^\bullet \in \mathrm{C}(\mathcal{B})$, *the following results hold.*

(a) $\mathrm{Hom}^\bullet_{\mathcal{A}}(X^\bullet, Z^\bullet) \to \mathrm{Hom}^\bullet_{\mathcal{C}}(\mathrm{H}X^\bullet, \mathrm{H}Z^\bullet)$ *induces an* $\mathrm{End}_{K(\mathcal{A})}(X^\bullet)$-$\mathrm{End}_{K(\mathcal{C})}(Z^\bullet)$-*homomorphism* $\mathrm{Hom}_{K(\mathcal{A})}(X^\bullet, Z^\bullet[i]) \to \mathrm{Hom}_{K(\mathcal{C})}(\mathrm{H}X^\bullet, \mathrm{H}Z^\bullet[i])$ *for all i .*

(b) $\mathrm{Hom}^\bullet_{\mathcal{A}}(X^\bullet, X^\bullet) \to \mathrm{Hom}^\bullet_{\mathcal{C}}(\mathrm{H}X^\bullet, \mathrm{H}X^\bullet)$ *induces a ring homomorphism* $\mathrm{End}_{K(\mathcal{A})}(X^\bullet) \to \mathrm{End}_{K(\mathcal{C})}(\mathrm{H}X^\bullet)$.

(c) $\mathrm{Hom}^\bullet_{\mathcal{C}}(Y^\bullet, \mathrm{H}Z^\bullet) \to \mathrm{Hom}^\bullet_{\mathcal{C}}(Y^\bullet, \mathrm{L}Z^\bullet)$ *induces an* $\mathrm{End}_{K(\mathcal{A})}(Y^\bullet)$-$\mathrm{End}_{K(\mathcal{C})}(Z^\bullet)$-*homomorphism* $\mathrm{Hom}_{K(\mathcal{C})}(Y^\bullet, \mathrm{H}Z^\bullet[i]) \to \mathrm{Hom}_{K(\mathcal{C})}(Y^\bullet, \mathrm{L}Z^\bullet[i])$ *for all i .*

(d) $\mathrm{Hom}^\bullet_{\mathcal{B}}(\mathrm{G}X^\bullet, U^\bullet) \cong \mathrm{Hom}^\bullet_{\mathcal{A}}(X^\bullet, \mathrm{F}U^\bullet)$ *induces an* $\mathrm{End}_{K(\mathcal{A})}(X^\bullet)$-$\mathrm{End}_{K(\mathcal{B})}(U^\bullet)$-*isomorphism* $\mathrm{Hom}_{K(\mathcal{B})}(\mathrm{G}X^\bullet, U^\bullet[i]) \cong \mathrm{Hom}_{K(\mathcal{A})}(X^\bullet, \mathrm{F}U^\bullet[i])$ *for all i .*

Furthermore, these correspondences are functorial.

LEMMA 7.2. *Let* $A \to \Lambda$ *be a ring homomorphism and* $T^\bullet$ *a tilting complex for* A. *If* $\mathrm{Hom}_{D(\mathrm{Mod}\Lambda)}(T^\bullet, T^\bullet \otimes^L_A \Lambda_A[i]) = 0$ *for all* $i \neq 0$, *then* $T^\bullet \otimes^L_A \Lambda$ *is a tilting complex for* Λ .

Proof. We have $T^\bullet \otimes^L_A \Lambda \cong T^\bullet \otimes_A \Lambda$ in $D^b(\mathrm{Mod}\,\Lambda)$, and $T^\bullet \otimes_A \Lambda$ belongs to $K^b(\mathcal{P}_\Lambda)$. Since $\mathrm{Hom}^\bullet_\Lambda(T^\bullet \otimes_A \Lambda, T^\bullet \otimes_A \Lambda) \cong \mathrm{Hom}^\bullet_A(T^\bullet, T^\bullet \otimes_A \Lambda_A)$, we have the following isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{D(\mathrm{Mod}\Lambda)}(T^\bullet \otimes^L_A \Lambda, T^\bullet \otimes^L_A \Lambda[i]) &\cong \mathrm{H}^i\mathrm{Hom}^\bullet_\Lambda(T^\bullet \otimes_A \Lambda, T^\bullet \otimes_A \Lambda) \\
&\cong \mathrm{H}^i\mathrm{Hom}^\bullet_A(T^\bullet, T^\bullet \otimes_A \Lambda_A) \\
&\cong \mathrm{Hom}_{D(\mathrm{Mod}A)}(T^\bullet, T^\bullet \otimes^L_A \Lambda_A[i]) \\
&= 0 \text{ for all } i \neq 0.
\end{aligned}$$

Let $X^\bullet$ be an object of $K^-(\text{Proj-}\Lambda)$ such that $\text{Hom}_{D(\text{Mod}\Lambda)}(T^\bullet \otimes^L_A \Lambda, X^\bullet[i])$ $= 0$ for all i. Then we have the following isomorphisms:

$$\begin{aligned}\text{Hom}_{D(\text{Mod}A)}(T^\bullet, X^\bullet \otimes_\Lambda \Lambda_A[i]) &\cong \text{H}^i\text{Hom}^\bullet_A(T^\bullet, X^\bullet \otimes_\Lambda \Lambda_A) \\ &\cong \text{H}^i\text{Hom}^\bullet_\Lambda(T^\bullet \otimes_A \Lambda, X^\bullet) \\ &\cong \text{Hom}_{D(\text{Mod}\Lambda)}(T^\bullet \otimes^L_A \Lambda, X^\bullet[i]) \\ &= 0 \text{ for all } i.\end{aligned}$$

Since $T^\bullet$ is a tilting complex, $X^\bullet \otimes_\Lambda \Lambda_A \cong 0$ in $D(\text{Mod}\,A)$, that is, $\text{H}^i(X^\bullet \otimes_\Lambda \Lambda_A) = \text{H}^i(X^\bullet) = 0$ for all i. Therefore $X^\bullet \cong 0$ in $D(\text{Mod}\,\Lambda)$.

COROLLARY 7.3 (Miyashita [27]). *Let $A \to \Lambda$ be a ring homomorphism and T a tilting A-module. If $\text{Tor}_i^A(T,\Lambda) = \text{Ext}^i_A(T, T\otimes_A\Lambda_A) = 0$ for all $i > 0$, then $T\otimes_A\Lambda$ is a tilting Λ-module.*

Proof. Let $P^\bullet$ be a projective resolution of T. Since $\text{Tor}_i^A(T,\Lambda) = 0$ for all $i > 0$, $P^\bullet \otimes_A \Lambda$ is a projective resolution of $T\otimes_A\Lambda$ and $\text{Hom}_{D(\text{Mod}A)}(P^\bullet, P^\bullet \otimes^L_A \Lambda_A[i]) \cong \text{Ext}^i_A(T, T\otimes_A\Lambda_A)$ for all $i > 0$. Then we are done by Lemma 7.2.

In case of a finite dimensional algebra A over a field k, there exist a duality $\text{D}: D^b(\text{mod}A) \to D^b(A\ \text{mod})$, where $\text{D} = \text{Hom}_k(-,k)$. Then we can define a cotilting complex $T^\bullet$ as follows,

(i) $T^\bullet \in K^b(\mathcal{I}_A)$, where $\mathcal{I}_A$ is the category of finitely generated injective right A-modules,

(ii) $\text{Hom}_{D(\text{mod}A)}(T^\bullet, T^\bullet[i]) = 0$ for all $i \neq 0$,

(iii) $\text{D}A \in \mathcal{T}(\text{add}T^\bullet)$, where $\mathcal{T}(\text{add}T^\bullet)$ is the triangulated subcategory of $K^b(\mathcal{I}_A)$ generated by objects in $\text{add}T^\bullet$.

Happel showed that if $X^{\bullet}$ belongs to $K^{b}(\mathcal{P}_A)$, then there exists an Auslander-Reiten translation $\tau_A X^{\bullet}$ which is isomorphic to $\nu X^{\bullet}[-1]$, where $\nu_A = -\otimes^{L}_{A} DA$, and then there exists an Auslander-Reiten triangle $\tau_A X^{\bullet} \to Y^{\bullet} \to X^{\bullet} \to \tau_A X^{\bullet}[1]$ in $D^{b}(\mathrm{mod}\, A)$ (see [13]). Then $\tau_A T^{\bullet}$ is a cotilting complex for A if $T^{\bullet}$ is a tilting complex for A. As well as Proposition 1.2 in [22], we have the following result.

PROPOSITION 7.4. *Let $A \to \Lambda$ be a k-algebra homomorphism between finite dimensional k-algebras. If $X^{\bullet} \in K^{b}(\mathcal{P}_A)$, then $\tau_\Lambda(X^{\bullet} \otimes^{L}_{A} \Lambda)$ is isomorphic to $R\,\mathrm{Hom}_A({}_A\Lambda_\Lambda, \tau_A X^{\bullet})$ in $D^{b}(\mathrm{mod}\Lambda)$.*

Proof. We have the following isomorphisms:

$$\begin{aligned}
\tau_\Lambda(X^{\bullet} \otimes^{L}_{A} \Lambda) &\cong (X^{\bullet} \otimes_A \Lambda) \otimes_\Lambda D\Lambda_\Lambda [-1] \\
&\cong X^{\bullet} \otimes_A D\Lambda_\Lambda [-1] \\
&\cong D\mathrm{Hom}_A(X^{\bullet}, {}_A\Lambda_\Lambda)[-1] \\
&\cong D({}_A\Lambda \otimes_A \mathrm{Hom}_A(X^{\bullet}, A))[-1] \\
&\cong \mathrm{Hom}_A({}_A\Lambda_\Lambda, D\mathrm{Hom}_A(X^{\bullet}, A)[-1]) \\
&\cong R\,\mathrm{Hom}_A({}_A\Lambda_\Lambda, \tau_A X^{\bullet}).
\end{aligned}$$

COROLLARY 7.5. *Let $A \to \Lambda$ be a k-algebra homomorphism between finite dimensional k-algebras, and $T^{\bullet}$ a tilting complex for A. If $\mathrm{Hom}_{D(\mathrm{mod}A)}(T^{\bullet}, T^{\bullet} \otimes^{L}_{A} \Lambda_A[i]) = 0$ for all $i \neq 0$, then $R\,\mathrm{Hom}_A({}_A\Lambda_\Lambda, \tau_A T^{\bullet})$ is a cotilting complex for Λ.*

LEMMA 7.6. *Let $A \to \Lambda$ a ring homomorphism and $T^{\bullet}$ an object of $D^{-}(\mathrm{Mod}A)$. If $T^{\bullet} \otimes^{L}_{A} \Lambda$ is a tilting complex for Λ, then we have $\mathrm{Hom}_{D(\mathrm{Mod}A)}(T^{\bullet}, T^{\bullet} \otimes^{L}_{A} \Lambda_A[i]) = 0$ for all $i \neq 0$.*

Proof. We may assume that $T^{\bullet}$ is an object of $K^{-}(\text{Proj-}A)$, then we have $T^{\bullet}\otimes^{L}_{A}\Lambda \cong T^{\bullet}\otimes_{A}\Lambda$ in $D^{-}(\text{Mod}\,\Lambda)$, and $T^{\bullet}\otimes_{A}\Lambda$ belongs to $K^{b}(\mathcal{P}_{\Lambda})$. Since $\text{Hom}^{\bullet}_{\Lambda}(T^{\bullet}\otimes_{A}\Lambda, T^{\bullet}\otimes_{A}\Lambda) \cong \text{Hom}^{\bullet}_{A}(T^{\bullet}, T^{\bullet}\otimes_{A}\Lambda_{\Lambda})$, we have the following isomorphisms:

$$\begin{aligned}\text{Hom}_{D(\text{Mod}A)}(T^{\bullet}, T^{\bullet}\otimes_{A}\Lambda_{\Lambda}[i]) &\cong H^{i}\text{Hom}^{\bullet}_{A}(T^{\bullet}, T^{\bullet}\otimes_{A}\Lambda_{\Lambda})\\ &\cong H^{i}\text{Hom}^{\bullet}_{\Lambda}(T^{\bullet}\otimes_{A}\Lambda, T^{\bullet}\otimes_{A}\Lambda)\\ &\cong \text{Hom}_{D(\text{Mod}\Lambda)}(T^{\bullet}\otimes^{L}_{A}\Lambda, T^{\bullet}\otimes^{L}_{A}\Lambda[i])\\ &= 0 \text{ for all } i \neq 0.\end{aligned}$$

THEOREM 7.7. *Let* $\mu : A \to \Lambda$ and $\varepsilon : \Lambda \to A$ *be ring homomorphisms such that* $\varepsilon\circ\mu = \text{id}_{A}$, *and* $T^{\bullet}$ *an object of* $D^{-}(\text{Mod}A)$. *Then* $T^{\bullet}\otimes^{L}_{A}\Lambda$ *is a tilting complex for* Λ *if and only if* $T^{\bullet}$ *is a tilting complex for* A *and* $\text{Hom}_{D(\text{Mod}A)}(T^{\bullet}, T^{\bullet}\otimes^{L}_{A}\Lambda_{A}[i]) = 0$ *for all* $i \neq 0$.
In this case, there exist ring homomorphisms $\eta : B \to \Gamma$ *and* $\pi : \Gamma \to B$ *such that* $\pi\circ\eta = \text{id}_{B}$, *where* $B = \text{End}_{D(\text{Mod}A)}(T^{\bullet})$ *and* $\Gamma =$ $\text{End}_{D(\text{Mod}\Lambda)}(T^{\bullet}\otimes^{L}_{A}\Lambda)$.

Proof. The 'if' part has been proved in Lemma 7.2. We may assume that $T^{\bullet}$ is an object of $K^{-}(\text{Proj-}A)$, then we have $T^{\bullet}\otimes^{L}_{A}\Lambda \cong T^{\bullet}\otimes_{A}\Lambda$ in $D^{-}(\text{Mod}\,\Lambda)$. Since $T^{\bullet}\otimes_{A}\Lambda$ is isomorphic to an object in $K^{b}(\mathcal{P}_{\Lambda})$, there exists an object $Q^{\bullet}$ in $K^{b}(\mathcal{P}_{\Lambda})$ such that $T^{\bullet}\otimes_{A}\Lambda \cong Q^{\bullet}$ in $K^{-}(\text{Proj-}\Lambda)$. Applying $-\otimes_{\Lambda}A_{A}$ to it, we get $T^{\bullet}\otimes_{A}\Lambda\otimes_{\Lambda}A_{A} \cong Q^{\bullet}\otimes_{\Lambda}A_{A}$ in $K^{-}(\text{Proj-}A)$. Therefore $T^{\bullet}$ is isomorphic to an object in $K^{b}(\mathcal{P}_{A})$, because $\varepsilon\circ\mu = \text{id}_{A}$ and $\Lambda\otimes_{\Lambda}A_{A} \cong A_{A}$. By $\varepsilon\circ\mu = \text{id}_{A}$, $T^{\bullet}$ is a direct summand of $T^{\bullet}\otimes_{A}\Lambda_{A}$ in $D(\text{Mod}\,A)$. According to Lemma 7.6, we get $\text{Hom}_{D(\text{Mod}A)}(T^{\bullet}, T^{\bullet}[i]) = 0$ for all $i \neq 0$. Then we have the following isomorphisms:

$$\text{Hom}_{D(\text{Mod}\Lambda)}(T^{\bullet}\otimes_{A}\Lambda, T^{\bullet}\otimes_{A}\Lambda\otimes_{\Lambda}A_{\Lambda}[i]) \cong H^{i}\text{Hom}^{\bullet}_{\Lambda}(T^{\bullet}\otimes_{A}\Lambda, T^{\bullet}\otimes_{A}\Lambda\otimes_{\Lambda}A_{\Lambda})$$

$$\cong H^i\mathrm{Hom}^\bullet_A(T^\bullet, T \otimes_A \Lambda \otimes_\Lambda A_A)$$
$$\cong \mathrm{Hom}_{D(\mathrm{Mod}A)}(T^\bullet, T^\bullet[i])$$
$$= 0 \text{ for all } i \neq 0.$$

By Lemma 7.2, $T^\bullet \otimes_A \Lambda \otimes_\Lambda A_A$, that is, $T^\bullet$ is a tilting complex for A. By $\varepsilon \circ \mu = \mathrm{id}_A$, μ and ε induce $\mu^\bullet$: $\mathrm{Hom}^\bullet_A(T^\bullet, T^\bullet) \to \mathrm{Hom}^\bullet_\Lambda(T^\bullet \otimes_A \Lambda, T^\bullet \otimes_A \Lambda)$ and $\varepsilon^\bullet$: $\mathrm{Hom}^\bullet_\Lambda(T^\bullet \otimes_A \Lambda, T^\bullet \otimes_A \Lambda) \to \mathrm{Hom}^\bullet_A(T^\bullet, T^\bullet)$ such that $\varepsilon^\bullet \circ \mu^\bullet = \mathrm{id}$. By Lemma 7.1, we get $\eta : B \to \Gamma$ and $\pi : \Gamma \to B$ such that $\pi \circ \eta = \mathrm{id}_B$.

Corollary 7.8 (Hoshino [15]). *Let* $\mu : A \to \Lambda$ *and* $\varepsilon : \Lambda \to A$ *be ring homomorphisms such that* $\varepsilon \circ \mu = \mathrm{id}_A$, *and* T *an* A*-module. Assume that* $\mathrm{Tor}_i^A(T, \Lambda) = 0$ *for all* $i > 0$. *Then* $T \otimes_A \Lambda$ *is a tilting* Λ*-module if and only if* T *is a tilting* A*-module and* $\mathrm{Ext}^i_A(T, T \otimes_A \Lambda_A) = 0$ *for all* $i > 0$.

Proof. In the same way as the proof of Corollary 2.2, we are done by Theorem 7.7.

§8. Extensions of Rings and Tilting Complexes.

In this section, we consider a condition for an extension (not necessary split) of a ring to induce a extension of a ring. Furthermore, we show that a Frobenius extension of a ring induces a Frobenius extension of a ring. Next theorem is a generalization of Corollary 5.4 in [36].

THEOREM 8.1. *Let* Λ *be an extension of a ring* A *such that* $0 \to A \to \Lambda \to M \to 0$ *is an exact sequence as* A-A-*bimodules. Let* $T^{\bullet}$ *be a tilting complex for* A, and $B = \mathrm{End}_{D(\mathrm{Mod}A)}(T^{\bullet})$, $\Gamma = \mathrm{End}_{D(\mathrm{Mod}\Lambda)}(T^{\bullet}\otimes^{L}_{A}\Lambda)$ *and* $N = \mathrm{Hom}_{D(\mathrm{Mod}A)}(T^{\bullet}, T^{\bullet}\otimes^{L}_{A} M)$. *Then the following are equivalent.*

(a) (i) $T^{\bullet}\otimes^{L}_{A}\Lambda$ *is a tilting complex for* Λ, *and* (ii) Γ *is an extension of a ring* B *such that* $0 \to B \to \Gamma \to N \to 0$ *is an exact sequence as* B-B-*bimodules,*

(b) $\mathrm{Hom}_{D(\mathrm{Mod}A)}(T^{\bullet}, T^{\bullet}\otimes^{L}_{A} M[i]) = 0$ *for all* $i \neq 0$.

Proof. It is clear that $T^{\bullet}\otimes^{L}_{A}\Lambda$ belongs to $K^{b}(\mathcal{P}_{\Lambda})$. We have the following commutative diagram:

$$\begin{array}{ccccccc} & & \mathrm{Hom}^{\bullet}_{A}(T^{\bullet}, T^{\bullet}) & \to & \mathrm{Hom}^{\bullet}_{\Lambda}(T^{\bullet}\otimes_{A}\Lambda, T^{\bullet}\otimes_{A}\Lambda) & & \\ & & \| & & \downarrow & & \\ 0 & \to & \mathrm{Hom}^{\bullet}_{A}(T^{\bullet}, T^{\bullet}) & \to & \mathrm{Hom}^{\bullet}_{A}(T^{\bullet}, T^{\bullet}\otimes_{A}\Lambda_{A}) & \to & \mathrm{Hom}^{\bullet}_{A}(T^{\bullet}, T^{\bullet}\otimes_{A}M_{A}) \to 0, \end{array}$$

where the bottom row is exact and vertical arrows are isomorphisms.

(b) $\Rightarrow$ (a): Since $\mathrm{Hom}_{D(\mathrm{Mod}A)}(T^{\bullet}, T^{\bullet}[i]) = \mathrm{Hom}_{D(\mathrm{Mod}A)}(T^{\bullet}, T^{\bullet}\otimes^{L}_{A} M[i]) = 0$ for all $i \neq 0$, by taking homology of the above diagram, we have

$$\begin{aligned} \mathrm{Hom}_{D(\mathrm{Mod}\Lambda)}(T^{\bullet}\otimes^{L}_{A}\Lambda, T^{\bullet}\otimes^{L}_{A}\Lambda[i]) &\cong \mathrm{Hom}_{D(\mathrm{Mod}A)}(T^{\bullet}, T^{\bullet}\otimes^{L}_{A}\Lambda_{A}[i]) \\ &\cong \mathrm{Hom}_{D(\mathrm{Mod}A)}(T^{\bullet}, T^{\bullet}\otimes^{L}_{A} M_{A}[i]) \end{aligned}$$

$$= 0 \text{ for all } i \neq 0$$

And we get the following commutative diagram:

$$\begin{array}{ccccc} & \mathrm{Hom}_{D(\mathrm{Mod}\,A)}(T^\bullet, T^\bullet) \to \mathrm{Hom}_{D(\mathrm{Mod}\,A)}(T^\bullet \otimes^L_A \Lambda, T^\bullet \otimes^L_A \Lambda) & & & \\ & \| \qquad\qquad\qquad \downarrow & & & \\ 0 \to & \mathrm{Hom}_{D(\mathrm{Mod}\,A)}(T^\bullet, T^\bullet) \to \mathrm{Hom}_{D(\mathrm{Mod}\,A)}(T^\bullet, T^\bullet \otimes^L_A \Lambda_A) & \to & \mathrm{Hom}_{D(\mathrm{Mod}\,A)}(T^\bullet, T^\bullet \otimes^L_A M_A) & \to 0, \end{array}$$

where the arrow of the top row is a ring homomorphism, the bottom row is exact as B-B-bimodules and vertical arrows are isomorphisms as B-B-bimodules. By Lemma 7.2, $T^\bullet \otimes^L_A \Lambda$ is a tilting complex for Λ.

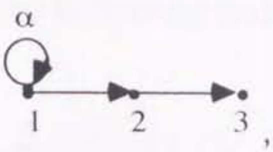
(a) $\Rightarrow$ (b): The condition (ii) implies that $\mathrm{Hom}_{D(\mathrm{Mod}\,A)}(T^\bullet, T^\bullet) \to \mathrm{Hom}_{D(\mathrm{Mod}\,A)}(T^\bullet, T^\bullet \otimes^L_A \Lambda_A)$ is injective. Since $T^\bullet \otimes^L_A \Lambda$ is a tilting complex for Λ, by Lemma 7.1, we have

$$\mathrm{Hom}_{D(\mathrm{Mod}\Lambda)}(T^\bullet \otimes^L_A \Lambda, T^\bullet \otimes^L_A \Lambda[i]) \cong \mathrm{Hom}_{D(\mathrm{Mod}A)}(T^\bullet, T^\bullet \otimes^L_A \Lambda_A[i])$$
$$= 0 \text{ for all } i \neq 0.$$

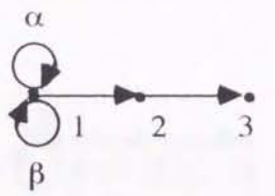
Then we have $\mathrm{Hom}_{D(\mathrm{Mod}A)}(T^\bullet, T^\bullet \otimes^L_A M[i]) = 0$ for all $i \neq 0$, by taking homology of the above exact sequence of complexes.

Remark. Let A and B be R-algebras which are projective as R-modules. If A and B are derived equivalent R-algebras, then M is just an A-bimodule which corresponds to a B-bimodule N under the induced equivalence $D^b(\mathrm{Mod}\, A^{\mathrm{op}} \otimes_R A) \to D^b(\mathrm{Mod}\, B^{\mathrm{op}} \otimes_R B)$ (see [36]).

Example. Let A be a finite dimensional algebra over a field k which has the following quiver with relations:



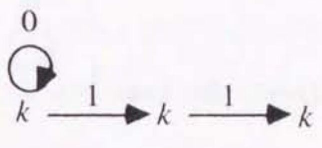
with $\alpha^3 = 0$, and Λ be a finite dimensional algebra over a field k which has the following quiver with relations:



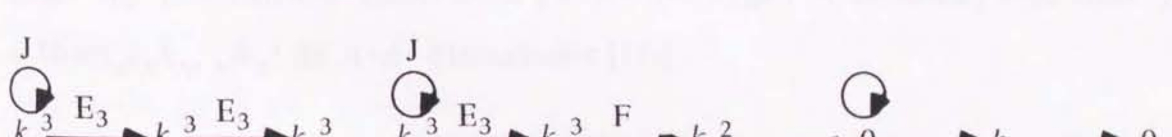
with $\alpha\beta = \beta\alpha = 0$ and $\alpha^2 = \beta^3$. Then Λ is a non split extension of A , and we have the following exact sequence as A-A-bimodules:

$$0 \to A \to \Lambda \to S(1)^{(2)} \otimes_k X \to 0,$$

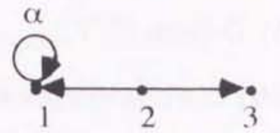
where $S(1)$ is simple left A-module corresponding to vertex 1 and X is a right A-module,



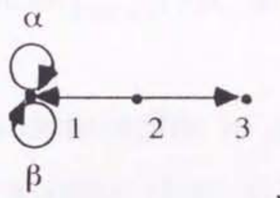
Let T_1 , T_2 and T_3 be the following right A-modules, respectively:



where $\mathrm{J} = \begin{pmatrix} 0&0&0 \\ 1&0&0 \\ 0&1&0 \end{pmatrix}$, $\mathrm{F} = \begin{pmatrix} 0&1&0 \\ 0&0&1 \end{pmatrix}$. Then $T = T_1 \oplus T_2 \oplus T_3$ satisfies the condition of Theorem 8.1, and B has the following quiver with a relation:



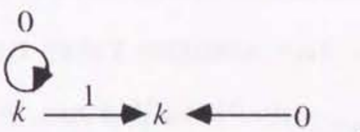
with $\alpha^3 = 0$, Γ has the following quiver with relations:



with $\alpha\beta = \beta\alpha = 0$ and $\alpha^2 = \beta^3$, and we get the following exact sequence as B-B-bimodules:

$$0 \to B \to \Gamma \to Y \otimes_k S'(1) \to 0,$$

where $S'(1)$ is a simple right B-module corresponding to vertex 1 and Y is a left B-module:



For a subring A of Λ, Λ is called a Frobenius extension of A provided that Λ_A is a finitely generated projective right A-module, and that ${}_A\Lambda_\Lambda \cong \mathrm{Hom}_A({}_\Lambda\Lambda_A, {}_AA_A)$ as A-Λ- bimodules [16].

LEMMA 8.2. *Let A be a ring and $T^\bullet$ a tilting complex for A. Given $X^\bullet \in K^b(\mathcal{P}_A)$, if $\mathrm{Hom}_{D(\mathrm{Mod}A)}(T^\bullet, X^\bullet[i]) = \mathrm{Hom}_{D(\mathrm{Mod}A)}(X^\bullet, T^\bullet[i]) = 0$ for all $i \neq 0$, then $X^\bullet$ is isomorphic to a direct summand of a finite direct sum of* *copies of* $T^{\bullet}$.

Proof. Let $B = \mathrm{End}_{D(\mathrm{Mod}A)}(T^{\bullet})$ and $G : D^{-}(\mathrm{Mod}A) \rightarrow D^{-}(\mathrm{Mod}B)$ a quasi-inverse functor of an equivalence functor induced by $T^{\bullet}$. Then we have the following isomorphisms,

$$\mathrm{Hom}_{D(\mathrm{Mod}A)}(T^{\bullet}, X^{\bullet}[i]) \cong \mathrm{Hom}_{D(\mathrm{Mod}A)}(B, GX^{\bullet}[i]) = 0 \text{ for all } i \neq 0 \qquad (1)$$

$$\mathrm{Hom}_{D(\mathrm{Mod}A)}(X^{\bullet}, T^{\bullet}[i]) \cong \mathrm{Hom}_{D(\mathrm{Mod}A)}(GX^{\bullet}, B[i]) = 0 \text{ for all } i \neq 0 \qquad (2).$$

Since $X^{\bullet} \in K^{b}(\mathcal{P}_A)$, $GX^{\bullet}$ is isomorphic to an object in $K^{b}(\mathcal{P}_B)$. Then, according to (1), we may assume that $GX^{\bullet}$ is a B-module of finite projective dimension, which has a finitely generated projective resolution. Therefore, according to (2), $GX^{\bullet}$ is a finitely generated projective B-module. Hence $X^{\bullet}$ is isomorphic to a direct summand of a finite direct sum of copies of $T^{\bullet}$.

The next theorem is the tilting complex version of a result of Miyashita [26].

THEOREM 8.3. *Let* Λ *be a Frobenius extension of a ring* A *such that* $0 \rightarrow A \rightarrow \Lambda \rightarrow M \rightarrow 0$ *is an exact sequence as* A*-*A*-bimodules. Let* $T^{\bullet}$ *be a tilting complex for* A *such that* $\mathrm{Hom}_{D(\mathrm{Mod}A)}(T^{\bullet}, T^{\bullet}\otimes^{L}_{A} M[i]) = 0$ *for all* $i \neq 0$, *and* $B = \mathrm{End}_{D(\mathrm{Mod}A)}(T^{\bullet})$, $\Gamma = \mathrm{End}_{D(\mathrm{Mod}\Lambda)}(T^{\bullet}\otimes^{L}_{A}\Lambda)$ *and* $N = \mathrm{Hom}_{D(\mathrm{Mod}A)}(T^{\bullet}, T^{\bullet}\otimes^{L}_{A} M)$.
Then $T^{\bullet}\otimes^{L}_{A}\Lambda$ *is a tilting complex for* Λ, *and* Γ *is a Frobenius extension of a ring* B *such that* $0 \rightarrow B \rightarrow \Gamma \rightarrow N \rightarrow 0$ *is an exact sequence as* B*-*B*-bimodules.*

Proof. By Theorem 8.1, it suffices to show that Γ is a Frobenius extension of B. Since $T^{\bullet}$ is a tilting complex for A and Λ_A is a finitely generated projective right Λ-module, $T^\bullet \otimes_\Lambda \Lambda_\Lambda$ belongs to $K^b(\mathcal{P}_\Lambda)$. Since ${}_\Lambda\Lambda_\Lambda \cong \mathrm{Hom}_\Lambda({}_\Lambda\Lambda_\Lambda, {}_\Lambda\Lambda_\Lambda)$ as Λ-Λ- bimodules, we have the following isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{D(\mathrm{Mod}\Lambda)}(T^\bullet \otimes^L_A \Lambda_\Lambda, T^\bullet[i]) &\cong \mathrm{H}^i\mathrm{Hom}^\bullet_\Lambda(T^\bullet \otimes_\Lambda \Lambda_\Lambda, T^\bullet)\\ &\cong \mathrm{H}^i\mathrm{Hom}^\bullet_\Lambda(T^\bullet, \mathrm{Hom}_\Lambda({}_\Lambda\Lambda_\Lambda, T^\bullet))\\ &\cong \mathrm{H}^i\mathrm{Hom}^\bullet_\Lambda(T^\bullet, T^\bullet \otimes_\Lambda \Lambda_\Lambda)\\ &\cong \mathrm{Hom}_{D(\mathrm{Mod}\Lambda)}(T^\bullet, T^\bullet \otimes^L_A \Lambda_\Lambda[i])\\ &= 0 \text{ for all } i \neq 0.\end{aligned}$$

By Lemma 8.2, $T^\bullet \otimes^L_A \Lambda_\Lambda$ is a direct summand of a finite direct sum of copies of $T^\bullet$. Since ${}_\Gamma\Gamma_B$ is isomorphic to ${}_\Gamma\mathrm{Hom}_{D(\mathrm{Mod}\Lambda)}(T^\bullet, T^\bullet \otimes^L_A \Lambda_\Lambda)_B$ as a Γ-B-bimodule, Γ_B is a finitely generated projective right B-module. And we get the following isomorphisms as B-Γ-bimodules:

$$\begin{aligned}\mathrm{Hom}_B({}_\Gamma\Gamma_B, {}_BB_B) &\cong \mathrm{Hom}_B({}_\Gamma\mathrm{Hom}_{D(\mathrm{Mod}\Lambda)}(T^\bullet, T^\bullet \otimes^L_A \Lambda_\Lambda)_B, {}_B\mathrm{Hom}_{D(\mathrm{Mod}\Lambda)}(T^\bullet, T^\bullet)_B)\\ &\cong \mathrm{Hom}_{D(\mathrm{Mod}\Lambda)}(T^\bullet \otimes^L_A \Lambda_\Lambda, T^\bullet)\\ &\cong \mathrm{Hom}_{D(\mathrm{Mod}\Lambda)}(T^\bullet \otimes^L_A \Lambda_\Lambda, T^\bullet \otimes^L_A \Lambda_\Lambda)\\ &\cong {}_B\Gamma_\Gamma\ .\end{aligned}$$

COROLLARY 8.4. *In the situation of Theorem 3.3,* $\mathrm{End}_{D(\mathrm{Mod}\Lambda)}(T^\bullet \otimes^L_A \Lambda_\Lambda)$ *is a Frobenius extension of* $\mathrm{End}_{D(\mathrm{Mod}\Lambda)}(T^\bullet \otimes^L_A \Lambda_\Lambda)$.

Proof. By theorem of Kasch [17], $\mathrm{End}_B(\mathrm{End}_{D(\mathrm{Mod}\Lambda)}(T^\bullet \otimes^L_A \Lambda_\Lambda)_B)$ is a Frobenius extension of $\mathrm{End}_{D(\mathrm{Mod}\Lambda)}(T^\bullet \otimes^L_A \Lambda_\Lambda)$. And we have

$$\begin{aligned}\mathrm{End}_B(\mathrm{End}_{D(\mathrm{Mod}\Lambda)}(T^\bullet \otimes^L_A \Lambda_\Lambda)_B) &\cong \mathrm{End}_B(\mathrm{Hom}_{D(\mathrm{Mod}\Lambda)}(T^\bullet, T^\bullet \otimes^L_A \Lambda_\Lambda)_B)\\ &\cong \mathrm{End}_{D(\mathrm{Mod}\Lambda)}(T^\bullet \otimes^L_A \Lambda_\Lambda).\end{aligned}$$

The next proposition is useful in exhibiting examples of Frobenius extensions of algebras which satisfy Theorem 8.3.

PROPOSITION 8.5. *Let* Λ *be a finite dimensional Frobenius algebra over a field k such that* $0 \to k \to \Lambda \to M \to 0$ *is an exact sequence in mod k. For a finite dimensional k- algebra A and a tilting complex* $T^\bullet$, *let* $B = \mathrm{End}_{D(\mathrm{Mod}A)}(T^\bullet)$, $\Gamma = \mathrm{End}_{D(\mathrm{Mod}A)}(T^\bullet \otimes^L_A (A \otimes_k \Lambda))$ *and* $\mathrm{N} = \mathrm{Hom}_{D(\mathrm{Mod}B)}(T^\bullet, T^\bullet \otimes^L_A (A \otimes_k M))$. *Then* $A \otimes_k \Lambda$ *is a Frobenius extension of A which satisfy theorem 3.3 with an exact sequence* $0 \to A \to A \otimes_k \Lambda \to A \otimes_k M \to 0$, *and* $B \otimes_k \Lambda$ *is a Frobenius extension of B such that an exact sequence* $0 \to B \to B \otimes_k \Lambda \to B \otimes_k M \to 0$ *is isomorphic to* $0 \to B \to \Gamma \to N \to 0$ *as a B-B-bimodule.*

Proof. It is well known that $A \otimes_k \Lambda$ is a Frobenius extension of A (see [29] for an example). Since $A \otimes_k M$ is a finite direct sum of copies of A as an A-A-module, $T^\bullet \otimes_A (A \otimes_k \Lambda)$ satisfies the condition of Theorem 8.3. We have the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 \to & \mathrm{Hom}^\bullet_A(T^\bullet, T^\bullet) & \to & \mathrm{Hom}^\bullet_A(T^\bullet, T^\bullet) \otimes_k \Lambda & \to & \mathrm{Hom}^\bullet_A(T^\bullet, T^\bullet) \otimes_k M & \to 0 \\
& \| & & \uparrow & & & \\
& \mathrm{Hom}^\bullet_A(T^\bullet, T^\bullet) & \to & \mathrm{Hom}^\bullet_{(A\otimes_k\Lambda)}(T^\bullet \otimes_k \Lambda, T^\bullet \otimes_k \Lambda) & & & \\
& \| & & \uparrow & & & \\
& \mathrm{Hom}^\bullet_A(T^\bullet, T^\bullet) & \to & \mathrm{Hom}^\bullet_{(A\otimes_k\Lambda)}(T^\bullet \otimes_A (A \otimes_k \Lambda), T^\bullet \otimes_A (A \otimes_k \Lambda)) & & & \\
& \| & & \downarrow & & & \\
0 \to & \mathrm{Hom}^\bullet_A(T^\bullet, T^\bullet) & \to & \mathrm{Hom}^\bullet_A(T^\bullet, T^\bullet \otimes_A (A \otimes_k \Lambda)_A) & \to & \mathrm{Hom}^\bullet_A(T^\bullet, T^\bullet \otimes_A (A \otimes_k M)_A) & \to 0,
\end{array}
$$

where the top and the bottom rows are exact sequences, and vertical arrows are isomorphisms. Then we have the following commutative diagram:

$$0 \to \mathrm{End}_{D(\mathrm{Mod}A)}(T^{\bullet}) \to \mathrm{End}_{D(\mathrm{Mod}A)}(T^{\bullet})\otimes_k \Lambda \to \mathrm{End}_{D(\mathrm{Mod}A)}(T^{\bullet})\otimes_k M \to 0$$

$$\| \qquad\qquad \wr\|$$

$$\mathrm{End}_{D(\mathrm{Mod}A)}(T^{\bullet}) \to \mathrm{End}_{D(\mathrm{Mod}A)}(T^{\bullet}\otimes^{L}_{A}(A\otimes_k \Lambda))$$

$$\| \qquad\qquad \wr\|$$

$$0 \to \mathrm{End}_{D(\mathrm{Mod}A)}(T^{\bullet}) \to \mathrm{Hom}_{D(\mathrm{Mod}A)}(T^{\bullet}, T^{\bullet}\otimes^{L}_{A}(A\otimes_k \Lambda)_{A}) \to \mathrm{Hom}_{D(\mathrm{Mod}A)}(T^{\bullet}, T^{\bullet}\otimes^{L}_{A}(A\otimes_k M)) \to 0,$$

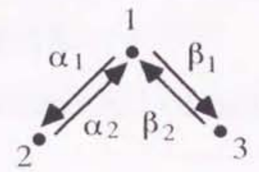
where the top and the bottom rows are exact sequences as B-B-bimodules, vertical arrows are isomorphisms as B-B-bimodules and arrows between the top row and the second row are isomorphisms as rings.

Remark. For a subring A of Λ, Λ is called a quasi-Frobenius extension of A provided that Λ_A is a finitely generated projective right A-module, and that ${}_A\Lambda_\Lambda$ is a direct summand of a finite direct sum of copies of $\mathrm{Hom}_A({}_\Lambda\Lambda_A, {}_AA_A)$ as A-Λ- bimodules and $\mathrm{Hom}_A({}_\Lambda\Lambda_A, {}_AA_A)$ is a direct summand of a finite direct sum of copies of ${}_A\Lambda_\Lambda$ as A-Λ- bimodules [28]. Then "a Frobenius extension" in Theorem 8.3 can be replaced by "a quasi-Frobenius extension".

Examples. (a) $k\,[X]\,/(X^n)$ and kG satisfy the condition of Proposition 8.5, where G is a finite group and k is a field.

(b) Let A be a finite dimensional k-algebra which has the quiver: 1 → 3 ← 2, and $\sigma : A \to A$ a k-algebra automorphism induced by interchanging vertex 1 with vertex 2. For a group $G = \{1, \sigma\}$, we define a strongly G- graded k-algebra $\Lambda = \oplus_{g\in G} A_g$ such that A_g has a natural left action of A and a right action of A which is through g (*i.e.* a crossed product of A with G which has a trivial factor set). Let T = $P(1)\oplus P(2)\oplus I(3)$, where $P(i)$ (resp., $I(i)$) is a projective (resp., injective) indecomposable right A-module corresponding to vertex i. Then Λ is a Frobenius extension of A, T satisfies Theorem 8.3.

(c) According to [20], we have the following example. Given positive integer n, let A be a finite dimensional algebra over a field k which has the following quiver with relations:

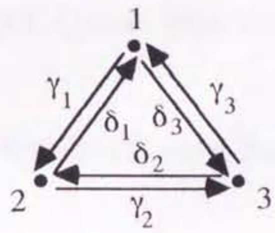


,

with $\alpha_2\alpha_1 = \beta_2\beta_1 = 0$ and $(\alpha_1\alpha_2\beta_1\beta_2)^n = (\beta_1\beta_2\alpha_1\alpha_2)^n$. Let $\sigma : A \to A$ be a k-algebra automorphism induced by interchanging vertex 2 with vertex 3. Let $\Lambda = \oplus_{g\in G} A_g$, where $G = \{1, \sigma\}$, and let $T^\bullet$ be the following complex:

$$P(2)^{(2)} \oplus P(3)^{(2)} \xrightarrow{M} P(1),$$

where $M = (0\ \alpha_2\ \beta_2\ 0)$. Then Λ is a Frobenius extension of A, and $T^\bullet$ satisfies Theorem 8.3. Then $B = \mathrm{End}_{K(\mathrm{Mod}\Lambda)}(T^\bullet)^{\mathrm{op}}$ is a finite dimensional algebra over a field k which has the following quiver with relations:



,

with $\gamma_1\gamma_2 = \gamma_2\gamma_3 = \gamma_3\gamma_1 = \delta_1\delta_3 = \delta_3\delta_2 = \delta_2\delta_1 = 0$, $\gamma_1\delta_1 = \delta_3\gamma_3$, $\delta_1\gamma_1 = (\gamma_2\delta_2)^n$ and $\gamma_3\delta_3 = (\delta_2\gamma_2)^n$. Let $\sigma: B \to B$ be a k-algebra automorphism induced by interchanging vertex 2 with vertex 3. Then Γ is ring-isomorphic to $\oplus_{g\in G} B_g$, where $G = \{1, \sigma\}$.

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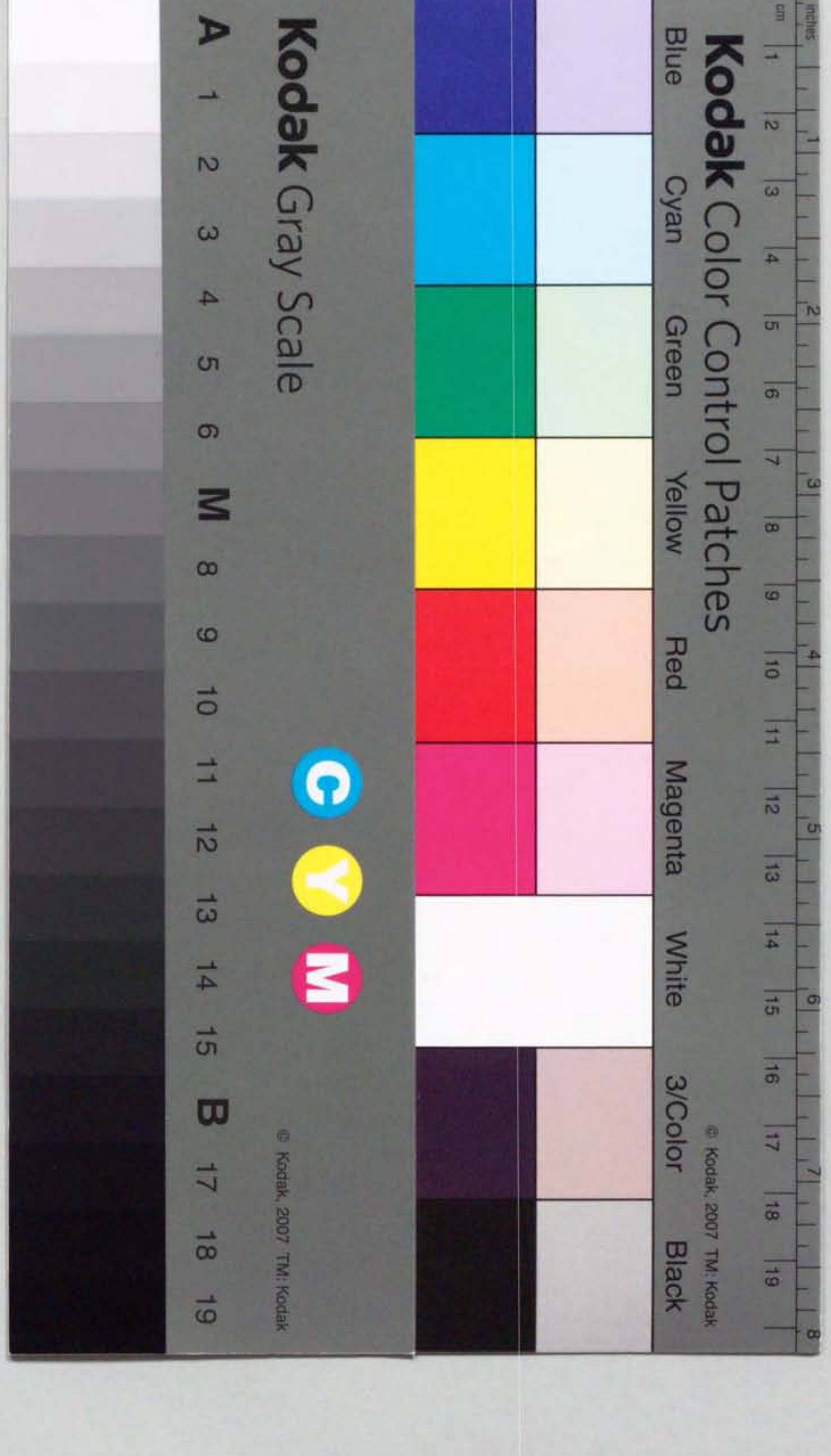
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