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Chapter 2

The construction of
\( P \)-expansive maps of regular continua: A geometric approach

In this chapter, we construct a new space \( Z \) from a continuous map \( f \) of a graph \( G \) and investigate the relationship between the dynamical behavior of \( f \) and the structure of \( Z \).

Let \( G \) be a graph, \( f : G \rightarrow G \) a continuous map and \( P \) a finite subset of \( G \) such that \( f(P) \subset P \). Put \( S(G, P) = P \cup \{C \mid C \text{ is a component of } G \setminus P \} \). Given \( x \in G \), the itinerary of \( x \) with respect to \( P \) and \( f \), written \( I_{P,f}(x) \) (or just \( I(x) \) if \( P \) and \( f \) are obvious from context), is defined to be the unique infinite sequence \( (C_n)_{n \geq 0} \) from \( S(G, P) \) given by the rule \( f^n(x) \in C_n \) for all \( n \geq 0 \). If no two points of \( G \) have the same itinerary, then \( f \) will be called \( P \)-expansive. And \( f \) is point-wise \( P \)-expansive if for each \( p, q \in P \), there exists some non-negative integer \( m \) such that \( A \cap (P \setminus \{f^m(p), f^m(q)\}) \neq \emptyset \) for each arc \( A \) in \( G \) between \( f^m(p) \) and \( f^m(q) \).

Let \( G \) be a graph, \( f : G \rightarrow G \) a continuous map and \( P \) a finite subset of \( G \)
such that \( f(P) \subseteq P \). We construct new spaces \( X_{\rightarrow} \) and \( X_{\leftarrow} \) from \( P \) and \( f \).

### 2.1 The constructions of \( X_{\rightarrow} \) and \( X_{\leftarrow} \)

First we want to define an equivalence relation \(~_1\) on \( P \). Let \( p, q \in P \). If for any non-negative integer \( i \), there exists an arc \( A_i \) in \( G \) between \( f^i(p) \) and \( f^i(q) \) such that \( A_i \cap P = \{ f^i(p), f^i(q) \} \), then we put \( p \sim^1_1 q \), where \( A_i \) may now consist of a single point. Now, if for \( p, q \in P \), there exist some points \( p_1, p_2, \ldots, p_k \) of \( P \) such that \( p \sim^1_1 p_1 \sim^1_1 p_2 \sim^1_1 \cdots \sim^1_1 p_k \sim^1_1 q \), then we set \( p \sim_1 q \). This relation \( \sim_1 \) is an equivalence relation on \( P \). Let \([p]_1\) be the equivalence class of \( p \), \( P_1 = \{ [p]_1 | p \in P \} \) and \( G_1 = G / \sim_1 \) the space obtained from \( G \) by identifying each equivalence class of \( P \). Then we define a continuous map \( f_1 : G_1 \rightarrow G_1 \) such that \( f_1|_{G_1 \setminus P_1} = f|_{G\setminus P} \) and \( f_1([p]_1) = [f(p)]_1 \) for \([p]_1 \in P_1 \). Similarly, if for any \( p, q \in P_1 \) and non-negative integer \( i \), there exists an arc \( A_i \) in \( G_1 \) between \( f^i_1(p) \) and \( f^i_1(q) \) such that \( A_i \cap P_1 = \{ f^i_1(p), f^i_1(q) \} \), then we put \( p \sim^2_2 q \). And if there exist some points \( p_1, p_2, \ldots, p_k \) of \( P_1 \) such that \( p \sim^2_2 p_1 \sim^2_2 p_2 \sim^2_2 \cdots \sim^2_2 p_k \sim^2_2 q \), then we set \( p \sim_2 q \). This relation \( \sim_2 \) is also an equivalence relation on \( P_1 \). Let \([p]_2 = \{ q | p \sim_2 q \text{ and } p, q \in P_1 \} \), \( P_2 = \{ [p]_2 | p \in P_1 \} \) and \( G_2 = G_1 / \sim_2 \) the space obtained from \( G_1 \) by identifying each equivalence class of \( P_1 \). Then we define a continuous map \( f_2 : G_2 \rightarrow G_2 \) such that \( f_2|_{G_2 \setminus P_2} = f_1|_{G_1 \setminus P_1} = f|_{G\setminus P} \) and \( f_2([p]_2) = [f_1(p)]_2 \) for \([p]_2 \in P_2 \). In the same way, we can obtain the space \( G_\ell \) and a continuous map \( f_\ell : G_\ell \rightarrow G_\ell \) for \( \ell \geq 1 \). Since \( P \) is finite, there is some natural number \( m \) such that \( f_m : G_m \rightarrow G_m \) is point-wise \( P \)-expansive. There exists a semi-conjugacy \( \pi_i \) between \( (G_{i-1}, f_{i-1}) \) and \( (G_i, f_i) \) for \( i = 1, 2, \ldots, m \), where \((G_0, f_0) = (G, f)\) (see Graph 2.1.1).
By the argument above, we may proceed with our construction, under the assumption that $f$ is point-wise $P$-expansive, in the rest part of this section.

Let $S(G, P) \setminus P = \{C_1, C_2, \ldots, C_n\}$ and $P = \{p_1, p_2, \ldots, p_k\}$. We will express the relation of elements of $S(G, P)$ as follows: If $p, q \in P$ and $f(p) = q$, then $p \to q$. This arrow $\to$ defines the Markov graph $P_\to$ on $P$ (Sec section 4). If $C_i, C_j \in S(G, P) \setminus P$ and $C_j \subseteq f(C_i)$, then $C_i \to C_j$. If $f(C_i) \cap C_j \neq \emptyset$, then $C_i \to C_j$. These arrows $\to$ and $\to$ define the Markov graphs $M_\to$ and $M_\to$ of elements of $S(G, P) \setminus P$ respectively. Note that $\to$ implies $\to$.

Now we will construct a new space $X_\to$ by using the Markov graphs $M_\to$ and $P_\to$. First we will construct a subspace $X$ which is the union of 3-dimensional
balls $B_1, B_2, \ldots, B_n$ in the Euclidean 3-dimensional space $\mathbb{E}^3$ by regarding elements $C_1, C_2, \ldots, C_n$ of $S(G, P) \setminus P$ as 3-dimensional balls $B_1, B_2, \ldots, B_n$ of $\mathbb{E}^3$. That is to say, $X = \bigcup_{i=1}^{n} B_i$, where the relationship of $B_i$ and $B_j$ is decided as follows: If $\text{cl}(C_i) \cap \text{cl}(C_j) = \emptyset$ for $C_i, C_j \in S(G, P) \setminus P$, then $B_i \cap B_j = \emptyset$. And if $\text{cl}(C_i) \cap \text{cl}(C_j) = \{q_1, q_2, \ldots, q_\ell\} \subset P$, then $B_i \cap B_j = \text{Bd}(B_i) \cap \text{Bd}(B_j) = \{q'_1, q'_2, \ldots, q'_\ell\}$, where $\text{Bd}(B)$ is the boundary of $B$. Without confusion, we can express elements of $\text{cl}(C_i) \cap \text{cl}(C_j)$ and $B_i \cap B_j$ in a similar way. And for each $p \in (P \cap \text{cl}(C_i)) \setminus \bigcup \{\text{cl}(C_j) \cap \text{cl}(C_{j'})| j \neq j' \text{ and } 1 \leq j, j' \leq n\}$, we take a corresponding point $p' \in \text{Bd}(B_i) \setminus \bigcup \{B_j \cap B_{j'}| j \neq j' \text{ and } 1 \leq j, j' \leq n\}$. For simplicity, we set $p' = p \in P$ (see Figure 2.1.2).

Figure 2.1.2

Put $X_0 = X$. We will construct a subspace $X_1$ contained in $X_0$ by using the Markov graph $M_\circ$ and $P_\circ$. For each $i = 1, 2, \ldots, n$, we have an embedding $h_i : X \rightarrow B_i$ such that

1. $h_i(X) \cap \text{Bd}(B_i) \subset P$, and
2. for each $p, q \in P$ with $p \in \text{Bd}(B_i)$ and $p \rightarrow q$, $h_i(q) = p \in \text{Bd}(B_i)$.

If $C_i \rightarrow C_j$ ($C_i, C_j \in S(G, P) \setminus P$) in the Markov graph $M_\circ$, then let $B_{i,j} = h_i(B_j)$ which is a copy of $B_j$. If $C_i \not\rightarrow C_j$, then $B_{i,j} = \emptyset$. Let $Y_i = \bigcup_{j=1}^{n} B_{i,j}$, $B_i = \{B_j| C_i \rightarrow C_j\}$ and $(\bigcup B_i) \cap P = \{p_{t(i;1)}, p_{t(i;2)}, \ldots, p_{t(i;k(i))}\}$, where $t(i : \ell)$
and \(k(i)\) are natural numbers with \(1 \leq t(i : \ell), k(i) \leq k\ (1 \leq \ell \leq k(i))\). And put \(h_i(p_{(i,j)}) = p_{i,(i,j)}\). Then we obtain a connected subset \(X_1 = Y_1 \cup Y_2 \cup \cdots \cup Y_n\) (see Figure 2.1.3).

Similarly, we will construct a subspace \(X_2\) in \(X_1\). Let \(h_{i_0,i_1} : X \hookrightarrow B_{i_0,i_1}\) be an embedding such that

1. \(h_{i_0,i_1}(X) \cap Bd(B_{i_0,i_1}) \subset h_{i_0}(P)\), and
2. for each \(p_{i_0,j} \in Bd(B_{i_0,i_1}) \cap h_{i_0}(P)\) and \(q \in P\) with \(p_j \to q\),
   
   \(h_{i_0,i_1}(q) = p_{i_0,j} \in Bd(B_{i_0,i_1})\).

If \(C_{i_1} \prec C_j\) in the Markov graph \(M_{--}\), then let \(B_{i_0,i_1,j} = h_{i_0,i_1}(B_j)\). And if \(C_{i_1} \not\prec C_j\), then \(B_{i_0,i_1,j} = \emptyset\). Let \(Y_{i_0,i_1} = \bigcup_{j=1}^{n} B_{i_0,i_1,j}, B_{i_1} = \{B_j | C_{i_1} \prec C_j\}\) and \((\bigcup B_{i_1}) \cap P = \{p_{t(i_0,i_1;1)}, p_{t(i_0,i_1;2)}, \ldots, p_{d(i_0,i_1;k(i_0,i_1))}\}\). Put \(h_{i_0,i_1}(p_{t(i_0,i_1;j)}) = p_{i_0,i_1,t(i_0,i_1;j)}\) (\(1 \leq j \leq t(i_0,i_1 : k(i_0,i_1))\)). Then we obtain \(X_2 = \bigcup \{Y_{i_0,i_1} | 1 \leq i_0, i_1 \leq n\}\) (see Figure 2.1.4).
When this operation is repeated inductively, we obtain $X_0 \supset X_1 \supset X_2 \supset \cdots$ and a subspace $X_\infty = \bigcap_{i=0}^\infty X_i$ of $\mathbb{B}^3$. Note that $X_\infty$ is connected.

Next let $X'_1, X'_2, \ldots$ be subspaces constructed in a similar way on basis of the Markov graph $M_\infty$. Then we obtain a subspace $X_\infty = \bigcap_{i=1}^\infty X'_i$ of $\mathbb{B}^3$. Note that $X_\infty$ is not always connected.

By using the construction of $X_\infty$, we show that $\lim_{m \to \infty} \text{diam}(B_{i_0,i_1,\ldots,i_m}) = 0$. Since we have assumed that $f$ is point-wise $P$-expansive, for any distinct points $p, q$ of $P$ there exists a non-negative integer $N_{p,q}$ such that $A \cap (P \setminus \{f^{N_{p,q}}(p), f^{N_{p,q}}(q)\}) \neq \emptyset$ for any arc $A$ in $G$ between $f^{N_{p,q}}(p)$ and $f^{N_{p,q}}(q)$. Let $N = \max\{N_{p,q} | p, q \in P$ and $p \neq q\}$. Let $m$ be a natural number and $B_{i_0,i_1,\ldots,i_m}$ a 3-dimensional ball from the construction of $X_m$. For any natural numbers $j_{m+1}, j_{m+2}, \ldots, j_{m+N}$ ($1 \leq j_{m+1}, j_{m+2}, \ldots, j_{m+N} \leq n$), where $n = \text{Card}(S(G, P) \setminus P)$, the 3-dimensional ball $B_{i_0,i_1,\ldots,i_m,j_{m+1},j_{m+2},\ldots,j_{m+N}}$ from the constructing of $X_{m+N}$ cannot contain two or more points of $\bigcup \{B_{i_0,i_1,\ldots,i_m} \cap B_{\ell_0,\ell_1,\ldots,\ell_m} | 1 \leq \ell_0, \ell_1, \ldots, \ell_m \leq n$ and $(\ell_0, \ell_1, \ldots, \ell_m) \neq (i_0, i_1, \ldots, i_m)\}$. Sup-
pose that there exist distinct points \( x, y \) of \( \bigcup \{ B_{\ell_0, i_0, \ldots, i_m} \cap B_{\ell_0, i_1, \ldots, i_m} | 1 \leq \ell_0, \ell_1, \ldots, \ell_m \} \) and \((\ell_0, \ell_1, \ldots, \ell_m) \neq (i_0, i_1, \ldots, i_m)\) such that \( x, y \in B_{i_0, i_1, \ldots, i_m} \cap B_{j_0, j_1, \ldots, j_m} \). Put \( x = p_{i_0, i_1, \ldots, i_m} = h_{i_0, i_1, \ldots, i_m}(p_i) \) and \( y = p_{j_0, j_1, \ldots, j_m} = h_{i_0, i_1, \ldots, i_m}(p_t) \). Then \( x, y \in P \cap B_{i_m} \). By the construction, for each \( i = 0, 1, \ldots, N \), there exists an arc \( A_i \) in \( G \) between \( f^i(p_s) \) and \( f^i(p_t) \) such that \( A_i \cap P = \{ f^i(p_s), f^i(p_t) \} \). This contradicts the definition of \( N \). Thus any two points \( x, y \in \bigcup \{ B_{\ell_0, i_0, \ldots, i_m} \cap B_{\ell_0, i_1, \ldots, i_m} | 1 \leq \ell_0, \ell_1, \ldots, \ell_m \leq n \) and \((\ell_0, \ell_1, \ldots, \ell_m) \neq (i_0, i_1, \ldots, i_m)\) are connected by the union of two or more 3-dimensional balls \( B_{\ell_0, i_0, \ldots, i_m}, B_{\ell_0, i_1, \ldots, i_m}, \ldots, B_{\ell_0, i_N, \ldots, i_m} \) (\( 1 \leq j_{m+1}, \ldots, j_{m+N} \leq n \)). Hence we may assume that \( \text{diam}(B_{i_0, i_1, \ldots, i_m}, B_{j_0, j_1, \ldots, j_m}) \) \( \leq \frac{1}{2} \text{diam}(B_{i_0, i_1, \ldots, i_m}) \). Thus we can suppose that \( \lim_{n \to \infty} \text{diam}(B_{i_0, i_1, \ldots, i_m}) = 0 \) (see Figure 2.1.5).

\[ \text{Figure 2.1.5} \]

### 2.2 The construction of \( Z \)

Let \( G \) be a graph, \( f : G \to G \) a continuous map, \( P = \{ p_1, p_2, \ldots, p_k \} \) a finite subset of \( G \) such that \( f(P) \subset P \) and \( S(G, P) \setminus P = \{ C_1, C_2, \ldots, C_n \} \). We may also assume that \( f \) is point-wise \( P \)-expansive in this section from the argument in section 2. And let \( X_-, X_- \) be the above spaces constructed by the Markov graphs \((M_-, P_-), (M_-, P_-)\) on \( S(G, P) \) respectively.
Theorem 2.2.1 The subspace $X_\omega$ of $E^3$ is a regular continuum.

Proof. Let $\epsilon > 0$ and $x \in X_\omega$. As $\lim_{m \to \infty} \text{diam}(B_{i_0,i_1,\ldots,i_m}) = 0$, for an $\epsilon$-neighbourhood $U_\epsilon(x)$ of $x$ in $X_\omega$, there exists a non-negative integer $\ell$ such that $B_{i_0,i_1,\ldots,i_\ell} \cap X_\omega \subset U_\epsilon(x)$ for any 3-dimensional ball $B_{i_0,i_1,\ldots,i_\ell}$ containing $x$. Let $B = \bigcup \{ B_{i_0,i_1,\ldots,i_\ell} \mid x \in B_{i_0,i_1,\ldots,i_\ell} \cap X_\omega \subset U_\epsilon(x) \}$. Then $B \cap X_\omega$ is a neighbourhood of $x$ in $X_\omega$ such that $B \cap X_\omega \subset U_\epsilon(x)$. By the construction of $X_\omega$, the boundary of $B$ has finite cardinality. Thus $X_\omega$ is a regular continuum. □

We define a map $\pi : G \to X_\omega$ as follows: Given $x \in G$, if $f^\ell(x) \in \text{cl}(C_i)$ for any $\ell = 0, 1, 2, \ldots$, then $\pi(x) = \bigcap_{\ell=0}^\infty B_{i_0,i_1,i_\ell,\ldots}$. We will investigate the uniqueness of $\pi(x)$ for each $x \in G$. Let $\{i_\ell\}_{\ell \geq 0}, \{j_\ell\}_{\ell \geq 0}$ be sequences of natural numbers such that $\{i_\ell\}_{\ell \geq 0} \neq \{j_\ell\}_{\ell \geq 0}$, $f^\ell(x) \in \text{cl}(C_{i_\ell}) \cap \text{cl}(C_{j_\ell})$ for each $\ell \geq 0$ and $1 \leq i_\ell, j_\ell \leq n$. We will show that $\bigcap_{\ell=0}^\infty B_{i_0,i_1,\ldots,i_\ell} = \bigcap_{\ell=0}^\infty B_{j_0,j_1,\ldots,j_\ell}$. Let $m = \min\{\ell | C_{i_\ell} \neq C_{j_\ell}\}$, then $f^m(x) \in P$. We put $x' = p_{0,i_1,\ldots,i_{m-1},i_\ell}$, where $p_{i_0,i_1,\ldots,i_{m-1},i_\ell} \in B_{i_0,i_1,\ldots,i_{m-1},i_\ell} \cap B_{j_0,j_1,\ldots,j_{m-1},j_\ell}$ and $p_\ell \in P$. Since $f(p_\ell) \in \text{cl}(C_{i_{m+1}}) \cap \text{cl}(C_{j_{m+1}})$, $p_{i_0,i_1,\ldots,i_{m-1},i_\ell} \in B_{i_0,i_1,\ldots,i_{m-1},i_\ell} \cap B_{j_0,j_1,\ldots,j_{m-1},j_\ell}$ by the construction of $X_{m+1}$. Similarly, since $f^2(p_\ell) \in \text{cl}(C_{i_{m+1}}) \cap \text{cl}(C_{j_{m+2}})$, $p_{i_0,i_1,\ldots,i_{m-1},i_\ell} \in B_{i_0,i_1,\ldots,i_{m+2}} \cap B_{j_0,j_1,\ldots,j_{m+2}}$. Inductively, for each $\ell \geq 0$, $p_{i_0,i_1,\ldots,i_{m-1},i_\ell} \in B_{i_0,i_1,\ldots,i_{m+\ell}} \cap B_{j_0,j_1,\ldots,j_{m+\ell}}$. As $\bigcap_{\ell=0}^\infty B_{i_0,i_1,\ldots,i_{m+\ell}}$ and $\bigcap_{\ell=0}^\infty B_{j_0,j_1,\ldots,j_{m+\ell}}$ are degenerate, $\{p_{0,i_1,\ldots,i_{m-1},i_\ell}\} = \bigcap_{\ell=0}^\infty B_{i_0,i_1,\ldots,i_{m+\ell}} = \bigcap_{\ell=0}^\infty B_{j_0,j_1,\ldots,j_{m+\ell}}$. Thus we can define $\{\pi(x)\} = \bigcap_{\ell=0}^\infty B_{i_0,i_1,\ldots,i_{m+\ell}} = \bigcap_{\ell=0}^\infty B_{j_0,j_1,\ldots,j_{m+\ell}} = x'$ (see Figure 2.2.1).
Lemma 2.2.2 \( \pi : G \rightarrow X_- \) is continuous.

Proof. Let \( x \in G \) and \( V \) be a neighbourhood of \( \pi(x) \) in \( X_- \).

Case 1. Assume that \( f^\ell(x) \notin P \) for any \( \ell = 0, 1, 2, \ldots \) and \( \{\pi(x)\} = \bigcap_{\ell=0}^{\infty} B_{i_0, i_1, \ldots, i_\ell} \). There exists a non-negative integer \( \ell \) such that \( B_{i_0, i_1, \ldots, i_\ell} \cap X_- \subset V \), where \( B_{i_0, i_1, \ldots, i_\ell} \) is a 3-dimensional ball containing \( \pi(x) \). Since \( C_{i_0, i_1, \ldots, i_\ell} = \{x \in G | x \in C_{i_0}, f(x) \in C_{i_1}, \ldots, f^\ell(x) \in C_{i_\ell}\} \) is an open set containing \( x \) and \( \pi(C_{i_0, i_1, \ldots, i_\ell}) \subset B_{i_0, i_1, \ldots, i_\ell} \cap X_- \), \( \pi \) is continuous at \( x \).

Case 2. Assume that there exists \( m = \min\{\ell | f^\ell(x) \in P\} < \infty \). There exists \( \ell > m \) such that \( B_{i_0, i_1, \ldots, i_\ell} \cap X_- \subset V \) for each \( B_{i_0, i_1, \ldots, i_\ell} \) containing \( \pi(x) \). Let \( C_{\ell} = \{C_{i_0, i_1, \ldots, i_\ell} | 1 \leq i_0, i_1, \ldots, i_\ell \leq Card(S(G, P))\} \) and \( U = \bigcup\{C \in C_{\ell} | x \in cl(C)\} \). Since \( f \) is continuous, \( U \) is a neighbourhood of \( x \) such that \( \pi(U) \subset V \).

Thus \( \pi \) is continuous. \( \square \)

Now we will put \( Z = \pi(G) \). Then \( X_- \subset Z \subset X_- \). In general it is difficult to recognize the precise structure of \( Z \), but by the above relation \( X_- \subset Z \subset X_- \), we can realize the approximate structure of \( Z \). Since \( X_- \) is regular, \( Z \) is also regular.

Note that by the construction, if for any element \( C \in S(G, P) \setminus P \), there exist finitely many elements \( C_1, C_2, \ldots, C_m \) of \( S(G, P) \) such that \( f(C) = \bigcup_{i=1}^{m} C_i \),
then $X_\pi = Z = X_\pi$.

Define a map $g : X_\pi \rightarrow X_\pi$ as follows: If $\{x\} = \bigcap_{\ell=0}^\infty B_{b_{\ell},i_1,\ldots,i_\ell}$, then
$$
\{g(x)\} = g(\bigcap_{\ell=0}^\infty B_{b_{\ell},i_1,\ldots,i_\ell}) = \bigcap_{\ell=1}^\infty B_{b_{\ell},i_1,\ldots,i_\ell}.
$$
We can investigate the uniqueness of $g$ as we did that of $\pi$. Note that $g(Z) \subset Z$.

**Lemma 2.2.3** $g : X_\pi \rightarrow X_\pi$ is continuous.

**Proof.** Let $x \in X_\pi$ and $V$ be a neighbourhood of $g(x)$ in $X_\pi$. Then there exists a non-negative integer $\ell$ such that $B_{i_0,i_1,\ldots,i_{\ell}} \cap X_\pi \subset V$ for any 3-dimensional ball $B_{i_0,i_1,\ldots,i_{\ell}}$ containing $g(x)$. Let $B = \bigcup\{B_{j_{0},j_1,\ldots,j_{\ell+1}} \mid x \in B_{j_{0},j_1,\ldots,j_{\ell+1}}\}$. Then $B$ is a neighbourhood of $x$ and $g(B \cap X_\pi) \subset V$. Thus $g$ is continuous. \(\Box\)

The following is the main theorem in this paper.

**Theorem 2.2.4** Let $G$ be a graph, $f : G \rightarrow G$ a continuous map and $P$ a finite subset of $G$ such that $f(P) \subset P$. Then there exist a regular continuum $Z$, a continuous map $g : Z \rightarrow Z$ and a semi-conjugacy $\pi : G \rightarrow Z$ such that

1. $g$ is $\pi(P)$-expansive, and
2. if $p, q \in P$ and $Q$ is a subset of $P$ with $A \cap Q \neq \emptyset$ for any arc $A$ in $G$ between $p$ and $q$, then $A' \cap \pi(Q) \neq \emptyset$ for any arc $A'$ in $Z$ between $\pi(p)$ and $\pi(q)$.

In addition, $f$ is point-wise $P$-expansive if and only if $\pi|_P$ is one-to-one.

**Proof.** Let $\pi$ and $g$ be the above maps. Let $x \in G$ with $f'(x) \in \text{cl}(C_t)$ for $\ell = 0, 1, 2, \ldots$. Then $\{\pi(x)\} = \bigcap_{\ell=0}^\infty B_{b_{\ell},i_1,\ldots,i_\ell}$ and $\{g \circ \pi(x)\} = \bigcap_{\ell=1}^\infty B_{b_{\ell},i_1,\ldots,i_\ell} = \{\pi \circ f(x)\}$. Thus $\pi$ is a semi-conjugacy between $(G, f)$ and $(Z, g)$.
We will show that (1) \( g \) is \( \pi(P) \)-expansive. Let \( x, y \) be distinct points of \( Z \). There exists a 3-dimensional ball \( B_{i_0,i_1,\ldots,i_t} \) such that \( x, y \in B_{i_0,i_1,\ldots,i_{t-1}}, \ x \in B_{i_0,i_1,\ldots,i_t} \) and \( y \not\in B_{i_0,i_1,\ldots,i_t} \). Then \( g^t(x) \in B_{i_t} \) and \( g^t(x) \not\in B_{i_t} \). Thus \( I_{\pi(P),g}(x) \neq I_{\pi(P),g}(y) \).

By the construction of \( X_\ast \), we can easily check (2). \( \square \)

**Proposition 2.2.5** Let \( G \) be a graph, \( f : G \longrightarrow G \) a continuous map and \( P \) the set of vertices of \( G \) with \( f(P) \subset P \). If \( f \) is point-wise \( P \)-expansive and \( f|_{[p,q]} \) is one-to-one for each edge \([p,q]\) between \( p \) and \( q \), then \( Z \) is homeomorphic to \( G \).

**Proof.** Let \( p, q \in P \) and \([p,q]\) be the edge between \( p \) and \( q \). Since \( f|_{[p,q]} \) is one-to-one, \( f([p,q]) \) is an arc between \( f(p) \) and \( f(q) \). Let \( \{C_{m_1}, C_{m_2}, \ldots, C_{m_\ell}\} \) be the set of elements of \( S(G,P) \setminus P \) which is contained in \( f([p,q]) \). As \( f \) is point-wise \( P \)-expansive, by the construction of \( X \) the 3-dimensional balls \( B_{m_1}, B_{m_2}, \ldots, B_{m_\ell} \) corresponding to \( C_{m_1}, C_{m_2}, \ldots, C_{m_\ell} \) form a chain between \( \pi(p) \) and \( \pi(q) \), i.e., \( B_{m_i} \cap B_{m_j} \neq \emptyset \) if and only if \( |i - j| \leq 1 \). Similarly, by the construction of \( X_1 \), finitely many smaller balls form a chain in each ball \( B_{m_i} \) \((i = 1, 2, \ldots, \ell)\), too. When we repeat this operation, \( \pi([p,q]) \) is an arc between \( \pi(p) \) and \( \pi(q) \). Thus \( Z \) is homeomorphic to \( G \). \( \square \)

### 2.3 Appendix

Let \( K \) be a continuum and \( P \) a finite subset of \( K \). Then we say that \( P \) **graph-separates** \( K \) if and only if there exists a finite set \( S(K,P) \) of subsets of \( K \) such that
(1) the element of $S(K, P)$ partition $K$, i.e., every point of $K$ is in exactly one member of $S(K, P)$,

(2) for each $p \in P$, $\{p\} \in S(K, P)$,

(3) for each $A \in S(K, P)$, the closure of $A$ in $K$ is arc-wise connected, and

(4) if $A, B \in S(K, P)$, then the closure of $A$ and $B$ either have empty intersection or intersect in only elements of $P$.

Note that we can also define $P$-expansive for a graph-separated continuum in a similar way.

Remark. We can obtain the same result in Theorem 2.2.4 by using a graph-separated continuum instead of a graph. □

2.4 Examples

In this section, a few concrete examples will be given to clarify the explanation given so far.

Example 2.4.1 Let $G$ be the unit interval $[0, 1]$. And denote $P = \{\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\}$.

We define a continuous map $f$ of $G$ into itself such that $f(x) = 2x$ (if $0 \leq x \leq \frac{1}{2}$) and $f(x) = -2x + 2$ (if $\frac{1}{2} \leq x \leq 1$). This map $f$ is point-wise $P$-expansive. Then $S(G, P) = \{(0, \frac{2}{7}), (\frac{2}{7}, \frac{4}{7}), (\frac{4}{7}, \frac{6}{7}), (\frac{6}{7}, 1)\}$, where put $C_1 = [0, \frac{2}{7}), C_2 = (\frac{2}{7}, \frac{4}{7}), C_3 = (\frac{4}{7}, \frac{6}{7}), C_4 = (\frac{6}{7}, 1], p_1 = \frac{2}{7}, p_2 = \frac{4}{7}$ and $p_3 = \frac{6}{7}$.

The Markov graph of $S(G, P) \setminus P$ and $P$ is as in Graph 2.4.1:
The above Markov graphs \((M_-, P_-)\) will give information useful in constructing the space \(Z\). Let \(X = B_1 \cup B_2 \cup B_3 \cup B_4\) and \(h_i : X \hookrightarrow B_i\) \((i = 1, 2, 3, 4)\) be an embedding such that (1) \(h_i(X) \cap Bd(B_i) \subset P\), and (2) for each \(p, q \in P\) with \(p \in Bd(B_i)\) and \(p \rightarrow q\), \(h_i(q) = p \in Bd(B_i)\). Then we describe the union \(Y_i\) of finitely many balls in each ball \(B_i\). For example when \(i = 2\), \(Y_2 = B_{2.3} \cup B_{2.4} \subset h_2(X)\), \(p_1 = h_2(p_2)\) and \(p_2 = h_2(p_3)\), since \(C_2 \rightarrow C_3\), \(C_2 \rightarrow C_4\), \(p_1 \rightarrow p_2\) and \(p_2 \rightarrow p_3\). In this way, we obtain a subspace \(X_1 = Y_1 \cup Y_2 \cup Y_3 \cup Y_4\) of \(E^3\) (see Figure 2.4.2).

Next we describe finitely many 3-dimensional balls in \(X_1\). Let \(h_{i,j} : X \hookrightarrow B_{i,j}\) be an embedding such that (1) \(h_{i,j}(X) \cap Bd(B_{i,j}) \subset h_i(P)\), and (2) for each \(p, q \in P\) with \(h_i(p) \in Bd(B_{i,j})\) and \(f^2(p) = q\), \(h_{i,j}(q) = h_i(p)\). Then we describe
the union $Y_{i,j}$ of finitely many balls in $B_{i,j}$. For example, when $i = 2$ and $j = 3$, $h_{2,3}(X) = B_{2,3,2} \cup B_{2,3,3} = Y_{2,3}$, $h_2(p_2) = h_{2,3}(p_1)$ and $h_2(p_3) = h_{2,3}(p_2)$, since $C_2 \rightarrow C_2$, $C_3 \rightarrow C_3$, $f^2(p_2) = p_1$ and $f^2(p_3) = p_2$. Put $X_2 = \bigcup_{i,j=1}^3 Y_{i,j}$ (see Figure 2.4.3).

Similarly, we can describe $X_i(i = 3, 4, \ldots)$. Finally the space $Z = \bigcap_{i=1}^\infty X_i$ is the universal dendrite (see Figure 2.4.4).

**Example 2.4.2** Let $G = [0, 1]$ be the unit interval, $P = \{0, \frac{1}{2}, 1\}$ and $f$ the same continuous map of $G$ as in Example 2.4.1. Then $X_- = Z = X_-$ and $Z$ is homeomorphic to $G = [0, 1]$. This implies that the structure of $Z$ depends on the way of selecting the points of $P$ (see Proposition 2.2.5).
Example 2.4.3 Let $G = [0, 1]$ be the unit interval and $P = \{\frac{1}{2}, \frac{3}{4}, 1\}$. We will define a continuous map $f$ of $G$ as follows: $f(x) = 4x(0 \leq x \leq \frac{1}{4})$, $f(x) = -2x + \frac{3}{2} (\frac{1}{2} \leq x \leq \frac{1}{2})$, $f(x) = 2x - \frac{1}{2} (\frac{1}{2} \leq x \leq \frac{3}{4})$ and $f(x) = -2x + \frac{5}{2} (\frac{3}{4} \leq x \leq 1)$. Then $S(G, P) = \{(0, \frac{1}{2}), (\frac{1}{2}, \frac{3}{4}), (\frac{3}{4}, 1), (1, \{1\})\}$, where put $C_1 = (0, \frac{1}{2})$, $C_2 = (\frac{1}{2}, \frac{3}{4})$, $C_3 = (\frac{3}{4}, 1)$, $p_1 = \frac{1}{2}$, $p_2 = \frac{3}{4}$ and $p_3 = 1$.

The Markov graph of $S(G, P) \setminus P$ and $P$ is as in Figure 2.4.5.

From the above Markov graph of $S(G, P) \setminus P$, we know that $B_{2,1} = \emptyset$ and $B_{3,1} = \emptyset$. Furthermore, the Markov graph of $P$ suggests the way of connection of each ball $B_{i,j}$, where $i, j = 1, 2, 3$ (see Figure 2.4.6).
Finally, $Z$ is the following dendrite (see Figure 2.4.7).

![Figure 2.4.7](image)

**Figure 2.4.7**

Example 2.4.4 Let $G$ be the following graph, $P = \{p_1, p_2, p_3, p_4, p_5, p_6\}$ a finite subset of $G$ and $f : G \rightarrow G$ a continuous map. And assume that $f(\text{cl}(C)) = G$ for any $C \subseteq S(G, P) \setminus P$, $f(p_1) = p_1 = f(p_4)$, $f(p_3) = p_2 = f(p_6)$ and $f(p_2) = p_3 = f(p_5)$. Note that $f$ is point-wise $P$-expansive (see Figure 2.4.8).

![Figure 2.4.8](image)

**Figure 2.4.8**

Then $Z$ is the triangular Sierpinski curve (see Figure 2.4.9).
Example 2.4.5 Let $f$ be a continuous map of $[0, 3]$ into itself and $P$ a periodic orbit $\{1, 2, 3\}$ of $f$ such that $f(x) = 6x (0 \leq x \leq \frac{1}{2}), f(x) = -2x + 4 (\frac{1}{2} \leq x \leq 1), f(x) = x + 1 (1 \leq x \leq 2)$ and $f(x) = -2x + 7 (2 \leq x \leq 3)$. Denote $C_0 = [0, 1), C_1 = (1, 2)$ and $C_2 = (2, 3]$ (see Graph 2.4.10).

The process of the construction of $Z$ is as in Figure 2.4.11.
Then we see that the structure of $Z$ is as in Figure 2.4.12.

Example 2.4.6 Let $f$ be a continuous map of $[0, 3]$ into itself and $P$ a periodic orbit $\{1, 2, 3\}$ of $f$ such that $f(x) = \frac{11}{3}x + \frac{1}{4}(0 \leq x \leq \frac{3}{4})$, $f(x) = -4x + 6(\frac{3}{4} \leq x \leq 1)$, $f(x) = x + 1(1 \leq x \leq 2)$ and $f(x) = -2x + 7(2 \leq x \leq 3)$. Denote $C_0 = [0, 1), C_1 = (1, 2)$ and $C_2 = (2, 3]$ (see Graph 2.4.13).
The process of the construction of $Z$ is as in Figure 2.4.14.

Then we see that the structure of $Z$ is as in Figure 2.4.15.