STRUCTURE OF MENGER MANIFOLDS AND TRANSFORMATION GROUPS

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THESIS

Submitted in partial fulfillment of

the requirements for the degree of

Doctor of Philosophy (Science)

in Doctoral Program in Mathematics

at the

UNIVERSITY OF TSUKUBA

JANUARY

1996

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to Professors Yukihiro Kodama, Ryosuke Nakagawa and Katsuro Sakai, my supervisors, for their kindness and constant encouragement and especially for thier valuable advice on mathematics as well as on doing research. I am indebted to Professor Katsuro Sakai for preparing this dissertation.

I am very grateful to Professors Takao Hoshina and Hisao Kato for their encouragement and to Professor Kazuhiro Kawamura for his comments, including some references which are unknown to me.

Tsukuba, January, 1996. YUTAKA IWAMOTO

CONTENTS

Introduction

The Menger compacta were introduced by K. Menger [Me]. They were generalizations of the Cantor set and of the Sierpiński's universal curve [Si]. The n-dimensional Menger compactum μ^n is known as a universal space for at most n-dimensional compacta and is very impotant in Dimension Theory. The 1-dimensional Menger compactum is called the universal curve and is characterized topologically by R. D. Anderson [An1]. In 1984, M. Bestvina [Be] established the topological characterizations for Menger compacta in all dimensions, which give a new point of view that the n-dimensional Menger compactum is the n-dimensional (finite-dimensional) analogue of the Hilbert cube. He also build up the foundation of the Menger manifold theory parallel to the theory of Hilbert cube manifolds. After his work, A. N. Dranishnikov, A. Chigogidze and the others have been established many important theorems of Menger manifolds ([Dr2], [Ch1, 2, 3, 4, 5, 6], [GHW], etc). For the history and related topics of Menger manifolds, see [CKT2].

In this paper, we first study some stable Menger manifolds (μ^n_∞-manifolds) and give their topological characterization. Besides, we consider some mapping properties and homeomorphism groups of Menger manifolds. Then we reconstruct Menger manifolds to consider their product structure. Finally, we study group actions on Menger manifolds.

In Chapter I, we give terminology and notations, and present some basic properties of Menger manifolds which will be needed in the sequel.

In Chapter II, we define μ_∞^{n+1}-manifolds which is a class of non-compact Menger manifolds and give a characterization theorem for μ_∞^{n+1}-manifolds. A. Chigogidze [Ch3] introduced the notion of the n-homotopy kernel of μ^{n+1}-manifolds and established the stability of n-homotopy kernels. Our characterization theorem implies that the class of μ_∞^{n+1}-manifolds coinside with the class of μ^{n+1}-manifolds which are homeomorphic to their n-homotopy kernels.

Brown and Cassler [Br] proved that each compact connected n-manifolds can be obtained from the n-cube by making identifications on the boundary. This was generalizes by Berlanga [Ber] to non-compact connected n-manifolds. In Chapter III, we give a mapping theorem of Brown-Cassler type for μ^{n+1}-manifolds. Roughly speaking, it is shown that each compact connected μ^{n+1}-manifolds can be obtained by makind identifications on some thin set.

It was proved by R. D. Anderson [An4] that the homeomorphism group of the 1-dimensional Menger compactum is algebraically simple. Chapter IV is devoted to extend this result to all dimensions.

In Chapter V, we introduce the infinite coordinate systems for μ^n-manifolds, called μ^n-coordinate systems. Using μ^n-coordinate systems, we characterize Z-sets in terms of infinite deficiency. Then we discuss how to define a kind of the Cartesian product in the category of μ^n-manifolds. It should be noted that the Cartesian product of μ^n-manifolds (e.g. $\mu^n \times \mu^n$) is neither a μ^n-manifold nor a μ^{2n}-manifold. However, μ^n-coordinate systems allow us to define the Δ_n-product which plays the role of the Cartesian product in the category of μ^n-manifolds.

In Chapter VI, we consider group actions on Menger manifolds and their fixed piont sets. The main purpose of this chapter is to show that each μ^n-

manifold M has the complete invariance property with respect to homeomorphisms, that is, if each non-empty closed subset of M is the fixed point set of some autohomeomorphism of M. This gives the affirmative answers to the questions [S3] and [CKT2, Problems 6.4.3, 6.4.4] in full generality. More generally, we can prove that any closed set of a Menger manifold can be the fixed point set of some semi-free G-action, where G is a compacr zero-dimensional topological abelian group. Using the notion of infinite deficiency which will be introduced in Chapter V, the theorem above can be generalized to the pseudo-interiors and pseudo-boundaries of Menger manifolds. Moreover, it is shown that every μ^n-manifold admits a free G-action with a G-invariant pseudo-interior and a G-invariant pseudo-boundary.

I. Preliminaries

The purpose of this chapter is to introduce basic notations and terminologies and to present some basic properties of Menger manifolds and absorbers which will be needed in the sequel.

§1.1. General definitions

All spaces considered in this dissertation are assumed to be separable metrizable and all maps are assumed to be continuous. By the letter d, we denote the metric of any spaces under consideration unless otherwise stated.

For a subset A of a space X, $\mathrm{Cl}_X A$, $\mathrm{Int}_X A$ and $\mathrm{Fr}_X A$ denote the topological closure, interior and boundary of A in X respectively.

Let $\mathcal{U}$ be a cover of a space X consisting of subsets of X and let A be a subset of X. The *star of A with respect to $\mathcal{U}$* is the set

$$\mathrm{St}(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} | A \cap U \neq \emptyset\}.$$

Let X be a space, A a subspace of X and let $0 \leq n < \infty$. We say that X is *connected in dimension n*, abbreviated C^n, provided that for every $0 \leq m \leq n$ every map $f : \mathbb{S}^m \to X$ extends to a map $\hat{f} : \mathbb{B}^{m+1} \to X$. Also, we say that X is *locally connected in dimension n*, abbreviated LC^n, provided that for every $x \in X$, for every neighborhood U of x and for every $0 \leq m \leq n$ and for every neighborhood U of x in X, there exists a neighborhood V of x in X such that every map $f : \mathbb{S}^m \to V$ extends to a map $\hat{f} : \mathbb{B}^{m+1} \to U$. We say that A is a *retract* of X provided that there is a map $r : X \to A$ such that $r|_A = \mathrm{id}_A$. We say that A is a *neighborhood retract* of X provided that there exists a neighborhood U of A in X such that A is a retract of U. A space X is called a *absolute neighborhood retract*, abbreviated ANR provided that X is a neighborhood retract of evvery space Y containing X as a closed subspace.

A compactum X is called a UV^n-compactum provided that there is an embedding of X into the Hilbert cube Q such that every neighborhood U of X in Q has a smaller neighborhood V of X in Q with the following properties: each map $f : \partial\mathbb{B}^i \to V$ can be extended to a map $\widehat{f} : \mathbb{B}^i \to U$ for $i = 0, 1, \ldots, n$. A map is called a *UV^n-map* if each fibre is a UV^n-compactum.

A map $f : X \to Y$ is called an *n-soft* if for every at most n-dimensional space A, every closed subspace B of A and every maps $\alpha : A \to Y$ and $\beta : B \to X$ with $\alpha|B = f\beta$, there exists a map $\gamma : A \to X$ such that $f\gamma = \alpha$ and $\gamma|B = \beta$. In casde $B = \emptyset$, then the map f is called an *n-invertible.*

Let $\{X_i,\, p_i^{i+1}\}_{i=0}^{\infty}$ be an inverse sequence and let $X = \varprojlim\{X_i,\, p_i^{i+1}\}_{i=0}^{\infty}$ be the inverse limit. We denote the projection onto the ith coordinate X_i by $p_i : X \to X_i$, and denote $p_i^j : X_j \to X_i$, $j > i$ for the map induced by the bonding maps. We assume that X_i is metrized by a metric d_i with $\mathrm{diam}(X_i) < 2^{-i}$ and endow the product space $\prod_{i=0}^{\infty} X_i$ with the metric $\mathrm{d}(x,\, y) = \sum_{i=0}^{\infty} \mathrm{d}_i(x_i,\, y_i)$. For each $n \geq 0$, we consider $\prod_{i=0}^{n} X_i$ as a subspace of $\prod_{i=0}^{\infty} X_i$.

Let K and L be simplicial complexes. The barycentric subdivision of K is denoted by $\beta(K)$. The n-skeleton of a simplicial complex K is denoted by $K^{(n)}$. By $K \times L$, we mean the simplicial complex obtained as the barycentric subdivision of the cell complex $\{\sigma \times \tau \mid \sigma \in K,\, \tau \in L\}$. We remark that for a vertex v of L, $K \times \{v\} \equiv \beta(K)$.

§1.2. Basic properties of Menger manifolds

In this section, we present some basic properties of Menger manifolds which will be needed in the sequel.

Let $\mathbb{I}^k$ be the k-cell in $\mathbb{R}^k$ and let $\mathcal{K}_0 = \{\mathbb{I}^k\}$. For $i = 0, 1, 2, \ldots$, let $\mathcal{K}_i$ be the cell complex whose k-cells are cubes obtained by dividing the k-cube $\mathbb{I}^k$ by all linear $(k-1)$-varieties in $\mathbb{R}^k$ determined by equations of the form $x_j = l/3^j$, $j = 1, 2, 3, \ldots$, $0 \leq l \leq 3^j$.

Let $M_0 = |\mathcal{K}_0|$ and $0 \leq n \leq k$. We inductively define M_i, $i \geq 1$ as follows:

$$M_i = \mathrm{St}(|\mathcal{K}_{i-1}^n|, \mathcal{K}_i) \cap M_{i-1}.$$

Then the intersection $M_n^k = \cap\{M_i\}_{i=0}^{\infty}$ is called the *Menger compactum of type* (k, n). The *n-dimensional universal Menger compactum* μ^n is the Menger compactum of type $(2n+1, n)$, that is, $\mu^n = M_n^{2n+1}$. There are many constructions of the universal Menger compactum. For example, Lefschetz's construction, Pasynkov's construction, Bestvina's construction, etc., see [Be], [Dr3] and [CKT2].

An *n-dimensional Menger manifold* (μ^n-manifold) is a topological manifold modeled on the n-dimensional universal Menger compactum μ^n.

A space X satisfies the *disjoint k-disks property* (DD^kP, for short), if for each open cover $\mathcal{U}$ of X and each pair of maps f_1, $f_2 : \mathbb{B}^k \to X$, there are maps g_1, $g_2 : \mathbb{B}^k \to X$ with disjoint images such that f_i and g_i are $\mathcal{U}$-close, $i = 1, 2$.

The following is the Bestvina's characterization theorem for μ^n-manifolds [Be].

Theorem 1.2.1 (Characterization). *An n-dimensional space (respectively, compactum) X is a μ^n-manifold (respectively, homeomorphic to μ^n) if and* *only if X satisfies the following conditions:*

(i) *X is locally compact (respectively, compact),*

(ii) *X is LC^{n-1} (respectively, $LC^{n-1} \cap C^{n-1}$) and*

(iii) *X satisfies DD^nP.*

We say two (proper) maps $f, g : X \to Y$ are (*properly*) *n-homotopic* (notation: $f \simeq^n g$, $f \simeq^n_p g$, respectively) if, for any (proper) map $\alpha : Z \to X$ from a space Z with $\dim Z \leq n$ into X, the compositions $f\alpha$ and $g\alpha$ are (properly) homotopic in the usual sense. The notion of n-homotopy equivalence is defined in obvious way.

Proposition 1.2.1 [Hu]. *Let $f : X \to Y$ be a map, where $\dim X \leq n$ and Y is LC^n. Then for any open cover $\mathcal{U}$ of Y, there are maps $\varphi : X \to P$ and $\psi : P \to Y$ such that f and $\psi\varphi$ are $\mathcal{U}$-homotopic, where P is a locally finite polyhedron with $\dim P \leq n$. In particular, we can choose ψ as a proper map.*

Let us recall that a map $f : X \to Y$ is said to be *n-invertible* if for any space Z with $\dim Z \leq n$ and any map $\alpha : Z \to Y$, there exists a map $\beta : Z \to X$ such that $f\beta = \alpha$.

Proposition 1.2.2 [Ch2]. *Every μ^n-manifold admits a proper n-invertible UV^{n-1}-surjection onto a Q-manifold.*

Proposition 1.2.3 [Ch3]. *Two μ^n-manifolds admitting proper UV^{n-1}-surjections onto the same LC^{n-1}-space are homeomorphic.*

A closed subset A of X is called a *Z-set* in X provided that for every open cover $\mathcal{U}$ of X there is a map $f : X \to X \setminus A$ such that f and id_X are $\mathcal{U}$-close. For a locally compact LC^{n-1}-space X with $\dim X \leq n$, this definition is equivalent to the following: for any map $f : \mathbb{I}^n \to X$ and any $\varepsilon > 0$, there is a map $g : \mathbb{I}^n \to X \setminus A$ which is ε-close to f (cf. [Be, Proposition 2.3.6]).

The following theorem is due to Bestvina [Be], where it is stated in terms of μ-homotopy. However, as is known [Ch1], the notion of μ-homotopy coincides with one of n-homotopy for maps between locally compact LC^n-spaces of dimension at most $n+1$.

Theorem 1.2.2 (Z-set unknotting theorem). *Let M be a μ^{n+1}-manifold and $f : A \to B$ be a homeomorphism between Z-sets in M. If $f \simeq^n_p \mathrm{id}_A$ in M, then f extends to a homeomorphism $h : M \to M$.*

An *n-homotopy kernel* of a μ^{n+1}-manifold M is defined to be the complement $M \setminus f(M)$ of the image of an arbitrary Z-embedding $f : M \to M$ with $f \simeq^n_p \mathrm{id}_M$. Using the Z-set unknotting theorem, two n-homotopy kernels are homeomorphic by an ambient homeomorphism of M onto itself. By $\mathrm{Ker}(M)$, we denote a representative of n-homotopy kernels of M.

Let us recall that a map $f : X \to Y$ is said to be *n-invertible* if for any space Z with $\dim Z \leq n$ and any map $\alpha : Z \to Y$, there exists a map $\beta : Z \to X$ such that $f\beta = \alpha$.

The following proposition is actually proved in [Ch3].

Proposition 1.2.4. *For each μ^{n+1}-manifold M there exists a proper $(n+1)$-invertible UV^n-surjection $f : M \to M \times [0,1]$ such that $f^{-1}(M \times [0,1)) =$ $\mathrm{Ker}(M)$.*

Theorem 1.2.3 [Dr2]. *There exists an $(n+1)$-invertible UV^n-surjection $f_n :$ $\mu^{n+1} \to Q$ satisfying the following condition:*

(∗) *$f_n^{-1}(X)$ is a μ^{n+1}-manifold for any locally compact LC^n-space $X \subset Q$.*

Theorem 1.2.4 [Ch4]. *For each locally finite polyhedron K, there exists a proper $(n+1)$-invertible UV^n-surjection $f_K : M_K \to K$ from a μ^{n+1}-manifold M_K onto K satisfying the following conditions:*

(*a*) *$f_K^{-1}(L)$ is a μ^{n+1}-manifold for any closed subpolyhedron L of K;*

(b) $f_K^{-1}(Z)$ *is a Z-set in* $f_K^{-1}(L)$ *for any Z-set* Z *in a closed subpolyhedron* L *of* K.

Let $f : X \to Y$ be a proper map. We say that f *induces an epimorphism of* j^{th} *homotopy groups of ends* if for every compactum $C \subset Y$ there exists a compactum $K \subset Y$ such that for each point $x \in X \setminus f^{-1}(K)$ and every map $\alpha : (\mathbb{S}^j, *) \to (Y \setminus K, f(x))$ there exist a map $\tilde{\alpha} : (\mathbb{S}^j, *) \to (X \setminus f^{-1}(C), x)$ and a homotopy $f\tilde{\alpha} \simeq \alpha$ rel $.*$ in $Y \setminus C$. Also we say that f *induces a monomorphism of* j^{th} *homotopy groups of ends* if for every compactum $C \subset Y$ there exists a compactum $K \subset Y$ such that for every map $\tilde{\alpha} : \mathbb{S}^j \to X \setminus f^{-1}(K)$ with $f\tilde{\alpha} \simeq *$ in $Y \setminus K$ it follows that $\tilde{\alpha} \simeq *$ in $X \setminus f^{-1}(C)$. It is said that f *induces an isomorphism of* j^{th} *homotopy groups of ends* if f induces both epimorphism and monomorphism of j^{th} homotopy groups of ends.

Theorem 1.2.5 (Classification)[Be]. *Let* $f : M \to N$ *be a proper map between* μ^n*-manifolds. If* f *induces isomorphisms of homotopy groups of dimension* $\leq n-1$ *and of ends of dimension* $\leq n-1$, *then* f *is properly* $(n-1)$*-homotopic to a homeomorphism.*

§1.3. Absorbers

This section is devoted to present some basic facts concerning absorbers of Polish spaces. The facts stated in this section will be used only in chapter IV. Let X be a Polish space, i.e., a complete separable metric space. By $\mathrm{Auth}(X)$, we denote the space of all homeomorphisms of X endowed with the limitation topology. Let Γ be a closed subgroup of $\mathrm{Auth}(X)$ and $\mathcal{K}$ be a closed hereditary, additive and Γ-invariant collection consisting of closed subsets of X, i.e., if D is a closed subset of some member of $\mathcal{K}$ then $D \in \mathcal{K}$ and if $A, B \in \mathcal{K}$ and $f \in \Gamma$ then $f(A \cup B) \in \mathcal{K}$. Let $\{A_i\}_{i=1}^{\infty}$ be a tower of members of $\mathcal{K}$. Then $\{A_i\}_{i=1}^{\infty}$ is called a Γ-$\mathcal{K}$*-skeleton* provided that for each open cover $\mathcal{U}$ of X, for each

$A, B \in \mathcal{K}$, for each $f \in \Gamma$ with $f(B) \subset A_i$, there is $h \in \Gamma$ which is $\mathcal{U}$-close to f such that $h|B = f|B$ and $h(A) \subset A_j$ for some $j \geq i$. The union $\cup_{i=1}^{\infty} A_i$ is called a Γ-$\mathcal{K}$-*skeletoid* if the collection $\{A_i\}_{i=1}^{\infty}$ is a Γ-$\mathcal{K}$-skeleton. A subset A of X is called a Γ-$\mathcal{K}$-*absorber* if there is a family $\{K_i\}_{i=1}^{\infty}$ of members of $\mathcal{K}$ with $A = \cup_{i=1}^{\infty} K_i$ such that for each open collection $\mathcal{U}$ of X, for each $B \in \mathcal{K}$, there is $h \in \Gamma$ such that $h|X \setminus \cup\mathcal{U} = \mathrm{id}$, $h|\cup\mathcal{U}$ is $\mathcal{U}$-close to id and $h(B \cap (\cup\mathcal{U})) \subset A$. In case $\Gamma = \mathrm{Auth}(X)$, Γ-$\mathcal{K}$-skeletons, Γ-$\mathcal{K}$-skeletoids and Γ-$\mathcal{K}$-absorbers are called $\mathcal{K}$-skeletons, $\mathcal{K}$-skeletoids and $\mathcal{K}$-absorbers respectively.

A subset $A \subset X$ is called a *thin* set if for each open cover $\mathcal{U}$ of X and for each open set $V \supset A$, there is $f \in \mathrm{Auth}(X)$ which is $\mathcal{U}$-close to id such that $f|X \setminus V = \mathrm{id}$ and $f(A) \cap A = \emptyset$. A closed hereditary, additive and Auth(X)-invariant collection $\mathcal{K}$ of a Polish space X is called a *perfect collection* provided that

(1) each member of $\mathcal{K}$ is a compact thin set in X,

(2) for each $A \in \mathcal{K}$, for each neighborhood V of A and each open cover $\mathcal{U}$ of X, there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\forall B \in \mathcal{K}$ with $B \subset V$, $\forall$ homeomorphism $f : A \to B$ which is $\mathcal{V}$-close to id, $\exists F \in \mathrm{Auth}(X)$ which is $\mathcal{U}$-close to id with $F|A = f$ and $F|X \setminus V = \mathrm{id}$.

For each open subset U of X, we put $\mathcal{K}(U) = \{K \in \mathcal{K} \mid K \subset U\}$, $\mathrm{Auth}(X||X \setminus U) = \{F \in \mathrm{Auth}(X) \mid F|X \setminus U = \mathrm{id}\}$.

Theorem 1.3.1 [BP, Chap. IV, Theorem 4.1]. *Let $\mathcal{K}$ be a perfect collection, A a $\mathcal{K}$-skeletoid in X and U an open subset of X. Then $A \cap U$ is an Auth($X||X \setminus U$)-$\mathcal{K}(U)$-skeletoid in X.*

Hence any $\mathcal{K}$-skeletoid is a $\mathcal{K}$-absorber if $\mathcal{K}$ is a perfect collection. The uniqueness of Γ-$\mathcal{K}$-absorbers follows from the next theorem.

Theorem 1.3.2 [We]. *Let A and B be two Γ-$\mathcal{K}$-absorbers in X. Then for* *each open collection $\mathcal{U}$ of X, there is $f \in \Gamma$ which is $\mathcal{U}$-close to id such that $f(A \cap (\cup\mathcal{U})) = B \cap (\cup\mathcal{U})$ and $f|X \setminus (\cup\mathcal{U}) = id$.*

Proposition 1.3.1 [BP, Chap. IV, Proposition 4.1]. *Let $\mathcal{K}$ be a perfect collection in a space X and let $\{A_i\}_{i=1}^{\infty}$ be a collection of members of $\mathcal{K}$ such that $A_i \subset A_{i+1}$ for each $i \in \mathbb{N}$. Then $\{A_i\}_{i=1}^{\infty}$ is a $\mathcal{K}$-skeleton if and only if, for every $Z \in \mathcal{K}$, $k \in \mathbb{N}$, $\varepsilon > 0$, there exists an embedding $f : Z \to X$ such that $d(f, id) < \varepsilon$, $f|Z \cap A_k = id_{Z \cap A_k}$ and $f(Z) \subset A_j$ for some $j \geq k$.*

Theorem 1.3.3 [Ch4]. *For each locally finite polyhedron K, there exists a proper n-invertible UV^{n-1}-surjection $f_K : M_K \to K$ from a μ^n-manifold M_K onto K satisfying the following conditions:*

(a) $f_K^{-1}(L)$ is a μ^n-manifold for any closed subpolyhedron L of K;

(b) $f_K^{-1}(Z)$ is a Z-set in $f_K^{-1}(L)$ for any Z-set Z in a closed subpolyhedron L of K.

II. Stable Menger Manifolds

In [Ch3], Chigogidze introduced the notion of the n-homotopy kernel of a μ^{n+1}-manifold and proved the following classification theorem for μ^{n+1}-manifolds: Two μ^{n+1}-manifolds have the same n-homotopy type if and only if their n-homotopy kernels are homeomorphic. There are close relations between Hilbert cube manifold (Q-manifold) theory and Menger manifold theory and the n-homotopy kernel of a μ^{n+1}-manifold plays the role of the product $X \times [0,1)$ of a Q-manifold X with $[0,1)$. It is said that X is $[0,1)$-*stable* if it is homeomorphic to ($\cong$) $X \times [0,1)$.

Wong [Wo] showed that a Q-manifold X is $[0,1)$-stable if and only if X is properly contractible to ∞, that is, for any compactum K in X there is a proper map $j_K : X \to X \setminus K$ which is properly homotopic to id_X. Replacing a proper homotopy with a proper n-homotopy, we have the notion of *properly* n*-contractible to* ∞. Moreover we say that X is *properly locally* (n-)*contractible at* ∞ if for any compactum $K \subset X$ there is a compactum $L \subset X$ with $K \subset L$ such that for each compactum $L' \subset X$ with $L \subset L'$ there exists a proper map $j_{L'} : X \setminus L \to X \setminus L'$ which is properly (n-)homotopic to $\mathrm{id}_{X \setminus L}$ in $X \setminus K$. In this chapter we define μ^{n+1}_∞*-manifolds* as μ^{n+1}-manifolds which are properly

n-contractible to ∞ and properly locally n-contractible at ∞ and show the following characterization theorem for μ_∞^{n+1}-manifolds (Theorem 2.1.1).

Theorem I. *Let M be a μ^{n+1}-manifold. Then M is a μ_∞^{n+1}-manifold if and only if M is homeomorphic to its n-homotopy kernel* $\mathrm{Ker}(M)$.

We will show that two n-homotopic proper maps into a μ_∞^{n+1}-manifold are properly n-homotopic (see Lemma 2.2.1). Thus we can remove the requirement of an n-homotopy between μ_∞^{n+1}-manifold to be proper, whence we obtain the following Z-set unknotting theorem for μ_∞^{n+1}-manifolds. Then we can obtain the following (Theorem 2.2.1).

Theorem II. *Each homeomorphism between two Z-sets in a μ_∞^{n+1}-manifold M extends to an ambient homeomorphism of M onto itself if it is n-homotopic to* id *in M.*

From Theorem 2.2 in [Ch3], it follows that two μ_∞^{n+1}-manifolds of the same n-homotopy type are homeomorphic. Similarly to [C1, Theorem 5], we can clarify the relation between n-homotopy equivalences and homeomorphisms (Theorem 2.2.2), that is,

Theorem III. *An n-homotopy equivalence between two μ_∞^{n+1}-manifolds is n-homotopic to a homeomorphism.*

Moreover as same as $[0,1)$-stable Q-manifolds [C1, Lemma 3.6], we can strengthen the open embedding theorem (Theorem 2.2.3), see [Ch2,3].

Theorem IV. *Each map from a μ_∞^{n+1}-manifold into a μ^{n+1}-manifold is n-homotopic to an open embedding.*

§2.1. Characterization of μ_∞^{n+1}-manifolds

A space X is said to be *properly (n-)contractible to ∞* if for any compactum K in X there exists a proper map $j_K : X \to X \setminus K$ which is properly (n-

)homotopic to id_X. If for any compactum $K \subset X$ there exists a compactum $L \subset X$ with $K \subset L$ such that for each compactum $L' \subset X$ with $L \subset L'$ there exists a proper map $j_{L'} : X \setminus L \to X \setminus L'$ which is properly (n-)homotopic to $\mathrm{id}_{X \setminus L}$ in $X \setminus K$ then a space X is said to be *properly locally (n-)contractible at* ∞. It is easy to see that for any space X, $X \times [0,1)$ is properly contractible to ∞ and properly locally contractible at ∞.

Lemma 2.1.1. *Let X be properly n-contractible to ∞ and properly locally n-contractible at ∞. Then for each compact cover $\{X_i\}_{i\in\omega}$ of X with $X_i \subset \mathrm{int}X_{i+1}$, there exist a subcover $\{X_{i_k} | k \in \omega,\ 0 = i_0 < i_1 < i_2 < \cdots\}$ and a collection of proper maps $\{f_k : X \to X \setminus X_{i_k}\}_{k\in\omega}$ such that $f_0 = \mathrm{id}_X$ and $f_{k-1} \simeq_p^n f_k$ in $X \setminus X_{i_{k-2}}$ for $k \geq 1$, where $X_{i_{-1}} = \emptyset$.*

Proof. For technical reasons we assume that $X_0 = \emptyset$. Let $L_{-2} = L_{-1} = L_0 = \emptyset$. We shall inductively choose integers $0 = i_{-2} = i_{-1} = i_0 < i_1 < i_2 < \cdots$ and construct compacta $L_{k-1} \subset X_{i_k} \subset L_k$ and proper maps $j_k : X \setminus L_{k-2} \to X \setminus X_{i_k}$, $k \in \omega$, satisfying the following conditions:

(1) $j_0 = \mathrm{id}_X$.

(2) For each compactum $M \supset L_k$ there is a proper map $j_M : X \setminus L_k \to X \setminus M$ such that $j_M \simeq_p^n \mathrm{id}_{X\setminus L_k}$ in $X \setminus X_{i_{k-2}}$.

(3) $j_k \simeq_p^n \mathrm{id}_{X\setminus L_{k-2}}$ in $X \setminus X_{i_{k-2}}$.

Let $i_1 = 1$. Being X properly n-contractible to ∞ and properly locally n-contractible at ∞, there exist a proper map $j_1 : X \to X \setminus X_{i_1}$ with $j_1 \simeq_p^n \mathrm{id}$ and a compactum $L_1 \supset X_1$ satisfying (2). Since $X = \cup_{i\in\omega} X_i$ and $X_i \subset \mathrm{int}X_{i+1}$ there exists $i_2 > i_1$ such that $X_{i_2} \supset L_1$. As in the above arguments there exist a proper map $j_2 : X \to X \setminus X_{i_2}$ with $j_2 \simeq_p^n \mathrm{id}_X$ and a compactum $L_2 \supset X_{i_2}$ satisfying (2).

Assume that, for $k \geq 2$, $i_0 < i_1 < \cdots < i_k$, L_k, and $j_k : X \setminus L_{k-2} \to$

$X \setminus X_{i_k}$ have been constructed. Choose $i_{k+1} > i_k$ so that $X_{i_{k+1}} \supset L_k$. Since $X_{i_{k+1}} \supset L_{k-1}$, by the property (2) of L_{k-1}, there exists a proper map $j_{X_{i_{k+1}}}$: $X \setminus L_{k-1} \to X \setminus X_{i_{k+1}}$ such that $j_{X_{i_{k+1}}} \simeq_p^n \mathrm{id}_{X \setminus L_{k-1}}$ in $X \setminus X_{i_{k-1}}$. Then put $j_{k+1} = j_{X_{i_{k+1}}}$. Since X is properly locally n-contractible at ∞, there exists a compactum $L_{k+1} \supset X_{i_{k+1}}$ satisfying (2).

Now define $f_k = j_k \cdots j_0 : X \to X \setminus X_{i_k}$ for $k \in \omega$ and observe that the collections of compacta $\{X_{i_k}\}_{k \in \omega}$ and maps $\{f_k\}_{k \in \omega}$ are as desired. □

A μ^{n+1}-manifold is called a μ_∞^{n+1}-*manifold* if it is properly n-contractible to ∞ and properly locally n-contractible at ∞. Theorem I is contained in the following.

Theorem 2.1.1 (Characterization). *For a μ^{n+1}-manifold M the following conditions are equivalent:*

(1) *M is a μ_∞^{n+1}-manifold.*

(2) $M \cong \mathrm{Ker}(M)$.

(3) *There is a proper $(n+1)$-invertible UV^n-surjection $f : M \to X$ onto some $[0,1)$-stable Q-manifold X.*

(4) *There is a proper $(n+1)$-invertible UV^n-surjection $g : M \to Y$ onto a space Y which is properly n-contractible to ∞ and properly locally n-contractible at ∞.*

Proof. We shall prove that (1) ⇒ (2). First we shall choose a compact cover $\{M_i\}_{i \in \omega}$ of M with $M_i \subset \mathrm{int}M_{i+1}$, $i \in \omega$ such that the topological frontier $\mathrm{Fr}M_i$ is a Z-set in $M \setminus \mathrm{int}M_i$. To this end, fix a proper UV^n-surjection $g : M \to X$ onto a Q-manifold X. Then choose a compact cover $\{X_i\}_{i \in \omega}$ of X consisting of Q-manifold with $X_i \subset \mathrm{int}X_{i+1}$ such that $\mathrm{Fr}X_i$ is a Z-set in both X_i and $X \setminus \mathrm{int}X_i$, $i \in \omega$ (see [C2], [CS]). For each $i \in \omega$, by the relative triangulation theorem for Q-manifolds [C3], we may assume that $X = P \times Q$, $X_i = P_1^i \times Q$ and

$X \setminus \mathrm{int} X_i = P_2^i \times Q$ for a locally finite polyhedron P and closed subpolyhedra $P_1^i, P_2^i \subset P$. Note that $P_1^i \cap P_2^i$ is a Z-set in P_2^i. Let $f_P : M_P \to P$ be a proper UV^n-surjection from a μ^{n+1}-manifold M_P onto P satisfying the condition (b) in Theorem 1.2.4. Since the composition $\pi_P g : M \to P$ is proper UV^n (where $\pi_P : P \times Q \to P$ is the canonical projection), there is a homeomorphism $k : M_P \to M$ by Proposition 1.2.3. Then by the property (b) of f_P, $f_P^{-1}(P_1^i \cap P_2^i)$ is a Z-set in $f_P^{-1}(P_2^i)$ and so is the topological frontier of $f_P^{-1}(P_1^i)$. Now let $M_i = k f_P^{-1}(P_1^i)$, $i \in \omega$. Then the compact cover $\{M_i\}_{i \in \omega}$ of M is the required one.

By Lemma 2.1.1, there is a collection of maps $\{f_i : M \to M \setminus M_i\}_{i \in \omega}$ such that $f_0 = \mathrm{id}_M$, $f_i \simeq_p^n f_{i+1}$ in $M \setminus M_{i-1}$ for $i \in \omega$. Using the Z-embedding approximation theorem for μ^{n+1}-manifolds [Be, 2.3.8], we can choose f_i as a Z-embedding for each $i \in \omega$. Put $K_i = M \setminus f_i(M)$ for $i \geq 1$. Then since $f_i \simeq_p^n \mathrm{id}_M$, by the definition of n-homotopy kernels, we have $K_i \cong \mathrm{Ker}(M)$. By Theorem 1.1, being $f_i \simeq_p^n f_{i+1}$ in $M \setminus M_{i-1}$ and $\mathrm{Fr} M_{i-1}$ a Z-set in $M \setminus \mathrm{int}\, M_{i-1}$, there exists a homeomorphism $h_i : M \to M$ such that $h_i f_i = f_{i+1}$ and $h_i|_{M_{i-1}} = \mathrm{id}$. Note that $h_i(K_i) = K_{i+1}$. Now we define $h : K_1 \to M$ by $h = \lim_{i \to \infty} h_i \cdots h_1$. Then $h|_{h^{-1}(\mathrm{int} M_i)} = h_{i+2} \cdots h_1|_{h^{-1}(\mathrm{int} M_i)}$. In fact, suppose that $h(x) \neq h_{i+2} \cdots h_1(x)$ for some $x \in h^{-1}(\mathrm{int} M_i)$. Then there is an open subset U of $\mathrm{int} M_i$ such that $h(x) \in U \subset \bar{U}$ and $h_{i+2} \cdots h_1(x) \notin \bar{U}$. Since $h_j|_{\mathrm{int} M_i} = \mathrm{id}$ for $j \geq i+2$, $h_j \cdots h_1(x) = h_{i+2} \cdots h_1(x) \notin \bar{U}$ for all $j \geq i+2$. This contradicts the definition of h.

One can easily see that h is injective. Moreover, since $M = \cup_{i \in \omega} M_i$ and $h_i \cdots h_1(K_1) = K_i \supset M_i$, it follows that h is surjective. To finish the proof, it only remains to note that h is open. Thus h is a homeomorphism.

To prove (2) $\Rightarrow$ (3), assume $M \cong \mathrm{Ker}(M)$. Then, by Proposition 1.2.4, there is a proper $(n+1)$-invertible UV^n-surjection $g : M \to M \times [0, 1)$. Let $h : M \to Y$

be a proper UV^n-surjection onto a Q-manifold Y (Proposition 1.2.2). Since $Y \times [0,1)$ is a $[0,1)$-stable Q-manifold, the composition $(h \times \mathrm{id}_{[0,1)})g : M \to Y \times [0,1)$ is the required one.

(3) $\Rightarrow$ (4) is trivial.

Finally we shall show that (4) $\Rightarrow$ (1). Let $h : M \to X$ be a proper $(n+1)$-invertible UV^n-surjection onto a space X properly n-contractible to ∞ and properly locally n-contractible at ∞. Let K be a compactum in M. Then there exists a compactum L' in X with $h(K) \subset L'$ such that for each compactum F' with $L' \subset F'$ there exist proper maps $i'_{h(K)} : X \to X \setminus h(K)$ and $j'_{F'} : X \setminus L' \to X \setminus F'$ such that $i'_{h(K)} \simeq^n_p \mathrm{id}_X$ in X and $j'_{F'} \simeq^n_p \mathrm{id}_{X \setminus L'}$ in $X \setminus h(K)$. Let $L = h^{-1}(L')$ and F be a compactum containing L. Since h is proper $(n+1)$-invertible, there exist proper maps $i_K : M \to M \setminus K$ and $j_F : M \setminus L \to M \setminus F$ such that $hi_K = i'_{h(K)}h$ and $hj_F = j'_{h(F)}h$.

Consider a proper map $\alpha : Z \to M \setminus L$ ($\subset M \setminus h^{-1}h(K)$), where $\dim Z \leq n$. We shall now show that $j_F\alpha$ is properly homotopic to α in $M \setminus K$. From Proposition 1.2.1, we may assume without loss of generality that Z is a locally finite polyhedron. Let $H : (X \setminus L') \times [0,1] \to X \setminus h(K)$ be a proper homotopy from $\mathrm{id}_{X \setminus L'}$ to $j'_{h(F)}$. Then $H(h\alpha \times \mathrm{id}) : Z \times [0,1] \to X \setminus h(K)$ is a proper homotopy from $h\alpha$ to $j'_{h(F)}h\alpha = hj_F\alpha$. Being $h|_{M \setminus h^{-1}h(K)} : M \setminus h^{-1}h(K) \to X \setminus h(K)$ is proper UV^n, by [La, §3, Lemma A], there exists a proper homotopy $F : Z \times [0,1] \to M \setminus h^{-1}h(K)$ from α to $j_F\alpha$. Thus $j_F \simeq^n_p \mathrm{id}_{M \setminus L}$ in $M \setminus K$. Similarly, we can conclude $i_F \simeq^n_p \mathrm{id}_M$. □

§2.2. Properties of μ^{n+1}_∞-manifolds

Lemma 2.2.1. *Let $f : X \to Y$ be a map from a locally compact space X into a LC^n-space Y admitting a proper $(n+1)$-invertible UV^n-surjection onto a space $Y \times [0,1)$. Then f is n-homotopic to a proper map whenever $\dim X \leq n+1$.*

Moreover, if f is a proper map n-homotopic to a proper map $g : X \to Y$ then $f \simeq_p^n g$.

Proof. Fix a proper map $p : X \to [0, 1)$ and let $h : Y \to Y \times [0, 1)$ be a proper $(n+1)$-invertible UV^n-surjection. Let $q : X \to Y \times [0, 1)$ be the map defined by $q(x) = (h_1 f(x), p(x))$, where $h(x) = (h_1(x), h_2(x))$, $x \in X$. Then q is proper and homotopic to hf. By the $(n+1)$-invertibility of h, there is a map $f' : X \to Y$ such that $hf' = q$. Note that f' is proper and $hf' \simeq hf$. Thus by the lifting property of h [La, §3, Lemma A], we conclude that $f \simeq^n f'$.

Next suppose that f is a proper map n-homotopic to a proper map $g : X \to Y$. Let $\alpha : Z \to X$ be a proper map, where $\dim Z \leq n$. We shall show that $f\alpha \simeq_p g\alpha$. By Proposition 1.2.1, we may assume without loss of generality that Z is a locally finite polyhedron. Let $\{Y_i\}_{i\in\omega}$ be a compact cover of Y with $Y_0 = \emptyset$ and $Y_i \subset \text{int}Y_{i+1}$, $i \in \omega$. Then for each $i \geq 1$, let Z_i be a compact subpolyhedron of Z such that

$$(hf\alpha)^{-1}(W_i) \cup (hg\alpha)^{-1}(W_i) \subset Z_i \subset \text{int}Z_{i+1},$$

where $Z_0 = \emptyset$ and $W_i = Y_i \times [0, 1 - 2^{-i}]$. Since $f \simeq^n g$, we can fix a homotopy $G_0 : Z \times [0, 1] \to Y$ from $f\alpha$ to $g\alpha$. For $k \geq 1$, we shall inductively construct a homotopy

$$G_k : (Z \setminus \text{int}Z_{2k-2}) \times [0, 1] \to Y \setminus h^{-1}(W_{2k-5})$$

from the restriction $f\alpha|$ of $f\alpha$ to the one $g\alpha|$ of $g\alpha$ satisfying the following conditions:

$(1)_k$ $G_k((Z \setminus \text{int}Z_{2k}) \times [0, 1]) \subset Y \setminus h^{-1}(W_{2k-2})$;

$(2)_k$ $G_k = G_{k-1}$ on $\text{Fr}Z_{2k-2} \times [0, 1]$.

Let $F_i : [0, 1) \to [1 - 2^{-i}, 1)$ be the map defined by

$$F_i(t) = 1 + (t - 1)2^{-i}$$

for each $i \geq 1$. Suppose that a homotopy

$$G_k : (Z \setminus \mathrm{int} Z_{2k-2}) \times [0,1] \to Y \setminus h^{-1}(W_{2k-5})$$

has been constructed for $k \in \omega$. Then let

$$A_{k+1} = (Z \setminus \mathrm{int} Z_{2k}) \times \{0,1\} \cup \mathrm{Fr} Z_{2k} \times [0,1]$$

and

$$B_{k+1} = (hG_k)^{-1}(W_{2k+1}) \cap (Z \setminus \mathrm{int} Z_{2k+2}) \times [0,1].$$

Since A_{k+1} and B_{k+1} are disjoint closed, we can choose $\beta : (Z \setminus \mathrm{int} Z_{2k}) \times [0,1] \to [0,1]$ such that $\beta(A_{k+1}) = 0$ and $\beta(B_{k+1}) = 1$. Define

$$G'_{k+1} : (Z \setminus \mathrm{int} Z_{2k}) \times [0,1] \to Y \times [0,1) \setminus W_{2k-2}$$

by

$$G'_{k+1}(w) = (s_k(w), (1 - \beta(w))t_k(w) + \beta(w)F_{2k+2}t_k(w)),$$

where $hG_k(w) = (s_k(w), t_k(w))$, $w \in (Z \setminus \mathrm{int} Z_{2k}) \times [0,1]$. By the lifting property [La], there is a homotopy

$$G_{k+1} : (Z \setminus \mathrm{int} Z_k) \times [0,1] \to Y \setminus W_{2k-3}$$

from $f\alpha|$ to $g\alpha|$ with $hG_{k+1} = G'_{k+1}$ and $G_{k+1} = G_k$ on A_{k+1} (i.e. satisfying $(2)_{k+1}$) such that G_{k+1} satisfies $(1)_{k+1}$.

We define $H : Z \times [0,1] \to Y$ by $H = G_k$ on each $(Z_{2k} \setminus \mathrm{int} Z_{2k-2}) \times [0,1]$. Then H is a well-defined homotopy from $f\alpha$ to $g\alpha$. Note that since h is proper, $\{h^{-1}(W_i)\}_{i \in \omega}$ is a compact cover of Y with $h^{-1}(W_i) \subset \mathrm{int}\, h^{-1}(W_{i+1})$. Thus it follows from our construction that H is proper. The proof is finished. □

Theorem 2.2.1. *Each homeomorphism between two Z-sets in a μ_∞^{n+1}-manifold M extends to an ambient homeomorphism of M onto itself if it is n-homotopic to* id *in M.*

Proof. The theorem directly follows from Theorem 1.2.2 and Lemma 2.2.1. □

Lemma 2.2.2. *If $f : M \to N$ is a proper n-homotopy equivalence between μ_∞^{n+1}-manifolds then f induces an isomorphism of homotopy groups of ends of* dim $\leq n$.

Proof. By Theorem 2.1.1, we can fix proper $(n+1)$-invertible UV^n-surjections $g : M \to X \times [0,1)$ and $h : N \to Y \times [0,1)$, where X and Y are some Q-manifolds. Let C be a compactum in N. Then there is a compactum $C'' \subset Y$ such that $C'' \times [0,t'] \supset h(C)$ for some $t' \in (0,1)$. Since h is proper, $C' = h^{-1}(C'' \times [0,t'])$ is a compactum with $C' \supset C$. Note that, since f is proper, $g(f^{-1}(C'))$ is a compactum in $X \times [0,1)$. Thus there exists $t_1 \in (0,1)$ such that

$$L = \pi_X(g(f^{-1}(C'))) \times [0,t_1] \supset g(f^{-1}(C')),$$

where $\pi_X : X \times [0,1) \to X$ is the canonical projection. Similarly, being g proper, there exists $t_2 \in (0,1)$ such that

$$K' = \pi_Y(hf(g^{-1}(L))) \times [0,t_2] \supset hf(g^{-1}(L)),$$

where $\pi_Y : Y \times [0,1) \to Y$ is the canonical projection. Put $K = h^{-1}(K')$ and let $x_0 \in M \setminus f^{-1}(K)$, $j \leq n$, and $\alpha : (S^j, *) \to (N \setminus K, f(x_0))$. Since f is an n-homotopy equivalence there exists $\alpha_1 : (S^j, *) \to (M, x_0)$ such that $f\alpha_1 \simeq \alpha$ rel .*. Being $\alpha_1^{-1}(x_0)$ and $\alpha_1^{-1}(g^{-1}(L))$ are disjoint closed sets in S^j, we can choose a map $\beta : S^j \to [0,1]$ such that $\beta(\alpha_1^{-1}(x_0)) = 0$ and $\beta(\alpha_1^{-1}(g^{-1}(L))) = 1$. Say $g\alpha_1(x) = (\pi_X g\alpha_1(x), t(x)) \in X \times [0,1)$, $x \in S^j$. Define $\alpha_2 : (S^j, *) \to (X \times [0,1), g(x_0))$ by

$$\alpha_2(x) = (\pi_X g\alpha_1(x), ((1-t_1)\cdot t(x) + t_1)\beta(x) + (1-\beta(x))\cdot t(x)),\ x \in S^j.$$

Clearly $\alpha_2 \simeq g\alpha_1$ rel $.*$ and $\alpha_2(S^j) \cap L = \emptyset$. Using the lifting property [La] of the proper UV^n-surjection g, there exists $\tilde{\alpha} : (S^j, *) \to (M, x_0)$ such that $\mathrm{img}\tilde{\alpha} \cap L = \emptyset$ and $\tilde{\alpha} \simeq \alpha_1$ rel $.*$. Hence we have $f\tilde{\alpha} \simeq \alpha$ rel $.*$ and $f\tilde{\alpha}(S^j) \cap C' = \emptyset$. By the same technique we performed above, we can choose a homotopy so that $f\tilde{\alpha} \simeq \alpha$ rel $.*$ in $N \setminus C$.

Next let $\gamma : S^j \to M \setminus f^{-1}(K)$ be a map such that $f\gamma \simeq *$ in $N \setminus K$. Since f is an n-homotopy equivalence, $g\gamma \simeq *$ in $X \times [0,1)$. By sliding the $[0,1)$-factor of the homotopy upward as the above, we have $g\gamma \simeq *$ in $X \times [0,1) \setminus L$. By the lifting property of g [La], it follows that $\gamma \simeq *$ in $X \setminus f^{-1}(C)$. Thus we conclude that f induces an isomorphism of homotopy groups of ends of dim $\leq n$. □

Theorem 2.2.2. *An n-homotopy equivalence between two μ_∞^{n+1}-manifolds is n-homotopic to a homeomorphism.*

Proof. Let $f : M \to N$ be an n-homotopy equivalence between μ_∞^{n+1}-manifolds. Then by Lemma 2.2.1 there is a proper map $h : M \to N$ such that $f \simeq^n h$; consequently, h is a proper n-homotopy equivalence. By Lemma 2.2.2 and Theorem 1.2.5, h is properly n-homotopic to a homeomorphism. Thus f is n-homotopic to a homeomorphism. □

Theorem 2.2.3. *Each map from a μ_∞^{n+1}-manifold into a μ^{n+1}-manifold is n-homotopic to an open embedding.*

Proof. Let $f : M \to N$ be a map from a μ_∞^{n+1}-manifold to a μ^{n+1}-manifold. By replacing N with $\mathrm{Ker}(N)$, we may assume that N is also a μ_∞^{n+1}-manifold. By the triangulation theorem for μ^{n+1}-manifold [Dr2], we can fix proper $(n+1)$-invertible UV^n-surjections $g : M \to K$ and $h : N \to L$, where K and L are locally finite polyhedra of dimension at most $n+1$. Then by the $(n+1)$-invertibility, g has a section $p : K \to M$ (i.e. $gp = \mathrm{id}_K$). Since N is a μ_∞^{n+1}-manifold, by Lemma 2.2.1, f is n-homotopic to a proper map $f' : M \to N$.

Then $\varphi = hf'p : K \to L$ is a proper map. Let $M(\varphi)$ be the mapping cylinder of φ, that is a space obtained from the disjoint union $K \times [0,1] \oplus L$ by identifying $(x,1)$ with $\varphi(x)$, $x \in K$. Define $c_\varphi : M(\varphi) \to L$ by $c_\varphi(x,t) = \varphi(x)$, $x \in K$. Let $f_n : \mu^{n+1} \to Q$ be a proper $(n+1)$-invertible UV^n-surjection satisfying the condition $(*)$ in Theorem 1.2.3. Embed $M(\varphi)$ into Q, whence $f_n^{-1}(M(\varphi))$ is a μ^{n+1}-manifold. We denote the restriction of f_n to $f_n^{-1}(M(\varphi))$ by $f_n|$. Observe that $f_n^{-1}(K \times \{0\}) \cong M$ and $f_n^{-1}(L) \cong N$ by Proposition 1.2.3. We identify $f_n^{-1}(K \times \{0\})$, $f_n^{-1}(L)$ with M, N respectively. Abusing notations, by $g : M \to K \times \{0\}$, $h : N \to L$ we denote the restrictions of f_n to M, N respectively. Using the $(n+1)$-invertibility of h, we can fix a section $q : L \to N$ of h. Note that since $c_\varphi f_n| : f_n^{-1}(M(\varphi)) \to L$ and $h : N \to L$ are proper UV^n-surjections, $f_n^{-1}(M(\varphi)) \cong N$ by Proposition 1.2.3. Observe that the map $qc_\varphi f_n|$ is an n-homotopy equivalence between μ_∞^{n+1}-manifolds $f_n^{-1}(M(\varphi))$ and N. Then by Theorem 2.2.2, there is a homeomorphism $s : f_n^{-1}(M(\varphi)) \to N$ such that $s \simeq^n qc_\varphi f_n|$. Note that $M' = f_n^{-1}(K \times [0,1))$ is open in $f_n^{-1}(M(\varphi))$ and is a μ_∞^{n+1}-manifold by Theorem 2.1.1. Since the inclusion $i : M = f_n^{-1}(K \times \{0\}) \hookrightarrow M'$ is an n-homotopy equivalence, by Theorem 2.2.2, we can choose a homeomorphism $r : M \to M'$ with $r \simeq^n i$. Then the map $sr : M \to N$ is an open embedding which is n-homotopic to $qc_\varphi(f_n|)i = q\varphi g = qhf'pg$. Since $pg \simeq_p^n \mathrm{id}_M$ and $qh \simeq_p^n \mathrm{id}_N$, we have $qhf'pg \simeq_p^n f' \simeq^n f$. The proof is finished. □

III. A Mapping Theorem

Brown and Cassler [Br] proved that each compact connected n-manifold M can be obtained from the n-cube $\mathbb{I}^n$ by making identifications on the boundary $\partial\mathbb{I}^n$, that is, there is a map $\varphi\colon \mathbb{I}^n \to M$ such that $\varphi(\mathbb{I}^n \setminus \partial\mathbb{I}^n) = M \setminus \varphi(\partial\mathbb{I}^n)$ is dense in M and $\varphi|\mathbb{I}^n \setminus \partial\mathbb{I}^n$ is an embedding. This was generalized by Berlanga [Ber] to a non-compact connected n-manifold M, that is, there exists a map $\varphi\colon \mathbb{I}^n \to \widetilde{M}$ such that $E = \varphi(\mathcal{E}(M)) \subset \partial\mathbb{I}^n$, $\varphi|E$ is a homeomorphism of E onto $\mathcal{E}(M)$, $\varphi(\mathbb{I}^n \setminus \partial\mathbb{I}^n) = M \setminus \varphi(\partial\mathbb{I}^n)$ is dense in M and $\varphi|\mathbb{I}^n \setminus \partial\mathbb{I}^n$ is an embedding into M, where $\mathcal{E}(X)$ is the *space of ends* of X and $\widetilde{X} = X \cup \mathcal{E}(X)$ is the *Freudenthal compactification* of X. For $\mathcal{E}(X)$, refer to [Ber, §1].

Hilbert cube manifolds (*Q-manifolds*) or *$(n+1)$-dimensional Menger manifolds* (*μ^{n+1}-manifolds*) are paracompact topological manifolds modeled on the Hilbert cube $Q = \mathbb{I}^\omega$ or the $(n+1)$-dimensional universal Menger compactum μ^{n+1}, respectively. The Q-manifold version of the above Brown-Cassler mapping theorem was established by Prasad [Pr]. Some other infinite-dimensional versions were proved in [S1]. In this chapter, we prove the μ^{n+1}-manifold version of the above Berlanga's mapping theorem. The Q-manifold version is proved in [IS] using the mapping cylinder technique used in [S1], which is an elegant approach but could not be applied to μ^n-manifolds. We take another approach for μ^{n+1}-manifolds, which is also valid for Q-manifolds. One should remark that our approach simplifies the Berlanga's proof in [Ber].

§3.1. A Mapping Theorem for μ^{n+1}-manifolds

Given a Z-set μ_0^{n+1} in μ^{n+1} which is homeomorphic to μ^{n+1}. Then $\mu^{n+1} \setminus \mu_0^{n+1}$ is an *n-homotopy kernel* of μ^{n+1}, which plays the role of $Q \times [0,1)$ (cf. [Ch3]). Let $(N, \delta N)$ be a pair of closed sets in a μ^{n+1}-manifold M. According to [Ch5], the pair $(N, \delta N)$ is said to be *n-clean* in M provided the following conditions are satisfied

(1) N, δN and $(M \setminus N) \cup \delta N$ are μ^{n+1}-manifolds;

(2) δN is a Z-set in both N and $(M \setminus N) \cup \delta N$;

(3) $N \setminus \delta N$ is open in M (i.e., $\mathrm{bd}_M N \subset \delta N$).

Lemma 3.1.1. *Each point x of a μ^{n+1}-manifold M has an arbitrarily small neighborhood W with $\delta W \subset W \setminus \{x\}$ such that the pair $(W, \delta W)$ is n-clean in M and homeomorphic to (μ^{n+1}, μ_0^{n+1}).*

Proof. By [Ch4, Theorem 1.3], there exists a proper $(n+1)$-invertible UV^n-map $f\colon N \to [0,2)$ from a μ^{n+1}-manifold N onto $[0,2)$ such that $f^{-1}([0,1])$ and $f^{-1}(\{1\})$ are μ^{n+1}-manifolds and $f^{-1}(\{1\})$ is a Z-set in both $f^{-1}([0,1])$ and $f^{-1}([0,2))$. Observe that N is homeomorphic to its n-homotopy kernel $\mathrm{Ker}(N)$ by [Iw1, Theorem 2.1]. Using [Iw1, Theorem IV], we have an open embedding $h\colon N \to U$. By the Z-set Unknotting Theorem [Be, p.102], we may assume that $x \in h(f^{-1}([0,1)))$. Let $W = hf^{-1}([0,1])$ and $\delta W = hf^{-1}(\{1\})$. Then $W \cong \delta W \cong \mu^{n+1}$ by the Bestvina's characterization of μ^{n+1} [Be, 5.2.3]. By the Z-set Unknotting Theorem [Be, 3.1.5], we have $(W, \delta W) \cong (\mu^{n+1}, \mu_0^{n+1})$. Since δW is a Z-set in $(h(N) \setminus W) \cup \delta W$, it is also a Z-set in $(M \setminus W) \cup \delta W$. Hence $(W, \delta W)$ is n-clean in M. □

The following is the μ^{n+1}-manifold version of Berlanga's theorem:

Theorem 3.1.1. *For each connected μ^{n+1}-manifold M, there exists a map $\varphi\colon \mu^{n+1} \to \widetilde{M}$ such that*

(i) $E = \varphi^{-1}(\mathcal{E}(M)) \subset \mu_0^{n+1}$;

(ii) $\varphi|E$ *is a homeomorphism of* E *onto* $\mathcal{E}(M)$;

(iii) $\varphi(\mu^{n+1} \setminus \mu_0^{n+1}) = M \setminus \varphi(\mu_0^{n+1})$ *is dense in* M;

(iv) $\varphi|\mu^{n+1} \setminus \mu_0^{n+1}$ *is an embedding into* M.

To prove Theorem 3.1.1, we show the following homeomorphism extension lemma:

Lemma 3.1.2. *Let $f\colon A \to B$ be a bijection between finite sets A and B in a μ^{n+1}-manifold M such that each $a \in A$ and $f(a) \in B$ are contained in a connected open set U_a in M. Then f extends to a homeomorphism $\tilde{f}\colon M \to M$ such that $\tilde{f}(U_a) = U_a$ for each $a \in A$ and $\tilde{f}|M \setminus \bigcup_{a\in A} U_a = \mathrm{id}$.*

Proof. Without loss of generality, we may assume that U_a ($a \in A$) are disjoint. By Lemma 3.1.1, there is an n-clean pair $(W_a, \delta W_a)$ in U_a such that $a \in W_a \setminus \delta W_a$. By the Z-set Unknotting Theorem [Be, p.102], we may assume that a, $f(a) \in W_a \setminus \delta W_a$. Since δW_a is a Z-set in W_a, using the Z-set Unknotting Theorem [Be, 3.1.5], there is a homeomorphism $f_a\colon W_a \to W_a$ such that $f_a(a) = f(a)$ and $f_a|\delta W_a = \mathrm{id}$. Then the required homeomorphism $\tilde{f}\colon M \to M$ can be defined by $\tilde{f}|M \setminus \bigcup_{a\in A} W_a = \mathrm{id}$ and $\tilde{f}|W_a = f_a$ for each $a \in A$. □

Proof of Theorem 3.1.1. Since $\widetilde{M}$ is metrizable, we may assume that $\widetilde{M}$ is a metric space given a metric d. In the following, we construct $\varphi\colon \mu^{n+1} \to \widetilde{M}$ as the composition of three maps:

$$\mu^{n+1} \xrightarrow{f} \widetilde{M} \xrightarrow{g} \widetilde{M} \xrightarrow{h} \widetilde{M}.$$

Step 1: By the analogue of [Be, 5.1.3] (p.103), there exists a locally finite simplicial complex K with $\dim K \leqslant n+1$ and a proper map $q\colon |K| \to M$ which induces isomorphisms of homotopy groups of dimension $\leqslant n$ and of homotopy groups of ends of dimension $\leqslant n$. We define

$$X = |K| \times [-1,0] \cup |T| \times [0,1],$$

where T be a maximal tree of the 1-skeleton $K^{(1)}$. Let $p\colon X \to |K|$ be the projection. By [Ch4, Theorem 1.3] (cf. [Ch3, Theorem 1.6]), there exists an $(n+1)$-invertible UV^n-map $f_X : M_X \to X$ of a μ^{n+1}-manifold M_X onto X such that $N = f_X^{-1}(|T| \times [0,1])$, $N_0 = f_X^{-1}(|T| \times \{0\})$ and $N^* = f_X^{-1}(|K| \times [-1,0])$ are μ^{n+1}-manifolds and $N_0 = N \cap N^*$ is a Z-set in both N and N^*. Observe that N and N_0 are n-connected and $N \setminus N_0 = f_X^{-1}(|K| \times (0,1])$ is open in M_X. And $pf_X|N\colon N \to |T|$ extends to a map $k\colon \widetilde{N} \to \widetilde{|T|}$ such that $k(\mathcal{E}(N)) = \mathcal{E}(|T|)$. It is easy to see that $\mathcal{E}(|T|)$ is a Z-set in $\widetilde{|T|}$. Since $f_X|N\colon N \to |T|$ is $(n+1)$-invertible and $\dim \widetilde{N} = n+1$, it follows that $\mathcal{E}(N)$ is a Z-set in $\widetilde{N}$. Then $\widetilde{N}$ is an $(n+1)$-dimensional n-connected LC^n compactum which has the disjoint $(n+1)$-cells property. Hence $\widetilde{N} \cong \mu^{n+1}$ by the Bestvina's characterization of μ^{n+1} [Be, 5.2.3]. Similarly $\widetilde{N}_0 \cong \mu^{n+1}$. By the Z-set Unknotting Theorem [Be, 3.1.5], we have a homeomorphism $f\colon \mu^{n+1} \to \widetilde{N}$ such that $f(\mu_0^{n+1}) = \widetilde{N}_0$.

Since $qpf_X : M_X \to M$ induces isomorphisms of homotopy groups of dimension $\leqslant n$ and of homotopy groups of ends of dimension $\leqslant n$, we have $M_X \cong M$ by [Be, Ch.6, Theorem]. Therefore we identify $M_X = M$. Then $N \setminus N_0$ is an open set in M and N_0 is a Z-set in both N and $N^* = (M \setminus N) \cup N_0$. The inclusion $N \subset M$ induces a homeomorphism between the spaces of ends because so is $|T| \subset |K|$. Hence we can regard $\mathcal{E}(N_0) = \mathcal{E}(N) = \mathcal{E}(M)$ and $\widetilde{N}_0 \subset \widetilde{N} \subset \widetilde{M}$.

We can write $K = \bigcup_{i \in \mathbb{N}} K_i$, where each K_i is a finite subcomplexes of K and $|K_i| \subset \mathrm{int}_{|K|}|K_{i+1}|$. For each $i \in \mathbb{N}$, let S_i be a simplicial neighborhood of $\mathrm{Sd}\, K_i$ in $\mathrm{Sd}\, K$ and $T_i = S_i \cap \mathrm{Sd}\, T$, where $\mathrm{Sd}\, K$ is the barycentric subdivision of K. Then

$$|K| = \bigcup_{i\in\mathbb{N}} |S_i|, \ |T| = \bigcup_{i\in\mathbb{N}} |T_i|, \ |S_i| \subset \mathrm{int}_{|K|}|S_{i+1}|, \ |T_i| \subset \mathrm{int}_{|K|}|T_{i+1}|,$$

and each $\mathrm{bd}_{|K|}|S_i|$ is a Z-set in both $|S_i|$ and $\mathrm{cl}_{|K|}(|K| \setminus |S_i|)$, each $\mathrm{bd}_{|T|}|T_i|$ is a Z-set in both $|T_i|$ and $\mathrm{cl}_{|T|}(|T| \setminus |T_i|)$ and each $|T_i|$ meets all components of $|S_i| \setminus |S_{i-1}|$ ($S_0 = \emptyset$). For each $i \in \mathbb{N}$, let

$$M_i = f_X^{-1}(|S_i| \times [-1,0] \cup |T_i| \times [0,1]) = f_X^{-1}p^{-1}(|S_i|),$$
$$\delta M_i = f_X^{-1}p^{-1}(\mathrm{bd}_{|K|}|S_i|) \quad \text{and} \quad N_i = f_X^{-1}(|T_i| \times [2^{-i},1]).$$

Then as is easily observed, $M = \bigcup_{i\in\mathbb{N}} M_i$, $N \setminus N_0 = \bigcup_{i\in\mathbb{N}} N_i$, each pair $(M_i, \delta M_i)$ is n-clean in M and each N_i meets all components of $M_i \setminus (\delta M_i \cup M_{i-1})$, where $M_0 = \emptyset$. (Each $(N_i, \delta N_i)$ is also n-clean in M, where

$$\delta N_i = f_X^{-1}(|T_i| \times \{2^{-i}\} \cup \mathrm{bd}_{|T|}|T_i| \times [2^{-i},1]).$$

But this fact is not used.)

Step 2: By local path-connectedness of M_i, we can choose $\varepsilon_1 > \varepsilon_2 > \cdots > 0$ so that each $x, y \in M_i$ can be connected by a path with diameter less than 2^{-i} if $d(x,y) < \varepsilon_i$. For each $i \in \mathbb{N}$, choose a finite ε_i-dense set A_i in $M_i \setminus (\delta M_i \cup M_{i-1})$ ($M_0 = \emptyset$), that is, for each point $x \in M_i \setminus (\delta M_i \cup M_{i-1})$ there is a point $a \in A_i$ such that $d(a,x) < \varepsilon$. Note that $(M_i \setminus M_{i-1}) \cup \delta M_{i-1}$ is a compact μ^{n+1}-manifold and $\delta M_{i-1} \cup \delta M_i$ is a Z-set in $(M_i \setminus M_{i-1}) \cup \delta M_{i-1}$. Since N_i meets each component of $M_i \setminus (\delta M_i \cup M_{i-1})$, we can apply the Z-set Unknotting Theorem [Be, 3.1.4] to construct a homeomorphism

$$g_i\colon (M_i \setminus M_{i-1}) \cup \delta M_{i-1} \to (M_i \setminus M_{i-1}) \cup \delta M_{i-1}$$

so that $A_i \subset g_i(N_i \setminus M_{i-1})$ and $g_i|\delta M_i \cup \delta M_{i-1} = \mathrm{id}$. We define a homeomorphism $g\colon M \to M$ by $g|(M_i \setminus M_{i-1}) \cup \delta M_{i-1} = g_i$. Then $g(M_i) = M_i$,

$g(\delta M_i) = \delta M_i$ and $g(N_i) \setminus (\delta M_i \cup M_{i-1})$ is ε_i-dense in $M_i \setminus (\delta M_i \cup M_{i-1})$ for each $i \in \mathbb{N}$.

Step 3: Let B_1 be a finite ε_2-dense set in $M_1 \setminus (\delta M_1 \cup g(N_1))$. Since $g(N_1) \setminus \delta M_1$ is a ε_1-dense in $M_1 \setminus \delta M_1$ and $g(N_1) \subset \mathrm{int}_M g(N_2)$, it follows that $g(N_2) \cap (M_1 \setminus (\delta M_1 \cup g(N_1)))$ is ε_1-dense in $M_1 \setminus (\delta M_1 \cup g(N_1))$, whence we have an injection

$$j_1 \colon B_1 \to g(N_2) \cap (M_1 \setminus (\delta M_1 \cup g(N_1)))$$

which is ε_1-close to id. Then we can assume that each $b \in B_1$ and $j_1(b)$ can be connected by a path in $M_1 \setminus (\delta M_1 \cup g(N_1))$ with diameter less than 2^{-1}. We choose a connected open set U_b in M_1 such that $\mathrm{diam}\, U_b < 2^{-1}$ and

$$b, j_1(b) \in U_b \subset M_1 \setminus (\delta M_1 \cup g(N_1)).$$

By Lemma 3.1.2, we have a homeomorphism $h_1 \colon \widetilde{M} \to \widetilde{M}$ by $h_1 | j_1(B) = j_1^{-1}$ and $h_1 | \widetilde{M} \setminus \bigcup_{b \in B} U_b = \mathrm{id}$. Then h_1 is 2^{-1}-close to id, $h_1 | g(N_1) \cup (\widetilde{M} \setminus M_1) = \mathrm{id}$ and $h_1 g(N_2)$ is ε_2-dense in M_2 (indeed $h_1 g(N_2) \setminus \delta M_2$ is ε_2-dense in $M_2 \setminus \delta M_2$) because $B_1 \cup g(N_1) \cup g(N_2) \subset h_1 g(N_2)$.

Similarly choosing a finite ε_3-dense set in $M_2 \setminus (\delta M_2 \cup h_1 g(N_2))$, we can construct a homeomorphism $h_2 \colon \widetilde{M} \to \widetilde{M}$ so that $h_2 | h_1 g(N_2) \cup (\widetilde{M} \setminus M_2) = \mathrm{id}$, h_2 is 2^{-2}-close to id and $h_2 h_1 g(N_3)$ is ε_3-dense in M_3 (indeed $h_2 h_1 g(N_3) \setminus \delta M_3$ is ε_3-dense in $M_3 \setminus \delta M_3$).

Inductively homeomorphisms $h_i \colon \widetilde{M} \to \widetilde{M}$ ($i \in \mathbb{N}$) can be constructed so that

$$h_i | h_{i-1} \cdots h_1 g(N_i) \cup (\widetilde{M} \setminus M_i) = \mathrm{id},$$

h_i is 2^{-i}-close to id and $h_i h_{i-1} \cdots h_1 g(N_{i+1})$ is ε_{i+1}-dense in M_{i+1}. Then $(h_i \cdots h_1)_{i \in \mathbb{N}}$ converges to a map $h \colon \widetilde{M} \to \widetilde{M}$ such that $h | \mathcal{E}(M) = \mathrm{id}$ and $h | g(N_i) = h_{i-1} \cdots h_1 | g(N_i)$ for each $i \in \mathbb{N}$, whence it follows that $hg | N \setminus N_0$

is an embedding into M. Since each $h(g(N_i)) = h_{i-1} \cdots h_1 g(N_i)$ is ε_i-dense in M_i, it follows that $h(g(N))$ is dense in M.

It is easy to verify that $\varphi = hgf \colon \mu^{n+1} \to \widetilde{M}$ is the desired map. $\square$

Remarks. In the above Step 1, we can take a different approach as follows: First choose n-clean pairs $(M_i, \delta M_i)$ in M so that $M_i \subset M_{i+1}$ and $M = \bigcup_{i \in \mathrm{N}} M_i$. As remarked before Lemma 3.1.1, there is an embedding $f_0 \colon \mu^{n+1} \to M_1$ such that $f_0(\mu^{n+1} \setminus \mu_0^{n+1})$ is open in M. Let $\mu^{n+1} \setminus \mu_0^{n+1} = \bigcup_{i \in \mathrm{N}} W_i$, where each W_i is compact and contained in $\operatorname{int} W_{i+1}$. Similarly as Step 3, we have homeomorphisms $f_i \colon M \to M$ such that $f_i \cdots f_0(\mu^{n+1}) \subset M_{i+1}$, $f_i \cdots f_0(W_{i+1})$ meets every component of $M_{i+1} \setminus M_i$ and

$$f_i | (M \setminus M_{i+1}) \cup M_{i-1} \cup f_{i-1} \cdots f_0(W_i) = \mathrm{id}.$$

In fact, connecting a point of each component of $M_{i+1} \setminus M_i$ and a point of $f_{i-1} \cdots f_0(W_{i+1} \setminus W_i) \setminus M_i$ by a connected open set in $M_{i+1} \setminus M_{i-1}$ and applying Lemma 3.1.2, we can inductively construct f_i. Then as the limit of $f_i \cdots f_1 f_0$, we can obtain an embedding $f \colon \mu^{n+1} \to \widetilde{M}$ such that $f(\mu^{n+1} \setminus \mu_0^{n+1})$ is open in M, $E = f^{-1}(\mathcal{E}(M)) \subset \mu_0^{n+1}$ and $f(E) = \mathcal{E}(M)$. Let $\widetilde{N} = f(\mu^{n+1})$, $N = f(\mu^{n+1} \setminus E)$, $\widetilde{N}_0 = f(\mu_0^{n+1})$, $N_0 = f(\mu_0^{n+1} \setminus E)$ and $N_i = f(W_i) = f_{i-1} \cdots f_0(W_i)$.

By using [BE, Lemma 5] (cf. [Br, Lemma 1]) instead of Lemma 3.1.2, the above arguments, Steps 2 and 3 are valid for an n-manifold M, where δM_i is replaced by the boundary ∂M_i of an n-manifold M_i which is bicollared in M. This approach simplifies the Berlanga's proof in [Ber].

The proof of Theorem 3.1.1 is also valid for Q-manifolds. (In Step 1, $M_X = X \times Q$ and $f_X \colon M_X \to X$ is the projection.)

IV. Homeomorphism Groups

By $H(X)$, we denote the group of homeomorphisms of X onto itself. In [Mc] and [Wo], it is proved that the group $H(X)$ is simple in the algebraic sense in case X is a normed linear space E which is homeomorphic to the countable infinite product E^ω of E and in case X is the Hilbert cube Q. Let $H_0(X)$ be the subgroup of $H(X)$ consisting of all homeomorphisms which are isotopic to the identity. In case X is a (finite-dimensional) manifold without the boundary, $H_0(X)$ is the smallest normal subgroup of $H(X)$ and is simple by [EC] and [Fi]. In this chapter, we prove this result in the case X is the $(n+1)$-dimensional universal Menger compactum μ^{n+1}.

§4.1. Algebraic simplicity of homeomorphism groups

Theorem 4.1.1. *The group $H(\mu^{n+1})$ is simple.*

As a corollary of this theorem, we have the following:

Corollary 4.1.1. *Let $k > 1$ be a natural number. Every homeomorphism $h \in H(\mu^{n+1})$ is can be written as a finite composition $h = h_n \cdots h_1$ of homeomorphisms $h_i \in H(\mu^{n+1})$ of period k.*

A subset A of X is said to be *deformable* in X if for each nonempty open set U in X there is a homeomorphism $h \in H(X)$ such that $h(A) \subset U$. A homeomorphism $h \in H(X)$ is said to be *supported* by $A \subset X$ if $h|X \setminus A = \mathrm{id}$. Let U be an open set in X, $B_0, B_1, \cdots$ pairwise disjoint open sets in U and $\varphi \in H(X)$. Following [Wo] (cf. [Nu]), we call $(B_i, \varphi)_{i \in \mathbb{Z}_+}$ a *dilation system* in U if $B_0, B_1, \cdots$ converges to a point $p \in U$, $\varphi|X \setminus U = \mathrm{id}$ (i.e., φ is supported by U) and $r(B_i) = B_{i-1}$ for each $i \in \mathbb{N}$. To prove Theorem 3.1.1, we apply the following theorem [Wo, Theorem 6] (cf. [Fi]).

Theorem (Fisher-Wong). *Suppose that X is a metrizable space in which every open set contains a dilation syatem. Let G be the normal subgroup of $H(X)$ generated by all homeomorphisms which are supported by deformable subsets of X. Then G is simple.*

In [Ch6], the following was shown.

Theorem (Chigogidze). *Every homeomorphism $h \in H(\mu^{n+1})$ is stable, that is, there are homeomorphisms $h_i \in H(\mu^{n+1})$, $i = 1, \cdots, n$, such that $h = h_n \cdots h_1$ and each h_i is the identity on some nonempty open set in μ^{n+1}.*

Thus we can reduce the Main Theorem to the following two lemmas:

Lemma 4.1.1. *Every open set in μ^{n+1} contains a dilation system.*

Lemma 4.1.2. *Every proper subset of μ^{n+1} is deformable in μ^{n+1}.*

To prove the above lemmas, we recall the definition of n-clean pairs in a μ^{n+1}-manifold M introduced by Chigogidze [Ch5]. A pair $(N, \delta N)$ of closed sets in M is said to be *n-clean* if the following conditions are satisfied

(1) N, δN and $(M \setminus N) \cup \delta N$ are μ^{n+1}-manifolds;

(2) δN is a Z-set in both N and $(M \setminus N) \cup \delta N$;

(3) $N \setminus \delta N$ is open in M (i.e., $\mathrm{bd}_M N \subset \delta N$).

Now we shall prove Lemma 4.1.1.

Proof of Lemma 4.1.1. By [IS, Lemma 1], every open set in μ^{n+1} contains an n-clean pair $(W, \delta W)$ in M such that $W \cong \delta W \cong \mu^{n+1}$. It suffices to show that W contains a dilation system. Choose disjoint open sets U_i, $i \in \mathbb{Z}$, in $W \setminus \delta W$ so that both $U_1, U_2, \cdots$ and $U_{-1}, U_{-2}, \cdots$ converge to the same point $p \in W \setminus \delta W$. Again by [IS, Lemma 1], each U_i contains an n-clean pair $(B_i, \delta B_i)$ such that $B_i \cong \delta B_i \cong \mu^{n+1}$. Let

$$V = (W \setminus \{p\}) \setminus \bigcup_{i \in \mathbb{Z}} (B_i \setminus \delta B_i).$$

Each $(U_i \setminus B_i) \cup \delta B_i$ is a μ^{n+1}-manifold open set in V and $(W \setminus \{p\}) \setminus \bigcup_{i \in \mathbb{Z}} B_i$ is also a μ^{n+1}-manifold open set in V. Then it follows that V is a μ^{n+1}-manifold. Since $\bigcup_{i \in \mathbb{Z}} \delta B_i$ is closed in V and a countable union of Z-sets, it is a Z-set in V (cf. [vM, 6.2.2]). Since δW is a Z-set in $(W \setminus \{p\}) \setminus \bigcup_{i \in \mathbb{Z}} B_i$, it is also a Z-set in V. By using the Z-set Unknotting Theorem [Be], we can construct a homeomorphism $r' \colon V \to V$ such that $r'|\delta W = \mathrm{id}$ and $r'(B_i) = B_{i-1}$ for each $i \in \mathbb{Z}$. Since W is the one-point compactification of $W \setminus \{p\}$ and $r'|\delta W = \mathrm{id}$, we can extend r' to a homeomorphism $r \in H(\mu^{n+1})$ by $r|\mu^{n+1} \setminus W = \mathrm{id}$ and $r(p) = p$. Then it is clear that $(B_i, r)_{i \in \mathbb{Z}_+}$ is a dilation system in W. □

To prove Lemma 4.1.2, we need the following

Lemma 4.1.3. *Let $(W, \delta W)$ be an n-clean pair in a μ^{n+1}-manifold M such that $W \cong \delta W \cong \mu^{n+1}$. Then $(M \setminus W) \cup \delta W \cong M$.*

Proof. First note that the inclusion $j \colon (M \setminus W) \cup \delta W \subset M$ induces a homeomorphism between the spaces of ends, whence j induces an isomorphism of homotopy groups of ends. By [Be, §6, Theorem], it suffices to show that j induces an isomorphism of homotopy groups of dimension $\leqslant n$.

Epi: Let $f \colon \mathbb{S}^i \to M$ $(i \leqslant n)$ be a map of the i-sphere, $A = \mathrm{cl}\, f^{-1}(W \setminus \delta W)$ and $B = \mathrm{bd}\, A$. By [Hu, Ch.V, Theorem 10.1], $f|B \colon B \to \delta W$ extends to a map $g'\colon A \to \delta W$ and we have a homotopy $h'\colon A \times \mathbb{I} \to W$ such that $h'_0 = f|A$, $h'_1 = g'$ and $h'_t|B = f|B$ for each $t \in \mathbb{I}$. We can extend g' and h' to a map $g\colon \mathbb{S}^i \to (M \setminus W) \cup \delta W$ and a homotopy $h\colon \mathbb{S}^i \times \mathbb{I} \to M$ by $g|\mathbb{S}^i \setminus A = h_t|\mathbb{S}^i \setminus A = f|\mathbb{S}^i \setminus A$. Then $h_0 = f$ and $h_1 = g$. This means that j induces an epimorphism of homotopy groups of dimension $\leqslant n$.

Mono: Suppose that a map $g\colon \mathbb{S}^i \to (M \setminus W) \cup \delta W$ $(i \leqslant n)$ extends to a map $f\colon \mathbb{B}^{i+1} \to M$ of the $(i+1)$-ball. Let $C = \operatorname{cl} f^{-1}(W \setminus \delta W)$ and $D = \operatorname{bd} C$. Similarly as the above, $f|D\colon D \to \delta W$ extends to a map $g'\colon C \to \delta W$ by [Hu, Ch.V, Theorem 10.1]. We can extend g' to a map $\widetilde{g}\colon \mathbb{B}^{i+1} \to (M \setminus W) \cup \delta W$ by $\widetilde{g}|\mathbb{B}^{i+1} \setminus C = f|\mathbb{B}^{i+1} \setminus C$. Since $\mathbb{S}^i \subset \mathbb{B}^{i+1} \setminus C$, $\widetilde{g}|\mathbb{S}^i = f|\mathbb{S}^i = g$. This implies that j induces a monomorphism of homotopy groups of dimension $\leqslant n$. □

Proof of Lemma 4.1.2. Let A be a proper subset of μ^{n+1} and U an open set in μ^{n+1}. Choose an open set V in μ^{n+1} with $A \cap V = \emptyset$. By [IS, Lemma 1], we have n-clean pairs $(W_i, \delta W_i)$ in μ^{n+1} $(i = 1, 2)$ such that $W_i \cong \delta W_i \cong \mu^{n+1}$, $W_1 \subset U$ and $W_2 \subset V$. Then $(\mu^{n+1} \setminus W_1) \cup \delta W_1 \cong \mu^{n+1} \cong W_2$ and $(\mu^{n+1} \setminus W_2) \cup \delta W_2 \cong \mu^{n+1} \cong W_1$ by Lemma 4.1.3. Using the Z-set Unknotting Theorem [Be], we can obtain a homeomorphism $h \in H(\mu^{n+1})$ such that $h(\delta W_2) = \delta W_1$, $h(W_2) = (\mu^{n+1} \setminus W_1) \cup \delta W_1$ and $h((\mu^{n+1} \setminus W_2) \cup \delta W_2) = W_1$. Then

$$h(A) \subset h(\mu^{n+1} \setminus V) \subset h(\mu^{n+1} \setminus W_2) = W_1 \subset U.$$

Hence A is deformable in μ^{n+1}. □

Proof of Corollary 4.1.1. Let G_k be the subgroup of $H(\mu^{n+1})$ generated by homeomorphisms of period k. We show that $G_k = H(\mu^{n+1})$. Since G_k is clearly a normal subgroup of $H(\mu^{n+1})$, it suffices to show that G_k is nontrivial, i.e., there exists a nontrivial homeomorphism $h_k \in H(\mu^{n+1})$ of period k. By using Garity-Henderson-Wright's model of μ^{n+1} in [GHW], such an h_k can be easily constructed as follows:

Let $P_0 = \bigcup_{i=1}^{k} \langle v_0, v_i \rangle$ be the one-point union of k many one-simplexes. We define a homeomorphism $h \in H(P_0)$ of period k by

$$h((1-t)v_0 + tv_i) = (1-t)v_0 + tv_{i+1},$$

where $v_{k+1} = v_1$. Note that h is simplicial with respect to the natural triangulation K_0 of P_0. For each $i \in \mathbf{N}$, we denote $\mathbf{I}_i = [0, 2^{-i}]$ and $Q_{i+1} = \prod_{j>i} \mathbf{I}_j$. We inductively define polyedra P_i in $P_0 \times \mathbf{I}_1 \times \cdots \times \mathbf{I}_i$ as follows: Triangulate $P_{i-1} \times \mathbf{I}_i$ by K_i so that $\operatorname{mesh} K_i \leqslant 2^{-i}$ and $h \times \mathrm{id}$ is simplicial. Let $P_i = |K_i^{(n+1)}|$ be the polyhedron of the $(n+1)$-skeleton of K_i. Then by [GHW, Theorem 2], $X = \bigcap_{i \in \mathbf{N}} P_i \times Q_{i+1}$ ($\subset P_0 \times Q_1$) is homeomorphic to μ^{n+1}. Observe $h \times \mathrm{id}|X \in H(X)$, which induces the required homeomorphim $h_k \in H(X)$ of period k. □

V. Product Structure

It is known that there are many similarities between μ^n-manifolds and Q-manifolds (Hilbert cube manifolds). But it should be observed that the Cartesian product of μ^n-manifolds, for example $\mu^n \times \mu^n$, is neither a μ^n-manifold nor a μ^{2n}-manifold. To avoid this phenomenon, Dranishnikov [Dr2] has constructed a universal map $g_n : \mu^n \to \mu^n$ which has the following property: $(*)$ *for any μ^n-manifold X embedded in μ^n, $g_n^{-1}(X) \cong X$ and there is an n-dimensional polyhedron $K \subset \mu^n$ such that $g_n^{-1}(K) \cong X$.* The map g_n corresponds to the projection $\pi : Q \times Q \to Q$ in the Q-manifolds theory and the polyhedron K above is called the triangulation of a μ^n-manifold X. Recall that the triangulation of a Q-manifold Y is a polyhedron L such that $\pi^{-1}(L) = L \times Q \cong Y$. One of the differences between the triangulations of Q-manifolds and μ^n-manifolds above is that, using the triangulations, every Q-manifolds can be represented as a space with infinite coordinates though any μ^n-manifolds cannot be so represented.

The first part of this chapter is devoted to defining the infinite coordinate systems for μ^n-manifolds, called μ^n-coordinate systems. And we prove the triangulation theorem by means of μ^n-coordinate systems. Moreover, using μ^n-coordinate systems, we can characterize Z-sets in terms of infinite deficiency.

In the second part, we discuss how to define a kind of the Cartesian product in the category of μ^n-manifolds. To do this, we use μ^n-coordinate systems and define the Δ_n-product which plays the role of the Cartesian product in the category of μ^n-manifolds. Concerning the Δ_n-product, we prove the stability theorem, that is, the Δ_n-products of a μ^n-manifold M with μ^n (resp. $[0,1)$) is homeomorphic to M (resp. the $(n-1)$-homotopy kernel of M) (notation: $M\Delta_n\mu^n \cong M$ (resp. $M\Delta_n[0,1) \cong \mathrm{Ker}(M)$)). One should note that the formulation $M\Delta_n[0,1) \cong \mathrm{Ker}(M)$ is quite natural since $(n-1)$-homotopy kernels were introduced by Chigogidze [Ch3] as the corresponding notion of "$[0,1)$-stable" Q-manifolds, where a $[0,1)$-stable Q-manifold is a Q-manifold X homeomorphic to $X \times [0,1)$.

§5.1. Infinite deficiency

Let K and L be simplicial complexes. By $K \times L$, we mean the simplicial complex defined by the barycentric subdivision of the cell complex $\{\sigma \times \tau | \sigma \in K, \tau \in L\}$. We denote the n-skeleton of the simplicial complex $K \times L$ by $K \times_n L$ (i.e. $K \times_n L = (K \times L)^{(n)}$).

Let $\{K_i\}_{i=0}^{\infty}$ be a sequence of (locally finite) simplicial complexes. Then we define the $\overset{n}{\nabla}$-product of simplicial complexes as follows:

$$\overset{n}{\nabla}{}_{i=0}^{1} K_i = K_0 \times_n K_1,$$

and inductively for $l > 1$,

$$\overset{n}{\nabla}{}_{i=0}^{l} K_i = (\overset{n}{\nabla}{}_{i=0}^{l-1} K_i) \times_n K_l.$$

We define $\overset{n}{\nabla}{}_{i=0}^{\infty} K_i$ as the limit (space) of the following inverse sequence:

$$|K_0| \xleftarrow{p_0^1} |\overset{n}{\nabla}{}_{i=0}^{1} K_i| \xleftarrow{p_1^2} |\overset{n}{\nabla}{}_{i=0}^{2} K_i| \xleftarrow{p_2^3} |\overset{n}{\nabla}{}_{i=0}^{3} K_i| \xleftarrow{p_3^4} \cdots,$$

where $p_i^{i+1} : |\overset{n}{\nabla}{}_{j=0}^{i+1} K_j| \to |\overset{n}{\nabla}{}_{j=0}^{i} K_j|$ is the restriction of the canonical projection $|\overset{n}{\nabla}{}_{j=0}^{i} K_j| \times |K_{i+1}| \to |\overset{n}{\nabla}{}_{j=0}^{i} K_j|$, $i \geq 0$. If $n = \infty$, we denote $\overset{n}{\nabla}{}_{j=0}^{i} K_j$ by $\nabla_{j=0}^{i} K_j$. Note that $\overset{n}{\nabla}{}_{i=0}^{\infty} K_i \cong \mathrm{cl}\big(\bigcup_{j=0}^{\infty}(\overset{n}{\nabla}{}_{k=0}^{j} K_k)\big)$ by [vM, 6.7.2]. If we set $P_i = |\overset{n}{\nabla}{}_{j=0}^{i} K_j|$, $i \geq 0$, then the inverse sequence $\{P_i, p_i^{i+1}\}_{i=0}^{\infty}$ is called the *defining sequence* with respective to $\{K_i\}_{i=0}^{\infty}$.

Let $\{T_i\}_{i=0}^{\infty}$ be a sequence of simplicial complexes. We say $\{T_i\}_{i=0}^{\infty}$ a μ^n-*coordinate system* provided that each T_i is a non-degenerate locally finite simplicial complex with dimension$\leq n$ such that the underlying polyhedron $|T_i|$ is a C^{n-1}-compactum for $i \geq 1$.

Lemma 5.1.1. *Let $\{P_i, p_i^{i+1}\}_{i=0}^{\infty}$ be the defining sequence with respective to a μ^n-coor-dinate system $\{T_i\}_{i=0}^{\infty}$. Then for each $i \geq 0$, each $k \leq n-1$, each map $f : \mathbb{B}^{k+1} \to P_i$, and each map $g : S^k \to P_{i+1}$ with $p_i^{i+1} \circ g = f|S^k$, there exists an extension $h : \mathbb{B}^{k+1} \to P_{i+1}$ of g such that $p_m^{i+1} \circ h$ and $p_m^i \circ f$ are $Sd^{(i+1-m)}(\nabla_{j=0}^{m} T_j)$-close for each $m \leq i$.*

Proof. Let $\pi_1 : P_i \times |T_{i+1}| \to P_i$ and $\pi_2 : P_i \times |T_{i+1}| \to |T_{i+1}|$ be the canonical projections. Since $|T_{i+1}| \in C^{n-1}$, there is an extension $\alpha : \mathbb{B}^{k+1} \to |T_{i+1}|$ of $\pi_2 \circ g$. Then the map $f' : \mathbb{B}^{k+1} \to P_i \times |T_{i+1}|$ defined by $f'(x) = (f(x), \alpha(x))$, $x \in \mathbb{B}^{k+1}$ is an extension of g with $\pi_1 \circ f' = f$. For each simplex $\sigma \in (\overset{n}{\nabla}{}_{j=0}^{i} T_j) \times T_{i+1}$, we can take a map $h_\sigma : (f')^{-1}(|\sigma|) \to |\sigma^n|$ so that $h_\sigma|(f')^{-1}(|\sigma^n|) = f'|(f')^{-1}(|\sigma^n|)$ since $k+1 \leq n$ [HW]. Define $h : \mathbb{B}^{k+1} \to P_{i+1}$ by $h|(f')^{-1}(|\sigma|) = h_\sigma$ for each simplex $\sigma \in (\overset{n}{\nabla}{}_{j=0}^{i} T_j) \times T_{i+1}$. Then the definition of defining sequences implies that $p_m^{i+1} \circ h$ and $p_m^i \circ f$ are $\mathrm{Sd}^{(i+1-m)}(\nabla_{j=0}^{m} T_j)$-close for each $m \leq i$. $\square$

Proposition 5.1.1. *For each μ^n-coordinate system $\{T_i\}_{i=0}^{\infty}$, $\overset{n}{\nabla}{}_{i=0}^{\infty} T_i$ is a μ^n-manifold. In particular, $\overset{n}{\nabla}{}_{i=0}^{\infty} T_i \cong \mu^n$ if $|T_0|$ is a C^{n-1}-compactum.*

Proof. Let $X = \overset{n}{\nabla}{}_{i=0}^{\infty} T_i$ and let $\{P_i, p_i^{i+1}\}_{i=0}^{\infty}$ be the defining sequence with respect to a μ^n-coordinate system $\{T_i\}_{i=0}^{\infty}$. First note that X is locally compact since each $|T_i|$is compact for $i \geq 1$. Thus all we have to do is to check the conditions of Bestvina's characterization. Let $x \in X$ be a point and let $U \subset X$ be a neighborhood of x. Then there is an open neighborhood U_N of $x_N = p_N(x_N)$ in P_N such that $p_N^{-1}(U_N) \subset U$ for some $N \in \mathbb{N}$. Take $a \geq N$ and a neighborhood V of x_a so that

$$\mathrm{St}^{(2)}(p_N^a(V), \mathrm{Sd}^{(a-N)}(\nabla_{j=0}^{N} T_j)) \subset U_N.$$

Since P_a is ANR, there is a neighborhood $W \subset V$of x_a such that any map from S^k $(k \leq n-1)$ into W can be extended to a map from $\mathbb{B}^{k+1}$ to V. Let $f : S^k \to X$ be a map such that $\mathrm{im} f \subset p_a^{-1}(W)$. Then there is an extension $g_a : \mathbb{B}^{k+1} \to V$ of $p_a \circ f$. For $i < a$, we put $g_i = p_i^a \circ g_a : \mathbb{B}^{k+1} \to P_i$. Using Lemma 5.1.1, we can inductively construct an extension $g_i : \mathbb{B}^{k+1} \to P_i$ $(i \geq a)$ of $p_i \circ f$ such that $p_m^i \circ g_i$ and $p_m^{i+1} \circ g_{i+1}$ are $\mathrm{Sd}^{(i+1-m)}(\nabla_{j=0}^{m} T_j)$-close. Since the sequence $\{p_m^i \circ g_i\}_{i \geq m}$ of maps is uniformly convergence, the map $h_i = \lim_{m \to \infty} p_i^m \circ g_m : \mathbb{B}^{k+1} \to X_i$ is continuous and clearly satisfies $p_i^{i+1} \circ h_{i+1} = h_i$ and $h_i|S^k = p_i \circ f$. Consider the map $h = \varprojlim h_i : \mathbb{B}^{k+1} \to X$. Then h is an extension of f. Since h_a and g_a are $\mathrm{St}^{(2)}(\mathrm{Sd}^{(a-N)}(\nabla_{j=0}^{N} T_j))$-close and $\mathrm{im} g_a \subset p_N^a(V)$, we have $\mathrm{im} h_a \subset U_N$. Thus $\mathrm{im} h \subset p_N^{-1}(U_N) \subset U$. Hence X is an LC^{n-1}-space. The other parts are the same as [GHW, Theorem 1]. □

Let $\{T_i\}_{i=0}^{\infty}$ be a μ^n-coordinate system and let $\bar{x} = (x_1, x_2, \dots)$ be a point such that $x_i \in T_i^{(0)}$ for each $i \geq 1$. Let $j_{x_i} : |\overset{n}{\nabla}_{j=0}^{i-1} T_j| \to |\overset{n}{\nabla}_{j=0}^{i} T_j| \times \{x_i\} \subset |\overset{n}{\nabla}_{j=0}^{i} T_j|$ be the inclusion. Then the inclusion $j_{\bar{x}} : |T_0| \to |\overset{n}{\nabla}_{j=0}^{\infty} T_j|$ is defined by $p_i \circ j_{\bar{x}} = j_{x_i} \circ j_{x_{i-1}} \circ \cdots \circ j_{x_1}$, $i \geq 0$.

Proposition 5.1.2. *Let $\{T_i\}_{i=0}^{\infty}$ be a μ^n-coordinate system such that $|T_0|$ is connected and let $\bar{x} = (x_1, x_2, \dots)$ be a point such that $x_i \in T_i^{(0)}$ for each $i \geq 1$. Then both the inclusion $j_{\bar{x}} : |T_0| \to |\overset{n}{\nabla}_{j=0}^{\infty} T_j|$ and the projection p_0 :*

$|\overset{n}{\nabla}_{j=0}^{\infty} T_j| \to |T_0|$ induce isomorphisms of homotopy groups of dimension$\leq n-1$ and of ends of dimension$\leq n-1$.

Proof. Note that both $j_{\bar{x}}$ and p_0 are proper maps. First we shall show that p_0 induces monomorphism of homotopy group of ends of dimension $\leq n-1$. Let C be a compactum in $|T_0|$ and let $K = |\mathrm{St}^{(2)}(C, T_0)|$. Let $\alpha : S^k \to \overset{n}{\nabla}_{j=0}^{\infty} T_j \setminus p_0^{-1}(K)$, $k \leq n-1$ be a map with $p_0 \circ \alpha \simeq *$ in $|T_0| \setminus K$ and let $\beta : \mathbb{B}^{k+1} \to |T_0| \setminus K$ be the contraction. By Lemma 5.1.1, we can construct extensions $\beta_i : \mathbb{B}^{k+1} \to |\overset{n}{\nabla}_{j=0}^{i} T_j|$ of $p_i \circ \alpha$ such that $p_m^i \circ \beta_i$ and $p_m^{i+1} \circ \beta_{i+1}$ are $\mathrm{St}^{(i+1-m)}(T_0)$-close for each i, where $p_i : |\overset{n}{\nabla}_{j=0}^{\infty} T_j| \to |\overset{n}{\nabla}_{j=0}^{i} T_j|$ and $p_m^i : |\overset{n}{\nabla}_{j=0}^{i} T_j| \to |\overset{n}{\nabla}_{j=0}^{m} T_j|$ are canonical projections. Since $\{p_i^m \circ \beta_m\}_{m \geq i}$ is uniformly Cauchy,

$$\alpha_i = \lim_{m \to \infty} p_i^m \circ \beta_m : \mathbb{B}^{k+1} \to |\overset{n}{\nabla}_{j=0}^{i} T_j|$$

is continuous for each $i \geq 1$ and clearly an extension of $p_i \circ \alpha$ with $p_i^{i+1} \circ \alpha_{i+1} = \alpha_i$. Thus the map $\tilde{\alpha} = \varprojlim \alpha_i : \mathbb{B}^{k+1} \to \overset{n}{\nabla}_{j=0}^{\infty} T_j$ is an extension of α. Since $p_0 \circ \tilde{\alpha} = \alpha_0$ and β are $\mathrm{St}^{(2)}(T_0)$-close, $\tilde{\alpha}(\mathbb{B}^{k+1}) \cap p_0^{-1}(C) = \emptyset$. Thus p_0 induce the monomorphisms of homotopy groups of dimension $\leq n-1$.

For epimorphism, let $\bar{a} \in \overset{n}{\nabla}_{j=0}^{\infty} T_j \setminus p_0^{-1}(K)$ be a point and let $\gamma : (S^k, *) \to (|T_0| \setminus K, p_0(a))$ be a map. We may assume that $a_i = q_i(a) \in T_i^{(0)}$ for each $i \geq 1$, where $q_i : \overset{n}{\nabla}_{j=0}^{\infty} T_j \to |T_i|$ is the canonical projection. Then the map $\tilde{\gamma} = j_{\bar{a}} \circ \gamma : (S^k, *) \to (\overset{n}{\nabla}_{j=0}^{\infty} T_j, y)$ satisfies $p_0 \circ \tilde{\gamma} = \gamma$.

Next we shall show that $j_{\bar{x}}$ induces epimorphism of homotopy groups of ends of dimension $\leq n-1$. Let $C' \subset \overset{n}{\nabla}_{j=0}^{\infty} T_j$ be a compactum and let $K' = p_0^{-1}(\mathrm{St}^{(2)}(p_0(C'), T_0))$. Let $y \in |T_0| \setminus p_0(K')$ be a point and let $\xi : (S^k, *) \to (\overset{n}{\nabla}_{j=0}^{\infty} T_j \setminus K', j_{\bar{x}}(y))$ be a map. Then the map $\tilde{\xi} = p_0 \circ \xi : (S^k, *) \to (|T_0|, y)$ is such that $\tilde{\xi}(S^k) \subset |T_0| \setminus j_{\bar{x}}^{-1}(K')$. Put $y_i = p_i \circ j_{\bar{x}}(y)$, $i \geq 1$ and $\xi_i = j_{x_i} \circ \cdots \circ j_{x_1} \circ \tilde{\xi}$. Then, as in the proof of Lemma 5.1.1, we can take a sequence of homotopies $\{h_t^i : (S^k, *) \to (|\overset{n}{\nabla}_{j=0}^{i} T_j|, y_i)\}$ such that h_t^i is a homotopy from ξ_i to $p_i \circ \xi$ rel. y_i

with $h_t^0 = p_0 \circ \xi$, $t \in [0,1]$, and $p_m^{i+1} \circ h_t^{i+1}$ and $p_m^i \circ h_t^i$ are $\mathrm{St}^{(i+1-m)}(\nabla_{j=0}^m T_j)$-close for each $m \leq i$. Then we define $H_t^i : (S^k, *) \to (|\overset{n}{\nabla}{}_{j=0}^i T_j|, y_i)$ by the uniform limit map $\lim_{l \to \infty} p_i^l \circ h_t^l$. Then H_t^i is a homotopy from ξ_i to $p_i \circ \xi$ rel. y_i with $p_i^{i+1} \circ H_t^{i+1} = H_t^i$. Thus the map $\bar{H}_t = \varprojlim H_t^i$ is a homotopy from $j_{\bar{x}} \circ \tilde{\xi}$ to ξ rel. $j_{\bar{x}}(y)$ and the image is contained in $\overset{n}{\nabla}{}_{j=0}^{\infty} T_j \setminus C'$. The rest is trivial since $p_0 \circ j_{\bar{x}} = \mathrm{id}$. □

Corollary 5.1.1. *Let $\{T_i\}_{i=0}^{\infty}$ and $\{T'_i\}_{i=0}^{\infty}$ be μ^n-coordinate systems. If there exists a proper map $h : |T_0| \to |T'_0|$ which induces isomorphisms of homotopy groups of dimension $\leq n-1$ and of ends of dimension $\leq n-1$ then $\overset{n}{\nabla}{}_{i=0}^{\infty} T_i \cong \overset{n}{\nabla}{}_{i=0}^{\infty} T'_i$.*

Proof. We may assume without loss of generality that both $|T_0|$ and $|T'_0|$ are connected. Let $\bar{x} = (x_1, x_2, \dots)$ be a point such that $x_i \in T_i^{(0)}$. Let $j_{\bar{x}} : |T_0| \to \overset{n}{\nabla}{}_{j=0}^{\infty} T_j$ be the inclusion map and let $p'_0 : \overset{n}{\nabla}{}_{j=0}^{\infty} T'_j \to |T'_0|$ be the projection. Then the map $j_{\bar{x}} \circ h \circ p'_0 : \overset{n}{\nabla}{}_{j=0}^{\infty} T'_j \to \overset{n}{\nabla}{}_{j=0}^{\infty} T_j$ induces isomorphisms of homotopy groups of dimension $\leq n-1$ and of ends of dimension $\leq n-1$ by Proposition 5.1.2. Since $\overset{n}{\nabla}{}_{j=0}^{\infty} T_j$ and $\overset{n}{\nabla}{}_{j=0}^{\infty} T'_j$ are μ^n-manifolds, by Theorem 1.2.5, we have $\overset{n}{\nabla}{}_{j=0}^{\infty} T_j \cong \overset{n}{\nabla}{}_{j=0}^{\infty} T'_j$. □

It is known that for each μ^n-manifold M, there is a locally finite polyhedron K with $\dim K \leq n$ and a proper UV^{n-1}-surjection $f : M \to K$ [Dr2]. Since proper UV^{n-1}-surjections induce isomorphisms of homotopy groups of dimension $\leq n-1$ and of ends of dimension $\leq n-1$, by Theorem 1.2.5, we obtain the following.

Theorem 5.1.1 (Triangulation). *For each μ^n-manifold M, there exists a μ^n-coordinate system $\{T_i\}_{i=0}^{\infty}$ such that $M \cong \overset{n}{\nabla}{}_{i=0}^{\infty} T_i$.*

A subset A of a μ^n-manifold M is said to have *infinite deficiency* (cf. [An3], [Cu]) provided that there exist a μ^n-coordinate system $\{S_i\}_{i=0}^{\infty}$ of M, a cofinal subset $N \subset \mathbf{N}$ and a homeomorphism $h : M \to \overset{n}{\nabla}_{i=0}^{\infty} S_i$ such that $q_i \circ h|A$ is constant for each $i \in N$, where $q_i : \overset{n}{\nabla}_{i=0}^{\infty} S_i \to |S_i|$ is the canonical projection.

Theorem 5.1.2. *A closed subset A of a μ^n-manifold M is a Z-set if and only if A has infinite deficiency.*

Proof. Suppose that A has infinite deficiency. Then there must exists a μ^n-coordinate system $\{S_i\}_{i=0}^{\infty}$ of M, a cofinal subset N of $\mathbf{N}$ and a homeomorphism $h : M \to \overset{n}{\nabla}_{i=0}^{\infty} S_i$ such that $q_i \circ h|A$ is constant for each $i \in N$, where $q_i : \overset{n}{\nabla}_{i=0}^{\infty} S_i \to |S_i|$ is the canonical projection. Assume that $q_i \circ h(A) = \{x_i\}$ for each $i \in N$. Choose a point $y = (y_1, y_2, \dots)$ so that $y_i \in S_i^{(0)}$ and $x_i \neq y_i$ if $i \in N$. Let $f : \mathbb{I}^n \to \overset{n}{\nabla}_{i=0}^{\infty} S_i$ be a map and let $\varepsilon > 0$ be given. Let $k > 0$ be an integer such that$\sum_{i=k}^{\infty} 2^{-i} < \varepsilon$. Let $s_t^m : |\overset{n}{\nabla}_{j=0}^{m} S_j| \to |S_t|$ be the projection $m \geq t$. We define $g_m : |\overset{n}{\nabla}_{j=0}^{m} S_j| \to |\overset{n}{\nabla}_{j=0}^{m} S_j|$ by $s_t^m \circ g_m = s_t^m$ if $t \leq k$ and $s_k^m \circ g_m = y_i$ if $t > k$. Let $\beta_v^u : |\overset{n}{\nabla}_{j=0}^{u} S_j| \to |\overset{n}{\nabla}_{j=0}^{v} S_j|$ be the projection. Since $\beta_m^{m+1} \circ g_{m+1} = g_m$, we can define the limit map $g = \varprojlim g_m \circ \beta_m^{\infty} \circ f : \mathbb{I}^n \to \overset{n}{\nabla}_{j=0}^{\infty} S_j$. Then g is a map with $d(f,g) < \varepsilon$ such that $\operatorname{im} g \cap h(A) = \emptyset$. Hence $h(A)$ is a Z-set in $\overset{n}{\nabla}_{j=0}^{\infty} S_j$, so A is a Z-set in M. Thus infinite deficient closed subsets of μ^n-manifolds are Z-sets.

Now we shall show that each Z-set has infinite deficiency. Let $\{D_j\}_{j=0}^{\infty}$ be a μ^n-coordinate system of M such that $D_j = [0,1]$ for each $j \geq 1$. Then let $\{D'_j\}_{j=0}^{\infty}$ be a sequence of simplicial complexes such that

$$D'_j = \begin{cases} D_j & j = 2k,\ k \geq 0 \\ \{0\} & \text{otherwise.} \end{cases}$$

Observe that $|\overset{n}{\nabla}_{j=0}^{2k} D'_j| \subset |\overset{n}{\nabla}_{j=0}^{2k} D_j|$ since $D_j = D_i = [0,1]$ for $i, j \geq 1$. We define a map $l_k : |\overset{n}{\nabla}_{j=0}^{k} D_j| \to |\overset{n}{\nabla}_{j=0}^{2k} D'_j|$ by

$$l_k(x_0, x_1, \dots, x_k) = (x_0, 0, x_1, 0, \cdots, x_k).$$

Then l_k is an embedding. Since $p_{2k}^{2k+2} \circ l_{k+1} = l_k$, we obtain a sliding map $l = \varprojlim l_k : \overset{n}{\nabla}{}_{j=0}^{\infty} D_j \to \overset{n}{\nabla}{}_{j=0}^{\infty} D_j$. Since each l_k is a closed embedding, l is also a closed embedding and clearly $\operatorname{im} l$ has infinite deficiency. Thus l is actually a Z-embedding.

Next we shall show that the sliding map l is properly $(n-1)$-homotopic to the identity map. Let $\alpha : P \to \overset{n}{\nabla}{}_{j=0}^{\infty} D_j$ be a proper map from an $(n-1)$-dimensional space P. Since $\overset{n}{\nabla}{}_{j=0}^{\infty} D_j$ is LC^{n-1}, we may assume that P is an $(n-1)$-dimensional polyhedron. We construct a homotopy $H^k : P \times [0,1] \to \overset{n}{\nabla}{}_{j=0}^{k} D_j$ such that

(1) $H_0^k = q_k^{\infty} \circ \alpha$, $H_1^k = q_k^{\infty} \circ l \circ \alpha$ and

(2) $q_m^{k+1} \circ H^{k+1}$ and $q_m^k \circ H^k$ are $\mathrm{St}^{(k+1-m)}(\nabla_{j=0}^{m} D_j)$-close for $m \leq k$,

where $q_v^u : |\overset{n}{\nabla}{}_{j=0}^{u} D_j| \to |\overset{n}{\nabla}{}_{j=0}^{v} D_j|$ is the projection. To this end, we put $H_t^0 = q_0^{\infty} \circ \alpha$ for $t \in [0,1]$ since $q_0^{\infty} \circ l \circ \alpha = l_0 \circ q_0^{\infty} \circ \alpha = q_0^{\infty} \circ \alpha$. Suppose that H^k has been constructed. Since D_{k+1} is C^{n-1}-compactum, we can take a proper homotopy $h^{k+1} : P \times [0,1] \to |(\overset{n}{\nabla}{}_{j=0}^{k} D_j) \times D_{k+1}|$ so that $pr \circ h^{k+1} = H^k$, $h_0^{k+1} = q_{k+1}^{\infty} \circ \alpha$ and $h_1^{k+1} = q_{k+1}^{\infty} \circ l \circ \alpha$, where $pr : |(\overset{n}{\nabla}{}_{j=0}^{k} D_j) \times D_{k+1}| \to |\overset{n}{\nabla}{}_{j=0}^{k} D_j|$ is the projection. As in the proof of Lemma 5.1.1, we can take a proper homotopy $H^{k+1} : P \times [0,1] \to |\overset{n}{\nabla}{}_{j=0}^{k+1} D_j|$ such that $H_0^{k+1} = q_{k+1}^{\infty} \circ \alpha$, $H_1^{k+1} = q_{k+1}^{\infty} \circ l \circ \alpha$, and $q_m^{k+1} \circ H^{k+1}$ and $q_m^k \circ H^k$ are $\mathrm{St}^{(k+1-m)}(\nabla_{j=0}^{m} D_j)$-close for $m \leq k$ (recall the definition of the product of simplicial complexes).

Then we define

$$G^k = \lim_{j \to \infty} q_k^j \circ H^j : P \times [0,1] \to |\overset{n}{\nabla}{}_{j=0}^{k} D_j|.$$

Since $\{q_k^j \circ H^j\}$ is uniformly Cauchy, G^k is continuous and clearly satisfies $G_0^k = q_k^{\infty} \circ \alpha$, $G_1^k = q_k^{\infty} \circ l \circ \alpha$ and $q_k^{k+1} \circ G^{k+1} = G^k$. Moreover, since each H^k satisfies the condition (2), G^k is a proper homotopy. Then the map $G = \varprojlim G^k : P \times [0,1] \to \overset{n}{\nabla}{}_{j=0}^{\infty} D_j$ is a proper homotopy between α and $l \circ \alpha$.

Thus the sliding map l is properly $(n-1)$-homotopic to the identity.

Let A be a Z-set in M and let $f : M \to \overset{n}{\nabla}_{j=0}^{\infty} D_j$ be a homeomorphism. Since the restriction of the sliding map $l|f(A) : f(A) \to l(f(A))$ is a homeomorphism between Z-sets $f(A)$ and $l(f(A))$ with $l|f(A) \simeq_p^{n-1} \mathrm{id}_{f(A)}$, using the Z-set unknotting theorem, we have a homeomorphism $g : \overset{n}{\nabla}_{j=0}^{\infty} D_j \to \overset{n}{\nabla}_{j=0}^{\infty} D_j$ such that $g(f(A)) = l(f(A))$. This means that A has infinite deficiency since $l(f(A))$ has infinite deficiency. □

§5.2. Product structure

Let $\{S_i\}_{i=0}^{\infty}$ and $\{T_i\}_{i=0}^{\infty}$ be sequences of simplicial complexes. We define the Δ_n-product of $\overset{n}{\nabla}_{j=0}^{\infty} S_j$ and $\overset{n}{\nabla}_{j=0}^{\infty} T_j$ as the limit of the following inverse sequence

$$|S_0 \times_n T_0| \overset{\alpha_1}{\leftarrow} |(\overset{n}{\nabla}_{j=0}^{1} S_j) \times_n (\overset{n}{\nabla}_{j=0}^{1} T_j)| \overset{\alpha_2}{\leftarrow} |(\overset{n}{\nabla}_{j=0}^{2} S_j) \times_n (\overset{n}{\nabla}_{j=0}^{2} T_j)| \overset{\alpha_3}{\leftarrow} \cdots,$$

where $\alpha_i : |(\overset{n}{\nabla}_{j=0}^{i-1} S_j) \times_n (\overset{n}{\nabla}_{j=0}^{i-1} T_j)| \to |(\overset{n}{\nabla}_{j=0}^{i} S_j) \times_n (\overset{n}{\nabla}_{j=0}^{i} T_j)|$ is the canonical projection. We denote the limit space by $(\overset{n}{\nabla}_{j=0}^{\infty} S_j)\Delta_n(\overset{n}{\nabla}_{j=0}^{\infty} T_j)$. For a simplicial complex K, we define $(\overset{n}{\nabla}_{i=0}^{\infty} S_i)\Delta_n K$ by $(\overset{n}{\nabla}_{i=0}^{\infty} S_i)\Delta_n(\overset{n}{\nabla}_{i=0}^{\infty} K_i)$, where the sequence $\{K_i\}_{i=0}^{\infty}$ is such that $K_0 = K$ and K_i is a point for each $i \geq 1$.

As in the proof of Proposition 5.1.1, we obtain the following.

Proposition 5.2.1. *Let $\{S_i\}_{i=0}^{\infty}$ and $\{T_i\}_{i=0}^{\infty}$ be μ^n-coordinate systems and K a simplicial complex. Then both $(\overset{n}{\nabla}_{i=0}^{\infty} S_i)\Delta_n(\overset{n}{\nabla}_{i=0}^{\infty} T_i)$ and $(\overset{n}{\nabla}_{i=0}^{\infty} S_i)\Delta_n K$ are μ^n-manifolds.*

Lemma 5.2.1. *Let M and N be μ^n-manifolds and let $\{S_i\}_{i=0}^{\infty}$ and $\{T_i\}_{i=0}^{\infty}$ be μ^n-coordinate systems of M and N respectively. Then the topological type of the μ^n-manifold $(\overset{n}{\nabla}_{i=0}^{\infty} S_i)\Delta_n(\overset{n}{\nabla}_{i=0}^{\infty} T_i)$ depends only on M and N.*

Proof. As is stated in the proof of Proposition 5.1.2, both the projection $pr : (\overset{n}{\nabla}_{i=0}^{\infty} S_i)\Delta_n(\overset{n}{\nabla}_{i=0}^{\infty} T_i) \to |S_0 \times_n T_0|$ and the inclusion $i : |S_0 \times_n T_0| \hookrightarrow |S_0| \times |T_0|$

induce isomorphism of homotopy groups of dimension $\leq n-1$ and of ends of dimension $\leq n-1$. By [Dr2], there exists an proper n-invertible UV^{n-1}-surjection $h : V \to M \times N$ of some μ^n-manifold V. Note that V is unique up to homeomorphism [Ch3]. Let $f : M \to |S_0|$, $g : N \to |T_0|$ be proper n-invertible UV^{n-1}-surjections (cf. [Dr2]). Then the composition $(f \times g) \circ h : V \to |S_0| \times |T_0|$ is also a proper n-invertible UV^{n-1}-surjection. Using the n-invertibility of $(f \times g) \circ h$ there is a proper map $\alpha : (\overset{n}{\nabla}_{i=0}^{\infty} S_i)\Delta_n(\overset{n}{\nabla}_{i=0}^{\infty} T_i) \to V$ such that $\alpha \circ (f \times g) \circ h = i \circ pr$. Now it is easy to see that α is a proper map between μ^n-manifolds which induces isomorphism of homotopy groups of dimension $\leq n-1$ and of end of dimension $\leq n-1$. Thus V is homeomorphic to $(\overset{n}{\nabla}_{i=0}^{\infty} S_i)\Delta_n(\overset{n}{\nabla}_{i=0}^{\infty} T_i)$. $\square$

We denote the topological type of the μ^n-manifold $(\overset{n}{\nabla}_{i=0}^{\infty} S_i)\Delta_n(\overset{n}{\nabla}_{i=0}^{\infty} T_i)$ by $M\Delta_n N$. For a simplicial complex K, the topological type of the μ^n-manifold $(\overset{n}{\nabla}_{i=0}^{\infty} S_i)\Delta_n K$ is unique and we denoted the topological type by $M\Delta_n K$.

Recall that an $(n-1)$-*homotopy kernel* of a μ^n-manifold M is defined to be the complement $M \setminus f(M)$ of the image of an arbitrary Z-embedding $f : M \to M$ with $f \simeq_p^{n-1} \mathrm{id}_M$. Using the Z-set unknotting theorem, two $(n-1)$-homotopy kernels of a μ^n-manifold are homeomorphic. By $\mathrm{Ker}(M)$, we denote a representative of $(n-1)$-homotopy kernels of M.

Theorem 5.2.1 (Stability). *Let M be a μ^n-manifold. Then we have the following:*

(1) $M\Delta_n\mu^n \cong M \cong M\Delta_n[0,1]$;

(2) $M\Delta_n[0,1) \cong \mathit{Ker}(M)$.

Proof. Let $\{S_i\}_{i=0}^{\infty}$ and $\{T_i\}_{i=0}^{\infty}$ be μ^n-coordinate systems of M and μ^n respectively. Take a proper n-invertible UV^{n-1}-surjection $f : M \to |S_0|$. Let $pr : (\overset{n}{\nabla}_{i=0}^{\infty} S_i)\Delta_n(\overset{n}{\nabla}_{i=0}^{\infty} T_i) \to |S_0 \times_n T_0|$ and $q : |S_0 \times_n T_0| \to |S_0|$ be the projections. Note that q induces isomorphisms of homotopy groups of dimension $\leq n-1$ and of ends of dimension $\leq n-1$ since $|T_0|$ is a C^{n-1}-compactum. Using the n-invertibility of f, there is a proper map $g : (\overset{n}{\nabla}_{i=0}^{\infty} S_i)\Delta_n(\overset{n}{\nabla}_{i=0}^{\infty} T_i) \to M$ with $q \circ pr = f \circ g$. Since f and $q \circ pr$ induce isomorphisms of homotopy groups of dimension $\leq n-1$ and of ends of dimension $\leq n-1$, also g does. So we have $M\Delta_n\mu^n \cong M$. Similarly we have $M \cong M\Delta_n[0,1]$.

By [Ch3] (cf. [Iw1, Proposition 1.4]), there is a proper n-invertible UV^{n-1}-surjection $h : \mathrm{Ker}(M) \to M \times [0,1)$. Let $k : M \to |S_0|$ be a proper n-invertible UV^{n-1}-surjection (cf. [Dr2]). Then the composition $(k \times \mathrm{id}_{[0,1)}) \circ h : \mathrm{Ker}(M) \to |S_0| \times [0,1)$ is also a proper n-invertible UV^{n-1}-surjection. Let $p : (\overset{n}{\nabla}_{i=0}^{\infty} S_i)\Delta_n(\overset{n}{\nabla}_{i=0}^{\infty} T_i) \to |S_0 \times_n [0,1)|$ and $i : |S_0 \times_n [0,1)| \hookrightarrow |S_0 \times T_0|$ be the projection and the inclusion respectively. Note that $i \circ p : (\overset{n}{\nabla}_{i=0}^{\infty} S_i)\Delta_n(\overset{n}{\nabla}_{i=0}^{\infty} T_i) \to |S_0 \times [0,1)|$ induces the isomorphisms of homotopy groups of dimension $\leq n-1$ and of ends of dimension $\leq n-1$. The n-invertibility of $(k \times \mathrm{id}_{[0,1)}) \circ h$ allows us to take a proper map $k : (\overset{n}{\nabla}_{i=0}^{\infty} S_i)\Delta_n(\overset{n}{\nabla}_{i=0}^{\infty} T_i) \to \mathrm{Ker}(M)$ which induces the isomorphisms of homotopy groups of dimension $\leq n-1$ and of ends of dimension $\leq n-1$. Thus $M\Delta_n[0,1)$ is homeomorphic to $\mathrm{Ker}(M)$. □

A space X is called *properly k-contractible to ∞* provided that for any compactum K in X, there is a proper map $j_K : X \to X \setminus K$ which is properly k-homotopic to id_X. We say X is *properly locally k-contractible at ∞* if for any compactum $K \subset X$, there is a compactum $L \subset X$ with $K \subset L$ such that for each compactum $L' \subset X$ with $L \subset L'$, there exists a proper map $j_{L'} : X \setminus L \to X \setminus L'$ which is properly k-homotopic to $\mathrm{id}_{X \setminus L}$ in $X \setminus K$.

A μ^n-manifold M is called a *μ^n_∞-manifold* if M is properly $(n-1)$-contractible to ∞ and properly locally $(n-1)$-contractible at ∞. μ^n_∞-manifolds were characterized topologically in [Iw1] as follows: *A μ^n-manifold M is a μ^n_∞-manifold if and only if $M \cong \mathrm{Ker}(M)$.* Thus Theorem 5.2.1 characterizes μ^n_∞-

manifolds in terms of Δ_n-product, that is:

Corollary 5.2.1. *M is a μ^n_∞-manifold if and only if $M \cong M\Delta_n[0,1)$.*

The formulation above is quite natural because the $(n-1)$-homotopy kernels were defined as the corresponding notion of "[0,1)-stable" Q-manifolds for μ^n-manifolds [Ch3].

VI. Group Actions and Fixed Points

Let G be a compact zero-dimensional topological group with the unit element e. An action of G on a space X is called *free* (resp. *semi-free*) if, for each $x \in X$, the isotropy group $G_x = \{g \in G \mid gx = x\}$ is trivial (resp. either trivial or is all of G). It is known that each n-dimensional Menger manifold (μ^n-manifold) admits a free G-action [Dr3], [S2]. Also in his paper [S3], K. Sakai has constructed a semi-free G-action on the n-dimensional universal Menger compactum μ^n and has obtained the following: *For each Z-set X in μ^n, there exists a semi-free G-action on μ^n such that X is the fixed point set of any $g \in G \setminus \{e\}$*. In the same paper, he asked whether the result above is still true for any closed subset X of μ^n.

On the other hand, it is known [M] that the Hilbert cube Q has the complete invariance property with respect to homeomorphisms, where a space X has *the complete invariance property with respect to homeomorphisms* (CIPH) if each non-empty closed subset of X is the fixed point set of some autohomeomorphism of X. Since μ^n is recognized as a finite dimensional analogue of Q, the following question naturally arose [CKT2, Problem 6.4.3]: *Is it true that μ^n has CIPH ?*

In the present chapter, we construct semi-free G-actions on μ^n-manifolds, on their pseudo-interiors and on their pseudo-boundaries. The main purpose of this chapter is to give the affirmative answers to the questions above as follows: *For each closed subset X of a μ^n-manifold M, there exists a semi-free G-action on M such that X is the fixed point set of any $g \in G \setminus \{e\}$* (Theorem 6.1.1). Using the idea of infinite deficiency, we can construct G-invariant Z-skeletoids in μ^n-manifolds, where a subspace A of a G-space X is called (G-)*invariant* provided that $A = \{ga \mid g \in G, a \in A\}$. This allows us to obtain the pseudo-interiors and the pseudo-boundaries versions of the theorem above as follows: *Let $\nu(M)$ (resp. $\Sigma(M)$) be a pseudo-interior (resp. a pseudo-boundary) of a μ^n-manifold M. Then for each closed subset X of $\nu(M)$ (resp. $\Sigma(M)$), there exists a semi-free G-action on $\nu(M)$ (resp. $\Sigma(M)$) such that X is the fixed points set of any $g \in G \setminus \{e\}$* (Corollary 6.2.2). As a consequence, every μ^n-manifold admits a free G-action with a G-invariant pseudo-interior and a G-invariant pseudo-boundary (Theorem 6.2.2).

§6.1. Fixed point sets of semi-free actions

In this section, we consider semi-free actions on Menger manifolds. Let X be a space and let $f : X \to X$ be a map. A closed subset A of X is called the *fixed point set of f* if $A = \{x \in X \mid f(x) = x\}$. The main purpose of this section is to prove the following theorem which gives the affirmative answer to the question of [S3].

Theorem 6.1.1. *Let G be a compact separable zero-dimensional group with the unit element e. For each closed subset X of a μ^n-manifold M, there exists a semi-free G-action on M such that X is the fixed point set of any $g \in G \setminus \{e\}$.*

Proof. By Pontryagin's theorem [Po, §46, C], G can be represented as the inverse limit of an inverse sequence

$$G_0 \xleftarrow{\varphi_0} G_1 \xleftarrow{\varphi_1} G_2 \xleftarrow{\varphi_2} G_3 \xleftarrow{\varphi_3} \cdots$$

consisting of non-trivial finite groups.

Step 1. Construction of a μ^n-manifold.

Let L_k be an $(n-1)$-connected finite simplicial complex with free G_k-action (cf. [S2]). Then $\mathbb{L}_k = L_k \times [0,1]$ is also an $(n-1)$-connected finite simplicial complex with free G_k-action. We identify $\beta(L_k)$ with $L_k \times \{0\}$. Then the free G_k-action on $\mathbb{L}_k$ induces the canonical semi-free G_k-action on the cone $v_k \star \mathbb{L}_k$ so that the vertex v_k is the one and only one fixed point. One should note that $\beta(L_k)$ (resp. $v_k \star \beta(L_k)$) is an invariant subset for each free (resp. semi-free) G_k-action on $\mathbb{L}_k$ (resp. $v_k \star \mathbb{L}_k$).

By Theorem 1.2.4 and the triangulation theorem for μ^n-manifolds [Be], we can take a PL-manifold $|M_0|$ with the triangulation M_0 and an n-invertible proper UV^{n-1}-surjection $f_{M_0} : M \setminus X \to |M_0|$ satisfying the following:

(i) $f_{M_0}^{-1}(L)$ is a μ^n-manifold for each closed subpolyhedron L of $|M_0|$ and

(ii) $f_{M_0}^{-1}(Z)$ is a Z-set in $f_{M_0}^{-1}(L)$ for each closed subcomplex L of $|M_0|$ and for each Z-set Z in L.

Take a tower $\{U_k\}_{k=0}^{\infty}$ of finite subcomplexes of M_0 so that

$$|U_k| \subset \mathrm{int}_{|M_0|}|U_{k+1}|, \quad |M_0| = \bigcup_{k=0}^{\infty} |U_k|$$

and $|U_k|$ is a compact PL-submanifold of $|M_0|$ such that

$$|W_k| = \mathrm{cl}_{|M_0|}(|M_0| \setminus |U_k|) \cap |U_k|$$

is a Z-set in each of $|U_k| \setminus \mathrm{int}_{|M_0|}|U_{k-1}|$ and $|U_{k+1}| \setminus \mathrm{int}_{|M_0|}|U_k|$ $(U_{-1} = \emptyset)$. Let W_k be the triangulation of $|W_k|$ induced by M_0. By K_k we denote the triangulation of $|U_k| \setminus \mathrm{int}_{|M_0|}|U_{k-1}|$ induced by M_0. Put $K_j^0 = K_j^{(n)}$ $W_j^0 = W_j^{(n)}$ and $B_j^0 = W_j^0$ for each $j \geq 0$, where $K_j^{(n)}$ denotes the n-skeleton of K_j. We define an n-dimensional simplicial complex P_0 as follows:

$$P_0 = K_0^0 \cup_{B_0^0} K_1^0 \cup_{B_1^0} K_2^0 \cup_{B_2^0} K_3^0 \cup_{B_3^0} \cdots .$$

Assume that K_j^{k-1}, W_j^{k-1} and B_j^{k-1} have been constructed for each $j \geq 0$. Note that for each integer $k \geq 0$, there are non-negative integers i and l such that $\sum_{\alpha=0}^{i} \alpha \leq k = \sum_{\alpha=0}^{i} \alpha + l < \sum_{\alpha=0}^{i+1} \alpha$ and that the integers i and l are uniquely defined by k. Hence we define functions $\lambda, \delta : \mathbb{Z}^{\geq 0} \to \mathbb{Z}^{\geq 0}$ so that

$$k = \sum_{\alpha=0}^{\lambda(k)} \alpha + \delta(k), \quad 0 \leq \delta(k) \leq \lambda(k).$$

Then we define K_j^k, W_j^k and B_j^k as follows:

$$K_j^k = \begin{cases} \left(K_j^{k-1} \times \mathbb{L}_{\delta(k)}^{\lambda(k)}\right)^{(n)} & j < \delta(k), \\ \left(\left(K_{\delta(k)}^{k-1} \times \mathbb{L}_{\delta(k)}^{\lambda(k)}\right) \\ \quad \bigcup_{W_{\delta(k)}^{k-1} \times \mathbb{L}_{\delta(k)}^{\lambda(k)}} \left(W_{\delta(k)}^{k-1} \times \left(v_{\delta(k)}^{\lambda(k)} \star \mathbb{L}_{\delta(k)}^{\lambda(k)}\right)\right)\right)^{(n)} & j = \delta(k), \\ \beta(K_j^{k-1}) & j > \delta(k), \end{cases}$$

$$W_j^k = \begin{cases} \left(W_j^{k-1} \times \mathbb{L}_{\delta(k)}^{\lambda(k)}\right)^{(n)} & j < \delta(k), \\ W_{\delta(k)}^{k-1} \times \{v_{\delta(k)}^{\lambda(k)}\} & j = \delta(k), \\ \beta(W_j^{k-1}) & j > \delta(k), \end{cases}$$

$$B_j^k = \begin{cases} \left(W_j^{k-1} \times \beta\big(L_{\delta(k)}^{\lambda(k)}\big)\right)^{(n)} & j < \delta(k), \\ W_{\delta(k)}^{k-1} \times \{v_{\delta(k)}^{\lambda(k)}\} = W_{\delta(k)}^k & j = \delta(k), \\ \beta(W_j^{k-1}) = W_j^k & j > \delta(k), \end{cases}$$

where $\mathbb{L}_j^i$ (resp. L_j^i) is a copy of $\mathbb{L}_j$ (resp. L_j) for each i. Note that W_j^k (resp. B_j^k) is a subcomplex of K_j^k (resp. W_j^k). Then we define an n-dimensional simplicial complex P_k as follows:

$$P_k = K_0^k \cup_{B_0^k} K_1^k \cup_{B_1^k} K_2^k \cup_{B_2^k} K_3^k \cup_{B_3^k} \cdots .$$

Let $r_{k-1}^k : |P_k| \to |P_{k-1}|$ be the map induced by the canonical projections $|K_j^k| \to |K_j^{k-1}|$. Note that each projections $r_{k-1}^k| : |K_j^k| \to |K_j^{k-1}|$ and $r_{k-1}^k|| : |B_j^k| \to |B_j^{k-1}|$ induce isomorphisms of homotopy groups of dim $< n-1$.

Put

$$N = \varprojlim\{|P_k|, r_k^{k+1}\}_{k=0}^{\infty},$$

$$N_j = \varprojlim\{|K_j^k|, r_k^{k+1}|\}_{k=0}^{\infty},$$

$$B_j = \varprojlim\{|B_j^k|, r_k^{k+1}||\}_{k=0}^{\infty}$$

for each j. By the construction, we have

$$N = N_0 \cup_{B_0} N_1 \cup_{B_1} N_2 \cup_{B_2} N_3 \cup_{B_3} \cdots .$$

Claim 1.1.1. *N_j and B_j are μ^n-manifolds for each j.*

Sublemma. *For each map $f : \mathbb{B}^n \to |K_j^{k-1}|$ and each map $g : \mathbb{S}^{n-1} \to |K_j^k|$ with $r_{k-1}^k g = f|\mathbb{S}^{n-1}$, there exist an extension $\hat{g} : \mathbb{B}^n \to |K_j^k|$ of g such that $r_{k-1}^k \hat{g}$ and f are $St(\beta(K_j^{k-1}), St(\beta^2(K_j^{k-1})))$-close. In particular, we can take h so that $r_m^k \hat{g}$ and $r_m^{k-1} f$ are $St(\beta^{k-m}(K_j^m), St(\beta^{k-m-1}(K_j^m)))$-close, for $m \leq k-1$.*

Proof of Sublemma. We show the case $j = \delta(k)$. Put

$$K = \left(K_j^{k-1} \times \mathbb{L}_j^{\lambda(k)}\right) \bigcup_{W_j^{k-1} \times \mathbb{L}_j^{\lambda(k)}} \left(W_j^{k-1} \times \left(v_j^{\lambda(k)} \star \mathbb{L}_j^{\lambda(k)}\right)\right),$$

$$\widehat{T} = \left|K_j^{k-1} \bigcup_{W_j^{k-1} \times \{0\}} \left(W_j^{k-1} \times [0,1]\right)\right|$$

and

$$T = \widehat{T} \setminus |W_j^{k-1} \times \{1\}|.$$

Note that K is a finite simplicial complex such that $K^{(n)} = K_j^k$. Since $|K| \setminus |W_j^{k-1} \times \{v_j^{\lambda(k)}\}| \cong T \times \mathbb{L}_j^{\lambda(k)}$, we identify these spaces. Let $g| = (g_1, g_2) : S' \to T \times |\mathbb{L}_j^{\lambda(k)}|$ be the restriction of g, where $S' = \mathbb{S}^{n-1} \setminus g^{-1}(W_j^{k-1} \times \{v_j^{\lambda(k)}\})$. Let

$p_1' : T \times |\mathbb{L}_j^{\lambda(k)}| \to T$ be the projection. Then p_1' can be extended to the map $p_1 : K \to \widehat{T}$ so that

$$p_1||T \times \mathbb{L}_j^{\lambda(k)}| = p_1' \text{ and } p_1(x, v_j^{\lambda(k)}) = (x, 0) \text{ for } x \in W_j^{k-1}.$$

Let $p_2 : \widehat{T} \to |K_j^{k-1}|$ be the canonical map such that

$$p_2||K_j^{k-1}| = \text{id and } p_2(x,t) = x \text{ for } (x,t) \in |W_j^{k-1} \times [0,1]|.$$

One should note that $p_2 p_1||K_j^k| = r_{k-1}^k$.

Let v be a vertex of $\mathbb{L}_j^{\lambda(k)}$. We identify $|K_j^{k-1}|$ with $|K_j^{k-1} \times \{v\}| \subset |K_j^k|$. Note that $|\mathbb{L}_j^{\lambda(k)}|$ is $AE(n)$ since it is $(n-1)$-connected ANR. Hence there exists a homotopy $h^1 : S' \times [0,1] \to |\mathbb{L}_j^{\lambda(k)}|$ such that $h_0^1 = g_2$ and $h_1^1 = v$. We define $h^2 : S' \times [0,1] \to T \times |\mathbb{L}_j^{\lambda(k)}|$ by $h^2(x,t) = (g(x), h^1(x,t))$. Then h^2 is a homotopy such that

$$h_0^2 = g|,\ h_1^2 = p_1 g| \text{ and } p_1 h_t^2 = p_1 g| \text{ for any } t \in [0,1].$$

Let $H^1 : \mathbb{S}^{n-1} \times [0,1] \to |K|$ be the extension of h^2 such that

$$H^1(x,t) = g(x) \text{ for } x \in g^{-1}(W_j^{k-1} \times \{v_j^{\lambda(k)}\}),\ t \in [0,1].$$

Since $r_{k-1}^r g = f$ and $p_2 p_1 g = r_{k-1}^k g = f|\mathbb{S}^{n-1}$, using the $[0,1]$-factor, we obtain the canonical homotopy $H^2 : \mathbb{S}^{n-1} \times [0,1] \to \widehat{T}$ such that $H_0^2 = p_1 g$, $H_1^2 = f|\mathbb{S}^{n-1}$ and $p_2 H_t^2 = f|\mathbb{S}^{n-1}$ for $t \in [0,1]$. Define a homotopy $H : \mathbb{S}^{n-1} \times [0,1] \to K$ by

$$H(x,t) = \begin{cases} H^1(x, 2t) & 0 \le t \le 1/2, \\ H^2(x, 2t-1) & 1/2 \le t \le 1. \end{cases}$$

Then H is a homotopy such that $H_0 = g$, $H_1 = f|\mathbb{S}^{n-1}$ and $p_2 p_1 H_t = f|\mathbb{S}^{n-1}$ for $t \in [0,1]$. By the essentiality of maps [HW] (cf. [GHW]), there is a homotopy $h : \mathbb{S}^{n-1} \times [0,1] \to |K_j^k|$ such that $h_0 = g$, $h_1 = f|\mathbb{S}^{n-1}$ and h_t and H_t are

K_j^k-close for $t \in [0,1]$. Then $r_{k-1}^k h_t$ and $f|\mathbb{S}^{n-1}$ are $\beta(K_j^{k-1})$-close by our definition for $t \in [0,1]$.

Choose $\eta > 0$ so that $f(x)$ and $f(y)$ are $\mathrm{St}(\beta^2(K_j^{k-1}))$-close whenever $\|x - y\| < \eta$, $x, y \in \mathbb{B}^n$. Take $t > 0$ so that $1 - t < \eta$ and let $\varepsilon = (1-t)/2$. Define $f' : \mathbb{B}^n \to K$ and $\hat{g} : \mathbb{B}^n \to |K_j^k|$ by

$$f'(x) = \begin{cases} f(x) & 0 \le \|x\| \le t, \\ f\big((\frac{(1-t-\varepsilon)\|x\| + (\varepsilon+t)^2 - t}{\varepsilon(t+\varepsilon)})x\big) & t \le \|x\| \le t+\varepsilon, \\ H\big(\frac{x}{\|x\|}, \frac{1-\|x\|}{1-t-\varepsilon}\big) & t+\varepsilon \le \|x\| \le 1. \end{cases}$$

$$\hat{g}(x) = \begin{cases} f(x) & 0 \le \|x\| \le t, \\ f\big((\frac{(1-t-\varepsilon)\|x\| + (\varepsilon+t)^2 - t}{\varepsilon(t+\varepsilon)})x\big) & t \le \|x\| \le t+\varepsilon, \\ h\big(\frac{x}{\|x\|}, \frac{1-\|x\|}{1-t-\varepsilon}\big) & t+\varepsilon \le \|x\| \le 1. \end{cases}$$

Then f' and $\hat{g}$ are well-defined maps and are extensions of g. Observe that $r_{k-1}^k f'$ and $r_{k-1}^k \hat{g}$ are $\beta(K_j^{k-1})$-close and

$$r_{k-1}^k f'(x) = \begin{cases} f(x) & 0 \le \|x\| \le t, \\ f\big((\frac{(1-t-\varepsilon)\|x\| + (\varepsilon+t)^2 - t}{\varepsilon(t+\varepsilon)})x\big) & t \le \|x\| \le t+\varepsilon, \\ f\big(\frac{x}{\|x\|}\big) & t+\varepsilon \le \|x\| \le 1. \end{cases}$$

Since $\|x - (\frac{(1-t-\varepsilon)\|x\| + (\varepsilon+t)^2 - t}{\varepsilon(t+\varepsilon)})x\| < \eta$ for $t \le \|x\| \le t+\varepsilon$ and $\|x - \frac{x}{\|x\|}\| < \eta$ for $t+\varepsilon \le \|x\| \le 1$, $r_{k-1}^k f'$ and f are $\mathrm{St}(\beta^2(K_j^{k-1}))$-close, i.e.

$$r_{k-1}^k \hat{g} \overset{\beta(K_j^{k-1})}{\longleftrightarrow} r_{k-1}^k f' \overset{\mathrm{St}(\beta^2(K_j^{k-1}))}{\longleftrightarrow} f.$$

Thus $r_{k-1}^k \hat{g}$ and f are $\mathrm{St}(\beta(K_j^{k-1}), \mathrm{St}(\beta^2(K_j^{k-1})))$-close. The rest parts are now obvious from our definitions. □

Proof of Claim 1.1.1. Note that N_j and B_j are locally compact. Thus all we have to do is to check the conditions of Bestvina's characterization. The proofs of the n-dimensionality and the DD^nP of N_j are the same with [GHW, Theorem 1] and we left to the reader. We only show that N_j is LC^{n-1} because the proof for B_j is essentially the same.

Let $x \in N_j$ be a point and let $U \subset N_j$ be a neighborhood of x. Then there is an open neighborhood U_N of $x_N = r_N^\infty(x)$ in $|K_j^N|$ such that $(r_N^\infty)^{-1}(U_N) \subset U$ for some $N \in \mathbb{N}$. Take $a \geq N$ and a neighborhood V of x_a so that

$$\mathrm{St}^4(r_N^a(V), \beta^{a-N} K_j^a) \subset U_N.$$

Since $|K_j^a|$ is ANR, there is a neighborhood $W \subset V$ of x_a such that any map from $\mathbb{S}^{n-1}$ to W can be extended to a map from $\mathbb{B}^n$ to V. Let $f : \mathbb{S}^{n-1} \to N_j$ be a map such that $f(\mathbb{S}^{n-1}) \subset (r_a^\infty)^{-1}(W) \subset U$. Then there is an extension $g_a : \mathbb{B}^n \to V$ of $r_a^\infty f$. For $i < a$, let $g_i = r_i^a g_a : \mathbb{B}^n \to |K_j^i|$. Using Sublemma, we can inductively construct extensions $g_i : \mathbb{B}^n \to |K_j^i|$ of $r_i^\infty f$ so that $r_m^i g_i$ and $r_m^{i-1} g_{m-1}$ are $\mathrm{St}(\beta^{i-m}(K_j^m), \mathrm{St}(\beta^{i-m-1}(K_j^m)))$-close for $m \leq i-1$, $i \geq a$. Since the sequence $\{r_m^i g_i\}_{i=0}^\infty$ is uniformly convergence, the limit map $h_m = \lim_{i \to \infty} r_m^i g_i : \mathbb{B}^n \to X_m$ is continuous and clearly satisfies the conditions $r_{m-1}^m h_m = h_{m-1}$ and $h_m | \mathbb{S}^{n-1} = r_m^\infty f$. Then $h = \varprojlim h_m : \mathbb{B}^n \to N_j$ is an extension of f. Since h_a and g_a is $\mathrm{St}^4(\beta^{a-N} K_j^a)$-close,

$$r_a^N h_N(\mathbb{B}^n) = h_a(\mathbb{B}^n) \subset \mathrm{St}^4(r_N^a(V), \beta^{a-N} K_j^a) \subset U_N.$$

Thus we have $h(\mathbb{B}^n) \subset U$. Hence N_j is LC^{n-1}. □

Claim 1.1.2. *B_j is a Z-set in each of N_j and N_{j+1} for each j.*

Proof of Claim 1.1.2. Roughly speaking, the claim follows from the fact that B_j is *infinite deficient*[1] in each of N_j and N_{j+1}. We only show that B_j is a Z-set in N_j. Let $f : \mathbb{I}^n \to N_j$ be a map and let $\varepsilon > 0$ be given. Choose i_0 so that $\sum_{l=i_0}^\infty 2^{-l} < \varepsilon$ and $\delta(i_0)$, $\delta(i_0+1) \geq j+1$. For each $k \leq i_0$, let $g_k = r_k^\infty f : \mathbb{I}^n \to |K_j^k|$. Let v be a vertex of $L_{\delta(i_0+1)}^{\lambda(i_0+1)}$. Since $|K_j^{i_0}| \times \{(v,1)\} \subset |K_j^{i_0+1}|$, we

[1] The notion *infinite deficiency* in μ^n-manifolds was introduced in [Iw2] to characterize Z-sets in μ^n-manifolds in terms of infinite-deficiency (cf. [An3])

define a map $g_{i_0+1} : \mathbb{I}^n \to |K_j^{i_0+1}|$ as follows:

$$g_{i_0+1}(x) = (g_{i_0}(x), v, 1) \in |K_j^{i_0}| \times |L_{\delta(i_0+1)}^{\lambda(i_0+1)}| \times [0,1] \left(\equiv |K_j^{i_0}| \times |\mathbb{L}_{\delta(i_0+1)}^{\lambda(i_0+1)}| \right)$$

Using the fact that $r_k^{k+1} : |K_j^{k+1}| \to |K_j^k|$ is a retraction for each k, we can choose a map $g_k : \mathbb{I}^n \to |K_j^k|$ $(k \geq i_0 + 1)$ so that $g_k = r_k^{k+1} g_{k+1}$. Then $g = \varprojlim g_k : \mathbb{I}^n \to N_j$ is a map ε-close to f. By our construction of B_j, it is easy to see that $g(\mathbb{I}^n) \cap B_j = \emptyset$. This finishes the proof of Claim 1.1.2. □

Step 2. Construction of a homeomorphism between N and $M \setminus X$.

Let $\widetilde{N}_j = f_{M_0}^{-1}(|K_j|)$ and $\widetilde{B}_j = f_{M_0}^{-1}(|W_j|) = \widetilde{N}_j \cap \widetilde{N}_{j+1}$. Then $\widetilde{N}_j$ and $\widetilde{B}_j$ are compact μ^n-manifolds by (i) and $\widetilde{B}_j$ is a Z-set in each of $\widetilde{N}_j$ and $\widetilde{N}_{j+1}$ by (ii). Since f_{M_0} is n-invertible, there is a map $p_j : |K_j^0| \to \widetilde{N}_j$ such that $f_{M_0} p_j = \mathrm{id}_{|K_j^0|}$. Observe that p_j induces isomorphisms of homotopy groups of dim $\leq n-1$. Then $r_j = p_j r_0^{\infty}| : N_j \to \widetilde{N}_j$ and $r_j' = p_j r_0^{\infty}| : B_j \to \widetilde{B}_j$ are maps between compact μ^n-manifolds that induce isomorphisms of homotopy groups of dim $\leq n-1$.

By the classification theorem for μ^n-manifolds [Be, 2.8.6], there exist homeomorphisms $h_0 : N_0 \to \widetilde{N}_0$ and $s_0 : B_0 \to \widetilde{B}_0$ such that $h_0 \simeq^{n-1} r_0$ and $s_0 \simeq^{n-1} r_0'$. Then we have

$$h_0 s_0^{-1} \simeq^{n-1} r_0 s_0^{-1} = r_0' s_0^{-1} \simeq^{n-1} \mathrm{id}_{\widetilde{B}_0}.$$

Using the Z-set unknotting theorem [Be, 3.1.4], there is a homeomorphism $f_0' : \widetilde{N}_0 \to \widetilde{N}_0$ such that $f_0'|\widetilde{B}_0 = h_0 s_0^{-1}$. Then $(f_0')^{-1}|h_0(B_0) = s_0 h_0^{-1}|h_0(B_0)$. In fact, for each $x \in h_0(B_0)$, we can represent x as $h_0 s_0^{-1}(y)$ for some $y \in \widetilde{B}_0$ since $f_0'(\widetilde{B}_0) = h_0 s_0^{-1}(\widetilde{B}_0) = h_0(B_0)$. So we have

$$(f_0')^{-1}(x) = (f_0')^{-1} h_0 s_0^{-1}(y) = (f_0')^{-1} f_0'(y) = y = s_0 h_0^{-1}(x).$$

Thus $f_0 = (f_0')^{-1}h_0 : N_0 \to \widetilde{N}_0$ is a homeomorphism such that $f_0|B_0 = s_0 \simeq^{n-1} r_0'$.

Assume that $f_{j-1} : N_{j-1} \to \widetilde{N}_{j-1}$ has been constructed so that $f_{j-1}|B_{j-2} = f_{j-2}|B_{j-2}$ and $f_{j-1}|B_{j-1} \simeq^{n-1} r_{j-1}'$ $(B_{-1} = \emptyset)$. As before, there exist homeomorphisms $h_j : N_j \to \widetilde{N}_j$ and $s_j : B_j \to \widetilde{B}_j$ such that $h_j \simeq^{n-1} r_j$ and $s_j \simeq^{n-1} r_j'$. Then the map $\widetilde{s}_j = f_{j-1} \cup s_j : B_{j-1} \cup B_j \to \widetilde{B}_{j-1} \cup \widetilde{B}_j$ is such that $\widetilde{s}_j \simeq^{n-1} r_j'$ since $r_j'|B_{j-1} = r_{j-1}'$. Hence we have

$$h_j(\widetilde{s}_j)^{-1} \simeq^{n-1} r_j(\widetilde{s}_j)^{-1} \simeq^{n-1} \mathrm{id}_{\widetilde{B}_{j-1} \cup \widetilde{B}_j}.$$

By the Z-set unknotting theorem, there is a homeomorphism $f_j' : \widetilde{N}_j \to \widetilde{N}_j$ such that $f_j'|\widetilde{B}_{j-1} \cup \widetilde{B}_j = h_j(\widetilde{s}_j)^{-1}$. Note that $(f_j')^{-1}|h_j(\widetilde{B}_{j-1} \cup \widetilde{B}_j) = (\widetilde{s}_j)^{-1}h_j|h_j(\widetilde{B}_{j-1} \cup \widetilde{B}_j)$. Then $f_j = (f_j')^{-1}h_j : N_j \to \widetilde{N}_j$ is a homeomorphism such that $f_j|B_{j-1} = \widetilde{s}_j(h_j)^{-1}h_j|B_{j-1} = \widetilde{s}_j|B_{j-1} = f_{j-1}$ and $f_j|B_j = \widetilde{s}_j|B_j = s_j \simeq^{n-1} r_j'$.

Thus the map $f : N \to M \setminus X$ defined by $f|N_j = f_j$ is a well-defined homeomorphism.

Step 3. Construction of a semi-free G-action.

First we shall define a free G-action on N. Let

$$G_j' = \left\{ \big(\varphi_0(g), \ldots, \varphi_{j-1}(g), g\big) \;\Big|\; g \in G_j \right\}.$$

Since $f|N_j = f_j : N_j \to \widetilde{N}_j$ is a homeomorphism between compacta, we can inductively obtain an increasing sequence $1 < i(0) < i(1) < i(2) < i(3) < \cdots$ of natural numbers satisfying the following:

(A) if $x, y \in N_j$ and $\mathrm{d}(x,y) < 2^{-i(j)}$ then $d(f(x), f(y)) < 2^{-j}$.

In case $\sum_{\alpha=0}^{i(j)} \alpha + j \leq \xi \leq \sum_{\alpha=0}^{i(j+1)} \alpha + j$, $|P_\xi|$ has $\mathbb{L}_l^{i(l)}$(or $v_l^{i(l)} \star \mathbb{L}_l^{i(l)}$)-factor for each $l \leq j$ and does not have $\mathbb{L}_m^{i(m)}$(or $v_m^{i(m)} \star \mathbb{L}_m^{i(m)}$)-factor for any $m \geq j+1$.

We define a G'_j-action using the only G_l-actions of $\mathbb{L}_l^{i(l)}$ and $v_l^{i(l)} \star \mathbb{L}_l^{i(l)}$, $l \leq j$. Let $\zeta = \sum_{\alpha=0}^{i(j)} \alpha + j$. Then $\lambda(\zeta) = i(j)$ and $\delta(\zeta) = j$. Observe that G'_j acts freely on

$$|K_0^\xi| \cup_{|B_0^\xi|} \cdots \cup_{|B_{j-1}^\xi|} |K_j^\xi| \setminus (r_\zeta^\xi)^{-1}(W_j^\zeta)$$

and acts trivially on

$$(r_\zeta^\xi)^{-1}(W_j^\zeta) \cup (|K_{j+1}^\xi| \cup_{|B_{j+1}^\xi|} |K_{j+2}^\xi| \cup_{|B_{j+2}^\xi|} \cdots).$$

In particular

(B) G'_j acts freely on $|K_0^\xi| \cup_{|B_0^\xi|} \cdots \cup_{|B_{j-2}^\xi|} |K_{j-1}^\xi|$ and acts trivially on $|K_{j+1}^\xi| \cup_{|B_{j+1}^\xi|} |K_{j+2}^\xi| \cup_{|B_{j+2}^\xi|} \cdots$.

Considering G as the diagonal subgroup of $\Pi_{i=0}^{\infty} G_i$, we define a G-action on N as follows:

$$(g_0, g_1, g_2, \dots)(x_0, x_1, x_2, \dots) = (g'_0 x_0, g'_1 x_1, g'_2 x_2, \dots),$$

where $x_\xi \in |P_\xi|$ and $g'_\xi = (g_0, g_1, g_2, \dots, g_j)$ for each $\sum_{\alpha=0}^{i(j)} \alpha + j \leq \xi \leq \sum_{\alpha=0}^{i(j+1)} \alpha + j$. For $l \leq j-1$ and $\sum_{\alpha=0}^{i(l)} \alpha + l \leq \xi \leq \sum_{\alpha=0}^{i(l+1)} \alpha + l$, G'_l acts trivially on

$$|K_{l+1}^\xi| \cup_{|B_{l+1}^\xi|} |K_{l+2}^\xi| \cup_{|B_{l+2}^\xi|} \cdots \supset |K_j^\xi| \cup_{|B_j^\xi|} |K_{j+1}^\xi| \cup_{|B_{j+1}^\xi|} \cdots$$

by (B). Hence if $x = (x_0, x_1, x_2, \cdots) \in N_j$ and $g \in G$ then

$$gx = (g'_0 x_0, g'_1 x_1, g'_2 x_2, \cdots)$$

and $g'_k x_k = x_k$ for each $k \leq \sum_{\alpha=0}^{i(j)} \alpha + j - 1$. Since $\sum_{\alpha=0}^{i(j)} \alpha + j - 1 \geq i(j) + 1$, we have

(C) $\mathrm{d}(gx, x) < 2^{-i(j)}$ whenever $x \in N_j$, $g \in G$.

Now it is easy to see that the G-action on N is free.

We define a function $\Theta : G \times M \to M$ by

$$\Theta(g, x) = \begin{cases} fgf^{-1}(x) & x \in M \setminus X, \\ x & x \in X. \end{cases}$$

Then Θ is continuous. In fact, let $\{(g_i, x_i)\}_{i=1}^{\infty}$ be a sequence such that $x_i \in M \setminus X$ and $\lim_{i\to\infty}(g_i, x_i) = (g_0, x_0) \in X$. For a given $\varepsilon > 0$, take $j_1 > 0$ so that $2^{-j_1} < \varepsilon/2$ and $\mathrm{d}(x_0, x_i) < \varepsilon/2$ for $i \geq j_1$. Let $j_2 > j_1$ be such that $x_i \notin \cup_{j=0}^{j_1} N_j$ for $i \geq j_2$. Since $f(N_j) = f_j(N_j) = \widetilde{N}_j$, we have $f^{-1}(x_i) \notin \cup_{j=0}^{j_1} N_j$ for $i \geq j_2$. By (C), $\mathrm{d}(f^{-1}(x_i), g_i f^{-1}(x_i)) < 2^{-i(j_1)}$ for $i \geq j_2$. By (A), we have

$$\begin{aligned} \mathrm{d}\big(\Theta(g_i, x_i), x_i\big) &= \mathrm{d}\big(fg_if^{-1}(x_i), x_i\big) \\ &= \mathrm{d}\big(fg_if^{-1}(x_i), ff^{-1}x_i\big) \\ &< 2^{-j_1} \\ &< \varepsilon/2 \end{aligned}$$

for $i \geq j_2$. Then for each $i \geq j_2$,

$$\begin{aligned} \mathrm{d}\big(\Theta(g_i, x_i), \Theta(g_0, x_0)\big) &= \mathrm{d}\big(\Theta(g_i, x_i), x_0\big) \\ &\leq \mathrm{d}\big(\Theta(g_i, x_i), x_i\big) + \mathrm{d}(x_i, x_0) \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

Moreover, for $g, g' \in G$, $x \in M \setminus X$,

$$\begin{aligned} \Theta\big(g', \Theta(g, x)\big) &= fg'f^{-1}\big(fgf^{-1}(x)\big) \\ &= fg'gf^{-1}(x) \\ &= \Theta(g'g, x). \end{aligned}$$

Thus Θ defines a G-action on M. Clearly, the action Θ is semi-free and satisfies our required condition. The proof is finished. □

Recall that a space X has *the complete invariance property with respect to homeomorphisms* (CIPH) if each non-empty closed subset of X is the fixed point set of some autohomeomorphism of X. As a direct consequence of Theorem 2.1, we obtain the affirmative answer to the questions [CKT2, Problems 6.4.3, 6.4.4].

Corollary 6.1.1. *Every μ^n-manifold has CIPH.*

§6.2. Pseudo-interiors and pseudo-boundaries

Let M be a μ^n-manifold. By $\mathcal{Z}_M$ (resp. $\mathcal{Z}_M^c$), we denote the collection of all Z-sets (resp. all compact Z-sets) in M. A $\mathcal{Z}_M$-absorber A of M is called a *pseudo-boundary* of M and the complement $M \setminus A$ is called a *pseudo-interior* of M (cf. [Ch2], [CKT1]). In case M is compact, every $\mathcal{Z}_M$-skeletoid is a $\mathcal{Z}_M$-absorber, therefore a pseudo-boundary of M since $\mathcal{Z}_M (\equiv \mathcal{Z}_M^c)$ is a perfect collection. The uniqueness of topological types of pseudo-boundaries (pseudo-interiors) follows from [BP, Chap. IV, Theorem 2.1]. According to [CKT1], the topological type of a pseudo-interior $\nu(\mu^n)$ of μ^n is equal to the n-dimensional Nöbeling space ν^n. The following criterion is a modification of [CKT1, Proposition 3.3.11].

Proposition 6.2.1. *Let M be a compact μ^n-manifold and let $\{A_i\}_{i=1}^{\infty}$ be a tower of Z-sets in M with the following properties:*

(1) $\forall \varepsilon > 0, \exists m > 0$ *such that A_m is ε-dense in M,*

(2) *A_i is a Z-set in each of A_{i+1} and M,*

(3) *$\{A_i\}_{i=1}^{\infty}$ is equi-LC^{n-1} and*

(4) *A_i is a μ^n-manifold.*

Then $\{A_i\}_{i=1}^{\infty}$ is a $\mathcal{Z}_M$-skeleton of M, i.e., the union $\cup_{i=1}^{\infty} A_i$ is a pseudo-boundary of M.

Proof. Let $\varepsilon > 0$ be a positive number and let Z be a Z-set in M. We fix a member A_k of $\{A_i\}_{i=1}^{\infty}$. Since $\{A_i\}_{i=1}^{\infty}$ is equi-LC^{n-1}, there exists a positive number $\delta < \varepsilon/2$ such that for any δ-close two maps $f, g : Z \cap A_k \to A_i$ with an extension $\hat{f} : Z \to A_i$ of f, there exists an extension $\hat{g} : Z \to A_i$ of g such that $\hat{g}$ and $\hat{f}$ are $\varepsilon/2$-close for any $i \in \mathbb{N}$. As in the proof of [Dr1, Lemma 2.1], we can take a map $\gamma : Z \to A_j$ so that $\mathrm{d}(\gamma, \mathrm{id}_Z) < \delta$ for some $j > k$ by (1) and (3). Note that $Z \cap A_k$ is a Z-set in A_j by (2). Since $\mathrm{d}(\gamma|Z \cap A_k, \mathrm{id}_{Z\cap A_k}) < \delta$, there is a map $\gamma' : Z \to A_j$ such that $\gamma'|Z \cap A_k = \mathrm{id}_{Z\cap A_k}$ and $\mathrm{d}(\gamma', \gamma) < \varepsilon/2$. By (4), we may assume that γ' is a Z-embedding using the Z-embedding approximation theorem [Be]. Then $\mathrm{d}(\gamma', \mathrm{id}_Z) \leq \mathrm{d}(\gamma', \gamma) + \mathrm{d}(\gamma, \mathrm{id}_Z) < \delta + \varepsilon/2 < \varepsilon$. Thus the proposition follows from Proposition 1.1. □

The next proposition follows from the standard arguments using the fact that every μ^n-manifold is locally compact, so we omit the proof.

Proposition 6.2.2. *Every $\mathcal{Z}_M^c$-absorber in M is also a $\mathcal{Z}_M$-absorber in M, i.e., a pseudo-boundary of M.*

Theorem 6.2.1. *Let X be a closed subset of a μ^n-manifold M and let G be a compact separable zero-dimensional group with the unit element e. Then there exist a semi-free G-action of M and a G-invariant pseudo-boundary $\Sigma(M)$ of M such that X is the fixed point set of any $g \in G \setminus \{e\}$.*

Proof. As in the proof of Theorem 6.1.1, we represent the group G as the inverse limit of an inverse sequence

$$G_0 \xleftarrow{\varphi_0} G_1 \xleftarrow{\varphi_1} G_2 \xleftarrow{\varphi_2} G_3 \xleftarrow{\varphi_3} \cdots$$

consisting of non-trivial finite groups.

Let L_k be an $(n-1)$-connected finite simplicial complex with free G_k-action. Put $\mathbb{L}_k = L_k \times [0,1]$ and $\mathbf{L}_k = \mathbb{L}_k \times [0,1]$. Note that L_k, $\mathbb{L}_k$ and $\mathbf{L}_k$ are all $(n-1)$-connected finite simplicial complex with free G_k-action and that $\beta(\mathbb{L}_j) = \mathbb{L}_j \times \{0\}$ (resp. $\beta^2(L_j) = \beta(L_j) \times \{0\}$) is an invariant subset of $\mathbf{L}_j$ (resp. $\beta(\mathbb{L}_j)$) for each j.

The G-action on M we used here is essentially the same with the one constructed in the proof of Theorem 6.1.1. So, in what follows, we use the notation given in the proof of Theorem 6.1.1. The only difference is that we use $\mathbf{L}_j$ (resp. $\mathbb{L}_j$) in place of $\mathbb{L}_j$ (resp. L_j), i.e.,

$$K_j^k = \begin{cases} \left(K_j^{k-1} \times \mathbf{L}_{\delta(k)}^{\lambda(k)}\right)^{(n)} & j < \delta(k), \\ \Big((K_{\delta(k)}^{k-1} \times \mathbf{L}_{\delta(k)}^{\lambda(k)}) \\ \quad \bigcup_{W_{\delta(k)}^{k-1} \times \mathbf{L}_{\delta(k)}^{\lambda(k)}} (W_{\delta(k)}^{k-1} \times (v_{\delta(k)}^{\lambda(k)} \star \mathbf{L}_{\delta(k)}^{\lambda(k)})) \Big)^{(n)} & j = \delta(k), \\ \beta(K_j^{k-1}) & j > \delta(k), \end{cases}$$

$$W_j^k = \begin{cases} \left(W_j^{k-1} \times \mathbf{L}_{\delta(k)}^{\lambda(k)}\right)^{(n)} & j < \delta(k), \\ W_{\delta(k)}^{k-1} \times \{v_{\delta(k)}^{\lambda(k)}\} & j = \delta(k), \\ \beta(W_j^{k-1}) & j > \delta(k), \end{cases}$$

$$B_j^k = \begin{cases} \left(W_j^{k-1} \times \beta(\mathbb{L}_{\delta(k)}^{\lambda(k)})\right)^{(n)} & j < \delta(k), \\ W_{\delta(k)}^{k-1} \times \{v_{\delta(k)}^{\lambda(k)}\} = W_j^k & j = \delta(k), \\ \beta(W_j^{k-1}) = W_j^k & j > \delta(k), \end{cases}$$

where $\mathbf{L}_j^l$ (resp. $\mathbb{L}_j^l$) is a copy of $\mathbf{L}_j$ (resp. $\mathbb{L}_j$) for each l. And the G-action is induced by the G_l-action of $\mathbf{L}_l^{i(l)}$, $l \geq 0$.

Let $V_j = N_0 \cup_{B_0} N_1 \cup_{B_1} \cdots \cup_{B_{j-1}} N_j$. First we shall constructed a tower $\{A_j(i)\}_{i=1}^{\infty}$ of Z-sets of V_j satisfying the following:

(a) $\forall \varepsilon > 0$, $\exists m > 0$ such that $A_j(m)$ is ε-dense in V_j, (*b*) $A_j(i)$ is a Z-set in each of $A_j(i+1)$ and V_j,

(*c*) $\{A_j(i)\}_{i=1}^{\infty}$ is equi-LC^{n-1} and

(*d*) $A_j(i)$ is a μ^n-manifold.

Let $R_j^0(i) = K_j^0$, $S_j^0(i) = W_j^0$ and $C_j^0(i) = B_j^0$ for each $i \geq 1$, $j \geq 0$. Assume that $R_j^l(i)$, $S_j^l(i)$ and $C_j^l(i)$ have been constructed for $l \leq k-1$, $k \geq 1$. Then we define $R_j^k(i)$, $S_j^k(i)$ and $C_j^k(i)$ as follows:

$$R_j^k(i) = \begin{cases} \left(R_j^{k-1}(i) \times \mathfrak{L}_j^k(i)\right)^{(n)} & j < \delta(k), \\ \left(\left(R_{\delta(k)}^{k-1}(i) \times \mathfrak{L}_{\delta(k)}^k(i)\right) \displaystyle\bigcup_{S_{\delta(k)}^{k-1}(i) \times \mathfrak{L}_{\delta(k)}^k(i)} \left(S_{\delta(k)}^{k-1}(i) \times \left(v_{\delta(k)}^{\lambda(k)} \star \mathfrak{L}_{\delta(k)}^k(i)\right)\right)\right)^{(n)} & j = \delta(k), \\ \beta(R_j^{k-1}(i)) & j > \delta(k), \end{cases}$$

$$S_j^k(i) = \begin{cases} \left(S_j^{k-1}(i) \times \mathfrak{L}_j^k(i)\right)^{(n)} & j < \delta(k), \\ S_{\delta(k)}^{k-1}(i) \times \{v_{\delta(k)}^{\lambda(k)}\} & j = \delta(k), \\ \beta(S_j^{k-1}(i)) & j > \delta(k), \end{cases}$$

$$C_j^k(i) = \begin{cases} \left(S_j^{k-1}(i) \times \mathfrak{K}_j^k(i)\right)^{(n)} & j < \delta(k), \\ S_{\delta(k)}^{k-1}(i) \times \{v_{\delta(k)}^{\lambda(k)}\} = S_j^k(i) & j = \delta(k), \\ \beta(S_j^{k-1}(i)) = S_j^k(i) & j > \delta(k), \end{cases}$$

where

$$\mathfrak{L}_j^k(i) = \begin{cases} \beta^2\left(L_{\delta(k)}^{\lambda(k)}\right) & (\exists m \in \mathbb{Z})\ [k = m \cdot 2^i], \\ \mathbf{L}_{\delta(k)}^{\lambda(k)} & \text{otherwise}, \end{cases}$$

$$\mathfrak{K}_j^k(i) = \begin{cases} \beta^2\left(L_{\delta(k)}^{\lambda(k)}\right) & (\exists m \in \mathbb{Z})\ [k = m \cdot 2^i], \\ \beta\left(\mathbb{L}_{\delta(k)}^{\lambda(k)}\right) & \text{otherwise}. \end{cases}$$

We define a simplicial complex $A_j^k(i)$ as follows:

$$A_j^k(i) = R_0^k(i) \cup_{C_0^k(i)} R_1^k(i) \cup_{C_1^k(i)} \cdots \cup_{C_{j-1}^k(i)} R_j^k(i).$$

Note that for each $i \geq 1$, $R_j^k(i)$, $S_j^k(i)$ and $C_j^k(i)$ are subcomplex of K_j^k, W_j^k and B_j^k respectively and $R_j^k(i) \cap W_j^k = C_j^k(i)$. Hence $A_j^k(i)$ is a subcomplex of $K_0^k \cup_{B_0^k} \cdots \cup_{B_{j-1}^k} K_j^k$, i.e., $|A_j^k(i)| \subset P_k$. Moreover, since $|R_j^k(i)| \subset |R_j^k(i+1)|$ and $R_j^k(i) \cap W_j^k = C_j^k(i)$, $|A_j^k(i)|$ is a subset of $|A_j^k(i+1)|$. It is easy to see that $r_{k-1}^k(|A_j^k(i)|) \subset |A_j^{k-1}(i)|$. Thus we can define $A_j(i)$ as the inverse limit of the following inverse sequence

$$|A_j^0(i)| \xleftarrow{r_0^1|} |A_j^1(i)| \xleftarrow{r_1^2|} |A_j^2(i)| \xleftarrow{r_2^3|} |A_j^3(i)| \xleftarrow{r_3^4|} \cdots.$$

Since $|A_j^k(i)| = |K_j^k|$ for each $k \leq 2^i - 1$ and $r_{k-1}^k| : |A_j^k(i)| \to |A_j^{k-1}(i)|$ is a retraction, $\{A_j(i)\}_{i=1}^\infty$ satisfies the condition (a). Since $|A_j^k(i)| \subset |K_0^k \cup_{B_0^k} \cdots \cup_{B_{j-1}^k} K_j^k|$, $A_j(i)$ is a subset of V_j. Also, $A_j(i)$ is a subset of $A_j(i+1)$ since $|A_j^k(i)| \subset |A_j^k(i+1)|$. As in the proofs of Claim 1.1.1 and Claim 1.1.2, one can see that the tower $\{A_j(i)\}_{i=1}^\infty$ satisfies the conditions (b), (c) and (d). The reason that we use $\mathbf{L}_k$ instead of $\mathbb{L}_k$ is to construct $\{A_j(i)\}_{i=1}^\infty$ satisfying the condition (b). Thus $\{A_j(i)\}_{i=1}^\infty$ is a $\mathcal{Z}_{V_j}$-skeleton of V_j by Proposition 6.2.1. We remark that each $A_j(i)$ is an invariant subspace of V_j.

Next we shall construct an invariant pseudo-boundary of M. Put $A_j = \cup_{i=1}^\infty A_j(i)$.

Claim 2.1.1. *$A' = \cup_{j=0}^\infty A_j$ is a $\mathcal{Z}_{M\setminus X}^c$-skeletoid ($\equiv \mathcal{Z}_{M\setminus X}^c$-absorber) in $M \setminus X$.*

Proof of Claim 2.1.1. We note that $\{V_j\}_{j=0}^\infty$ is a compact tower of μ^n-manifolds such that $\cup_{j=0}^\infty V_j = M \setminus X$ and $V_j \subset \mathrm{int}_{M\setminus X} V_{j+1}$. Let $B \in \mathcal{Z}_{M\setminus X}^c$ and let $\mathcal{U}$ be an open collection of $M \setminus X$. Then there is $j_0 > 0$ such that $B \subset \mathrm{int}_{M\setminus X} V_{j_0}$. Note that B is a Z-set in V_{j_0}. Since A_{j_0} is a Z-skeletoid of V_{j_0}, there is a homeomorphism $h : V_{j_0} \to V_{j_0}$ such that $h|V_{j_0} \cap (\mathrm{cl}_{M\setminus X}((M \setminus X) \setminus V_{j_0}) = \mathrm{id}$, $h|(\cup\mathcal{U}) \cap V_{j_0}$ is $\mathcal{U}|V_{j_0}$-close to id and $h(B \cap (\cup\mathcal{U})) \subset A_{j_0}$. In particular, we may assume that h can be extended to a homeomorphism $\hat{h} : M \setminus X \to M \setminus X$ so that $\hat{h}|\mathrm{cl}_{M\setminus X}((M \setminus X) \setminus V_{j_0}) = \mathrm{id}$. Then the homeomorphism $\hat{h}$ is such that

$\hat{h}|(M \setminus X) \setminus (\cup\mathcal{U}) = \mathrm{id}$, $\hat{h}| \cup \mathcal{U}$ is $\mathcal{U}$-close to id and $\hat{h}(B \cap (\cup\mathcal{U})) \subset A'$. Thus A' is a $\mathcal{Z}^c_{M\setminus X}$-absorber in $M \setminus X$. □

Let A be a $\mathcal{Z}^c_M$-skeletoid in M. (The existence of a $\mathcal{Z}^c_M$-skeletoid ($\equiv \mathcal{Z}^c_M$-absorber) in M is assured. In fact, one can easily construct such a $\mathcal{Z}^c_M$-skeletoid as in Claim 2.1.1.) Since $\mathcal{Z}^c_M|M \setminus X = \mathcal{Z}^c_{M\setminus X}$ and $\mathcal{Z}^c_{M\setminus X}$ is a perfect collection, $A \cap (M \setminus X)$ is a $\mathcal{Z}^c_{M\setminus X}$-skeletoid in $M \setminus X$ by Theorem 1.1. By Theorem 1.3.2, there is a homeomorphism $\gamma : M \to M$ such that $\gamma(A') = A \cap (M \setminus X)$ and $\gamma|_X = \mathrm{id}$. Let $\Sigma(M) = A' \cup (X \cap A)$. Then $\gamma(\Sigma(M)) = A$ and $\Sigma(M)$ is an invariant subspace of M since A' is invariant and X is the fixed points set of any $g \in G \setminus \{e\}$. Thus $\Sigma(M)$ is an invariant $\mathcal{Z}^c_M$-skeletoid of M. By Proposition 6.2.2, $\Sigma(M)$ is an invariant pseudo-boundary of M. The proof is finished. □

The proof of Theorem 6.2.1 gives the following.

Corollary 6.2.1 (cf. [CKT1]). *Every μ^n-manifold has a pseudo-interior and a pseudo-boundary.*

Corollary 6.2.2. *Let $\nu(M)$ (resp. $\Sigma(M)$) be a pseudo-interior (resp. a pseudo-boundary) of a μ^n-manifold M. Let G be a compact separable zero-dimensional group with the unit element e. Then for each closed subset X of $\nu(M)$ (resp. $\Sigma(M)$), there exists a semi-free G-action on $\nu(M)$ (resp. $\Sigma(M)$) such that X is the fixed point set of any $g \in G \setminus \{e\}$.*

Proof. We shall give the proof only for $\nu(M)$ because the proof for $\Sigma(M)$ is similar. Let X be a closed subset of $\nu(M)$ and put $\widetilde{X} = \mathrm{cl}_M X$ and $B = M \setminus \nu(M)$. By Theorem 6.1.1, there exists a semi-free G-action $\Theta : G \times M \to M$ with a G-invariant pseudo-boundary A of M such that $\widetilde{X}$ is the fixed point set of any $g \in G \setminus \{e\}$. Since $A \cap (M \setminus \widetilde{X})$ and $B \cap (M \setminus \widetilde{X})$ are $\mathcal{Z}_{M \setminus \widetilde{X}}$-absorbers, there exists a homeomorphism $h : M \to M$ such that $h|\widetilde{X} = \mathrm{id}_{\widetilde{X}}$ and $h(B \cap (M \setminus \widetilde{X})) = A \cap (M \setminus \widetilde{X})$. Then the map $\Theta' : G \times M \to M$ defined by $\Theta'(g, x) = h^{-1}\Theta(g, h(x))$ redefines a semi-free G-action on M. In fact, for $x \in M$ and $g, g' \in G$,

$$\begin{aligned}
\Theta'(g', \Theta'(g, x)) &= \Theta'\Big(g', h^{-1}\Theta\big(g, h(x)\big)\Big) \\
&= h^{-1}\Theta\Big(g', hh^{-1}\Theta\big(g, h(x)\big)\Big) \\
&= h^{-1}\Theta\Big(g', \Theta\big(g, h(x)\big)\Big) \\
&= h^{-1}\Theta\big(g'g, h(x)\big) \\
&= \Theta'(g'g, x).
\end{aligned}$$

Since $\big\{\Theta(g, x) \mid g \in G,\ x \in M \setminus A\big\} = M \setminus A$ and $M \setminus A = h(\nu(M))$, it follows that $\Theta'(G \times \sigma(M)) = \sigma(M)$. It is easy to see that $\widetilde{X} = \big\{x \in M \mid \Theta'(g, x) = x\big\}$

for any $g \in G \setminus \{e\}$. Thus $\Theta_\nu = \Theta' | G \times \nu(M)$ is the required semi-free G-action on $\nu(M)$. $\square$

If we take $X = \emptyset$ in Theorem 3.1, the proof gives the following:

Theorem 6.2.2. *Every μ^n-manifold admits a free G-action with a G-invariant pseudo-boundary for any compact zero-dimensional group G.*

Corollary 6.2.3. *Every pseudo-interior $\nu(M)$ (resp. pseudo-boundary $\Sigma(M)$) of a μ^n-manifold M admits a free G-action for any compact zero-dimensional group G.*

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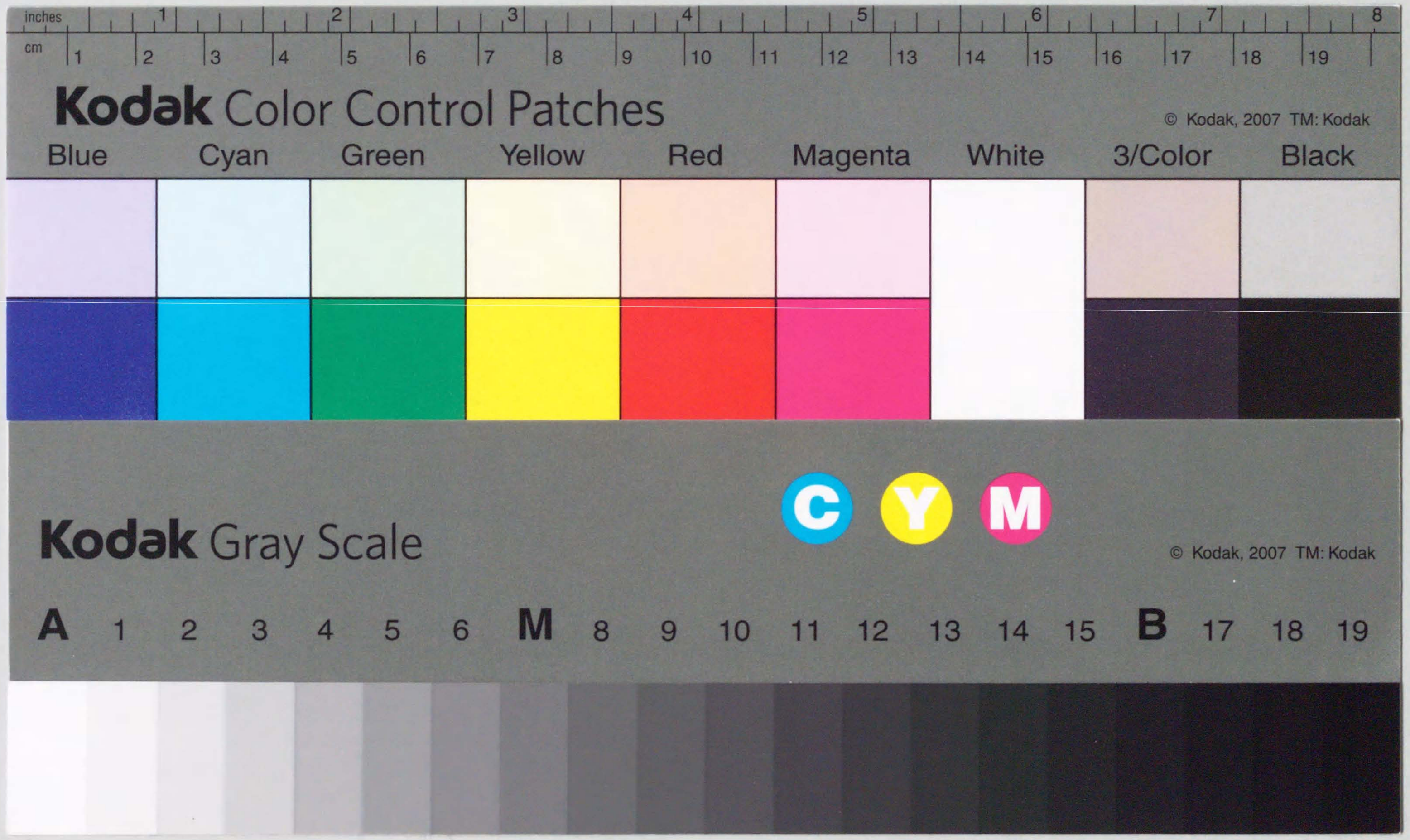
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