Crystal Bases, Path Models, and a Twining Character Formula for Demazure Modules

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0 Introduction.

In [FRS] and [FSS], they introduced new character-like quantities corresponding to a graph automorphism of a Dynkin diagram, called twining characters, for certain Verma modules and integrable highest weight modules over a symmetrizable Kac-Moody algebra, and gave twining character formulas for them. Recently, the notion of twining characters has naturally been extended to various modules, and formulas for them has been given ([KN], [KK], [N1]-[N4]).

The purpose of this paper is to give a twining character formula for Demazure modules over a symmetrizable Kac-Moody algebra. Our formula is an extension of one of the main results in [KN], which describes the twining characters of Demazure modules over a finite-dimensional semi-simple Lie algebra. While their proof is an algebro-geometric one, we give a combinatorial proof by using the theories of path models and crystal bases.

Let us explain our formula more precisely. Let \( g = g(A) = n_\pm \oplus h \oplus n_+ \) be a symmetrizable Kac-Moody algebra over \( \mathbb{Q} \) associated to a generalized Cartan matrix \( A = (a_{ij})_{i,j \in I} \) of finite size, where \( h \) is the Cartan subalgebra, \( n_+ \) the sum of positive root spaces, and \( n_- \) the sum of negative root spaces, and let \( \omega : I \to I \) be a (Dynkin) diagram automorphism, that is, a bijection \( \omega : I \to I \) satisfying \( a_{\omega(i),\omega(j)} = a_{ij} \) for all \( i, j \in I \). It is known that a diagram automorphism induces a Lie algebra automorphism \( \omega \in Aut(g) \) that preserves the triangular decomposition of \( g \). Then we define a linear automorphism \( \omega^* \in GL(h^*) \) by \( (\omega^*(\lambda))(h) = \lambda(\omega(h)) \) for \( \lambda \in h^*, h \in h \). We set \( (h^*)^0 = \{ \lambda \in h^* \mid \omega^*(\lambda) = \lambda \} \), and call its elements symmetric weights. We also set \( \tilde{W} := \{ w \in W \mid w \omega^* = \omega^* w \} \).

Further we define a "folded" matrix \( \tilde{A} \) associated to \( \omega \), which is again a symmetrizable GCM if \( \omega \) satisfies a certain condition, called the linking condition (we assume it throughout this paper). The Kac-Moody algebra \( \tilde{g} = \tilde{g}(\tilde{A}) \) associated to \( \tilde{A} \) is called the orbit Lie algebra. We denote by \( \tilde{h} \) the Cartan subalgebra of \( \tilde{g} \) and by \( \tilde{W} \) the Weyl group of \( \tilde{g} \). Then there exist a linear isomorphism \( P_{\omega}^* : \tilde{h}^* \to (h^*)^0 \) and a group isomorphism \( \Theta : \tilde{W} \to \tilde{W} \) such that \( \Theta(\tilde{w}) = P_{\omega}^* \circ \tilde{w} \circ (P_{\omega}^*)^{-1} \) for all \( \tilde{w} \in \tilde{W} \).

Let \( \lambda \) be a dominant integral weight. Denote by \( L(\lambda) = \bigoplus_{\chi \in \mathfrak{t}^*} L(\lambda)_{\chi} \) the irreducible highest weight \( g \)-module of highest weight \( \lambda \). Then, for \( w \in W \), we define the Demazure module \( L_w(\lambda) \) of lowest weight \( w(\lambda) \) in \( L(\lambda) \) by \( L_w(\lambda) := U(\mathfrak{b})L(\lambda)_{w(\lambda)} \), where \( U(\mathfrak{b}) \) is the universal enveloping algebra of the Borel subalgebra \( \mathfrak{b} := \mathfrak{h} \oplus n_+ \) of \( g \). If \( \lambda \) is symmetric, then we have a (unique) linear automorphism \( \tau_\omega : L(\lambda) \to L(\lambda) \) such that

\[ \tau_\omega(xv) = \omega^{-1}(x)\tau_\omega(v) \quad \text{for all } x \in g, v \in L(\lambda) \]

and \( \tau_\omega(u_\lambda) = u_\lambda \) with \( u_\lambda \) a (nonzero) highest weight vector of \( L(\lambda) \). Then it is easily seen that the Demazure module \( L_w(\lambda) \) with \( w \in \tilde{W} \) is \( \tau_\omega \)-stable. Here we define the twining
character \( \text{ch}^\omega(L_w(\lambda)) \) of \( L_w(\lambda) \) by:

\[
\text{ch}^\omega(L_w(\lambda)) := \sum_{\chi \in (\mathfrak{h}^*)^0} \text{tr}(\tau_\omega|_{L_w(\lambda)_\chi})e(\chi).
\]

Our main theorem is the following:

**Theorem.** Let \( \lambda \) be a symmetric dominant integral weight and \( w \in \widehat{W} \). Set \( \widehat{\lambda} := (P^w)^{-1}(\lambda) \) and \( \widehat{\omega} := \Theta^{-1}(w) \). Then we have

\[
\text{ch}^\omega(L_w(\lambda)) = P^\omega(\text{ch}L_{\widehat{\omega}}(\widehat{\lambda})),
\]

where \( L_{\widehat{\omega}}(\widehat{\lambda}) \) is the Demazure module of lowest weight \( \widehat{\omega}(\widehat{\lambda}) \) in the irreducible highest weight module \( L(\widehat{\lambda}) \) of highest weight \( \widehat{\lambda} \) over the orbit Lie algebra \( \widehat{\mathfrak{g}} \).

The starting point of this work is the main result in [NS1]. Denote by \( \mathbb{B}(\lambda) \) the set of Lakshmibai-Seshadri paths (L-S paths for short) of class \( \lambda \), where L-S paths of class \( \lambda \) are, by definition, piecewise linear, continuous maps \( \pi : [0,1] \to \mathfrak{h}^* \) parametrized by sequences of elements in \( \mathcal{W}(\lambda) \) and rational numbers with a certain condition, called the chain condition. In [Li1], Littelmann showed that there exists a subset \( \mathbb{B}_w(\lambda) \) of \( \mathbb{B}(\lambda) \) such that

\[
\sum_{\pi \in \mathbb{B}_w(\lambda)} e(\pi(1)) = \text{ch}L_w(\lambda).
\]

For \( \pi \in \mathbb{B}(\lambda) \), we define a path \( \omega^*(\pi) : [0,1] \to \mathfrak{h}^* \) by \( (\omega^*(\pi))(t) := \omega^*(\pi(t)) \). If \( \lambda \) is symmetric and \( w \in \widehat{W} \), then \( \mathbb{B}_w(\lambda) \) is \( \omega^*-\)stable. We denote by \( \mathbb{B}^0_w(\lambda) \) the set of all elements of \( \mathbb{B}_w(\lambda) \) fixed by \( \omega^* \). Then we see from the main result of [NS1] that

\[
\sum_{\pi \in \mathbb{B}^0_w(\lambda)} e(\pi(1)) = P^\omega(\text{ch}L_{\widehat{\omega}}(\widehat{\lambda})).
\]

In this paper, we prove that the left-hand side is, in fact, equal to \( \text{ch}^\omega(L_w(\lambda)) \).

In order to prove the equality \( \text{ch}^\omega(L_w(\lambda)) = \sum_{\pi \in \mathbb{B}^0_w(\lambda)} e(\pi(1)) \), we introduce a “quantum version” of twining characters, called \( q \)-twining characters. Let \( U_q(\mathfrak{g}) \) be the quantum group associated to the Kac-Moody algebra \( \mathfrak{g} \) over the field \( \mathbb{Q}(q) \) of rational functions in \( q \), and \( V(\lambda) = \bigoplus_{\chi \in \mathfrak{h}^*} V(\lambda)_\chi \) the irreducible highest weight \( U_q(\mathfrak{g}) \)-module of highest weight \( \lambda \). For \( w \in W \), the quantum Demazure module \( V_w(\lambda) \) is defined by \( V_w(\lambda) := U_q^+(\mathfrak{g})V(\lambda)|_{\mathfrak{h}(w)} \), where \( U_q^+(\mathfrak{g}) \) is the “positive part” of \( U_q(\mathfrak{g}) \). A diagram automorphism \( \omega \) induces a \( \mathbb{Q}(q) \)-algebra automorphism \( \omega_q \) of \( U_q(\mathfrak{g}) \). Assume that \( \lambda \) is symmetric. Then we get a \( \mathbb{Q}(q) \)-linear automorphism \( \tau_{\omega_q} \) of \( V(\lambda) \) that has the same properties as \( \tau_\omega \) in the Lie algebra case. Since \( V_w(\lambda) \) is stable under \( \tau_{\omega_q} \) if \( w \in \widehat{W} \), we can define the \( q \)-twining character \( \text{ch}^q_q(V_w(\lambda)) \) of \( V_w(\lambda) \) by

\[
\text{ch}^q_q(V_w(\lambda)) := \sum_{\chi \in (\mathfrak{h}^*)^0} \text{tr}(\tau_{\omega_q}|_{V_w(\lambda)_\chi})e(\chi),
\]
where the traces are naively elements of $\mathbb{Q}(q)$ (in fact, they are elements of $\mathbb{Q}[q, q^{-1}]$). We show that the specialization of the $q$-twining character $\text{ch}_q^\omega(V_w(\lambda))$ by $q = 1$ is equal to the (ordinary) twining character $\text{ch}^\omega(L_w(\lambda))$, that is,

$$\text{ch}_q^\omega(V_w(\lambda))\big|_{q=1} = \text{ch}^\omega(L_w(\lambda)).$$

The advantage of considering a quantum version is the existence of a basis of $V_w(\lambda)$ compatible with $\tau_{\omega_q}$. Let $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ be the (lower) crystal base of $V(\lambda)$. In [Kas3], Kashiwara showed that, for each $w \in W$, there exists a subset $\mathcal{B}_w(\lambda)$ of $\mathcal{B}(\lambda)$ such that

$$V_w(\lambda) = \bigoplus_{b \in \mathcal{B}_w(\lambda)} \mathbb{Q}(q)G_\lambda(b),$$

where $G_\lambda(b)$ denotes the (lower) global base introduced in [Kas2]. We prove that $\tau_{\omega_q}$ stabilizes the basis $\{G_\lambda(b) \mid b \in \mathcal{B}_w(\lambda)\}$ of $V_w(\lambda)$.

By combining these facts and the equivalence theorem between path models $\mathbb{B}(\lambda)$ and crystal bases $\mathcal{B}(\lambda)$, which was proved by Kashiwara [Kas5] et al., we can obtain the desired equality above, and hence the our main theorem.

This paper is organized as follows. In §1 we review some facts about Kac-Moody algebras, diagram automorphisms, orbit Lie algebras, quantum groups, crystal bases, and path models. There we also define an algebra automorphism of the quantum group $U_q(\mathfrak{g})$ induced from a diagram automorphism. In §2, we recall the definition of the twining characters of $L(\lambda)$ and $L_w(\lambda)$, and then introduce the $q$-twining characters of $V(\lambda)$ and $V_w(\lambda)$. Furthermore, we show that the $q$-twining characters of $V(\lambda)$ and $V_w(\lambda)$ are $q$-analogues of the twining characters of $L(\lambda)$ and $L_w(\lambda)$, respectively. In §3 we give a proof of our main theorem by calculating the $q$-twining character of $V_w(\lambda)$.

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1 Preliminaries.

1.1 Kac-Moody Algebras and Diagram Automorphisms. In this subsection, we review some basic facts about Kac-Moody algebras from [Kac] and [MP], and about diagram automorphisms from [FRS] and [FSS].

Let $A = (a_{ij})_{i,j \in I}$ be a symmetrizable generalized Cartan matrix (GCM for short) indexed by a finite set $I$. Then there exists a diagonal matrix $D = \text{diag}(\varepsilon_i)_{i \in I}$ with $\varepsilon_i \in \mathbb{Q}_{>0}$ such that $D^{-1}A$ is a symmetric matrix. Let $\omega : I \to I$ be a diagram automorphism of order $N$, that is, a bijection $\omega : I \to I$ of order $N$ such that $a_{\omega(i), \omega(j)} = a_{ij}$ for all $i, j \in I$.
Remark 1. Set
\[ D' = \text{diag}(\varepsilon'_i)_{i \in I} := \text{diag} \left( \frac{1}{\sum_{k=0}^{N-1} \varepsilon_{\omega^k(i)}} \right) \quad (i \in I). \]

Then we see that \( \varepsilon'_{\omega(i)} = \varepsilon'_i \) and \( (D')^{-1} A \) is a symmetric matrix. Hence, by replacing \( D \) with \( D' \) above if necessary, we may (and will henceforth) assume that \( \varepsilon_{\omega(i)} = \varepsilon_i \) (see also [N1, §3.1]).

We take a realization \((\mathfrak{h}, \Pi, \Pi^\vee)\) of the GCM \( A = (a_{ij})_{i,j \in I} \) over \( \mathbb{Q} \) and linear automorphisms \( \omega : \mathfrak{h} \to \mathfrak{h} \) and \( \omega^* : \mathfrak{h}^* \to \mathfrak{h}^* \) as follows (cf. [Kac, Exercises 1.15 and 1.16]).

Let \( \mathfrak{h}' \) be an \( n \)-dimensional vector space over \( \mathbb{Q} \) with \( \Pi^\vee := \{ \alpha^\vee_i \}_{i \in I} \) a basis. We define a \( \mathbb{Q} \)-linear automorphism \( \omega' : \mathfrak{h}' \to \mathfrak{h}' \) by \( \omega'(\alpha^\vee_i) = \alpha^\vee_{\omega(i)} \), and \( \omega'' : (\mathfrak{h}^*)' \to (\mathfrak{h}')^* \) by \( (\omega''(\lambda))(h) := \lambda((\omega')^{-1}(h)) \) for \( \lambda \in (\mathfrak{h}')^* \) and \( h \in \mathfrak{h}' \). We also define \( \varphi : \mathfrak{h}' \to (\mathfrak{h}')^* \) by \( \varphi(\alpha^\vee_i)(\alpha^\vee_j) = a_{ij} \). It can be readily seen that \( \omega'' \circ \varphi = \varphi \circ \omega' \). This means that \( \text{Im} \varphi \) is \( \omega'' \)-stable, and hence we can take a complementary subspace \( \mathfrak{h}'' \) of \( \text{Im} \varphi \) in \( (\mathfrak{h}')^* \) that is also \( \omega'' \)-stable. Now set \( \mathfrak{h} := \mathfrak{h}' \oplus \mathfrak{h}'' \), and \( \Pi := \{ \alpha_i \}_{i \in I} \), where \( \alpha_i \in \mathfrak{h}^* \) is defined by

\[
\alpha_i \left( \sum_{j \in I} c_j \alpha^\vee_j + h'' \right) := \sum_{j \in I} c_j (\varphi(\alpha^\vee_j))(\alpha^\vee_i) + h''(\alpha^\vee_i) \quad \text{for} \quad h'' \in \mathfrak{h}''. \quad (1.1)
\]

Then we see that \( \Pi \) is a linearly independent subset of \( \mathfrak{h}^* \). Furthermore, since \( \dim \mathfrak{h}'' = \#I - \dim \mathfrak{h} \), we have \( \dim \mathfrak{h} = 2\#I - \text{rank} A \). Hence \((\mathfrak{h}, \Pi, \Pi^\vee)\) is a (minimal) realization of the GCM \( A \). We define a \( \mathbb{Q} \)-linear automorphism \( \omega : \mathfrak{h} \to \mathfrak{h} \) by \( \omega(h' + h'') := \omega'(h') + \omega''(h'') \) for \( h' \in \mathfrak{h}' \) and \( h'' \in \mathfrak{h}'' \), and the transposed map \( \omega^* : \mathfrak{h}^* \to \mathfrak{h}^* \) by \( (\omega^*(\lambda))(h) = \lambda(\omega(h)) \) for \( \lambda \in \mathfrak{h}^* \) and \( h \in \mathfrak{h} \). Then we can check, by using (1.1), that \( \omega^*(\alpha_i) = \omega_{\omega^{-1}(i)} \) for each \( i \in I \).

Here, as in [Kac, §2.1], we define the (standard) nondegenerate symmetric bilinear form \((\cdot, \cdot)\) on \( \mathfrak{h} \) associated to the decomposition \( \mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}'' \) above. We set

\[
\begin{align*}
(\alpha^\vee_i, h) &:= \alpha_i(h) \varepsilon_i \quad \text{for} \quad i \in I, \ h \in \mathfrak{h}, \\
(h, h') &:= 0 \quad \text{for} \quad h, h' \in \mathfrak{h}''.
\end{align*}
\]

Then it follows from the construction above and Remark 1 that \( (\omega(h), \omega(h')) = (h, h') \) for all \( h, h' \in \mathfrak{h} \). We denote also by \((\cdot, \cdot)\) the nondegenerate symmetric bilinear form on \( \mathfrak{h}^* \) induced from the bilinear form on \( \mathfrak{h} \). Then \( (\omega^*(\lambda), \omega^*(\lambda')) = (\lambda, \lambda') \) for all \( \lambda, \lambda' \in \mathfrak{h}^* \).

We set

\[
(\mathfrak{h}^*)^0 := \{ \lambda \in \mathfrak{h}^* \mid \omega^*(\lambda) = \lambda \}, \quad \mathfrak{h}^0 := \{ h \in \mathfrak{h} \mid \omega(h) = h \}. \quad (1.2)
\]

Elements of \((\mathfrak{h}^*)^0\) are called symmetric weights. Note that \((\mathfrak{h}^*)^0\) can be identified with \((\mathfrak{h}^0)^*\) in a natural way.
Remark 2. Let $p$ be a Weyl vector, i.e., an element of $\mathfrak{h}^*$ such that $p(\alpha_i^\vee) = 1$ for all $i \in I$. Then, by replacing $p$ with $(1/N) \sum_{k=0}^{N-1} (\omega^*)^k(p)$ if necessary, we may (and will henceforth) assume that a Weyl vector $p$ is a symmetric weight.

Let $g = g(A)$ be the Kac-Moody algebra over $\mathbb{Q}$ associated to the GCM $A$ with $\mathfrak{h}$ the Cartan subalgebra, $\Pi = \{\alpha_i\}_{i \in I}$ the set of simple roots, and $\Pi^\vee = \{\alpha_i^\vee\}_{i \in I}$ the set of simple coroots. Denote by $\{x_i, y_i \mid i \in I\}$ the Chevalley generators, where $x_i$ (resp. $y_i$) spans the root space of $g$ corresponding to $\alpha_i$ (resp. $-\alpha_i$). The Weyl group $W$ of $g$ is defined by $W := \langle r_i \mid i \in I \rangle$, where $r_i$ is the simple reflection with respect to $\alpha_i$. The following lemma is obvious from the definitions of Kac-Moody algebras and the linear map $\omega : \mathfrak{h} \to \mathfrak{h}$ above (see also [FSS, §3.2]).

**Lemma 1.1.** The $\mathbb{Q}$-linear map $\omega : \mathfrak{h} \to \mathfrak{h}$ above can be extended to a Lie algebra automorphism $\omega \in \text{Aut}(g)$ of order $N$ such that $\omega(x_i) = x_{\omega(i)}$ and $\omega(y_i) = y_{\omega(i)}$.

Let $\lambda$ be a dominant integral weight. Denote by $L(\lambda) = \bigoplus_{\chi \in \mathfrak{h}^*} L(\lambda)_\chi$ the irreducible highest weight $g$-module of highest weight $\lambda$, where $L(\lambda)_\chi$ is the $\chi$-weight space of $L(\lambda)$. We set $b := \mathfrak{h} \oplus \mathfrak{n}_+$, where $\mathfrak{n}_+$ is the sum of positive root spaces of $g$. For $w \in W$, the Demazure module $L_w(\lambda) \subset L(\lambda)$ of lowest weight $w(\lambda)$ is defined by $L_w(\lambda) := U(b)L(\lambda)_{w(\lambda)}$, where $U(b)$ is the universal enveloping algebra of $b$. In addition, for each $i \in I$, we define the Demazure operator $D_i$ by

$$D_i(e(\lambda)) := \frac{e(\lambda + \rho) - e(r_i(\lambda + \rho))e(-\rho)}{1 - e(-\alpha_i)}$$

for $\lambda \in \mathfrak{h}^*$. (1.3)

By [Kas3], [Ku], and [M], we know the following character formula for Demazure modules.

**Theorem 1.2.** Let $\lambda$ be a dominant integral weight and $w \in W$. Assume that $w = r_{i_1}r_{i_2}\cdots r_{i_k}$ is a reduced expression of $w$. Then we have

$$\text{ch} L_w(\lambda) = D_{i_1} \circ D_{i_2} \circ \cdots \circ D_{i_k}(e(\lambda)).$$

(1.4)

**Remark 3.** The Demazure operators $\{D_i\}_{i \in I}$ satisfy the braid relations (see [D]). Hence the right-hand side of (1.4) above does not depend on the choice of a reduced expression of $w$.

### 1.2 Orbit Lie Algebras.

In this subsection, we review the notion of orbit Lie algebras. For details, see [FRS] and [FSS].

We set

$$c_{ij} := \sum_{k=0}^{N_i-1} a_{i,\omega^k(j)} \quad \text{for} \quad i, j \in I \quad \text{and} \quad c_i := c_{ii} \quad \text{for} \quad i \in I,$$

(1.5)

where $N_i$ is the number of elements of the $\omega$-orbit of $i \in I$ in $I$. From now on, we assume that a diagram automorphism $\omega$ satisfies

$$c_i = 1 \text{ or } 2 \quad \text{for each} \quad i \in I.$$
This condition is called the linking condition. Here we choose a complete set \( \hat{I} \) of representatives of the \( \omega \)-orbits in \( I \), and define a matrix \( \hat{A} = (\hat{a}_{ij})_{i,j \in \hat{I}} \) by
\[
\hat{A} = (\hat{a}_{ij})_{i,j \in \hat{I}} := \left( \frac{2c_{ij} / c_j}{c_i} \right)_{i,j \in \hat{I}}.
\]  (1.7)

**Proposition 1.3** ([FSS, §2.2]). The matrix \( \hat{A} \) is a symmetrizable GCM.

The Kac-Moody algebra \( \hat{\mathfrak{g}} := g(\hat{A}) \) over \( \mathbb{Q} \) associated to the GCM \( \hat{A} \) is called the orbit Lie algebra (associated to the diagram automorphism \( \omega \)). Denote by \( \hat{\mathfrak{h}} \) the Cartan subalgebra of \( \hat{\mathfrak{g}} \), and by \( \hat{\Pi} = \{ \hat{\alpha}_i \}_{i \in \hat{I}} \) and \( \hat{\Pi}^\vee = \{ \hat{\alpha}_i^\vee \}_{i \in \hat{I}} \) the set of simple roots and simple coroots of \( \hat{\mathfrak{g}} \), respectively.

As in [FRS, §2], we have a \( \mathbb{Q} \)-linear isomorphism \( P_\omega : \mathfrak{h}^0 \rightarrow \hat{\mathfrak{h}} \) such that
\[
\left\{ \begin{array}{l}
P_\omega \left( \frac{1}{N_i} \sum_{k=0}^{N_i-1} \alpha_{\omega^k(i)}^\vee \right) = \hat{\alpha}_i^\vee \quad \text{for each } i \in \hat{I}, \\
(P_\omega(h), P_\omega(h')) = (h, h') \quad \text{for all } h, h' \in \mathfrak{h}^0,
\end{array} \right.
\]
where we denote also by \( (\cdot, \cdot) \) the (standard) nondegenerate symmetric bilinear form on \( \hat{\mathfrak{h}} \). Let \( P_\omega^* : \hat{\mathfrak{h}}^* \rightarrow \mathfrak{h}^0 \) be the transposed map of \( P_\omega \) defined by
\[
(P_\omega^*(\lambda))(h) := \hat{\lambda}(P_\omega(h)) \quad \text{for } \lambda \in \hat{\mathfrak{h}}^*, h \in \mathfrak{h}^0.
\]  (1.8)

**Proposition 1.4** ([FRS, Proposition 3.3]). Set \( \tilde{W} := \{ w \in W \mid w^* = \omega^* w \} \). Then there exists a group isomorphism \( \Theta : \tilde{W} \rightarrow \tilde{W} \) such that \( \Theta(\hat{w}) = P_\omega^* \circ \hat{w} \circ (P_\omega^*)^{-1} \) for each \( \hat{w} \in \tilde{W} \).

### 1.3 Quantum Groups

From now on, we take the bilinear form \( (\cdot, \cdot) \) in such a way that \( (\alpha_i, \alpha_i) \in \mathbb{Z}_{>0} \) for all \( i \in I \). Let \( P \subset \mathfrak{h}^* \) be an \( \omega^* \)-stable integral weight lattice such that \( \alpha_i \in P \) for all \( i \in I \), and set \( P_+ := \{ \lambda \in P \mid \lambda(\alpha_i^\vee) \in \mathbb{Z}_{\geq 0} \text{ for all } i \in I \} \). Notice that the dual lattice \( P^\vee := \text{Hom}_\mathbb{Z}(P, \mathbb{Z}) \) is stable under \( \omega \). The quantum group (or quantized universal enveloping algebra) \( U_q(\mathfrak{g}) \) associated to \( \mathfrak{g} \) is, by definition, the algebra generated by the symbols \( X_i, Y_i \) and \( q^h \) (\( h \in P^\vee \)) over the field \( \mathbb{Q}(q) \) of rational functions in \( q \) with the following defining relations:
\[
\begin{align*}
q^0 &= 1, 
q^h q^{h'} &= q^{h + h'} \quad \text{for } h_1, h_2 \in P^\vee, 
q^h X_i q^{-h} &= q^{\alpha_i(h)} X_i, 
q^h Y_i q^{-h} &= q^{-\alpha_i(h)} Y_i \quad \text{for } i \in I, h \in P^\vee, 
[X_i, Y_i] &= \delta_{ij} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}} \quad \text{for } i \in I, 
\sum_{k=0}^{1-\alpha_{ij}} (-1)^k X_i^{(k)} X_j X_i^{(1-\alpha_{ij} - k)} &= 0 \quad \text{for } i, j \in I \text{ with } i \neq j, 
\sum_{k=0}^{1-\alpha_{ij}} (-1)^k Y_i^{(k)} Y_j Y_i^{(1-\alpha_{ij} - k)} &= 0 \quad \text{for } i, j \in I \text{ with } i \neq j.
\end{align*}
\]  (1.9)
Here we have used the following notation:

\[ q_i = q^{(\alpha_i, \alpha_i)} , \quad t_i = q^{(\alpha_i, \alpha_i)\gamma} , \]

\[ [n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{k=1}^{n} [k]_i, \quad \text{and} \quad X_i^{(n)} = \frac{X_i^n}{[n]_i!} , \quad Y_i^{(n)} = \frac{Y_i^n}{[n]_i!} . \]

**Lemma 1.5.** There exists a unique \( \mathbb{Q}(q) \)-algebra automorphism \( \omega_q \) of \( U_q(g) \) such that

\[ \omega_q(X_i) = X_{\omega(i)} , \quad \omega_q(Y_i) = Y_{\omega(i)} , \quad \text{and} \quad \omega_q(q_h) = q^{\omega(h)} . \]

**Proof.** We need only show that the images of the generators by \( \omega_q \) also satisfy the defining relations (1.9). However it can be easily checked by using the equalities \( q_{\omega(i)} = q_i , \quad [n]_{\omega(i)} = [n]_i , \quad \text{and} \quad t_{\omega(i)} = t_i . \)

Let \( \lambda \in \mathcal{P}_+ \). Denote by \( V(\lambda) = \bigoplus_{\chi \in \mathcal{P}^+} V(\lambda)_\chi \) the irreducible highest weight \( U_q(g) \)-module of highest weight \( \lambda \), where \( V(\lambda)_\chi \) is the \( \chi \)-weight space of \( V(\lambda) \). It is known (cf. [Kas1, (1.2.7)]) that

\[ V(\lambda) \cong U_q^- (g) / \left( \sum_{i \in I} U_q^- (g) Y_i^{1+\lambda(\alpha_i^\vee)} \right) , \tag{1.10} \]

where \( U_q^- (g) \) is the \( \mathbb{Q}(q) \)-subalgebra of \( U_q(g) \) generated by \( \{Y_i\}_{i \in I} \). For each \( w \in W \), we define the quantum Demazure module \( V_w(\lambda) \) by \( V_w(\lambda) := U_q^+(g)V(\lambda)_{\omega(w)} \), where \( U_q^+(g) \) is the \( \mathbb{Q}(q) \)-subalgebra of \( U_q(g) \) generated by \( \{X_i\}_{i \in I} \).

### 1.4 Crystal Bases and Global Bases

In this subsection, we review the notions of (lower) crystal bases and (lower) global bases. For details, see [Ja] and [Kas1]–[Kas3].

First let us recall the definition of the Kashiwara operators \( E_i , F_i \) on \( V(\lambda) \). It is known that each element \( u \in V(\lambda)_\chi \) can be uniquely written as \( u = \sum_{k \geq 0} Y_i^{(k)} u_k \), where \( u_k \in (\ker X_i) \cap V(\lambda)_{\chi+k\alpha_i} \). We define the \( \mathbb{Q}(q) \)-linear operators \( E_i , F_i \) on \( V(\lambda) \) by

\[ E_i u := \sum_{k \geq 0} Y_i^{(k-1)} u_k , \quad F_i u := \sum_{k \geq 0} Y_i^{(k+1)} u_k . \tag{1.11} \]

Denote by \( A_0 \) the subring of \( \mathbb{Q}(q) \) consisting of the rational functions in \( q \) regular at \( q = 0 \), and by \( \mathcal{L}_0(\lambda) \) the \( A_0 \)-submodule of \( V(\lambda) \) generated by all elements of the form \( F_{i_1} F_{i_2} \cdots F_{i_k} u_\lambda \), where \( u_\lambda \) is a (nonzero) highest weight vector of \( V(\lambda) \). Let \( \mathcal{B}(\lambda) \subset \mathcal{L}_0(\lambda)/q\mathcal{L}_0(\lambda) \) be the set of nonzero images of \( F_{i_1} F_{i_2} \cdots F_{i_k} u_\lambda \) by the canonical map \( - : \mathcal{L}_0(\lambda) \rightarrow \mathcal{L}_0(\lambda)/q\mathcal{L}_0(\lambda) \). Then it is known from [Kas1, Theorem 2] that \( (\mathcal{L}_0(\lambda), \mathcal{B}(\lambda)) \) is a (lower) crystal base of \( V(\lambda) \), i.e.,

1. \( V(\lambda) = \mathbb{Q}(q) \otimes_{A_0} \mathcal{L}_0(\lambda) \),
2. \( \mathcal{L}_0(\lambda) = \bigoplus_{\chi \in \mathcal{P}^+} \mathcal{L}_0(\lambda)_\chi \), where \( \mathcal{L}_0(\lambda)_\chi = \mathcal{L}_0(\lambda) \cap V(\lambda)_\chi \),
3. \( E_i \mathcal{L}_0(\lambda) \subset \mathcal{L}_0(\lambda) \) and \( F_i \mathcal{L}_0(\lambda) \subset \mathcal{L}_0(\lambda) \),
4. for each \( w \in W \), \( \mathcal{L}_0(\lambda)_\chi \mathcal{L}_0(\lambda) \subset \mathcal{L}_0(\lambda)_\chi \).

...
(4) \( B(\lambda) \) is a basis of the \( \mathbb{Q} \)-vector space \( \mathcal{L}_0(\lambda)/q\mathcal{L}_0(\lambda) \),

(5) \( E_iB(\lambda) \subseteq B(\lambda) \cup \{0\} \) and \( F_iB(\lambda) \subseteq B(\lambda) \cup \{0\} \),

(6) \( B(\lambda) = \bigcup_{x \in \mathfrak{h}} B(\lambda)_x \) (disjoint union), where \( B(\lambda)_x = B(\lambda) \cap (\mathcal{L}_0(\lambda)_x/q\mathcal{L}_0(\lambda)_x) \),

(7) For \( b_1, b_2 \in B(\lambda) \), \( b_1 = F_ib_2 \) if and only if \( b_2 = E_ib_1 \).

Note that, by (3), we have the operators on \( \mathcal{L}_0(\lambda)/q\mathcal{L}_0(\lambda) \) induced from \( E_i, F_i \), which are also denoted by \( E_i, F_i \) (cf. (5), (7)).

Next we recall the notion of (lower) global bases. Set \( V_\mathbb{Q}(\lambda) := U_\mathbb{Q}(\mathfrak{g})u_\lambda \subset V(\lambda) \), where \( U_\mathbb{Q}(\mathfrak{g}) \) is the \( \mathbb{Q}[q, q^{-1}] \)-subalgebra of \( U_q(\mathfrak{g}) \) generated by all \( X_i^{(n)}, Y_i^{(n)}, q^h \), and

\[
\{q^h\} := \prod_{k=1}^{n} \frac{q^{1-k}q^{-h} - q^{k-1}q^{-h}}{q^k - q^{-k}}
\]

for \( i \in I, n \in \mathbb{Z}_{\geq 0}, h \in P^* \). We define a \( \mathbb{Q} \)-algebra automorphism \( \psi : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \) by

\[
\begin{align*}
\psi(X_i) := X_i, & \quad \psi(Y_i) := Y_i & \text{for } i \in I, \\
\psi(q) := q^{-1}, & \quad \psi(q^h) := q^{-h} & \text{for } h \in P^*.
\end{align*}
\]

By virtue of (1.10), we have a \( \mathbb{Q} \)-linear automorphism \( \psi \) of \( V(\lambda) \) defined by \( \psi(xu_\lambda) := \psi(x)u_\lambda \) for \( x \in U_\mathbb{Q}(\mathfrak{g}) \). Let \( \mathcal{L}_\infty(\lambda) \) be the image of \( \mathcal{L}_0(\lambda) \) by \( \psi \). Then it is known (see, for example, [Kas2]) that the restriction of the canonical map \( \tau \) to \( E(\lambda) := V_\mathbb{Q}(\lambda) \cap \mathcal{L}_0(\lambda) \cap \mathcal{L}_\infty(\lambda) \) is an isomorphism from \( E(\lambda) \) to \( \mathcal{L}_0(\lambda)/q\mathcal{L}_0(\lambda) \) as \( \mathbb{Q} \)-vector spaces. We denote by \( G_\lambda \) the inverse of this isomorphism. Then we have

\[
V(\lambda) = \bigoplus_{b \in B(\lambda)} \mathbb{Q}(q)G_\lambda(b). \tag{1.13}
\]

Moreover we have the following.

**Theorem 1.6** ([Kas3, Proposition 3.2.3]). Let \( \lambda \in P_+ \) and \( w \in W \). Then there exists a subset \( B_w(\lambda) \) of \( B(\lambda) \) such that

\[
V_w(\lambda) = \bigoplus_{b \in B_w(\lambda)} \mathbb{Q}(q)G_\lambda(b). \tag{1.14}
\]

1.5 *Path Models.* Let \( \lambda \in P_+ \). For \( \mu, \nu \in W\lambda \), we write \( \mu \geq \nu \) if there exist a sequence \( \mu = \lambda_0, \lambda_1, \ldots, \lambda_s = \nu \) of elements in \( W\lambda \) and a sequence \( \beta_1, \ldots, \beta_s \) of positive real roots such that \( \lambda_k = r_{\beta_k}(\lambda_{k-1}) \) and \( \lambda_{k-1}(\beta_k^\vee) < 0 \) for \( k = 1, 2, \ldots, s \), where for a positive real root \( \beta \), we denote by \( r_\beta \) the reflection with respect to \( \beta \), and by \( \beta^\vee \) the dual root of \( \beta \). Then we define \( \text{dist}(\mu, \nu) \) to be the maximal length \( s \) among all possible such sequences.

**Remark 4.** Assume that \( \lambda \in P_+ \cap (\mathfrak{h}^*)^0 \). It immediately follows that \( \mu \geq \nu \) if and only if \( \omega^*(\mu) \geq \omega^*(\nu) \). Moreover, we have \( \text{dist}(\omega^*(\mu), \omega^*(\nu)) = \text{dist}(\mu, \nu) \) when \( \mu \geq \nu \).
Let $\lambda \in P_+, \mu, \nu \in W\lambda$ with $\mu \geq \nu$, and $0 < a < 1$ a rational number. An $a$-chain for $(\mu, \nu)$ is, by definition, a sequence $\mu = \lambda_0 > \lambda_1 > \cdots > \lambda_r = \nu$ of elements in $W\lambda$ such that $\text{dist}(\lambda_i, \lambda_{i-1}) = 1$ and $\lambda_i = r_{\beta_i}(\lambda_{i-1})$ for some positive real root $\beta_i$, and such that $a\lambda_{i-1}(\beta_i^\vee) \in \mathbb{Z}$ for all $i = 1, 2, \ldots, r$.

Here let us consider a pair $\pi = (\nu; a)$ of a sequence $\nu : \nu_1 > \nu_2 > \cdots > \nu_s$ of elements in $W\lambda$ and a sequence $a : 0 = a_0 < a_1 < \cdots < a_s = 1$ of rational numbers such that for each $i = 1, 2, \ldots, s - 1$, there exists an $a_i$-chain for $(\nu_i, \nu_{i+1})$. Then we associate to $\pi = (\nu; a)$ the following path $\pi : [0, 1] \to \mathfrak{h}^*$:

$$\pi(t) := \sum_{i=1}^{j-1} (a_i - a_{i-1}) \nu_i + (t - a_{j-1}) \nu_j \quad \text{for} \quad a_{j-1} \leq t \leq a_j.$$  

Such a path is called a Lakshmibai-Seshadri path (L-S path for short) of class $\lambda$. Denote by $\mathcal{B}(\lambda)$ the set of L-S paths of class $\lambda$.

Let us recall the raising and lowering root operators (cf. [Li1]-[Li4]). For convenience, we introduce an extra element $\theta$ that is not a path. For $\pi \in \mathcal{B}(\lambda)$ and $i \in I$, we set

$$h_i^\pi(t) := (\pi(t))(\alpha_i^\vee), \quad m_i^\pi := \min \{h_i^\pi(t) \mid t \in [0, 1]\}. \quad (1.16)$$

First we define the raising root operator $e_i$ with respect to the simple root $\alpha_i$. We define $e_i\theta := \theta$, and $e_i\pi := \theta$ for $\pi \in \mathcal{B}(\lambda)$ with $m_i^\pi > -1$. If $m_i^\pi \leq -1$, then we can take the following points:

$$t_1 := \min \{t \in [0, 1] \mid h_i^\pi(t) = m_i^\pi\}, \quad t_0 := \max \{t' \in [0, t_1] \mid h_i^\pi(t) \geq m_i^\pi + 1 \text{ for all } t \in [0, t']\}. \quad (1.17)$$

We set

$$(e_i\pi)(t) := \begin{cases} 
\pi(t) & \text{if } 0 \leq t \leq t_0, \\
\pi(t) - (h_i^\pi(t) - m_i^\pi - 1)\alpha_i & \text{if } t_0 \leq t \leq t_1, \\
\pi(t) + \alpha_i & \text{if } t_1 \leq t \leq 1.
\end{cases} \quad (1.18)$$

The lowering root operator $f_i$ is defined in a similar fashion. We define $f_i\theta := \theta$, and $f_i\pi := \theta$ for $\pi \in \mathcal{B}(\lambda)$ with $h_i^\pi(1) - m_i^\pi < 1$. If $h_i^\pi(1) - m_i^\pi \geq 1$, then we can take the following points:

$$t_0 := \max \{t \in [0, 1] \mid h_i^\pi(t) = m_i^\pi\}, \quad t_1 := \min \{t' \in [t_0, 1] \mid h_i^\pi(t) \geq m_i^\pi + 1 \text{ for all } t \in [t', 1]\}. \quad (1.19)$$

We set

$$(f_i\pi)(t) := \begin{cases} 
\pi(t) & \text{if } 0 \leq t \leq t_0, \\
\pi(t) - (h_i^\pi(t) - m_i^\pi)\alpha_i & \text{if } t_0 \leq t \leq t_1, \\
\pi(t) - \alpha_i & \text{if } t_1 \leq t \leq 1.
\end{cases} \quad (1.20)$$

Then we know the following.
Theorem 1.7 ([Li1] and [Li2]). Let \( \pi \in \mathbb{B}(\lambda) \). If \( e_i\pi \neq \theta \) (resp. \( f_i\pi \neq \theta \)), then \( e_i\pi \in \mathbb{B}(\lambda) \) (resp. \( f_i\pi \in \mathbb{B}(\lambda) \)). Hence the set \( \mathbb{B}(\lambda) \cup \{\theta\} \) is stable under the action of the root operators. Moreover, every element \( \pi \in \mathbb{B}(\lambda) \) is of the form \( \pi = f_{i_1}f_{i_2} \cdots f_{i_k}\pi_\lambda \) for some \( i_1, i_2, \ldots, i_k \in I \), where \( \pi_\lambda := (\lambda; 0, 1) = t\lambda \) is the only element of \( \mathbb{B}(\lambda) \) such that \( e_i\pi_\lambda = \theta \) for all \( i \in I \). Furthermore, we have

\[
\sum_{\pi \in \mathbb{B}(\lambda)} e(\pi(1)) = ch \, L(\lambda), \quad \sum_{\pi \in \mathbb{B}_w(\lambda)} e(\pi(1)) = ch \, L_w(\lambda),
\]

where \( \mathbb{B}_w(\lambda) := \{ (\nu_1, \ldots, \nu_s; a) \in \mathbb{B}(\lambda) | \nu_1 \leq w(\lambda) \} \) for each \( w \in W \).

It is known from [Kas5] et al. that \( \mathbb{B}(\lambda) \) has a natural crystal structure isomorphic to \( \mathcal{B}(\lambda) \). Namely, we have the following theorem (see [La2] for the second assertion).

Theorem 1.8. There exists a unique bijection \( \Phi : \mathcal{B}(\lambda) \xrightarrow{\sim} \mathbb{B}(\lambda) \) such that

\[
\Phi(F_{i_1}F_{i_2} \cdots F_{i_k}b_\lambda) = f_{i_1}f_{i_2} \cdots f_{i_k}\pi_\lambda.
\]

Moreover, \( \Phi(\mathbb{B}_w(\lambda)) = \mathbb{B}_w(\lambda) \) for each \( w \in W \).

At the end of this subsection, we recall the main result of [NS1]. Let \( \lambda \in P_+ \cap (\mathfrak{h}^*)^0 \). For \( \pi \in \mathbb{B}(\lambda) \), we define a path \( \omega^*(\pi) : [0, 1] \rightarrow \mathfrak{h}^* \) by \( (\omega^*(\pi))(t) := \omega^*(\pi(t)) \). Then we deduce that \( \mathbb{B}(\lambda) \) and \( \mathbb{B}_w(\lambda) \) with \( w \in \tilde{W} \) are \( \omega^* \)-stable (cf. Remark 4 and [NS1, Lemma 3.1.1]). Denote by \( \mathbb{B}_0(\lambda) \) the set of L-S paths that are fixed by \( \omega^* \), and set \( \mathbb{B}_0^w(\lambda) := \mathbb{B}_w(\lambda) \cap \mathbb{B}_0(\lambda) \) for each \( w \in \tilde{W} \).

Theorem 1.9 ([NS1, Theorem 3.2.4]). Let \( \lambda \in P_+ \cap (\mathfrak{h}^*)^0 \), and \( w \in \tilde{W} \). Set \( \hat{\lambda} := (P^*_w)^{-1}(\lambda) \) and \( \hat{\omega} := \Theta^{-1}(w) \). Then we have

\[
\mathbb{B}_0^w(\lambda) = P^*_w(\mathbb{B}(\hat{\lambda})), \quad \mathbb{B}_w^0(\lambda) = P^*_w(\mathbb{B}(\hat{\lambda})),
\]

where we denote by \( \mathbb{B}(\hat{\lambda}) \) the set of all L-S paths of class \( \hat{\lambda} \) for the orbit Lie algebra \( \hat{\mathfrak{g}} \), and set \( \mathbb{B}(\hat{\lambda}) := \{ (\hat{\nu}_1, \ldots, \hat{\nu}_s; a) \in \mathbb{B}(\lambda) | \hat{\nu}_1 \leq \hat{\omega}(\hat{\lambda}) \} \) with \( \leq \) the relative Bruhat order on \( \tilde{W} \hat{\lambda} \). Here, for \( \hat{\pi} \in \mathbb{B}(\hat{\lambda}) \), we define a path \( P^*_w(\hat{\pi}) : [0, 1] \rightarrow (\mathfrak{h}^*)^0 \) by \( (P^*_w(\hat{\pi}))(t) := P^*_w(\hat{\pi}(t)) \).

2 Twining Characters and q-twining Characters.

2.1 The Twining Characters. From now on, we always assume that \( \lambda \in P_+ \cap (\mathfrak{h}^*)^0 \) and \( \hat{\omega} \in \tilde{W} \). First we consider the linear automorphism \( \omega^{-1} \otimes id \) of the Verma module \( M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} Q(\lambda) \) of highest weight \( \lambda \) over \( \mathfrak{g} \), where \( Q(\lambda) \) is the one-dimensional \( \mathfrak{b} \)-module on which \( h \in \mathfrak{h} \) acts by the scalar \( \lambda(h) \) and \( n_+ \) acts trivially. Since this map stabilizes the (unique) maximal proper \( \mathfrak{g} \)-submodule \( N(\lambda) \) of \( M(\lambda) \), we obtain an induced
\(\mathbb{Q}\)-linear automorphism \(\tau_\omega : L(\lambda) \to L(\lambda)\), where \(L(\lambda) = M(\lambda)/N(\lambda)\). It is easily seen that \(\tau_\omega\) has the following properties:

\[
\tau_\omega(xv) = \omega^{-1}(x)\tau_\omega(v) \quad \text{for} \quad x \in \mathfrak{g}, v \in L(\lambda)
\]

and \(\tau_\omega(u_\lambda) = u_\lambda\), where \(u_\lambda\) is a (nonzero) highest weight vector of \(L(\lambda)\).

**Remark 5.** From [N1, Lemma 4.1] (or [NS2, Lemma 2.2.3]), we know that \(\tau_\omega\) is a unique endomorphism of \(L(\lambda)\) with the properties above.

The twining character \(\text{ch}_\omega(L(\lambda))\) of \(L(\lambda)\) is defined to be the formal sum

\[
\text{ch}_\omega(L(\lambda)) := \sum_{\chi \in \mathfrak{h}^*} \text{tr}(\tau_\omega|_{L(\lambda)_\chi}) e(\chi).
\]  

Since \(\tau_\omega(L(\lambda)_\chi) = L(\lambda)_{\omega^*(\chi)}\) for all \(\chi \in \mathfrak{h}^*\) and \(\dim L(\lambda)_{w(\lambda)} = 1\) for all \(w \in W\), we see that the Demazure module \(L_w(\lambda)\) is \(\tau_\omega\)-stable for all \(w \in \widetilde{W}\). Hence we can define the twining character \(\text{ch}_\omega(L_w(\lambda))\) of \(L_w(\lambda)\) by

\[
\text{ch}_\omega(L_w(\lambda)) := \sum_{\chi \in \mathfrak{h}^*} \text{tr}(\tau_\omega|_{L_w(\lambda)_\chi}) e(\chi).
\]

### 2.2 The q-twining Characters

In this subsection, we introduce the \(q\)-twining characters of \(V(\lambda)\) and \(V_w(\lambda)\), which are, in fact, \(q\)-analogues of \(\text{ch}_\omega(L(\lambda))\) and \(\text{ch}_\omega(L_w(\lambda))\), respectively (see Proposition 2.1 below).

By \((1.10)\), we have a \(\mathbb{Q}(q)\)-linear automorphism \(\tau_{\omega_q} : V(\lambda) \to V(\lambda)\) induced from \(\omega_q^{-1} : U_q^{-}(\mathfrak{g}) \to U_q^{-}(\mathfrak{g})\). As in the usual Lie algebra case in \(\S 2.1\), \(\tau_{\omega_q}\) has the following properties:

\[
\tau_{\omega_q}(xv) = \omega_q^{-1}(x)\tau_{\omega_q}(v) \quad \text{for} \quad x \in U_q(\mathfrak{g}), v \in V(\lambda)
\]

and \(\tau_{\omega_q}(u_\lambda) = u_\lambda\), where \(u_\lambda\) is a (nonzero) highest weight vector of \(V(\lambda)\).

**Remark 6.** In a similar way to the proof of [N1, Lemma 4.1], we can show that \(\tau_{\omega_q}\) is a unique endomorphism of \(V(\lambda)\) with the properties above.

The \(q\)-twining character \(\text{ch}_q^\omega(V(\lambda))\) of \(V(\lambda)\) is defined to be the formal sum

\[
\text{ch}_q^\omega(V(\lambda)) := \sum_{\chi \in \mathfrak{h}^*} \text{tr}(\tau_{\omega_q}|_{V(\lambda)_\chi}) e(\chi).
\]

We easily see that the quantum Demazure module \(V_w(\lambda)\) is \(\tau_{\omega_q}\)-stable for every \(w \in \widetilde{W}\). Hence we can define the \(q\)-twining character \(\text{ch}_q^\omega(V_w(\lambda))\) of \(V_w(\lambda)\) by

\[
\text{ch}_q^\omega(V_w(\lambda)) := \sum_{\chi \in \mathfrak{h}^*} \text{tr}(\tau_{\omega_q}|_{V_w(\lambda)_\chi}) e(\chi).
\]
It is clear that all $V(\lambda)_{X,Q}$ are finitely generated, torsion free $Q[q,q^{-1}]$-modules. Therefore they are free $Q[q,q^{-1}]$-modules of finite rank because $Q[q,q^{-1}]$ is a principal ideal domain. We also know that the natural map $Q(q) \otimes_{Q[q,q^{-1}]} V(\lambda)_Q \to V(\lambda)$ (given by $a \otimes v \to av$) is a $Q(q)$-linear isomorphism.

Now we consider $Q$ as a $Q[q,q^{-1}]$-module by the evaluation at $q = 1$. Set $V := Q \otimes_{Q[q,q^{-1}]} V(\lambda)_Q$ and $V_x := Q \otimes_{Q[q,q^{-1}]} V(\lambda)_x,Q$. It follows from [Ja, Lemma 5.12] that $V(\lambda)_Q$ is stable under the actions of $X_i, Y_i$, and $(q^h - q^{-h})/(q - q^{-1})$ for $i \in I, h \in P^\vee$. Thus we obtain endomorphisms $x_i, y_i, h$ of $V$ defined by

$$x_i := 1 \otimes X_i, \quad y_i := 1 \otimes Y_i, \quad h := 1 \otimes (q^h - q^{-h})/(q - q^{-1}),$$

respectively. From [Ja, Lemmas 5.13 and 5.14], we know that the endomorphisms $x_i, y_i, h$ of $V$ satisfy the Serre relations, and hence that these endomorphisms make $V$ into a $g$-module. Moreover, $V \cong L(\lambda)$ as $g$-modules, and the image of $V_x$ by this $g$-module isomorphism is $L(\lambda)_X$ for all $\chi \in \mathfrak{h}^*$. Taking these facts into account, we show the following proposition.

**Proposition 2.1.** Let $\chi \in (\mathfrak{h}^*)^0$ and $w \in \widetilde{W}$. Then $\tr(\tau_{\omega_q}|V(\lambda)_\chi)$ and $\tr(\tau_{\omega_q}|V_{w(\lambda)}_\chi)$ are elements of $Q[q,q^{-1}]$. Moreover, we have

$$\tr(\tau_{\omega_q}|V(\lambda)_\chi)|_{q=1} = \tr(\tau_{\omega_q}|L(\lambda)_\chi),$$

and hence

$$\chi^\omega_q(V(\lambda))|_{q=1} = \chi^\omega(L(\lambda)), \quad \chi^\omega_q(V_w(\lambda))|_{q=1} = \chi^\omega(L_w(\lambda)).$$

**Proof.** It can be easily checked that $V(\lambda)_Q$ is $\tau_{\omega_q}$-stable, and the following diagram commutes:

$$\begin{array}{ccc}
Q(q) \otimes_{Q[q,q^{-1}]} V(\lambda)_Q & \xrightarrow{\sim} & V(\lambda) \\
\downarrow_{1 \otimes (\tau_{\omega_q}|V(\lambda)_Q)} & & \downarrow_{\tau_{\omega_q}} \\
Q(q) \otimes_{Q[q,q^{-1}]} V(\lambda)_Q & \xrightarrow{\sim} & V(\lambda).
\end{array}$$

Since $V(\lambda)_X,Q$ is a free $Q[q,q^{-1}]$-module, we can define the trace of $\tau_{\omega_q}|V(\lambda)_X,Q$ for each $\chi \in (\mathfrak{h}^*)^0$. Note that a basis of $V(\lambda)_X,Q$ over $Q[q,q^{-1}]$ is also a basis of $V(\lambda)_X$ over $Q(q)$. We obtain from the commutative diagram above that

$$\tr(\tau_{\omega_q}|V(\lambda)_\chi) = \tr(\tau_{\omega_q}|V(\lambda)_{X,Q}) \in Q[q,q^{-1}] \quad \text{for all } \chi \in (\mathfrak{h}^*)^0.$$

Now let $w \in \widetilde{W}$, and take $u_{w(\lambda)} \in V(\lambda)_{w(\lambda),Q} \setminus \{0\}$. Here we remark that the rank of the free $Q[q,q^{-1}]$-module $V(\lambda)_{w(\lambda),Q}$ is one. We define $V_w(\lambda)_Q$ to be the $Q[q,q^{-1}]$-submodule of $V(\lambda)$ generated by the elements of the form $X_{i_1}X_{i_2} \cdots X_{i_k} u_{w(\lambda)}$. It is clear that $V_w(\lambda)_Q$ is
\(\tau_{\omega q}\)-stable. Since \(V(\lambda)_{Q}\) is stable under the action of \(X_{i}\), we see that \(V_{w}(\lambda)_{Q}\) is a \(\mathbb{Q}[q, q^{-1}]\)-submodule of \(V(\lambda)_{Q}\). We set \(V_{w}(\lambda)_{x, Q} := V_{w}(\lambda)_{Q} \cap V(\lambda)_{x, Q}\). Then we immediately obtain the following commutative diagram:

\[
\begin{array}{c}
\mathbb{Q}(q) \otimes_{\mathbb{Q}[q, q^{-1}]} V_{w}(\lambda)_{Q} \xrightarrow{\sim} V_{w}(\lambda) \\
\downarrow_{\tau_{\omega q}} \\
\mathbb{Q}(q) \otimes_{\mathbb{Q}[q, q^{-1}]} V(\lambda)_{Q} \xrightarrow{\sim} V(\lambda).
\end{array}
\]

Hence, in the same way as above, we have

\[
\text{tr}(\tau_{\omega q}|_{V_{w}(\lambda)_{x}}) = \text{tr}(\tau_{\omega q}|_{V_{w}(\lambda)_{x, Q}}) \in \mathbb{Q}[q, q^{-1}] \quad \text{for all} \quad \chi \in (\mathfrak{h}^*)^0,
\]

thereby completing the proof of the first assertion.

Next we show the equalities (2.5). Note that the \(\mathbb{Q}\)-linear automorphism \(\tau'_{\omega} := 1 \otimes (\tau_{\omega q})_{V(\lambda)_{Q}}\) of \(V := \mathbb{Q} \otimes_{\mathbb{Q}[q, q^{-1}]} V(\lambda)_{Q}\) satisfies \(\tau'_{\omega}(xv) = \omega^{-1}(x)\tau'_{\omega}(v)\) for \(x \in \mathfrak{g}, v \in V\), and \(\tau'_{\omega}(1 \otimes u_{\lambda}) = 1 \otimes u_{\lambda}\). Hence it follows from Remark 5 that the following diagram commutes:

\[
\begin{array}{c}
\mathbb{Q} \otimes_{\mathbb{Q}[q, q^{-1}]} V(\lambda)_{Q} \xrightarrow{\sim} L(\lambda) \\
\downarrow_{\tau'_{\omega}} \\
\mathbb{Q} \otimes_{\mathbb{Q}[q, q^{-1}]} V_{w}(\lambda)_{Q} \xrightarrow{\sim} L(\lambda).
\end{array}
\]

Remark that, for all \(\chi \in (\mathfrak{h}^*)^0\),

\[
\text{tr}(\tau_{\omega}|_{L(\lambda)_{x}}) = \text{tr}(\tau'_{\omega}|_{V(\lambda)_{x, Q}}) = 1 \otimes \text{tr}(\tau_{\omega q}|_{V(\lambda)_{x, Q}}) = \text{tr}(\tau_{\omega q}|_{V(\lambda)_{x, Q}})|_{q=1},
\]

since we regard \(\mathbb{Q}\) as a \(\mathbb{Q}[q, q^{-1}]\)-module by the evaluation at \(q = 1\). Combining (2.8) with (2.7), we obtain

\[
\text{tr}(\tau_{\omega}|_{L(\lambda)_{x}}) \equiv (2.8) \quad \text{tr}(\tau_{\omega q}|_{V(\lambda)_{x, Q}})|_{q=1} = (2.7) \quad \text{tr}(\tau_{\omega q}|_{V(\lambda)_{x}})|_{q=1} \quad \text{for all} \quad \chi \in (\mathfrak{h}^*)^0,
\]

which proves the first equality of (2.5). By considering \(V_{w} := \mathbb{Q} \otimes_{\mathbb{Q}[q, q^{-1}]} V_{w}(\lambda)_{Q}\) for \(w \in \widehat{W}\), we also obtain

\[
\text{tr}(\tau_{\omega q}|_{V_{w}(\lambda)_{x}})|_{q=1} = \text{tr}(\tau_{\omega}|_{L_{w}(\lambda)_{x}}) \quad \text{for all} \quad \chi \in (\mathfrak{h}^*)^0
\]

in the same way. This completes the proof of Proposition 2.1. \(\Box\)

### 3 Twining Character Formula for Demazure Modules.

The main result of this paper is the following.

**Theorem 3.1.** Let \(\lambda \in P_{+} \cap (\mathfrak{h}^*)^0\) and \(w \in \widehat{W}\). Set \(\hat{\lambda} := (P_{\omega}^*)^{-1}(\lambda)\) and \(\hat{w} := \Theta^{-1}(w)\). Then we have

\[
\text{ch}^\omega(L_{w}(\lambda)) = P_{\omega}^*(\text{ch} \hat{L}_{\hat{\omega}}(\hat{\lambda})),
\]

where \(\hat{L}_{\hat{\omega}}(\hat{\lambda})\) is the Demazure module of lowest weight \(\hat{w}(\hat{\lambda})\) in the irreducible highest weight module \(\hat{L}(\hat{\lambda})\) of highest weight \(\hat{\lambda}\) over the orbit Lie algebra \(\hat{\mathfrak{g}}\).
We need some lemmas in order to prove this theorem.

**Lemma 3.2.** For each \( i \in I \), we have \( \tau_{\omega_q} \circ E_i = E_{\omega^{-1}(i)} \circ \tau_{\omega_q} \) and \( \tau_{\omega_q} \circ F_i = F_{\omega^{-1}(i)} \circ \tau_{\omega_q} \).

**Proof.** We show only \( \tau_{\omega_q} \circ E_i = E_{\omega^{-1}(i)} \circ \tau_{\omega_q} \) since the proof of \( \tau_{\omega_q} \circ F_i = F_{\omega^{-1}(i)} \circ \tau_{\omega_q} \) is similar. Let \( u = \sum_{k \geq 0} Y_{\omega^{-1}(i)}^{(k)} u_k \in V(\lambda) \), where \( u_k \in (\ker X_i) \cap V(\lambda)_{x+k\alpha_i} \). Since \( \omega_q^{-1}(Y_{\omega^{-1}(i)}^{(k)}) = Y_{\omega^{-1}(i)}^{(k)} \), we have

\[
\tau_{\omega_q} \circ E_i(u) = \sum_{k \geq 0} Y_{\omega^{-1}(i)}^{(k)} \tau_{\omega_q}(u_k).
\]

On the other hand, \( \tau_{\omega_q}(u) = \sum_{k \geq 0} Y_{\omega^{-1}(i)}^{(k)} \tau_{\omega_q}(u_k) \in V(\lambda)_{x+k\alpha_i} \). Here we note that \( \tau_{\omega_q}(u_k) \in (\ker X_{\omega^{-1}(i)}) \cap V(\lambda)_{x+k\alpha_{\omega^{-1}(i)}} \). Hence, by the uniqueness of the expression of \( \tau_{\omega_q}(u) \), we have

\[
E_{\omega^{-1}(i)} \circ \tau_{\omega_q}(u) = \sum_{k \geq 0} Y_{\omega^{-1}(i)}^{(k)} \tau_{\omega_q}(u_k).
\]

Therefore we obtain \( \tau_{\omega_q} \circ E_i(u) = E_{\omega^{-1}(i)} \circ \tau_{\omega_q}(u) \) for all \( u \in V(\lambda) \), thereby completing the proof. \( \square \)

This lemma implies that \( L_0(\lambda) \) is \( \tau_{\omega_q} \)-stable. Hence we have the \( \mathbb{Q} \)-linear automorphism \( \overline{\tau}_{\omega_q} \) of \( L_0(\lambda)/qL_0(\lambda) \) induced from \( \tau_{\omega_q} \). Then, by the definition of \( \overline{\tau}_{\omega_q} \) and Lemma 3.2, we can easily check that the set \( B(\lambda) \) is \( \overline{\tau}_{\omega_q} \)-stable. Moreover, by Theorem 1.8, we have the following commutative diagram:

\[
\begin{array}{ccc}
B(\lambda) & \xrightarrow{\Phi} & B(\lambda) \\
\overline{\tau}_{\omega_q} \downarrow & & \downarrow \omega^* \\
B(\lambda) & \xrightarrow{\Phi} & B(\lambda).
\end{array}
\]  \hspace{1cm} (3.2)

Here we have used the fact that \( \omega^* \circ e_i = e_{\omega^{-1}(i)} \circ \omega^* \) and \( \omega^* \circ f_i = f_{\omega^{-1}(i)} \circ \omega^* \) (see [NS1, Lemma 3.1.1]). The next lemma immediately follows from the commutative diagram (3.2) and Theorem 1.8, since \( B_w(\lambda) \) is \( \omega^* \)-stable for all \( w \in \overline{W} \).

**Lemma 3.3.** Let \( w \in \overline{W} \). Then \( B_w(\lambda) \) is stable under \( \overline{\tau}_{\omega_q} \). Hence we obtain the following commutative diagram:

\[
\begin{array}{ccc}
B_w(\lambda) & \xrightarrow{\Phi} & B_w(\lambda) \\
\overline{\tau}_{\omega_q} \downarrow & & \downarrow \omega^* \\
B_w(\lambda) & \xrightarrow{\Phi} & B_w(\lambda).
\end{array}
\]  \hspace{1cm} (3.3)

Because \( \psi \circ \tau_{\omega_q} = \tau_{\omega_q} \circ \psi \), we see that \( L_{\omega_q}(\lambda) \) is also \( \tau_{\omega_q} \)-stable. Since \( V_0(\lambda) \) is obviously \( \tau_{\omega_q} \)-stable, we deduce that \( E(\lambda) \) is \( \tau_{\omega_q} \)-stable.

**Lemma 3.4.** \( \tau_{\omega_q} \circ G^*_\lambda = G^*_\lambda \circ \overline{\tau}_{\omega_q} \).
Proof. Remark that \( \{G_a(b) \mid b \in B(\lambda)\} \) is a basis of the \( \mathbb{Q} \)-vector space \( E(\lambda) \). Hence, for \( b \in B(\lambda) \), we have \( \tau_{\omega_q}(G_a(b)) = \sum b' \in B(\lambda) c_{b'} G_a(b') \) for some \( c_{b'} \in \mathbb{Q} \) since \( E(\lambda) \) is \( \tau_{\omega_q} \)-stable. Then we obtain \( \tau_{\omega_q}(b) = \sum b' \in B(\lambda) c_{b'} b' \) in \( \mathcal{L}_0(\lambda) / q \mathcal{L}_0(\lambda) \). Put \( b'' := \tau_{\omega_q}(b) \in B(\lambda) \). Because \( B(\lambda) \) is a basis of the \( \mathbb{Q} \)-vector space \( \mathcal{L}_0(\lambda) / q \mathcal{L}_0(\lambda) \), we see that \( c_{b''} = 1 \) and \( c_{b'} = 0 \) for all \( b' \in B(\lambda) \), \( b' \neq b'' \). Hence we obtain \( \tau_{\omega_q}(G_a(b)) = G_a(b'') = G_a(\tau_{\omega_q}(b)) \), as desired.

Proof of Theorem 3.1. By combining Lemmas 3.3 and 3.4, we see that the set \( \{G_a(b) \mid b \in B_w(\lambda) \cap B(\lambda)_x\} \) is \( \tau_{\omega_q} \)-stable. Because \( \{G_a(b) \mid b \in B_w(\lambda)\} \) is a basis of \( V_w(\lambda)_x \) over \( \mathbb{Q} (q) \) (see (1.14)), we obtain

\[
\text{tr}(\tau_{\omega_q} | V_w(\lambda)_x) = \# \{G_a(b) \mid \tau_{\omega_q}(G_a(b)) = G_a(b), b \in B_w(\lambda) \cap B(\lambda)_x\}
\]

for \( \chi \in (\mathfrak{h})^0 \) (note that if an endomorphism \( f \) on a finite-dimensional vector space \( V \) stabilizes a basis of \( V \), then the trace of \( f \) on \( V \) is equal to the number of basis elements fixed by \( f \)). By Lemma 3.4 again, we get

\[
\text{tr}(\tau_{\omega_q} | V_w(\lambda)_x) = \# \{b \in B_w(\lambda) \cap B(\lambda)_x \mid \tau_{\omega_q}(b) = b\},
\]

and hence

\[
\text{ch}^\omega_q(V_w(\lambda)) = \sum_{b \in B_w^0(\lambda)} e(\text{wt}(b)), \tag{3.4}
\]

where \( \text{wt}(b) := \chi \) if \( b \in B(\lambda)_x \), and \( B_w^0(\lambda) \) is the set of elements of \( B_w(\lambda) \) fixed by \( \tau_{\omega_q} \). The commutative diagram (3.3) implies that

\[
\text{ch}^\omega_q(V_w(\lambda)) \overset{(3.4)}{=} \sum_{b \in B_w^0(\chi)} e(\text{wt}(b)) \overset{(3.3)}{=} \sum_{\pi \in \mathfrak{B}_w^0(\chi)} e(\pi(1)).
\]

We see from Theorems 1.7 and 1.9 that the right-hand side of the above equality coincides with \( P_w^\ast(\text{ch} \tilde{L}_\theta(\lambda)) \), where \( \tilde{\lambda} := (P_w^\ast)^{-1}(\lambda) \) and \( \tilde{\omega} := \Theta^{-1}(w) \). Therefore we obtain

\[
\text{ch}^\omega_q(V_w(\lambda)) = P_w^\ast(\text{ch} \tilde{L}_\theta(\lambda)).
\]

Notice that the right-hand side is independent of \( q \). Hence we find that \( \text{ch}^\omega_q(V_w(\lambda)) \bigg|_{q=1} = P_w^\ast(\text{ch} \tilde{L}_\theta(\lambda)) \). Combining this with (2.6), we finally arrive at the conclusion that

\[
\text{ch}^\omega(L_w(\lambda)) = P_w^\ast(\text{ch} \tilde{L}_\theta(\lambda)).
\]

Thus we have proved Theorem 3.1. \( \square \)

Remark 7. By replacing \( V_w(\lambda) \) by \( V(\lambda) \) and \( L_w(\lambda) \) by \( L(\lambda) \) in the arguments above, we can give another proof of the twining character formula for the integrable highest weight module \( L(\lambda) \), which is the main result of [FSS] ([FRS]).
References.


