Eigenvalues of the hermitian Wilson-Dirac operator and chiral properties of the domain-wall fermion*

CP-PACS Collaboration : A. Ali Khan[a], S. Aoki[b], Y. Aoki[a,b†], R. Burkhalter[a,b], S. Ejiri[a], M. Fukugita[c], S. Hashimoto[d], N. Ishizuka[a,b], Y. Iwasaki[a,b], T. Izubuchi[e], K. Kanaya[b], T. Kaneko[d], Y. Kuramashi[d], T. Manke[a,‡], K.-I. Nagai[a], J. Noaki[a], M. Okawa[d], H.P. Shanahan[a,§], Y. Taniguchi[b], A. Ukawa[a,b] and T. Yoshié[a,b]

[a]Center for Computational Physics, University of Tsukuba, Tsukuba, Ibaraki 305-8577, Japan

[b]Institute of Physics, University of Tsukuba, Tsukuba, Ibaraki 305-8571, Japan

[c]Institute for Cosmic Ray Research, University of Tokyo, Kashiwa 277-8582, Japan

[d]High Energy Accelerator Research Organization (KEK), Tsukuba, Ibaraki 305-0801, Japan

[e]Institute of Theoretical Physics, Kanazawa University, Ishikawa 920-1192, Japan

Chiral properties of QCD formulated with the domain-wall fermion (DWQCD) are studied using the anomalous quark mass m_{5q} and the spectrum of the 4-dimensional Wilson-Dirac operator. Numerical simulations are made with the standard plaquette gauge action and a renormalization-group improved gauge action. Results are reported on the density of zero eigenvalue obtained with the accumulation method, and a comparison is made with the results for m_{5q}.

1. Introduction

Formulation of chiral fermions on the lattice has been one of long-standing problems in lattice field theories. Several years ago, the domain-wall fermion (DWF) formalism [1,2], which is a Wilson fermion in $D+1$ dimensions with Dirichlet boundary condition in the extra dimension, has been proposed as a new formulation of lattice chiral fermion. In the limit of large extra dimension size, $N_s \to \infty$, the spectrum of free domain-wall fermion contains massless modes at the edges in the extra dimension.

While the massless modes are shown to be stable in perturbation theory[3,4], their existence may be spoiled non-perturbatively in the presence of dynamical gauge fields. We studied this issue through an anomalous quark mass m_{5q} in Ref. [5]. This quantity measures the magnitude of chiral symmetry breaking with the domain-wall QCD (DWQCD).

In this article we make a status report of our attempt to understand the results on the N_s-dependence of m_{5q} obtained in Ref. [5] through measurements of the eigenvalue distribution of the 4-dimensional Wilson-Dirac operator.

2. Chiral Properties of DWQCD

We define the anomalous quark mass by[5]

$$m_{5q} = \lim_{t\to\infty} \frac{\sum_{\mathbf{x}} \left\langle J^a_{5q}(t,\mathbf{x}) P^b(0,\mathbf{0})\right\rangle}{\sum_{\mathbf{x}} \left\langle P^a(t,\mathbf{x}) P^b(0,\mathbf{0})\right\rangle}. \quad (1)$$

This quantity measures the chiral symmetry breaking effect in the axial Ward-Takahashi identity:

$$\sum_\mu \left\langle \nabla_\mu A^a_\mu(x) P^b(0)\right\rangle = 2m_f \left\langle P^a(x) P^b(0)\right\rangle$$

*Talk presented by K.-I. Nagai

†address after 1 May, 2000: RIKEN BNL Research Center, Brookhaven National Laboratory, Upton, NY 11973, USA

‡address after 1 Feb., 2000: Department of Physics, Columbia University, 538 W 120th St., New York, NY 10027, USA

§address after 15 Sept., 2000: Department of Biochemistry and Molecular Biology, University College London, London, England, UK

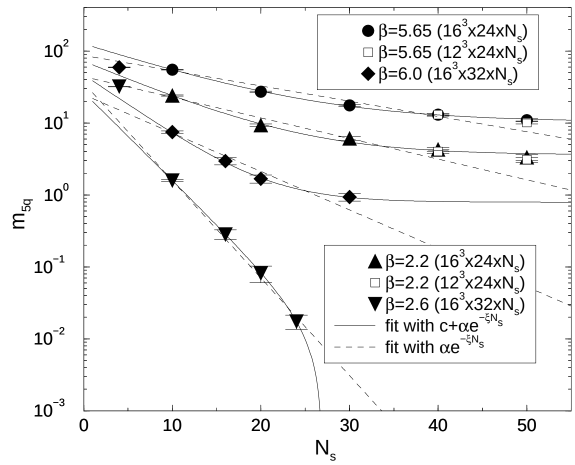

Figure 1. m_{5q} as a function of N_s at $a^{-1} \simeq 1$ GeV and $m_0 = 1.7$ and at $a^{-1} \simeq 2$ GeV and $m_0 = 1.8$, where m_0 is the domain-wall height, for the plaquette ($\beta = 5.65, 6.0$) and RG-improved ($\beta = 2.2, 2.6$) gauge action [5].

$$+2\left\langle J^a_{5q}(x)P^b(0)\right\rangle + i\left\langle \delta^a_x P^b(0)\right\rangle \ , \qquad (2)$$

where $A^a_\mu(x)$ is the axial-vector current, $P^a(x)$ is the pseudoscalar density, and $J^a_{5q}(x)$ represents the explicit breaking of chiral symmetry. On a smooth gauge field background, the anomalous contribution $\left\langle J^a_{5q}(x)P^b(y)\right\rangle$ vanishes as $\exp(-cN_s)$ at large N_s [6]. Therefore m_{5q} also vanishes exponentially in this case.

In Ref. [5], we carried out quenched simulations to study the N_s dependence in detail. Two values of lattice spacing, $a^{-1} \simeq 1$ and 2 GeV, are explored, using both the plaquette and an RG-improved gauge actions.

Our main results for m_{5q} are summarized in Fig. 1. The results for the plaquette action are obtained at $\beta = 5.65$ ($a^{-1} \simeq 1$ GeV) and 6.0 (2 GeV), and those for the RG-improved action at $\beta = 2.2$ (1 GeV) and 2.6 (2 GeV). Solid lines are fits to $c + \alpha e^{-\xi N_s}$, and dashed lines to $\alpha e^{-\xi N_s}$

Important points to note in Fig. 1 are: (i) comparing results for the spatial sizes 12^3 and 16^3 at $a^{-1} \simeq 1$ GeV, we find the finite volume effects in m_{5q} to be small, (ii) m_{5q} decreases with the lattice spacing, (iii) the magnitude of m_{5q} is smaller for the RG-improved action than for the plaquette action, and (iv) from data in the range of N_s we explore, m_{5q} seems to remain nonzero in the limit $N_s \to \infty$, in all cases except at $\beta = 2.6$ for the RG-improved action. If confirmed with studies at larger values of N_s, the last point means that DWQCD realizes chiral symmetry at $a^{-1} \simeq 2$ GeV only for the case of the RG-improved action.

3. Eigenvalues of the hermitian Wilson-Dirac operator and chiral property

Chiral symmetry of DWQCD can be studied also through the transfer matrix in the direction of the extra dimension[7,8]. When the transfer matrix has a unit eigenvalue, chiral symmetry is not realized in DWQCD because the left and right chiral modes on the two edges in the extra dimension couple with each other.

A unit eigenvalue of the transfer matrix is in one-to-one correspondence with a zero eigenvalue of the hermitian Wilson-Dirac operator defined by

$$H_W(m_0) = \gamma_5 D_W(-m_0) \ , \qquad (3)$$

which is much easier to calculate. Here, $D_W(-m_0)$ is the four dimensional Wilson-Dirac kernel with a bare mass $-m_0$. Therefore, a failure of exponential decay of m_{5q} would result if H_W develops a zero eigenvalue.

We calculate eigenvalues of H_W^2 by the Lanczos method using 50–100 configurations at several values of coupling in the range $a^{-1} \simeq$ 1–2 GeV using both plaquette and RG-improved actions. The results from the Lanczos method are checked by the Ritz functional method for H_W. We also study the dependence on the lattice size. The maximum lattice at $a^{-1} \simeq 1$ GeV is 12^4 for both actions, while the one at $a^{-1} \simeq 2$ GeV is 24^4 for the RG-improved action and $16^3 \times 32$ for the plaquette action.

3.1. Eigenvalue distributions

In Figs. 2 and 3 we plot Monte Carlo time histories for the six lowest eigenvalues of H_W^2 for the plaquette and RG-improved actions. In each figure the left panel shows results for $a^{-1} \simeq 1$ GeV and the right panel for $a^{-1} \simeq 2$ GeV. The lattice size at $a^{-1} \simeq 2$ GeV is the same as in the previous

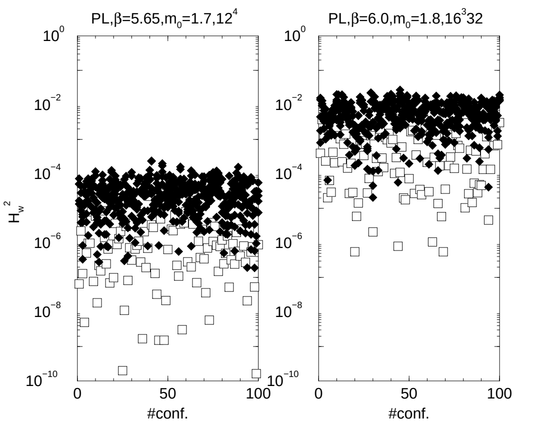

Figure 2. Monte Carlo time histories for the six lowest eigenvalues of H_W^2 obtained with the plaquette gauge action.

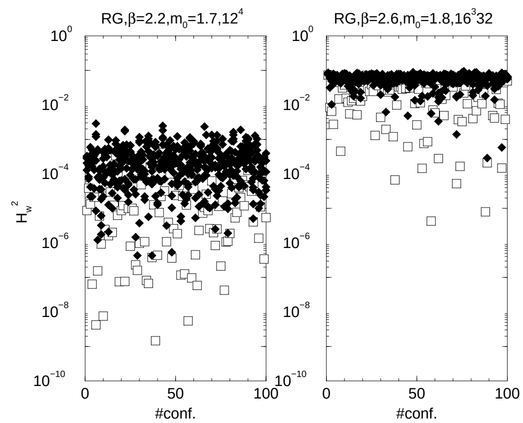

Figure 3. The same as Fig. 2 obtained with the RG-improved gauge action.

work of m_{5q} shown in Fig. 1. Open squares plot the minimum eigenvalue $\lambda^2_{\rm min}$ and filled diamonds are the five higher eigenvalues.

There is a clear trend that the minimum eigenvalues become larger for smaller lattice spacings. Another interesting point is that the RG-improved action gives larger values of $\lambda^2_{\rm min}$ than the plaquette action, which indicates that the RG-improved action has a better chiral behavior. These trends are parallel to the features we noted for m_{5q} in Sec. 2.

3.2. Spectral density

The spectral density of H_W is defined by

$$\rho(\lambda) = \lim_{V\to\infty} \frac{1}{3\cdot 4\cdot V} \sum_{\lambda'} \delta(\lambda' - \lambda), \tag{4}$$

where the summation is over the eigenvalues of H_W. We are interested in the density of zero eigenvalues, $\rho(0)$, since we expect this quantity to be related to the existence of unit eigenvalue of the transfer matrix. To calculate this quantity, we adopt the accumulation method proposed in [9], which is based on the relation

$$A(\lambda) \equiv \int_0^{\lambda^2} d{\lambda'}^2 \widetilde{\rho}({\lambda'}^2) = \frac{1}{3\cdot 4\cdot V} \sum_{|\lambda'|\le\lambda} \mathbf{1} \tag{5}$$

$$= \int_{-\lambda}^{\lambda} d\lambda' \rho(\lambda') \simeq 2\rho(0)\lambda + O(\lambda^2), \tag{6}$$

where $\widetilde{\rho}(\lambda^2)$ is the spectral density function for H_W^2. We note that, for the small-λ expansion of $A(\lambda)$ in (6), analyticity of $\rho(\lambda)$ at the origin is assumed.

In Fig. 4, we show typical examples of the accumulation $A(\lambda)$ from the eigenvalue distribution of H_W^2 for the case of the RG-improved action. Results for $\rho(0)$ obtained by a linear fitting following (6), normalized by the string tension, are summarized in Fig. 5.

Our results for $\rho(0)$ for the plaquette action are consistent with the previous data by Edwards *et al.* [9]. Results for the RG-improved action show a similar β dependence. A significant difference is that the RG-improved action leads to much smaller values of $\rho(0)$ than the plaquette action, roughly by an order of magnitude.

4. Discussions

We have applied the accumulation method to estimate the spectral density at zero eigenvalue of the hermitian Wilson-Dirac operator, $\rho(0)$. We found that this method leads to non-zero values of $\rho(0)$ at $a^{-1} \simeq$ 1–2 GeV for both the plaquette and RG-improved actions.

At $a^{-1} \simeq 1$ GeV, the non-zero result for $\rho(0)$

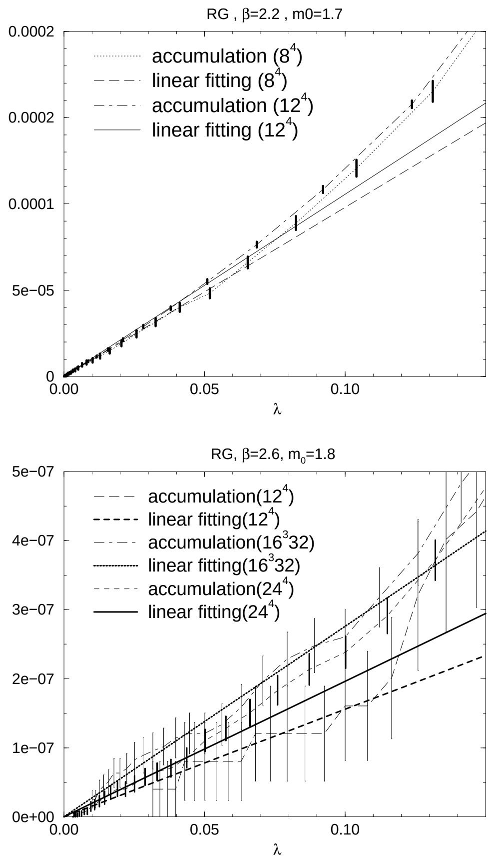

Figure 4. The accumulation $A(\lambda)$ at $\beta = 2.2$ ($a^{-1} \simeq 1$ GeV) and $\beta = 2.6$ ($a^{-1} \simeq 2$ GeV) from the RG-improved action.

is consistent with the finite m_{5q} in the large N_s limit observed in [5] with both the plaquette and RG-improved actions. At $a^{-1} \simeq 2$ GeV, while a consistency also holds with the plaquette action, there is an apparent contradiction for the case of the RG-improved action since m_{5q} seems to decay exponentially with N_s for this case.

In the accumulation data shown in Fig. 4 we observe that results are very noisy at $\beta = 2.6$ (2 GeV). Since the fit results for $\rho(0)$ fluctuates with volume, it is difficult to determine the size dependence. Therefore simulations with larger lattices are needed to check if the slope remains non-vanishing toward infinite volume. Another point to examine is if the analyticity assumption

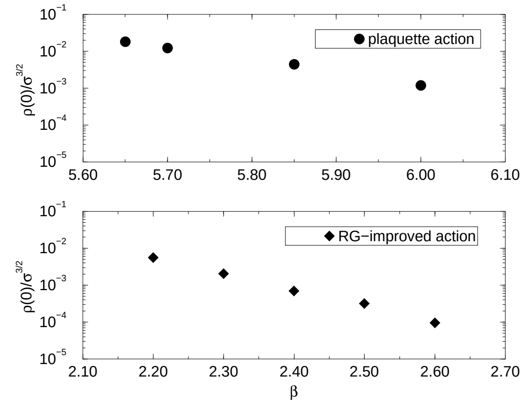

Figure 5. $\rho(0)/\sigma^{3/2}$ as a function of β for the plaquette and RG-improved gauge action. Data obtained on the largest lattices are shown.

for $\rho(\lambda)$ at the origin is justified if there is a spectral gap. Further studies are required to clarify these points.

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REFERENCES

1. D. Kaplan, Phys. Lett. **B288** (1992) 342.
2. Y. Shamir, Nucl. Phys. **B406** (1993) 90.
3. S. Aoki and Y. Taniguchi, Phys. Rev. **D59** (1999) 054510.
4. Y. Kikukawa, H. Neuberger and A. Yamada, Nucl. Phys. **B526** (1998) 572.
5. CP-PACS collaboration: A. Ali Khan *et al.*, hep-lat/0007014.
6. V. Furman and Y. Shamir, Nucl. Phys. **B439** (1995) 54.
7. H. Neuberger, Phys. Rev. **D57** (1998) 5417.
8. Y.Kikukawa and T.Noguchi,hep-lat/9902022.
9. R. Edwards, U. Heller and R. Narayanan, Phys. Rev. **D60** (1999) 034502.