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Analysis of fully developed turbulence in terms of Tsallis statistics

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The analysis of the fully developed turbulence with the help of the binomial multiplicative process is reinvestigated from the standpoint of Tsallis nonextensive statistics in order to provide some insight in understanding the underlying physical meaning of the Tsallis ensemble. The formula is derived which relates Tsallis index q with the intermittency exponent μ that is a manifestation of multifractality of the sizes of eddies. It is shown that the probability density function of the local dissipation of turbulent kinetic energy can be represented by a distribution function of the type in Tsallis statistics with the index q determined by the experimentally observable quantity μ through the derived formula.

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The study of fully developed turbulence was started by Kolmogoroff [1] with dimensional analysis in order to derive the exponent of the energy spectrum in the inertial range, called the Kolmogoroff spectrum, i.e.,

$$E_k \propto \epsilon^{2/3} k^{-5/3}. \tag{1}$$

Here, ϵ is the energy input rate, and k is the wave number representing the size of eddies in the inertial region. Within the log-normal theory [2–4], the Fourier transformation $\mathcal{T}$ of the dissipative correlation function defines the intermittency exponent μ, i.e.,

$$\mathcal{T}\langle \epsilon(\mathbf{r})\epsilon(\mathbf{r}+\ell)\rangle_r \propto \epsilon^2 k^{-3}(k/K)^{\mu}, \tag{2}$$

where K is the wave number corresponding to the largest scale L, e.g., the size of the grid which produces turbulence. The energy spectrum becomes

$$E_k \propto \epsilon^{2/3} k^{-5/3}(k/K)^{-\mu/9}, \tag{3}$$

which modifies the Kolmogoroff spectrum 5/3 to $5/3+\mu/9$.

Introduction of fractal dimensional analysis of the fully developed turbulence was started by the β model [5], where it is assumed that the smaller the size of eddies, the less space is filled with the same fractal dimension. It was further developed by the p model [6,7] with the help of multifractal theory. There, it was assumed that each size of eddies has its own space filling fractal dimension. The analysis was performed with the help of the binomial multiplicative process for the energy cascade in the inertial region.

Tsallis [8] introduced the nonextensive entropy

$$S_q = \left(\sum_i p_i^q - 1 \right) / (1-q), \tag{4}$$

to produce generalized Boltzmann-Gibbs statistics, by taking their extreme with the constraint indicating the conservation of probability: $\Sigma_i p_i = 1$, and with the one indicating the conservation of q-averaged internal energy [9]: $U_q = \Sigma_i p_i^q E_i / \Sigma_j p_j^q$. Then, one obtains the general form of the probability distribution function of the Tsallis ensemble in the form

$$p_i = \left[1 - \frac{(1-q)\beta(E_i - U_q)}{\sum_j p_j^q} \right]^{1/(1-q)} / \bar{Z}_q, \tag{5}$$

with the partition function

$$\bar{Z}_q = \sum_i \left[1 - \frac{(1-q)\beta(E_i - U_q)}{\sum_j p_j^q} \right]^{1/(1-q)}. \tag{6}$$

Note that Tsallis statistics reduce to Boltzmann-Gibbs statistics taking the limit $q \to 1$. Here, we are using the units where the Boltzmann constant is unity.

It was shown [10] that the value q of the parameter appearing in Tsallis statistics is related to the extremes $\alpha_{\max}$ and $\alpha_{\min}$ of the multifractal spectrum $f_d(\alpha)$ by

$$1/(1-q) = 1/\alpha_{\min} - 1/\alpha_{\max}. \tag{7}$$

In the derivation, this fact was used that for one-dimensional nonlinear maps the sensitivity to initial conditions becomes expressed as [11]

$$\eta(t) = [1 + (1-q)\lambda_q t]^{1/(1-q)}. \tag{8}$$

Here Eq. (8) is the solution of the time-evolution equation

$$\frac{d\eta(t)}{dt} = \lambda_q \eta(t)^q. \tag{9}$$

In this paper, we will show how the Tsallis index q is related to the intermittency exponent μ, and will try to extract a

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deeper physical understanding of the Tsallis ensemble from the multifractal point of view associated with the analysis of fully developed turbulence.

Now let us remember how the intermittency is introduced within the p model with the help of the multiplicative binomial process [6,7]. By making use of the scale invariance of the Navier-Stokes equation in the inertial range for high Reynolds number limit, it was introduced the scaling exponent α associated with the local dissipation of turbulent kinetic energy ϵ_r which is averaged over a domain of size r [7], i.e., $\epsilon_r \sim r^{\alpha-1}$. Then, the total dissipation energy E_r associated with this domain size scales as $E_r \sim r^{\alpha-1+d}$, where d represents the dimension of physical space [7]. The multifractal spectrum $f_d(\alpha)$, representing how the set with the scaling exponent lying between α and $\alpha+d\alpha$ distributes, is related to the mass exponent $\tau_d(\bar{q})$ by the Legendre transformation

$$f_d(\alpha)=\alpha\bar{q}+\tau_d(\bar{q}), \tag{10}$$

with the definition of the new variable $\bar{q}$:

$$df_d(\alpha)/d\alpha=\bar{q}. \tag{11}$$

We use here the notation $\bar{q}$ to avoid any confusion with the Tsallis index q. Note that the mass exponent is related to the fractal dimension $D_{\bar{q}}$ through

$$\tau_d(\bar{q})=(1-\bar{q})D_{\bar{q}}+(d-1)\bar{q}. \tag{12}$$

The variable α is defined by the mass exponent as

$$\alpha=-d\tau_d(\bar{q})/d\bar{q}. \tag{13}$$

Let us restrict ourselves here to the analysis on the measured time series of the streamwise velocity component of an isotropic turbulence behind grids. Then, the dimension of physical space d will be 1 [7]. At the nth stage of the cascade within the multiplicative binomial process model (p model) [6], the partition function $E_{r,\bar{q}}$ in a box of size $r \sim \ell_n = \delta_n L$ ($\delta_n = 2^{-n}$) is given by

$$E_{r,\bar{q}}=\sum_{0\leqslant\xi\leqslant 1} N_n(\xi)E_r(\xi)^{\bar{q}}, \tag{14}$$

with

$$N_n(\xi)=\begin{pmatrix} n \\ \xi n\end{pmatrix}, \quad E_r(\xi)=\mu_n(\xi)E_L, \tag{15}$$

where $\xi=k/n$ $(k=0,1,2,\cdots,n-1)$ and

$$\mu_n(\xi)=[p^{\xi}(1-p)^{1-\xi}]^n. \tag{16}$$

E_L is the total dissipation of turbulent energy in a box of the largest size L. $E_r(\xi)$ represents the turbulent dissipation energy in a box of size r due to the eddies labeled ξ.

The mass exponent $\tau(\bar{q})$ for the fully developed turbulence is defined by $E_{r,\bar{q}}=E_L^{\bar{q}}\delta_n^{-\tau(\bar{q})}$ [6]. Since Eq. (14) is calculated as $E_{r,\bar{q}}=E_L^{\bar{q}}[p^{\bar{q}}+(1-p)^{\bar{q}}]^n$, we have the expression of $\tau(\bar{q})$ which gives us [6]

$$D_{\bar{q}}=\log_2[p^{\bar{q}}+(1-p)^{\bar{q}}]^{1/(1-\bar{q})}. \tag{17}$$

The multifractal spectrum $f(\xi)$ and the exponent $\alpha(\xi)$ are defined, respectively, through

$$N_n(\xi)=L^{f(\xi)}/\ell_n^{f(\xi)}=\delta_n^{-f(\xi)}, \tag{18}$$

$$\mu_n(\xi)=\ell_n^{\alpha(\xi)}/L^{\alpha(\xi)}=\delta_n^{\alpha(\xi)}. \tag{19}$$

Comparison of the first equation in Eq. (15) with Eq. (18), and of Eq. (16) with (19) give us [12]

$$f(\xi)=-[\xi\log_2\xi+(1-\xi)\log_2(1-\xi)], \tag{20}$$

$$\alpha(\xi)=-[\xi\log_2 p+(1-\xi)\log_2(1-p)]. \tag{21}$$

It can be checked that $f(\xi),\alpha(\xi)$ and $D_{\bar{q}}$ satisfy the relations (10)–(13) for $d=1$. For $p>1/2$, Eq. (21) tells us that $\alpha_{\min}$ and $\alpha_{\max}$ are, respectively, given by

$$\alpha_{\min}=\alpha(\xi=1)=-\log_2 p=D_{\bar{q}=+\infty}, \tag{22}$$

$$\alpha_{\max}=\alpha(\xi=0)=-\log_2(1-p)=D_{\bar{q}=-\infty}, \tag{23}$$

with $f(\alpha_{\min})=f(\alpha_{\max})=0$. Note that

$$\bar{q}=[\ln\xi-\ln(1-\xi)]/[\ln p-\ln(1-p)]. \tag{24}$$

The intermittency exponent μ for $d=1$ is given by [7]

$$\mu=-2\frac{dD_{\bar{q}}}{d\bar{q}}\Big|_{\bar{q}=0}=2(\alpha_0-D_0), \tag{25}$$

with $\alpha_0=\alpha_{\bar{q}=0}$ and $D_0=D_{\bar{q}=0}$. As Eq. (24) tells us that $\bar{q}=0$ is realized by $\xi=\xi_0=1/2$, we see from Eqs. (21) and (17) that

$$\alpha_0=-\log_2\sqrt{p(1-p)}=(\alpha_{\min}+\alpha_{\max})/2, \tag{26}$$

and $D_0=1$. Then, Eq. (25) reduces to

$$\mu=-\log_2[p(1-p)]-2=\alpha_{\min}+\alpha_{\max}-2. \tag{27}$$

The same result can be derived by the direct differentiation of Eq. (17) as it should be.

Now we will show how to derive the relation Eq. (7) within the p model, and then we can obtain a formula which gives us the relation between the intermittency exponent μ and Tsallis index q.

Let us interpret the measure $\mu_n(\xi)$ by means of points distributed according to the relation

$$\mu_n(\xi)=B_n(\xi)/B_n, \tag{28}$$

where $B_n(\xi)$ represents a number of points labeled ξ, and B_n is the total number of points:

$$B_n=\sum_{\xi} N_n(\xi)B_n(\xi). \tag{29}$$

The mean distance $\ell_n(\xi)$ of $B_n(\xi)$ points can be defined by

$$B_n(\xi)=\ell_n^{\alpha(\xi)}/\ell_n(\xi)^{\alpha(\xi)}=(\delta_n/\delta_n(\xi))^{\alpha(\xi)}, \tag{30}$$

where we introduced $\delta_n(\xi)$ by $\ell_n(\xi)=\delta_n(\xi)L$. With the help of Eq. (19), we see that Eqs. (28) and (30) reduce to

$$1/B_n=\ell_n(\xi)^{\alpha(\xi)}/L^{\alpha(\xi)}=\delta_n(\xi)^{\alpha(\xi)}. \tag{31}$$

This indicates that each box of size $\ell_n(\xi)^{\alpha(\xi)}$ contains one point.

Differentiating Eq. (31), we have

$$dB_n=\frac{\ln B_n}{\alpha(\xi)\,\delta_n(\xi)^{\alpha(\xi)}}\,d\alpha(\xi)-\frac{\alpha(\xi)}{\delta_n(\xi)^{\alpha(\xi)+1}}\,d\delta_n(\xi). \tag{32}$$

For fixed B_n, Eq. (32) reduces to

$$\frac{d\delta_n(\xi)}{d\alpha(\xi)}=\frac{\ln B_n}{\alpha(\xi)^2}\,\delta_n(\xi), \tag{33}$$

which gives us

$$\delta_n(\xi=0)/\delta_n(\xi=1)=B_n^{\alpha_{\min}^{-1}-\alpha_{\max}^{-1}}, \tag{34}$$

by the integration from $\delta(\xi=1)$ to $\delta(\xi=0)$ (from $\alpha_{\min}$ to $\alpha_{\max}$). On the other hand, since $\alpha(\xi)$ does not depend on B_n, we obtain from Eq. (32) the *time-evolution* equation for $\delta_n(\xi)^{-1}$:

$$\frac{d\delta_n(\xi)^{-1}}{dB_n}=\frac{1}{\alpha(\xi)}\,(\delta_n(\xi)^{-1})^{1-\alpha(\xi)}. \tag{35}$$

This shows how the mean distance among the points labeled ξ changes as *time* B_n evolves. The solution of Eq. (35) is consistent with Eq. (34) for $B_n\gg 1$.

Changing the variable t to B_n through the relation $B_n=(1-q)\lambda_q t$, Eq. (9) reduces to

$$\frac{d\eta(B_n)}{dB_n}=\frac{1}{1-q}\,\eta(B_n)^q, \tag{36}$$

which has the same structure as Eq. (35). Although Eq. (35) describes the *time-evolution* of mean distances $\delta_n(\xi)$ among the points distributed according to the measure $\mu_n(\xi)$ when the number of points B_n are increased, Eq. (9) or (36) is the time-evolution equation derived by the original nonlinear equation which describes a real time-evolution of the system under consideration. As was inspected in Ref. [10], the smallest splitting between two nearby orbits whose distance is of the order of the minimum mean distance $\ell_n(\xi=1)=\delta_n(\xi=1)L$ should become at most a splitting of the order of the maximum mean distance $\ell_n(\xi=0)=\delta_n(\xi=0)L$. We put the time when this happens at $B_n\gg 1$. Equating the solution (8)

$$\eta(B_n)=\delta_n(\xi=0)/\delta_n(\xi=1)\sim B_n^{1/(1-q)} \tag{37}$$

with (34), we have the relation (7) derived in Ref. [10]. Note that in the case of turbulence larger eddies are broken into smaller eddies as time goes on. Here, we are investigating the situation in the reversed time direction.

Substituting the obtained $\alpha_{\min}$ and $\alpha_{\max}$ into Eq. (7) and with the help of Eq. (27), we have the formula which relates μ and q in the form:

$$q=1-\frac{1+\mu+\log_2(1+\sqrt{1-2^{-\mu}})\cdot\log_2(1-\sqrt{1-2^{-\mu}})}{\log_2(1+\sqrt{1-2^{-\mu}})-\log_2(1-\sqrt{1-2^{-\mu}})}. \tag{38}$$

By making use of the observed value of the intermittency exponent $\mu=0.235$ [7,13] into Eqs. (38) and (27), we obtain $q=0.237$ and $p=0.694$.

Now we will show how Tsallis distribution function describes the probability density function of the local dissipation ϵ_r of turbulent kinetic energy with the index q obtained by the measured intermittency exponent. The probability density function $P_\epsilon(\epsilon_r)$ of the local dissipation of turbulent kinetic energy is given by [7]

$$\begin{aligned}P_\epsilon(\epsilon_r)d(\epsilon_r/\epsilon)&\propto\left(\frac{r}{L}\right)^{D_0-f_d(\alpha)}\frac{\epsilon}{\epsilon_r\ln(r/L)}\,d(\epsilon_r/\epsilon)\\&=\delta_n^{D_0-f(\xi)}\frac{d\alpha}{d\xi}\,d\xi.\end{aligned} \tag{39}$$

Since $d\alpha/d\xi$ does not depend on ξ, we will inspect the probability density function $P_{\mathrm{B}}(\xi;n)$ defined by

$$\begin{aligned}P_{\mathrm{B}}(\xi;n)d\xi&=P_\epsilon(\epsilon_r)d(\epsilon_r/\epsilon)\\&=Z_n^{-1}[2\xi^\xi(1-\xi)^{1-\xi}]^{-n}d\xi\end{aligned} \tag{40}$$

based upon the binomial multiplicative process, where $Z_n=\int_0^1 d\xi[2\xi^\xi(1-\xi)^{1-\xi}]^{-n}$.

In their analysis, Meneveau and Sreenivasan [7] approximate $f(\xi)$ by

$$f(\xi)=D_0+(\xi-\xi_0)^2/(2\sigma_n^2), \tag{41}$$

with

$$\begin{aligned}\sigma_n^2&=-\mu(d\alpha/d\xi)^{-2}\\&=-n^{-1}\left(\ln\frac{p}{1-p}\right)^{-2}\ln[4p(1-p)],\end{aligned} \tag{42}$$

which was derived by fitting Eq. (41) at $\xi=\xi_1=p$ where

$$\frac{df(\alpha)}{d\alpha}\Big|_{\alpha=\alpha_1}=1, \tag{43}$$

with $\alpha_1=\alpha(\xi_1)$. We see from Eqs. (20), (21), (24), and (17) that $f(\xi_1)=\alpha_1=D^-_{q=1}$. Then, the probability density function reduces to the Gaussian form [7]

$$P_{\mathrm{G}}(\xi;n)=Z_{n,G}^{-1}\exp[-(\xi-\xi_0)^2/2\sigma_n^2], \tag{44}$$

which gives us $\langle(\ln\epsilon_r)^2\rangle_r-\langle\ln\epsilon_r\rangle_r^2=\mu\ln(L/r)$, which is the definition of the intermittency exponent μ introduced in Eq. (2) within the log-normal model [2–4]. The partition function $Z_{n,G}$ for $n\gg 1$ is given by $Z_{n,G}=\sqrt{2\pi}\sigma_n$.

Now, we propose that the probability density function (40) can be well approximated by the Tsallis type distribution function of the form:

$$P_{\mathrm{T}}(\xi;n)=Z_{n,T}^{-1}\left[1-\left(\frac{1-q}{n}\right)\frac{(\xi-\xi_0)^2}{2\sigma_n^2}\right]^{n/(1-q)}, \tag{45}$$

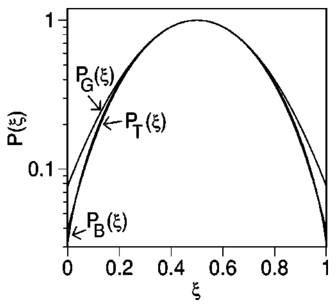

FIG. 1. The probability density functions of the dissipation of turbulent kinetic energy for $n=5$. $P_{\mathrm{B}}(\xi)$ is the one based on the binomial multiplicative process, $P_{\mathrm{T}}(\xi)$ on Tsallis statistics, and $P_{\mathrm{G}}(\xi)$ on the Gaussian approximation. They are normalized by the peak height, and have, respectively, the smallest, middle, and the largest values at the origin ($\xi=0$). Parameters are $q=0.237$ and $p=0.694$. $P_{\mathrm{B}}(\xi)$ and $P_{\mathrm{T}}(\xi)$ intersect each other near $\xi=0.05$ and $\xi=0.95$.

with the Tsallis index q given by Eq. (38). For $n \gg 1$, the partition function $Z_{n,T}$ satisfies $Z_{n,T}=Z_{n,G}$. In Fig. 1, we put the probability density functions (40), (44), and (45) for $n=5$ with the parameters q and p given below Eq. (38). There is only a negligible n dependence for the difference among these functions besides their values in logarithmic scale, i.e., the larger the value n, the values of the probability functions at both ends ($\xi=0$ and $\xi=1$) become smaller. We see the superiority of the Tsallis distribution function (45) in the analysis of the probability density (40) of dissipative kinetic energy within the present model.

In this paper, we succeeded in deriving formula (38) to determine the Tsallis index q from the experimentally observable quantity μ, the intermittency exponent, and proposed the explicit form (45) of the probability density function of the local dissipative kinetic energy in turbulence. Reinvestigation of the data observed by experiments and numerical simulations [7,13,14] based on the analysis given in this paper is greatly required. Although the proposed probability density function $P_{\mathrm{T}}(\xi;n)$ describes the *true* probability density $P_{\mathrm{B}}(\xi;n)$ quite accurately for all the region $0 \leqslant \xi \leqslant 1$, we do not know yet its detailed background mechanism. In order to see the underlying dynamics supporting Tsallis statistics (45) in connection with multifractality, we are investigating dynamical systems whose time evolution is driven by measure preserving maps having $\mu_n(\xi)$ as an invariant measure. This might give us some relationship between Eqs. (35) and (36). Note that the value of the Tsallis index $q=0.237$ derived in this paper is very close to the value 0.2445 [10] associated with the chaos threshold of the logistic map. There is a possibility that the background mechanism of the fully developed turbulence is in fact based upon the Tsallis ensemble characterized by the probability density function (45) instead of Eq. (40) which is the one based upon the binomial multiplicative process. These future problems will be reported elsewhere.

Let us close this paper by noting the case $\mu=0$ ($p=1/2$) for the present model. In this case, we see that $D_q^-=1$, $\alpha=1$, and $f(\alpha)=1$. Therefore, the multifractal spectrum consists of a single point meaning that no multifractality appears. The same feature of the spectrum comes out for the Kolmogoroff model [1] and even for the β model [5]. Note that the latter model gives $\mu \neq 0$. In these cases, the analysis of the present paper is not applicable because Eq. (7) has been deduced under the (implicit) assumption of continuity of the $f(\alpha)$ multifractal spectrum, which is not true if $\mu=0$ [15].

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