Chapter 3

Selfinjective semiprimary rings

In this chapter, we shall give some characterizations of selfinjectivity of semiprimary rings. Using the characterizations, we shall investigate Faith's conjecture (Problem B) in Section 3.3.

Throughout this chapter, for a ring $R$, we always denote $J = \text{Rad}(R)$, $S = \text{Soc}(R_R)$ and $S_k = \text{Soc}_k(R_R)$ for each non-negative integer $k$. Thus $\text{Soc}(eR) = eS$ for $e \in \pi(R)$. The terminology "a connected ring" means a ring that is indecomposable as a ring.

3.1 Selfinjectivity of semiprimary rings

In this section we characterize the selfinjectivity of semiprimary rings by using terms of chains of right ideals such that each factor is semisimple.

We first recall that a module $Y$ is $X$-injective for a module $X$ in case every homomorphism from any submodule of $X$ to $Y$ can be extended to a homomorphism from $X$ to $Y$.

The following two lemmas are essential in this section.
Lemma 3.1.1. Let $R$ be a ring and let

$$0 \to X' \to X \to X'' \to 0$$

be a short exact sequence of right $R$-modules. Then a right $R$-module $Y$ is $X$-injective if and only if $Y$ satisfies the following two conditions:

1. $Y$ is $X'$- and $X''$-injective.
2. The induced sequence

$$0 \to \text{Hom}_R(X'', Y) \to \text{Hom}_R(X, Y) \to \text{Hom}_R(X', Y) \to 0$$

is exact.

Proof. ($\Rightarrow$). This is clear from [1, Proposition 16.13(1)].

($\Leftarrow$). Suppose that $Y$ satisfies the conditions (1) and (2). We may assume that $X' \leq X$ and $X'' = X/X'$. Let $Z$ be a submodule of $X$ and $\alpha : Z \to Y$ a homomorphism. Since $Y$ is $X'$-injective by the assumption (1), the restriction map $\alpha|_{Z \cap X'}$ can be extended to a homomorphism $X' \to Y$. By the assumption (2), this extension can also be extended to a homomorphism $X \to Y$. Therefore there exists a homomorphism $\alpha' : X \to Y$ such that $\alpha'|_{Z \cap X'} = \alpha|_{Z \cap X'}$. Then the homomorphism $\beta : (Z + X')/X' \to Y$ defined by $\beta\pi(z) = \alpha(z) - \alpha'(z)$ for each $z \in Z$ is well-defined, where $\pi : X \to X/X'$ is the canonical epimorphism. Since $Y$ is $X''$-injective by the assumption (1), there exists an extension $\beta' : X'' = X/X' \to Y$ of $\beta$. Then $\alpha' + \beta'\pi : X \to Y$ becomes an extension of $\alpha$. Thus $Y$ is $X$-injective as required. □

Lemma 3.1.2. Let $R$ be a ring and let $X$ be a right $R$-module with a chain of submodules

$$0 = X_0 \leq X_1 \leq \cdots \leq X_{n-1} \leq X_n = X$$

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such that each factor $X_{i+1}/X_i$ is semisimple. Then a right $R$-module $Y$ is $X$-injective if and only if the canonical homomorphism

$$\text{Hom}_R(X/X_i, Y) \to \text{Hom}_R(X_{i+1}/X_i, Y)$$

is surjective for each $i = 0, 1, \ldots, n - 2$.

Proof. Since each $X_{i+1}/X_i$ is semisimple, $Y$ is $X_{i+1}/X_i$-injective. Thus by induction the lemma follows from Lemma 3.1.1. □

Let $n$ be a non-negative integer. We recall that a module $X$ has Loewy length $n$ in case $\text{Soc}_n(X) = X$ and $\text{Soc}_{n-1}(X) \neq X$. As a corollary of Lemma 3.1.2, we have

Corollary 3.1.3. Let $R$ be a semiprimary ring of Loewy length $n$ and let $Y$ be a right $R$-module. Then the following statements are equivalent:

(1) $Y$ is injective.

(2) The canonical homomorphism

$$\text{Soc}_{i+1}(Y)/\text{Soc}_i(Y) \to \text{Hom}_R(J^i/J^{i+1}, Y)$$

is an isomorphism for each $i = 1, 2, \ldots, n - 1$.

(3) The canonical homomorphism

$$l_Y(S_{i-1})/l_Y(S_i) \to \text{Hom}_R(S_i/S_{i-1}, Y)$$

is an isomorphism for each $i = 1, 2, \ldots, n - 1$.

Proof. For a right ideal $I$ of $R$ and a right $R$-module $Y$, we have the canonical isomorphism $l_Y(I) \cong \text{Hom}_R(R/I, Y)$. Therefore, by applying Lemma 3.1.2 to the lower and upper Loewy series

$$0 = J^n < J^{n-1} < \cdots < J < J^0 = R$$

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and

\[ 0 = S_0 < S_1 < \cdots < S_{n-1} < S_n = R, \]

it follows from \( \text{Soc}_R(Y) = l_Y(J^k) \) that the equivalences hold. \( \square \)

In particular, we have

**Corollary 3.1.4.** Let \( R \) be a semiprimary ring of Loewy length \( n \). Then the following statements are equivalent:

1. \( R \) is right selfinjective.

2. The canonical homomorphism

\[ S_{i+1}/S_i \to \text{Hom}_R(J^i/J^{i+1}, R) \]

is an isomorphism for each \( i = 1, 2, \ldots, n-1 \).

3. The canonical homomorphism

\[ l_R(S_{i-1})/l_R(S_i) \to \text{Hom}_R(S_i/S_{i-1}, R) \]

is an isomorphism for each \( i = 1, 2, \ldots, n-1 \).

By the definition of Loewy length, a right \( R \)-module \( X \) is of Loewy length 1 if and only if \( X \) is semisimple. Hence semiprimary rings of Loewy length 1 are just semisimple rings. For a right selfinjective semiprimary ring \( R \), if an indecomposable projective right \( R \)-module \( X \) is of Loewy length 2, then \( X \) is of (composition) length 2, because \( \text{Soc}(X) \) and \( X/\text{Rad}(X) \) are simple and \( \text{Soc}(X) = \text{Rad}(X) \). Thus right selfinjective semiprimary rings of Loewy length 2 are QF. Therefore, the case of Loewy length 3 is fundamental in Problem B.

In order to apply Lemma 3.1.2 to semiprimary rings of Loewy length 3, we need the next lemma. We recall that a ring \( R \) is a right Kasch ring in
case every simple right $R$-module is isomorphic to a minimal right ideal of $R$.

**Lemma 3.1.5.** Let $R$ be a connected right Kasch semiperfect ring such that $eS$ is simple for every $e \in \text{pi}(R)$.

1. For $e, f \in \text{pi}(R)$, $eR/eJ \cong fR/fJ$ iff $eR \cong fR$ iff $eS \cong fS$.

2. If $eR$ is simple for some $e \in \text{pi}(R)$, then $R$ is semisimple.

3. If $R$ is not semisimple, then $S \not\leq J$.

**Proof.** (1) Let $e_1, \ldots, e_n$ be a basic set of orthogonal primitive idempotents of $R$. Then $e_1R/e_1J, \ldots, e_nR/e_nJ$ is an irredundant set of representatives of the simple right $R$-modules. On the other hand, by assumption $e_1S, \ldots, e_nS$ must be also an irredundant set of representatives of the simple right $R$-modules. Thus the statement follows.

2. Assume $e = e_1$. Then, since $e_1R$ is simple,

$$\text{Hom}_R\left(\sum_{i=2}^n e_iR, e_1R\right) = \text{Hom}_R(e_1R, \sum_{i=2}^n e_iR) = 0$$

by (1). Thus, if $R$ is connected, then we have $n = 1$, i.e., $R$ is semisimple.

3. Assume $S \not\leq J$. Since $R$ is semiperfect, there exists $e \in \text{pi}(R)$ such that $eS \not\leq eJ$. Then $eR = eS$ and this is simple. However, since $R$ is not semisimple, this contradicts (2). \qed

We now give a characterization of connected right selfinjective semiprimary rings of Loewy length 3.

**Theorem 3.1.6.** A connected semiprimary ring $R$ of Loewy length 3 is right selfinjective if and only if $R$ satisfies the following three conditions:

1. $l_R(S) = J$ and $S \leq J$. 

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(2) \( R/J \cong \text{End}_R(S) \) canonically.

(3) \( J/S \cong \text{Hom}_R(J/S, S) \) canonically.

Proof. (\( \Rightarrow \)). Assume that \( R \) is right selfinjective. Then \( R \) is a right PF ring (Remark 1.2.3(3)). Thus it follows from [14, Thorem 6] that \( l_R(S) = J \). Also \( S \leq J \) by Lemma 3.1.5(3). Thus (1) holds. The conditions (2) and (3) follow from the injectivity of \( R_R \), \( l_R(S) = J \) and \( l_R(J) = S \).

(\( \Leftarrow \)). Assume that \( R \) satisfies the three conditions. Since \( J^3 = 0 \), we have \( J^2 \leq l_R(J) = S \). Thus each factor of the chain \( 0 < S < J < R \) is semisimple as a right \( R \)-module. Applying Lemma 3.1.2 to this chain, we observe that \( R \) is right selfinjective by assumption. \( \square \)

### 3.2 Ring decompositions of PF rings

In this section we investigate certain ring decompositions of PF rings. The results will be used in Section 3.3.

Let \( X \) and \( Y \) be right \( R \)-modules. Following Harada [12], we say that \( Y \) is \( X \)-simple-injective in case every homomorphism from a submodule of \( X \) to \( Y \) with simple image can be extended to a homomorphism from \( X \) to \( Y \).

**Lemma 3.2.1.** Let \( R \) be a semiperfect ring. Let \( e, f, g \in \text{pi}(R) \) such that \( eJ = eS \), \( eR \) is \( fR \)-simple-injective and \( fR \) is \( gR \)-simple-injective, and let \( X \) be a submodule of \( gJ \). If \( \text{Hom}_R(gR, eS) \neq 0 \) and \( \text{Hom}_R(X, fS) \neq 0 \), then \( fR \cong gR \) and \( X \leq gS \).

**Proof.** Since \( \text{Hom}_R(X, fS) \neq 0 \) and \( fS \) is semisimple, there exists a homomorphism \( \alpha : X \to fR \) with simple image. Then by assumption \( \alpha \) can be extended to a homomorphism \( \beta : gR \to fR \) and we have the following com-
Since \( gR \) is finitely generated, the canonical epimorphism \( gR/gJ \to \beta(gR)/\beta(gJ) \) induced by \( \beta \) is an isomorphism. Since \( \text{Hom}_R(gR, eS) \neq 0 \), there exists a monomorphism \( gR/gJ \to eR \). Making the composite of the canonical epimorphism \( \beta(gR) \to \beta(gR)/\beta(gJ) \), the inverse isomorphism \( \beta(gR)/\beta(gJ) \to gR/gJ \) and the monomorphism \( gR/gJ \to eR \), we have a homomorphism \( \gamma : \beta(gR) \to eR \) with \( \text{Im} \gamma = eS \). Then by assumption \( \gamma \) can be extended to a homomorphism \( \delta : fR \to eR \) and we have the following commutative diagram:

\[
\begin{array}{ccc}
\beta(gR) & \xrightarrow{\leq} & fR \\
\downarrow{\gamma} & & \downarrow{\delta} \\
\beta(gR)/\beta(gJ) & \cong & gR/gJ \to eR
\end{array}
\]

To show that \( \beta \) is an isomorphism, assume that \( \beta \) is not an isomorphism. Then \( \beta(gR) \leq fJ \). Therefore, since \( \gamma \neq 0 \) and \( eJ = eS \), \( \delta \) must be an isomorphism. Then \( \gamma \) is a monomorphism and \( \beta(gJ) = 0 \). Thus \( \alpha(X) = \beta(X) = 0 \). However, this contradicts \( \alpha \neq 0 \). Therefore \( \beta \) is an isomorphism and hence \( \alpha \) is a monomorphism. Thus \( fR \cong gR \) and \( X \leq gS \).

\[\square\]

**Lemma 3.2.2.** Let \( R \) be a right Kasch semiperfect ring such that \( eR \) is \( fR \)-simple-injective for every pair \( e, f \in \pi(R) \). Let \( e, g \in \pi(R) \) such that \( eJ = eS \) and \( \text{Hom}_R(gR, eS) \neq 0 \). Then \( gJ \leq gS \).

**Proof.** For any finitely generated nonzero submodule \( X \) of \( gJ \), since \( R \) is right Kasch, there exists \( f \in \pi(R) \) such that \( \text{Hom}_R(X, fS) \neq 0 \). Then by Lemma 3.2.1 \( X \leq gS \). Therefore \( gJ \leq gS \).

\[\square\]
The following two lemmas are dual to Lemmas 3.2.1 and 3.2.2, respectively.

**Lemma 3.2.3.** Let $R$ be a right Kasch semiperfect ring such that $eS$ is simple for every $e \in \pi(R)$ and $S_R$ is essential in $R_R$. Let $e, f, g \in \pi(R)$ such that $eJ = eS$ and let $X$ be a submodule of $gR$ such that $X \geq gS$. If $\text{Hom}_R(eR/eJ, gR) \neq 0$ and $\text{Hom}_R(fR/fJ, gR/X) \neq 0$, then

(a) $fR \cong gR$ and $fJ$ can be embedded in $X$,

or

(b) $fR \cong eR$.

Furthermore, if every monomorphism between indecomposable projective right $R$-modules is an isomorphism, then (a) necessarily occur and $gJ \leq X$ in (a).

**Proof.** Since $\text{Hom}_R(fR/fJ, gR/X) \neq 0$, there exists a monomorphism $\alpha : fR/fJ \to gR/X$. Since $gR$ is projective, there exists a homomorphism $\beta : fR \to gR$ such that $\alpha \pi = \pi' \beta$, where $\pi : fR \to fR/fJ$ and $\pi' : gR \to gR/X$ are the canonical epimorphisms. Thus we have the following commutative diagram:

\[
\begin{array}{ccc}
 fR & \longrightarrow & fR/fJ \\
 \downarrow \beta & & \downarrow \alpha \\
 gR & \longrightarrow & gR/X \\
\end{array}
\]

If $\beta$ is a monomorphism, then $fR \cong gR$ by assumption and Lemma 3.1.5(1). Since $\pi'(fJ) = \alpha(xJ) = 0$, we have $\beta(fJ) \leq X$. Thus $fJ$ can be embedded in $X$ by $\beta$. Therefore the case (a) occurs.

If $\beta$ is not a monomorphism, then we have $fS \leq \text{Ker} \beta$ and $gS \leq \text{Im} \beta$ since $S_R$ is essential in $R_R$ and $fS, gS$ are simple. Thus by $\text{Hom}_R(eR/eJ, gR) \neq 0$.
There exists a monomorphism $\gamma : eR/eJ \to \text{Im} \beta$. Since $fR$ is projective, there exists a homomorphism $\delta : eR \to fR$ such that $\gamma \pi'' = \tilde{\beta} \delta$, where $\pi'' : eR \to eR/eJ$ is the canonical epimorphism and $\tilde{\beta} : fR \to \text{Im} \beta$ is the epimorphism induced by $\beta$. Thus we have the following commutative diagram:

$$
\begin{array}{ccc}
    eR & \overset{\pi''}{\longrightarrow} & eR/eJ \\
      \downarrow \delta & & \downarrow \gamma \\
    fR & \overset{\beta}{\longrightarrow} & \text{Im} \beta
\end{array}
$$

If $\delta$ is not a monomorphism, then $\text{Im} \delta$ is semisimple by $eJ = eS$. So we have $\text{Im} \delta \leq fS$, which contradicts $fS \leq \text{Ker} \beta$ and $\gamma \pi'' \neq 0$. Thus $\delta$ is a monomorphism and hence $eR \cong fR$ by assumption and Lemma 3.1.5(1). Therefore the case (b) occurs.

To show the last statement of the lemma, assume that every monomorphism between indecomposable projective modules is an isomorphism and (a) does not occur. Then the homomorphism $\delta : eR \to fR$ is an isomorphism. Thus $\gamma : eR/eJ \to \text{Im} \beta$ must be an isomorphism. Then we have $\text{Im} \beta = gS$. Since $X \geq gS$, we have

$$\alpha \pi(fR) = \pi'(\text{Im} \beta) = \pi'(gS) = 0,$$

which contradicts $\alpha \pi \neq 0$. Therefore the case (a) necessarily occur. In the case (a), $\beta : fR \to gR$ is an isomorphism by assumption. Thus we have $gJ \leq X$. 

\[ \square \]

**Lemma 3.2.4.** Let $R$ be a right Kasch semiperfect ring such that $eS$ is simple for every $e \in \text{pi}(R)$ and $S_R$ is essential in $R_R$. Suppose that every monomorphism between indecomposable projective right $R$-modules is an isomorphism. Let $e, g \in \text{pi}(R)$ such that $eJ = eS$ and $\text{Hom}_R(eR/eJ, gR) \neq 0$. Then $gJ \leq gS$. 

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Proof. Since $gR$ is not simple by assumption, we have $gS \leq gJ$. If $gR \neq gJ$, then there exist submodules $X$, $Y$ of $gJ$ such that $gS \leq X < Y \leq gJ$ and $Y/X$ is simple. Let $f \in \pi(R)$ such that $Y/X \cong fR/fJ$. Then by Lemma 3.2.3 $gJ \leq X$, a contradiction. \qed

Lemma 3.2.5. Let $R$ be a connected right Kasch semiperfect ring such that $eS$ is simple for every $e \in \pi(R)$. Let $e \in \pi(R)$ such that $eJ = eS$. If $eS \cong eR/eJ$, then $e$ is a basic idempotent.

Proof. Let $e_1, \ldots, e_n$ be a basic set of orthogonal primitive idempotents of $R$ and assume that $e = e_1$. If $e_1S \cong e_1R/e_1J$, then by Lemma 3.1.5(1) we have

$$\text{Hom}_R\left(\sum_{i=2}^{n} e_iR, e_1R\right) = \text{Hom}_R(e_1R, \sum_{i=2}^{n} e_iR) = 0.$$

Since $R$ is connected, $n = 1$ and $e = e_1$ is a basic idempotent. \qed

We now prove the following proposition, which provides certain ring decompositions of right PF rings.

Proposition 3.2.6. Let $R$ be a connected right Kasch semiperfect ring such that $eS$ is simple for every $e \in \pi(R)$. Suppose that $R$ satisfies one of the following three conditions:

(1) $eR$ is $fR$-simple-injective for every pair $e, f \in \pi(R)$.

(2) $S_R$ is essential in $R_R$ and every monomorphism between indecomposable projective right $R$-modules is an isomorphism.

(3) $S_R$ is essential in $R_R$ and $\text{Soc}(eR/eS) \neq 0$ for every $e \in \pi(R)$.

If $eJ^2 = 0$ for some $e \in \pi(R)$, then the Loewy length of $R$ is less than or equal to 2.
Proof. Let \( e \in \pi(R) \) such that \( eJ^2 = 0 \). If \( eJ = 0 \), then \( R \) is semisimple by Lemma 3.1.5(2) and the Loewy length of \( R \) is equal to 1. Thus we assume that \( eJ \neq 0 \) and \( eS = eJ \). If \( eS \cong eR/eJ \), then by Lemma 3.2.5 the Loewy length of \( R \) is equal to 2. Therefore, furthermore we assume \( eS \not\cong eR/eJ \).

Assume that \( R \) satisfies (3). Let \( e_1, \ldots, e_n \) be a basic set of orthogonal primitive idempotents of \( R \) and let \( T_i = e_iR/e_iJ \) for each \( i \). We may assume that \( e = e_1 \) and \( n > 1 \). It suffices to show that \( e_1, \ldots, e_n \) can be renumbered so that \( T_k \cong e_{k+1}S \) (\( 1 \leq k < n \)) and \( e_kJ = e_kS \) (\( 1 \leq k \leq n \)). We prove this by induction. First, since \( e_1S \not\cong T_1 \), let \( T_1 \cong e_2S \). Next we assume that \( e_1, \ldots, e_i \) can be renumbered so that \( T_k \cong e_{k+1}S \) (\( 1 \leq k \leq i \)) \( e_kJ = e_kS \) (\( 1 \leq k \leq i \)) for some \( i \) with \( 1 \leq i < n \). Then by the assumption (3), there exists \( j \) such that \( \text{Hom}_R(e_jR/e_jJ, e_{i+1}R/e_{i+1}S) \neq 0 \). Then by Lemma 3.2.3, either (a) \( e_jR \cong e_{i+1}R \) and \( e_jJ = e_jS \), or (b) \( e_jR \cong e_iR \) holds. If the case (b) holds, since \( \text{Hom}_R(e_1R/e_1J, e_{i+1}R/e_{i+1}S) \neq 0 \), there exists a monomorphism \( \alpha : e_iR/e_iJ \to e_{i+1}R/e_{i+1}S \). Let \( \beta : e_iR \to e_{i+1}R \) be a lifting of \( \alpha \). By Lemma 3.1.5(1) \( \beta \) is not a monomorphism. Thus since \( e_iJ = e_iS \), \( \text{Im} \beta \) is semisimple and \( \text{Im} \beta \leq e_{i+1}S \). This contradicts \( \alpha \neq 0 \). Therefore the case (b) does not hold and hence the case (a) must do. Thus \( e_jR \cong e_{i+1}R \) and \( e_jJ = e_jS \). Hence \( j = i + 1 \) and \( e_{i+1}J = e_{i+1}S \). Therefore by induction \( e_1, \ldots, e_n \) can be renumbered as desired.

Similarly, in case \( R \) satisfies (1) or (2), we can prove the statements by using Lemmas 3.2.2 or 3.2.4. \qed

Theorem 3.2.7. Every right PF ring \( R \) has a ring decomposition

\[
R = R_1 \times R_2 \times R_3
\]

for which \( R_1 \) is a semisimple ring, \( R_2 \) is a QF ring such that all indecomposable projective modules have Loewy length 2, and \( R_3 \) is a right PF ring such that all indecomposable projective modules have Loewy length \( \geq 3 \).

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Proof. By Theorem 1.2.2 a right PF ring $R$ is a right Kasch semiperfect ring such that $eS$ is simple for every $e \in \text{pi}(R)$ and $S_R$ is essential in $R_R$. Thus we can apply Proposition 3.2.6 to each connected ring direct summand $R'$ of $R$. Then the type of $R'$ is one of the types of $R_1$, $R_2$ and $R_3$ of the theorem. Therefore the theorem follows. \hfill $\square$

Remark 3.2.8. (1) A ring $R$ is said to be a dual ring in case $R$ satisfies the double annihilator property, that is, for each left ideal $K$ of $R$ and for each right ideal $L$ of $R$,

$$l_Rr_R(K) = K \text{ and } r_Rl_R(L) = L$$

hold. The concept of dual rings is a generalization of two-sided PF rings. Similar to the case of right PF rings, it follows from [11] that every dual ring $R$ is a right Kasch semiperfect ring such that $eS$ is simple for every $e \in \text{pi}(R)$, $S_R$ is essential in $R_R$, and $eR$ is $fR$-simple-injective for every pair $e, f \in \text{pi}(R)$. Therefore, for dual rings, Proposition 3.2.6 is valid and we obtain a similar ring decomposition to one in Theorem 3.2.7.

(2) Theorem 3.2.7 states that for a right PF ring $R$, the sum of all indecomposable projective right modules $eR$ ($e \in \text{pi}(R)$) of Loewy length $i$ becomes a ring direct summand of $R$ for $i = 1, 2$. However this does not necessarily hold for all indecomposable projective modules of Loewy length $\geq 3$, as the following example shows.

Example 3.2.9. Let $K$ be a field, $Q$ the quiver

\[
\begin{array}{cccc}
1 \\
& \gamma & \alpha \\
2 \rightarrow & 3 \rightarrow & 4 \\
\beta & \delta & \epsilon
\end{array}
\]

and $\rho$ the set of relations $\{\beta\alpha\gamma - e\delta, \alpha\gamma\beta\alpha, \delta\beta, \gamma\epsilon\}$ for $Q$ over $K$. Let $R = KQ/\langle \rho \rangle$ the factor $K$-algebra of the path algebra $KQ$ and let $e_i$ be the

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primitive idempotent of $R$ corresponding to the vertex $i$ for $1 \leq i \leq 4$. Then it is routine to check that $R$ is a connected QF ring and the composition diagrams of the Loewy factors of the indecomposable projective right $R$-modules have the following form:

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
3 & 1 & 2 & 4 \\
2 & 3 & 1 & 4 \\
1 & 2 & 3 & 4
\end{array}
\]

where "$f$" denote a composition factor that is isomorphic to $e_i R/e_i J$ for $1 \leq i \leq 4$. Then $e_i R$ has Loewy length 4 for $1 \leq i \leq 3$ and $e_4 R$ has Loewy length 3. Thus the sum of all indecomposable right $R$-modules $eR$ ($e \in \pi(R)$) of Loewy length 3 is not a ring direct summand of $R$ even if $R$ is a QF ring.

### 3.3 Faith's conjecture

In this final section we reduce Faith's conjecture (Problem B) to some problems on bimodules over semisimple rings. First we construct rings of a special form by using bimodules and bimodule homomorphisms.

Let $T$ be any ring and let both $U$ and $V$ be nonzero $(T, T)$-bimodules. The notations $\text{Hom}_T(V, U)$ and $\text{End}_T(U)$ denote the sets of all right $T$-module homomorphisms. Then $\text{Hom}_T(V, U)$ becomes a $(T, T)$-bimodule naturally. Let $\phi : V \to \text{Hom}_T(V, U)$ be a $(T, T)$-bimodule homomorphism. We can define a ring structure on the additive group $R = T \oplus V \oplus U$ as the following way:

1. The multiplication by an element of $T$ is defined by using the $(T, T)$-bimodule structures of $T$, $U$ and $V$;

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(2) \( V \times U, U \times V \) and \( U \times U \) are defined by 0;

(3) \( V \times V \) is defined by \( vv' = [\phi(v)](v') \) for \( v, v' \in V \).

It is routine to check that \( R \) becomes a ring and \( T \) is a subring of \( R \) with the same identity.

**Lemma 3.3.1.** Let \( T, U, V, \phi \) and \( R \) be as above. Suppose that \( T \) is semisimple, \( T \cong \text{End}_T(U) \) canonically and \( \phi \) is an isomorphism. Then

1. \( R \) is a semiprimary ring of Loewy length 3.
2. \( J = \text{Rad}(R) = V \oplus U \) and \( S = \text{Soc}(R_R) = U \).
3. \( l_R(S) = J \) and \( l_R(J) = S \).
4. \( R \) is right selfinjective.

**Proof.** It follows from the definition of the multiplication of \( R \) and the assumption of \( \phi \) that \( (V \oplus U)^3 = 0 \) and \( (V \oplus U)^2 \neq 0 \). So, since \( R/(V \oplus U) \cong T \) is semisimple, \( V \oplus U = \text{Rad}(R) = J \) and \( R \) is a semiprimary ring of Loewy length 3. Then by assumption we have

\[
U = l_R(V \oplus U) = l_R(J) = \text{Soc}(R_R) = S
\]

and

\[
l_R(S) = l_R(U) = V \oplus U = J.
\]

Thus by assumption and Theorem 3.1.6, \( R \) is right selfinjective. \( \square \)

To state the next theorem, which shows that Faith's conjecture for certain semiprimary rings can be reduced to a problem on semisimple rings and their bimodules, we need to introduce a condition and a notation.

Let \( T \) be a ring. For a pair of \((T,T)\)-bimodules \((U,V)\), we consider the condition
\((*)\) \( T \cong \text{End}_T(U) \) canonically and \( V \cong \text{Hom}_T(V,U) \) as \((T,T)\)-bimodules.

For a \((T,T)\)-bimodule \( M \) and a ring endomorphism \( \psi \) of \( T \), we define a \((T,T)\)-bimodule \( \psi M \) as follows: the right \( T \)-module structure of \( \psi M \) is the same as \( M_T \) and the left \( T \)-operations is defined by \( t \cdot m = \phi(t)m \) for \( t \in T, \ m \in M \).

**Theorem 3.3.2.** The following statements are equivalent:

1. Every right selfinjective semiprimary ring \( R \) with \( l_R(S_2) = J^2 \) is QF.
2. Every right selfinjective semiprimary ring \( R \) with the same upper and lower Loewy series is QF.
3. Every right selfinjective semiprimary ring \( R \) of Loewy length 3 is QF.
4. For every basic semisimple ring \( T \), if a \((T,T)\)-bimodule \( V \) and a ring automorphism \( \psi \) of \( T \) satisfy \( V \cong \psi \text{Hom}_T(V,T) \) as \((T,T)\)-bimodules, then \( V_T \) is finitely generated.
5. For every semisimple ring \( T \), if a pair of \((T,T)\)-bimodules \((U,V)\) satisfies the condition \((*)\), then \( V_T \) is finitely generated.

**Proof.** (1) \( \Rightarrow \) (2). Let \( R \) be a right selfinjective semiprimary ring of Loewy length \( n \). If \( R \) has the same upper and lower Loewy series, that is, \( S_i = J^{n-i} \) for each \( i \), then

\[
l_R(S_2) = l_R(J^{n-2}) = S_{n-2} = J^2.
\]

Thus by the assumption of (1) \( R \) is QF.

(2) \( \Rightarrow \) (3). If \( R \) is a right selfinjective semiprimary ring of Loewy length 3, then \( R \) is a right PF ring. Thus by Theorem 3.2.7 we may assume \( eJ^2 \neq 0 \) for each \( e \in \pi(R) \). Then, since \( eJ^3 = 0 \), we have \( eJ^2 = eS \). Hence \( J^2 = S = S_1 \). Similarly \( J = S_2 \). Therefore \( R \) has the same upper and lower Loewy series, and \( R \) is QF by the assumption of (2).
(3) \Rightarrow (5). Let T be a semisimple ring and let (U, V) be a pair of (T, T)-bimodules satisfying (\ast). Then by Lemma 3.3.1 the ring \( R = T \oplus V \oplus U \) becomes a right selfinjective semiprimary ring of Loewy length 3 and \( J/S \cong V \). Thus by the assumption of (3) \( R \) is QF and hence \( (J/S)_{R} \) is finitely generated. Therefore \( V_{T} \) is finitely generated.

(5) \Rightarrow (1). Let \( R \) be a right selfinjective semiprimary ring \( R \) with \( l_{R}(S_{2}) = J^{2} \) and let \( T = R/J \). Then by Corollary 3.1.4 we have the canonical isomorphisms \( T \cong \text{End}_{T}(S), J/J^{2} \cong \text{Hom}_{T}(S_{2}/S, S) \) and \( S_{2}/S \cong \text{Hom}_{T}(J/J^{2}, S) \). Hence \( (J/J^{2}) \oplus (S_{2}/S) \cong \text{Hom}_{T}((J/J^{2}) \oplus (S_{2}/S), S) \). Thus, by letting \( U = S \) and \( V = (J/J^{2}) \oplus (S_{2}/S) \), the pair of \( (T, T) \)-bimodules \( (U, V) \) satisfies (\ast). Therefore by the assumption of (5) \( V_{T} \) is finitely generated and hence \( R \) is QF by Lemma 1.2.5.

(4) \Rightarrow (5). Let \( T \) be a semisimple ring and let \( (U, V) \) be a pair of \( (T, T) \)-bimodules satisfying (\ast). We may assume that \( T \) is basic. Then, since \( T \cong \text{End}_{T}(U) \) and \( T \) is basic semisimple, there exists a right \( T \)-isomorphism \( \alpha : U \to T \). Thus there exists a ring automorphism \( \psi \) of \( T \) that makes the following square commutative:

\[
\begin{array}{ccc}
T & \xrightarrow{\psi} & T \\
\downarrow \cong & & \downarrow \cong \\
\text{End}_{T}(U) \xrightarrow{\cong} \text{Hom}_{T}(U, \text{End}_{T}(T)) & \xrightarrow{\cong} & \text{End}_{T}(T)
\end{array}
\]

where the vertical isomorphisms are induced by left multiplications. Then it is easy to see that \( U \cong \psi T \) as \( (T, T) \)-bimodules by \( \alpha \) and hence we have

\[
V \cong \text{Hom}_{T}(V, U) \cong \text{Hom}_{T}(V, \psi T) \cong \psi \text{Hom}_{T}(V, T)
\]

as \( (T, T) \)-bimodules. Therefore by the assumption of (4) \( V_{T} \) is finitely generated.

(5) \Rightarrow (4). This follows from the proof of the implication (4) \Rightarrow (5). \qed
Restricting the theorem above, we have

**Theorem 3.3.3.** The following statements are equivalent:

1. Every right selfinjective semiprimary ring $R$ with $l_R(S_2) = J^2$ such that $R/J \cong S$ as $(R, R)$-bimodules is QF.

2. Every right selfinjective semiprimary ring $R$ with the same upper and lower Loewy series such that $R/J \cong S$ as $(R, R)$-bimodules is QF.

3. Every right selfinjective semiprimary ring $R$ of Loewy length 3 such that $R/J \cong S$ as $(R, R)$-bimodules is QF.

4. For every basic semisimple ring $T$, if a $(T, T)$-bimodule $V$ satisfies $V \cong \text{Hom}_T(V, T)$ as $(T, T)$-bimodules, then $V_T$ is finitely generated.

**Remark 3.3.4.** (1) Let $R$ be a connected semiprimary ring of Loewy length 3 and let $T = R/J$, $U = S$ and $V = J/S$. Let denote $R' = T \oplus V \oplus U$ the ring constructed above. Though $R$ and $R'$ are not isomorphic in general as the example below shows, Theorem 3.1.6 and Lemma 3.3.1 state that $R$ is right selfinjective (QF) if and only if so is $R'$. Therefore, to investigate Faith’s conjecture for semiprimary rings of Loewy length 3, we can concentrate on rings of the form $T \oplus V \oplus U$.

(2) For the statements of Theorems 3.3.3 and 3.3.2, “semiprimary rings” in (1) to (3) and “semisimple rings” in (4) (and (5)) can be replaced by “semiprimary local rings” and “division rings”, respectively. Analogous replacements for the statements of Corollary 3.3.6 and Theorems 3.3.8 and 3.3.9 are valid.

**Example 3.3.5.** Let $p$ be a prime number and let $R = \mathbb{Z}/\mathbb{Z}p^3$. Then $R$ is a QF ring of Loewy length 3. Let $T = R/J \cong \mathbb{Z}/\mathbb{Z}p$, $U = S \cong \mathbb{Z}/\mathbb{Z}p$ and
\( V = J/S \cong \mathbb{Z}/p \), and let \( R' = T \bigoplus V \bigoplus U \) be the ring constructed above. Then \( R' \) is also a QF ring of Loewy length 3. However \( R \) and \( R' \) are not isomorphic.

The following corollary shows that Faith's conjecture for general semiprimary rings can be reduced to that for semiprimary rings of Loewy length 3.

**Corollary 3.3.6.** The following statements are equivalent:

1. Every right selfinjective semiprimary ring \( R \) is QF.
2. The following two statements hold.
   
   (a) Every right selfinjective semiprimary ring \( R \) satisfies \( l_R(S_2) = J^2 \).
   
   (b) Every right selfinjective semiprimary ring \( R \) of Loewy length 3 is QF.
3. The following two statements hold.
   
   (a) Every right selfinjective semiprimary ring \( R \) satisfies \( l_R(S_2) = J^2 \).
   
   (b) For every basic semisimple ring \( T \), if a \((T,T)\)-bimodule \( V \) and a ring automorphism \( \psi \) of \( T \) satisfy \( V \cong \psi \text{Hom}_T(V,T) \) as \((T,T)\)-bimodules, then \( V_T \) is finitely generated.
4. The following two statements hold.
   
   (a) Every right selfinjective semiprimary ring \( R \) satisfies \( l_R(S_2) = J^2 \).
   
   (b) For every semisimple ring \( T \), if a pair of \((T,T)\)-bimodules \((U,V)\) satisfies (\( \ast \)), then \( V_T \) is finitely generated.
Proof. (1) ⇒ (2). If a right selfinjective semiprimary ring \( R \) is QF, by Lemma 2.1.7 \( R \) satisfies

\[
l_R(S_2) = l_R(\text{Soc}_R(R)) = l_Rr_R(J^2) = J^2.
\]

Thus this implication follows.

(2) ⇔ (3) ⇔ (4). These are clear from Theorem 3.3.2.

(4) ⇒ (1). Let \( R \) be a right selfinjective semiprimary ring. Then by assumption and Theorem 3.3.2 (and its proof), \( (J/J^2)_R \) is finitely generated. Thus \( R \) is QF by Lemma 1.2.5. \( \square \)

Remark 3.3.7. (1) As the proof of the implication (1) ⇒ (2) above shows, the condition (a) of Corollary 3.3.6 is equivalent to \( l_Rr_R(J^2) = J^2 \), that is, \( R(R/J^2) \) is cogenerated by \( R \). It follows from the proof of Theorem 3.3.2 that right selfinjective semiprimary rings of Loewy length 3 satisfy the condition (a).

(2) For the statements (1) and (2)-(4)(a) of Corollary 3.3.6 above, we can replace "right selfinjective semiprimary rings" by "right PF one-sided perfect rings". (See the proof of Theorem 3.3.2 and Lemma 1.2.5.)

Let \( T \) be a ring and let \( U, V \) be \( (T,T) \)-bimodules. To give several sufficient conditions for Problem B is affirmative, for the pair \( (U, V) \) we consider the following conditions

\((*)\) \( T \cong \text{End}_T(U) \) canonically and there exists a surjective \( (T,T) \)-homomorphism \( V \rightarrow \text{Hom}_T(V,U) \)

and

\((**')\) \( T \cong \text{End}_T(U) \) canonically, \( V_T \cong \text{Hom}_T(V,U)_T \) and \( T'V \cong _T\text{Hom}_T(V,U) \).

We note that if the pair \( (U, V) \) satisfies \((*)\), then \( (U,V) \) satisfies \((*')\) and \((**')\).
Theorem 3.3.8. If, for every semisimple ring $T$, $V_T$ is finitely generated whenever a pair of $(T, T)$-bimodules $(U, V)$ satisfies ($\ast'$), then every right PF one-sided perfect ring is QF.

Proof. Let $R$ be a right PF one-sided perfect ring. Then the canonical homomorphism $S_2/S \rightarrow \text{Hom}_R(J/J^2, S)$ is an isomorphism and by [14, Theorem 6] and Lemma 2.1.7 the canonical homomorphism $J/J^2 \rightarrow \text{Hom}_R(S_2/S, S)$ is an epimorphism. Thus, letting $T = R/J$, $U = S$ and $V = (J/J^2) \oplus (S_2/S)$, we see that the pair of $(T, T)$-bimodules $(U, V)$ satisfies ($\ast'$). Hence by assumption $V_T$ is finitely generated. Therefore, since $R$ is one-sided perfect, $R$ is right artinian by Lemma 1.2.5. \qed

Theorem 3.3.9. If, for every semisimple ring $T$, $V_T$ is finitely generated whenever a pair of $(T, T)$-bimodules $(U, V)$ satisfies ($\ast''$), then every right selfinjective semiprimary ring is QF.

Proof. Let $R$ be a right selfinjective semiprimary ring of Loewy Length $n$. Consider the lower Loewy series

\[(LS) \quad 0 = J^n < J^{n-1} < \cdots < J < J^0 = R\]

and the upper Loewy series

\[(US) \quad 0 = S_0 < S_1 < \cdots < S_{n-1} < S_n = R\]

of $R_R$. By Lemma 2.1.7 all factors of these series are semisimple on both sides. Let $T = R/J$, $U = S$, $V = (R/J) \oplus (J/J^2) \oplus \cdots \oplus J^{n-1}$ and $W = S_1 \oplus (S_2/S_1) \oplus \cdots \oplus (R/S_{n-1})$. Then (LS) and (US) have equivalent refinements on both sides by the Schreier refinement theorem. Thus, since $V$ and $W$ are semisimple on both sides, $V$ and $W$ are isomorphic on both sides as $T$-modules. Therefore, since $W \cong \text{Hom}_T(V, U)$ as $(T, T)$-bimodules by Corollary 3.1.4, the pair of $(T, T)$-bimodules $(U, V)$ satisfies ($\ast''$). Thus by assumption $V_T$ is finitely generated and hence $R$ is right artinian. \qed
At the last part of this chapter, we consider relations between the cardinality of selfinjective rings and Faith’s conjecture. Lawrence [22] proved that every countable right selfinjective ring is QF (Theorem 1.2.8). Concerning this, we state the following result as a corollary of Theorems 3.3.8 and 3.3.9.

We denote by $|X|$ the cardinality of a set $X$ and denote by $\aleph_0$ the countable cardinality.

**Corollary 3.3.10.** The following statements hold.

1. Every right PF one-sided perfect ring $R$ with $|R/J| \leq \aleph_0$ is QF.
2. Every right selfinjective semiprimary ring $R$ with $|R/J| < |R|$ is QF.

To prove the result above, we note

**Lemma 3.3.11.** Let $X$ be a right vector space over a division ring $D$. Then

1. $|X| = \begin{cases} |D|^\dim(X) & \text{if } \dim(X) < \aleph_0 \\ |D| \times \dim(X) & \text{if } \dim(X) \geq \aleph_0 \end{cases}$

2. $|\Hom_D(X, D)| = |D|^\dim(X)$.

3. $\dim(D\Hom_D(X, D)) = |D|^\dim(X)$.

**Proof.**
1. This follows from [1, p. 98, Exercise 16].
2. This is clear from $\Hom_D(X, D) \cong D^I$, the direct product of copies of $D$, as left $D$-vector spaces.
3. This is a theorem of Erdős and Kaplansky (see [5, p.400]).

We now prove Corollary 3.3.10.

**Proof of Corollary 3.3.10.** (1) In view of the proof of Theorem 3.3.8, it suffices to show that for a basic semisimple ring $T$ with $|T| \leq \aleph_0$, if a pair of $(T, T)$-bimodules $(U, V)$ satisfies (*), then $V_T$ is finitely generated.
Let $e_1, \ldots, e_n$ be a complete set of orthogonal primitive central idempotents of $T$. Let $D_i = e_i T e_i$ and $V_i = V e_i$ for $1 \leq i \leq n$. Then each $D_i$ is a division ring and each $V_i$ is a right $D_i$-vector space. Also $T = D_1 \times \cdots \times D_n$ as a ring direct sum and $V = V_1 \times \cdots \times V_n$. Since $T \cong \text{End}_T(U)$ canonically, $U_T \cong T_T$. Therefore, as $\mathbb{Z}$-modules,

$$\text{Hom}_T(V, U) \cong \text{Hom}_T(V, T) \cong \text{Hom}_{D_1}(V_1, D_1) \times \cdots \times \text{Hom}_{D_n}(V_n, D_n).$$

Let $\delta_i = |D_i|$ and $\alpha_i = \dim(V_i D_i)$ for each $i$. Then $\delta_i \leq \aleph_0$ for all $i$ by assumption. To show that all $\alpha_i$ are finite, assume that $\alpha_1, \ldots, \alpha_m \geq \aleph_0$ and $\alpha_{m+1}, \ldots, \alpha_n < \aleph_0$ for some $m$ with $1 \leq m \leq n$. Then by Lemma 3.3.11(2),

$$|\text{Hom}_T(V, U)| = |\text{Hom}_{D_1}(V_1, D_1)| \times \cdots \times |\text{Hom}_{D_n}(V_n, D_n)|$$

$$= \delta_1^{\alpha_1} \times \cdots \times \delta_n^{\alpha_n}$$

$$\geq 2^{\alpha_1} \times \cdots \times 2^{\alpha_m}$$

$$> \max\{\alpha_1, \ldots, \alpha_m\}.$$

On the other hand, by Lemma 3.3.11(1) and by the assumptions $\delta_i \leq \aleph_0$ ($1 \leq i \leq n$), $\alpha_i \geq \aleph_0$ ($1 \leq i \leq m$) and $\alpha_i < \aleph_0$ ($m+1 \leq i \leq n$), we have

$$|V| = |V_1| \times \cdots \times |V_n|$$

$$= \delta_1 \times \alpha_1 \times \cdots \times \delta_m \times \alpha_m \times \delta_{m+1}^{\alpha_{m+1}} \times \cdots \times \delta_n^{\alpha_n}$$

$$= \max\{\alpha_1, \ldots, \alpha_m\}.$$

Therefore $|V| < |\text{Hom}_T(V, U)|$. However this contradicts the existence of a surjection $V \to \text{Hom}_T(V, U)$. Therefore all $\alpha_i$ are finite and hence $V_T$ is finitely generated.

(2) In view of the proof of Theorem 3.3.9, it suffices to show that for a basic semisimple ring $T$, if a pair of $(T, T)$-bimodules $(U, V)$ satisfies $(\ast')$ and $|T| < |V|$, then $V_T$ is finitely generated. If $V_T$ is not finitely generated, by using a similar discussion to the proof of (1), we can show $|V| < |\text{Hom}_T(V, U)|$. 44
However this contradicts the fact that $\tau V \cong \tau \Hom_T(V, U)$. Hence $V_T$ is finitely generated. \hfill $\Box$

Finally we state some conditions for the examples which show that Problem B is negative, if there exist. To simplify the situation, we consider the problem (Faith's conjecture) for right selfinjective semiprimary local rings of Loewy length 3. By Theorem 3.3.3, the existence of a non-QF right selfinjective semiprimary local ring $R$ with $R/J \cong S$ as $(R, R)$-bimodules, is equivalent to the existence of a pair of a division ring $D$ and a $(D, D)$-bivector space $V$ such that $\dim(V_D)$ is infinite and $V \cong \Hom_D(V, D)$ as $(D, D)$-bivector spaces.

**Proposition 3.3.12.** Let $D$ be a division ring and let $V$ be a $(D, D)$-bivector space such that $\dim(V_D) \geq \aleph_0$ and $V \cong \Hom_D(V, D)$ as $(D, D)$-bivector spaces. Then

$$|D|^\dim(V_D) = |D| \quad \text{and} \quad \dim(DV) = |D|.$$ 

**Proof.** Let $\delta = |D|$, $\alpha = \dim(V_D)$ and $\beta = \dim(DV)$. By assumption $\alpha, \beta \geq \aleph_0$ and by Corollary 3.3.10(1), we have $\delta > \aleph_0$. It follows from Lemma 3.3.11(3) and assumption that

$$\beta = \dim(DV) = \dim(D\Hom_D(V, D)) = |D|^\dim(V_D) = \delta^\alpha.$$ 

Also by Lemma 3.3.11(1),

$$\delta \times \alpha = |V| = |\Hom_D(V, D)| = \delta \times \beta = \delta \times \delta^\alpha.$$ 

Thus, since $\max(\alpha, \delta) \leq \delta^\alpha$, we have $\delta^\alpha = \delta$. \hfill $\Box$

As the following example shows, for any pair of infinite cardinals $\alpha$, $\delta$ with $\delta^\alpha = \delta$, there exist a division ring $D$ and a $(D, D)$-bivector space $V$ such that $|D| = \delta$, $\dim(V_D) = \alpha$ and $\dim(DV) = \delta$. Thus, the argument only
by using the fact that $|D|^\dim(V_D) = |D|$ and $\dim(DV) = |D|^\dim(V_D) = |D|$ does not valid to solve Faith's conjecture.

**Example 3.3.13.** For any pair of infinite cardinals $\alpha, \delta$ with $\delta = \delta^\alpha$, there exist a division ring $D$ with $|D| = \delta$ and a $(D, D)$-bivector space $V$ with $\dim(V_D) = \alpha$ and $\dim(DV) = \delta$.

Let $K$ be a field with $|K| \leq \aleph_\theta$ and let $X$ be a set of indeterminates with $|X| = \delta$. Let $D = K(X)$ be a rational function field over $K$ with commutative indeterminates of $X$. Then it is easy to see that $|D| = \delta$. Since $X$ is an infinite set, there exists an injective map $\sigma : X \to X$ such that $|X \setminus \sigma(X)| = \delta$. Let $X_1 = \sigma(X)$ and $X_2 = X \setminus \sigma(X)$. Define an injective ring endomorphism $\psi$ of $D$ by $\psi(k) = k$ for $k \in K$ and $\psi(x) = \sigma(x)$ for $x \in X$.

Let $W = _D D$ be the $(D, D)$-bivector space by changing the left operation of the regular bimodule $_D D_D$ via $\psi$. Since $X_2$ is left linearly independent over $K(X_1)$ in $K(X)$, $\dim(_{K(X_1)}K(X)) \geq |X_2| = \delta$. Thus we have $\dim(_{K(X_1)}K(X)) = \delta$ by $|K(X)| = \delta$ and hence $\dim(_D W) = \delta$. Also, clearly $\dim(W_D) = 1$.

Let $A$ be a set of indices with $|A| = \alpha$ and let $V = W^{(A)}$ the direct sum of copies of the $(D, D)$-bivector space $W$. Then it follows from $\dim(_D W) = \delta$, $\dim(W_D) = 1$ and $\alpha \leq \delta$ that $\dim(_D V) = \delta$ and $\dim(V_D) = \alpha$. By Lemma 3.3.11(3) we see that $\dim(_D \Hom_D(V, D)) = \delta^\alpha = \delta$. Therefore, we have

\[ _D V \cong _D \Hom_D(V, D). \]

It should be noted that $V_D \not\cong \Hom_D(V, D)_D$. (Cf. Theorem 3.3.8.) Let $W' = _D D$ be the $(D, D)$-bivector space by changing the right operation of the regular bimodule $_D D_D$ via $\psi$. Then $\Hom_D(V, D) \cong (W')^A$ the direct product of copies of $W'$ as $(D, D)$-bivector spaces. Thus by Lemma 3.3.11(3), we have

\[ \dim(\Hom_D(V, D)_D) = \dim(((W')^A)_D) \geq |D|^{|A|} = \delta^\alpha = \delta. \]

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Therefore, $V_D \not\cong \text{Hom}_D(V,D)_D$.

As the last argument of the example above shows, we have

**Proposition 3.3.14.** Let $R$ be a right PF one-sided perfect local ring with $\alpha = \dim((J/J^2)(R/R))$. If $J/J^2$ is an $\alpha$ direct sum of nonzero $(R, R)$-subbimodules, then $\alpha$ is finite and $R$ is QF.

**Proof.** Since $R$ is right PF, by Lemma 2.1.7 and [14, Theorem 6], the canonical homomorphism $S_2/S \to \text{Hom}_R(J/J^2, S)$ is an isomorphism and the canonical homomorphism $J/J^2 \to \text{Hom}_R(S_2/S, S)$ is an epimorphism. Hence we have a surjective $(R, R)$-homomorphism $J/J^2 \to \text{Hom}_R(\text{Hom}_R(J/J^2, S), S)$.

Let $D = R/J$, $U = S$, $V = J/J^2$ and $\delta = |D|$. Then $D$ is a division ring, $U, V$ are $(D, D)$-bivector spaces, $D \cong \text{End}_D(U)$ canonically, and there exists a surjective $(D, D)$-homomorphism $V \to \text{Hom}_D(\text{Hom}_D(V, U), U)$. By assumption, $\dim(V_D) = \alpha$ and $V$ can be expressed as $V = \bigoplus_{\Lambda} V_{\lambda}$, where $V_{\lambda}$ are nonzero $(D, D)$-subbivector spaces and $|\Lambda| = \alpha$. Then we have

$$\text{Hom}_D(V, U) = \text{Hom}_D(\bigoplus_{\Lambda} V_{\lambda}, U) \cong \prod_{\Lambda} \text{Hom}_D(V_{\lambda}, U)$$

as $(D, D)$-bivector spaces.

To show that $\alpha$ is finite, assume that $\alpha$ is infinite. By Corollary 3.3.10(1) $\delta = |D|$ is infinite. Since $V_{\lambda} \neq 0$, there exists a monomorphism $(D^\alpha)_D \to (\prod_{\Lambda} \text{Hom}_D(V_{\lambda}, U))_D$. Thus, since $\dim((D^\alpha)_D) = \delta^\alpha$ by Lemma 3.3.11(3), we have

$$\dim(\text{Hom}_D(V, U)_D) = \dim((\prod_{\Lambda} \text{Hom}_D(V_{\lambda}, U))_D) \geq \delta^\alpha.$$ 

Hence by Lemma 3.3.11(3) again, $\dim(D \text{Hom}_D(\text{Hom}_D(V, U), U)) \geq \delta^{\delta^\alpha}$. Thus it follows from the existence of the epimorphism $V \to \text{Hom}_D(\text{Hom}_D(V, U), U)$ that $\dim(D V) \geq \delta^{\delta^\alpha}$. Therefore by Lemma 3.3.11(1), $|V| \geq \delta \times \delta^{\delta^\alpha} = \delta^{\delta^\alpha}$. 

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Then since $\delta^\alpha \geq \max(\delta, \alpha)$, we have $\delta^{(\delta^\alpha)} > \max(\delta, \alpha)$. However, this contradicts $|V| = \delta \times \alpha$, which follows from $\dim(V_D) = \alpha$ and Lemma 3.3.11(1). Therefore $\alpha$ must be finite and $R$ is QF by Lemma 1.2.5. \hfill \square