The classification using the weight equivalence relation is a generalization of Yonemura's classification.

The paper is organized as follows. In Section 2, some definitions and results about the hypersurface purely elliptic singularity are presented. We also prepare the term 'fundamental polynomial' which means the essential part of the defining polynomial. In Section 3, we introduce the weight equivalence relation among the defining polynomials and describe how to determine the weight equivalence class. Section 4 presents its applications to the two and the three-dimensional cases. Furthermore, the construction of defining polynomials and some considerations are given in Section 5. The last Section 6 treats the defining polynomials of three-dimensional hypersurface purely elliptic singularities of (0, 1)-type to prove Theorem 18.

Throughout this paper, the symbols \( \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R} \) denote the sets of natural numbers, integers, rational numbers, real numbers; for example, \( \mathbb{Q}_{>0} = \{ x \in \mathbb{Q} \mid x > 0 \} \). For a topological space \( X \), \( \text{int}X \) means the set of interior points of \( X \). \( \#A \) denotes the cardinality of a set \( A \).

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2 Preliminaries

Let \( z = (z_0, \cdots, z_n) \) be variable and \( f(z) = \sum m a_m z^m \in \mathcal{C}[z_0, \cdots, z_n], m = (m_0, \cdots, m_n) \in \mathbb{Z}_{>0}^{n+1}, \) and \( z^m = z_0^{m_0} \cdots z_n^{m_n}. \) The Newton diagram \( \Gamma_+(f) \) is the convex hull of \( \bigcup_{m \neq 0} (m + \mathbb{R}^{n+1}_{>0}) \) in \( \mathbb{R}^{n+1}_{>0} \) and the Newton boundary \( \Gamma(f) \) is the union of the compact faces of \( \Gamma_+(f) \). Set \( f_\Delta(z) = \sum_{m \in \Delta} a_m z^m \) for a face \( \Delta \) of \( \Gamma(f) \). We say that the polynomial \( f \) is nondegenerate if \( f_\Delta \) has no critical point in \( (\mathcal{C} - \{0\})^{n+1} \) for any face \( \Delta \) of \( \Gamma(f) \). Let \( \delta = (1, \cdots, 1) \in \mathcal{C}^{n+1}. \)

Then the hypersurface purely elliptic singularity is characterized in terms of the Newton boundary and its compact face as follows.

Theorem 1 (Watanabe [7]) Let \( f \) be a nondegenerate polynomial in \( \mathcal{C}[z_0, \cdots, z_n] \) and suppose that the hypersurface \( X = \{ f = 0 \} \) has an isolated singularity at \( x = 0 \in \mathcal{C}^{n+1}. \) Then,

(i) \( (X, x) \) is purely elliptic if and only if \( \delta \in \Gamma(f) \).

Let \( \Delta_0 \) be the compact face of the Newton boundary \( \Gamma(f) \) containing \( \delta \) in its relative interior and let \( s = \dim \Delta_0. \) Then,
(ii) $(X, x)$ is of $(0, s - 1)$-type if and only if $s \geq 2$ and
$(X, x)$ is of $(0, 0)$-type if and only if $s = 0$ or $1$.

In this paper, the above corresponding compact face $\Delta_0$ and the polynomial $f_{\Delta_0}$ are called fundamental face and fundamental polynomial, respectively. For simplicity, we say that $f$ is a $(0, s - 1)$-type polynomial in $C[z_0, \ldots, z_n]$ if $f$ is a nondegenerate polynomial defining a $n$-dimensional purely elliptic singularity of $(0, s - 1)$-type at $x = 0$.

3 Weight equivalence class

Let $S_{n+1}$ be the symmetric group. For $\sigma \in S_{n+1}$, this paper suggests the action of $\sigma$ for $f(z) = \sum a_m z^m$ as follows.

**Definition 2** $\sigma(f) = \sum a_m \sigma^\sigma(m)$, where $\sigma(m) = (\sigma(m_0), \ldots, \sigma(m_n))$.

For $s \in \mathbb{Z}_{\geq 0}$ with $0 \leq s \leq n$, let

$$\Phi^n_s = \left\{ f \in C[z_0, \ldots, z_n] \mid \begin{array}{l}
\text{There exists a compact face } \Delta(f) \text{ of } \Gamma(f) \\
\text{such that } \delta \in int(\Delta(f)) \text{ and } \dim(\Delta(f)) = s.
\end{array} \right\}.$$

Then we introduce an equivalent relation on $\Phi^n_s$ using the action $\sigma$ for $f \in \Phi^n_s$.

**Definition 3** For $f, g \in \Phi^n_s$, $f$ and $g$ are weight equivalent if there exists $\sigma \in S_{n+1}$ such that $\Delta(f)$ and $\Delta(\sigma(g))$ lie on the same $s$-dimensional hyperplane. Then we denote it $f \sim g$ and call its equivalence class a weight equivalence class.

**Remark 4** The weight equivalent classification is weaker than the analytical classification.

**Example 5** Let $f = x^2 + y^3 + z^6, g = y^2 + x^3 + x^6 + yz^3 \in \Phi^2_2$. For $\sigma(m) = (m_2, m_1, m_0), \sigma(g) = x^2 + y^3 + x^6 + xz^3$. Then $\Delta(f)$ and $\Delta(\sigma(g))$ lie on the same two-dimensional hyperplane whose normal vector is $(3, 2, 1)$. Hence $f \sim g$.

We consider the following set.

$$D\Phi^n_s = \{ f \in \Phi^n_s \mid f \text{ has an isolated singularity at } 0 \}.$$  

It follows from the definition of $\Phi^n_s$ and Theorem 1 that $D\Phi^n_s$ is the set of defining polynomials giving the $n$-dimensional hypersurface purely elliptic singularity of $(0, s - 1)$-type for $s \geq 2$ and $D\Phi^n_0 \cup D\Phi^n_1$ is the one of $(0, 0)$-type.