

## Singular integrals and Feller semigroups : Real Analysis Methods for Markov Processes

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# Singular Integrals and Feller Semigroups

Real Analysis Methods for Markov Processes

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# Preface

This book is an easy-to-read reference providing a link with functional analysis, real analysis, partial differential equations and probability. Most mathematicians working in partial differential equations are only vaguely familiar with the powerful ideas of stochastic analysis. On the other hand, the additional intuition which this book conveys might provide better insight and be helpful to their work. In addition, the book provides a compendium for a large variety of facts from functional analysis, real analysis, singular integral operators and Markov processes - for looking up quickly a theorem. This book gives better coverage of important examples and applications, and is amply illustrated and all figures and tables are provided with appropriate captions.

The purpose of this book is a self-contained account of the functional analytic approach to the problem of construction of Markov processes with Ventcel' (Wentzell) boundary conditions in probability. More precisely, we prove existence theorems for Feller semigroups with Dirichlet boundary condition, oblique derivative boundary condition and first-order Ventcel' boundary condition for second-order, uniformly elliptic differential operators with *discontinuous* coefficients. Our approach here is distinguished by the extensive use of the ideas and techniques characteristic of the recent developments in the Calderón–Zygmund theory of singular integral operators with non-smooth kernels.

It should be emphasized that singular integral operators with non-smooth kernels provide a powerful tool to deal with smoothness of solutions of partial differential equations, with minimal assumptions of regularity on the coefficients. The Calderón–Zygmund theory of singular integrals continues to be one of the most influential works in modern history of analysis, and is a very refined mathematical tool whose full power is yet to be exploited.

This book is addressed to advanced undergraduates or beginning-graduate students and also mathematicians with interest in real analysis, functional analysis and partial differential equations. For the former, it may serve as an effective introduction to these four interrelated fields of analysis. For the latter, it provides a method for the study of elliptic boundary value problems with *discontinuous* coefficients, a powerful method clearly capable of extensive further development. Bibliographical references are discussed primarily in Notes and Comments at the end of each chapter. These notes are intended to supplement the text and place it in better perspective. This book will lead to a better insight into the study of singular integrals and elliptic boundary value problems for graduate students about to enter the subject, and mathematicians in the field looking for a coherent overview.

The author is grateful to Professors Akihiko Miyachi and Yasushi Ishikawa for fruitful conversations while working on this book.

Last but not least, I owe a great debt of gratitude to my family who gave me moral support during the preparation of this book.

Tsuchiura,  
November 2020

Kazuaki Taira



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## Bibliography

- [1] Acquistapace, P. (1992). On BMO regularity for linear elliptic systems, *Ann. Mat. Pura Appl.* **161**, 231–269.
- [2] Adams, R. A. and Fournier, J. J. F. (2003). *Sobolev spaces*, second edition, Pure and Applied Mathematics, Vol. 140 (Academic Press, Amsterdam Heidelberg New York Oxford).
- [3] Agmon, S., Douglis, A. and Nirenberg, L. (1959). Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I, *Comm. Pure Appl. Math.* **12**, 623–727.
- [4] Amann, H. (1976). Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM Rev.* **18**, 620–709.
- [5] Aronszajn, N. and Smith, K. T. (1961). Theory of Bessel potentials I, *Ann. Inst. Fourier (Grenoble)* **11**, 385–475.
- [6] Axler, S., Bourdon, P. and Ramey, W. (2001). *Harmonic function theory*, second edition, Graduate Texts in Mathematics, No. 137 (Springer-Verlag, Berlin Heidelberg New York).
- [7] Bergh, J. and Löfström, J. (1976). *Interpolation spaces*, An introduction (Springer-Verlag, Berlin Heidelberg New York).
- [8] Blumenthal, R. M. and Gettoor, R. K. (1968). *Markov processes and potential theory* (Academic Press, New York London).
- [9] Bony, J.-M. (1967). Principe du maximum dans les espaces de Sobolev, *C. R. Acad. Sc. Paris Sér. A-B* **265**, A333–A336.
- [10] Bony, J.-M. (1967). Problème de Dirichlet et semi-groupe fortement fellerien associés à un opérateur intégro-différentiel, *C. R. Acad. Sc. Paris Sér. A-B* **265**, A361–A364.
- [11] Bony, J.-M., Courrège, P. et Priouret, P. (1968). Semi-groupes de Feller sur une variété à bord compacte et problèmes aux limites intégro-différentiels du second ordre donnant lieu au principe du maximum, *Ann. Inst. Fourier (Grenoble)* **18**, 369–521.
- [12] Bramanti, M. (1994). Commutators of integral operators with positive kernels, *Matematiche (Catania)* **49**, 149–168.
- [13] Bramanti, M. and Cerutti, M. C. (1993).  $W_p^{1,2}$  solvability for the Cauchy–Dirichlet problem for parabolic equations with VMO coefficients, *Comm. Partial Differential Equations* **18**, 1735–1763.
- [14] Brezis, H. (2011). *Functional analysis, Sobolev spaces and partial differential equations*, Universitext (Springer-Verlag, New York).
- [15] Calderón, A. P. and Zygmund, A. (1952). On the existence of certain

- singular integrals, *Acta Math.* **88**, 85–139.
- [16] Calderón, A. P. and Zygmund, A. (1961). Local properties of solutions of elliptic partial differential equations, *Studia Math.* **20**, 171–225.
- [17] Campanato, S. (1965). Equazioni ellittiche del II° ordine e spazi  $\mathfrak{L}^{(2,\lambda)}$ , *Ann. Mat. Pura Appl.* **69**, 321–382.
- [18] Chiarenza, F., Frasca, M. and Longo, P. (1991). Interior  $W^{2,p}$  estimates for nondivergence elliptic equations with discontinuous coefficients, *Ricerche di Matematica* **60**, 149–168.
- [19] Chiarenza, F., Frasca, M. and Longo, P. (1993).  $W^{2,p}$ -solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients, *Trans. Amer. Math. Soc.* **336**, 841–853.
- [20] Cohn, D. L. (2013). *Measure theory*, second edition, Birkhäuser Advanced Texts: Basler Lehrbücher (Birkhäuser/Springer-Verlag, New York).
- [21] Coifman, R., Rochberg, R. and Weiss, G. (1976). Factorization theorems for Hardy spaces in several variables, *Ann. of Math. (2)* **103**, 611–635.
- [22] Di Fazio, G. and Palagachev, D. K. (1996). Oblique derivative problem for elliptic equations in nondivergence form with VMO coefficients, *Comment. Math. Univ. Carolinae* **37**, 537–556.
- [23] Duoandikoetxea, J. (2001). *Fourier analysis* (American Mathematical Society, Providence, Rhode Island).
- [24] Dynkin, E. B. (1959). *Foundations of the theory of Markov processes*, (Fizmatgiz, Moscow (in Russian)); English translation: (Pergamon Press, Oxford London New York Paris, 1960); German translation: (Springer-Verlag, Berlin Göttingen Heidelberg, 1961); French translation: (Dunod, Paris, 1963).
- [25] Dynkin, E. B. (1965). *Markov processes I, II* (Springer-Verlag, Berlin Göttingen Heidelberg).
- [26] Feller, W. (1952). The parabolic differential equations and the associated semigroups of transformations, *Ann. of Math.* **55**, 468–519.
- [27] Feller, W. (1955). On second order differential equations, *Ann. of Math.* **61**, 90–105.
- [28] Folland, G. B. (1995). *Introduction to partial differential equations*, second edition (Princeton University Press, Princeton, New Jersey).
- [29] Folland, G. B. (1999). *Real analysis*, second edition (John Wiley & Sons, New York Chichester Weinheim Brisbane Singapore Toronto).
- [30] Friedman, A. (1970/1982). *Foundations of modern analysis* (Dover Publications Inc., New York).
- [31] Garcia-Cuerva, J. and Rubio de Francia, J. L. (1985). *Weighted norm inequalities and related topics* (North-Holland, Amsterdam New York Oxford).
- [32] Garnett, J. B. (2007). *Bounded analytic functions*, revised first edition, Graduate Texts in Mathematics, Vol. 236 (Springer-Verlag, New York).
- [33] Gilbarg, D. and Trudinger, N. S. (2001). *Elliptic partial differential equations of second order*, reprint of the 1998 edition, Classics in Mathematics (Springer-Verlag, New York Berlin Heidelberg Tokyo).
- [34] Hopf, E. (1952). A remark on linear elliptic differential equations of second order. *Proc. Amer. Math. Soc.* **3**, 791–793.
- [35] Hörmander, L. (2007). *The analysis of linear partial differential operators III*, Pseudo-differential operators, reprint of the 1994 edition, Classics in Mathematics (Springer-Verlag, Berlin Heidelberg New York Tokyo).
- [36] Itô, K. and McKean, H. P. Jr. (1996). *Diffusion processes and their sample*

- paths*, reprint of the 1974 edition, Classics in Mathematics (Springer-Verlag, Berlin Heidelberg New York).
- [37] John, F. and Nirenberg, L. (1961). On functions of bounded mean oscillation, *Comm. Pure and Appl. Math.* **14**, 415–426.
- [38] Kato, T. (1995). *Perturbation theory for linear operators*, reprint of the 1980 edition, Classics in Mathematics (Springer-Verlag, Berlin Heidelberg New York).
- [39] Kolmogorov, A. N. and Fomin, S. V. (1975). *Introductory real analysis*, Translated from the second Russian edition and edited by R. A. Silverman (Dover Publications, New York).
- [40] Ladyzhenskaya, O. A. and Ural'tseva, N. N. (1986). A survey of results on the solubility of boundary-value problems for second-order uniformly elliptic and parabolic quasi-linear equations having unbounded singularities, *Russian Math. Surveys* **41**, 1–31.
- [41] Lamperti, J. (1977). *Stochastic processes* (Springer-Verlag, New York Heidelberg Berlin).
- [42] Levi, E. E. (1907). Sulle equazioni lineari totalmente ellittiche, *Rend. Circ. Mat. Palermo* **24**, 275–317.
- [43] Lieberman, G. M. (1987). Local estimates for subsolutions and supersolutions of oblique derivative problems for general second order elliptic equations, *Trans. Amer. Math. Soc.* **304**, 343–353.
- [44] Lions, J.-L. et Magenes. E. (1968). *Problèmes aux limites non-homogènes et applications, 1, 2* (Dunod, Paris); English translation: *Non-homogeneous boundary value problems and applications, 1, 2* (Springer-Verlag, Berlin Heidelberg New York).
- [45] Malý, J. and Ziemer, W. P. (1997). *Fine regularity of solutions of elliptic partial differential equations* (American Mathematical Society, Providence, Rhode Island).
- [46] Maugeri, A. and Palagachev, D. K. (1998). Boundary value problem with an oblique derivative for uniformly elliptic operators with discontinuous coefficients, *Forum Math.* **10**, 393–405.
- [47] Maugeri, A., Palagachev, D. K. and Softova, L. G. (2000). *Elliptic and parabolic equations with discontinuous coefficients* Mathematical Research, **109** (Wiley-VCH, Berlin).
- [48] McLean, W. (2000). *Strongly elliptic systems and boundary integral equations* (Cambridge University Press, Cambridge).
- [49] Meyers, N. (1963). An  $L^p$ -estimate for the gradient of solutions of second order elliptic divergence equations, *Ann. Scuola Norm. Sup. Pisa* **17**, 189–206.
- [50] Miranda, C. (1963). Sulle equazioni ellittiche del secondo ordine di tipo non variazionale a coefficienti discontinui, *Ann. Mat. Pura Appl.* **63**, 353–386.
- [51] Neri, U. (1971). *Singular integrals*, Lecture Notes in Mathematics, No. 200 (Springer-Verlag, Berlin Heidelberg New York).
- [52] Neri, U. (1977). Some properties of functions with bounded mean oscillation, *Studia Math.* **61**, 63–75.
- [53] Nirenberg, L. (2001). *Topics in nonlinear functional analysis*, revised reprint of the 1974 original, Courant Lecture Notes in Mathematics, No. 6 (New York University, Courant Institute of Mathematical Sciences, New York); (American Mathematical Society, Providence, Rhode Island).
- [54] Oleïnik, O. A. (1952). On properties of solutions of certain boundary prob-



- lems for equations of elliptic type (in Russian). *Mat. Sbornik* **30**, 595–702.
- [55] Oleinik, O. A. and Radkevič, E. V. (1973). *Second order equations with nonnegative characteristic form* (Russian); (Itogi Nauki, Moscow, 1971). English translation; (American Mathematical Society, Providence, Rhode Island and Plenum Press, New York).
- [56] Peetre, J. (1960). Rectification à l'article “Une caractérisation des opérateurs différentiels”, *Math. Scand.* **8**, 116–120.
- [57] Ray, D. (1956). Stationary Markov processes with continuous paths, *Trans. Amer. Math. Soc.* **82**, 452–493.
- [58] Reed, M. and Simon, B. (1980). *Methods of modern mathematical physics I: Functional analysis*, revised and enlarged edition (Academic Press, New York).
- [59] Revuz, D. and Yor, M. (1999). *Continuous martingales and Brownian motion*, third edition (Springer-Verlag, Berlin New York Heidelberg).
- [60] Rudin, W. (1987). *Real and complex analysis*, third edition (McGraw-Hill, New York).
- [61] Sarason, D. (1975). Functions of vanishing mean oscillation, *Trans. Amer. Math. Soc.* **207**, 391–405.
- [62] Sato, K. and Ueno, T. (1965). Multi-dimensional diffusion and the Markov process on the boundary, *J. Math. Kyoto Univ.* **4**, 529–605.
- [63] Schaefer, H. H. (1991). *Topological vector spaces*, second edition (Springer-Verlag, New York Berlin Heidelberg).
- [64] Schauder, J. (1934). Über lineare elliptische Differentialgleichungen zweiter Ordnung, *Math. Z.* **38**, 257–282.
- [65] Schauder, J. (1935). Numerische Abschätzungen in elliptischen linearen Differentialgleichungen, *Studia Math.* **5**, 34–42.
- [66] Schechter, M. (2002). *Principles of functional analysis*, second edition, Graduate Studies in Mathematics, Vol. 36 (American Mathematical Society, Providence, Rhode Island).
- [67] Stein, E. M. (1962). The characterization of functions arising as potentials II, *Bull. Amer. Math. Soc.* **68**, 577–582.
- [68] Stein, E. M. (1970). *Singular integrals and differentiability properties of functions* (Princeton University Press, Princeton).
- [69] Stein, E. M. (1981). The differentiability of functions in  $\mathbf{R}^n$ , *Ann. of Math.* (2) **113**, 383–385.
- [70] Stein, E. M. (1993). *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals* (Princeton University Press, Princeton).
- [71] Stein, E. M. and Shakarchi, R. (2005). *Real analysis* (Princeton University Press, Princeton Oxford).
- [72] Taibleson, M. H. (1964). On the theory of Lipschitz spaces of distributions on Euclidean  $n$ -space I, *J. Math. Mech.* **13**, 407–479.
- [73] Taira, K. (1988). *Diffusion processes and partial differential equations* (Academic Press, San Diego New York London Tokyo).
- [74] Taira, K. (1996). Boundary value problems for elliptic integro-differential operators, *Math. Z.* **222**, 305–327.
- [75] Taira, K. (2002). *Singular integrals, Feller semigroups and Markov processes*. In the Proceedings of the Second International Conference on Semigroups of Operators, Theory and Applications (Rio de Janeiro, 2001), pp. 270–284, Optimization Software, Los Angeles, 2002,
- [76] Taira, K. (2003). Logistic Dirichlet problems with discontinuous coefficients, *J. Math. Pures Appl.* **82**, 1137–1190.

- [77] Taira, K. (2006). On the existence of Feller semigroups with discontinuous coefficients, *Acta Math. Sinica (English Series)* **22**, 595–606.
- [78] Taira, K. (2009). On the existence of Feller semigroups with discontinuous coefficients II, *Acta Math. Sinica (English Series)* **25**, 715–740.
- [79] Taira, K. (2014). *Semigroups, boundary value problems and Markov processes*, second edition, Springer Monographs in Mathematics (Springer-Verlag, Berlin Heidelberg New York).
- [80] Taira, K. (2016). *Analytic semigroups and semilinear initial boundary value problems*, second edition, London Mathematical Society Lecture Note Series, No. 434 (Cambridge University Press, Cambridge).
- [81] Taira, K. (2020). Dirichlet problems with discontinuous coefficients and Feller semigroups, *Rend. Circ. Mat. Palermo, II. Ser.* **69**, 287–323.
- [82] Taira, K. (2020). Ventcel' boundary value problems for elliptic Waldenfels operators, *Boll. Unione Mat. Ital.* **13**, 213–256.
- [83] Taira, K. (2020). *Boundary value problems and Markov processes: Functional analysis methods for Markov processes*, third edition, Lecture Notes in Mathematics, No. 1499 (Springer-Verlag, Berlin Heidelberg New York).
- [84] Taira, K. (2020). Logistic Neumann problems with discontinuous coefficients. *Ann. Univ. Ferrara, Sez. VII Sci. Mat.* **66**, 409–485.
- [85] Taira, K. Feller semigroups and degenerate elliptic operators III, *Math. Nachr.*, 2020;1-41. <https://doi.org/10.1002/mana.201800421>
- [86] Taira, K. Oblique derivative problems and Feller semigroups with discontinuous coefficients. *Ricerche Mat.* <https://doi.org/10.1007/s11587-020-00509-5>
- [87] Taira, K. Semilinear degenerate elliptic boundary value problems via the Semenov approximation. *Rend. Circ. Mat. Palermo, II. Ser.* <https://doi.org/10.1007/s12215-020-00560-z>
- [88] Talenti, G. (1966). Equazioni lineari ellittiche in due variabili, *Matematiche (Catania)* **21**, 339–376.
- [89] Tanabe, H. (1978/1981). *Functional analysis, I, II* (in Japanese) (Jikkyo-Shuppan, Tokyo).
- [90] Tanabe, H. (1997). *Functional analytic methods for partial differential equations* (Marcel Dekker, New York Basel).
- [91] Torchinsky, A. (2004). *Real-variable methods in harmonic analysis* (Dover Publications Inc., Mineola, New York).
- [92] Triebel, H. (1983). *Theory of function spaces* (Birkhäuser, Basel Boston Stuttgart).
- [93] Troianiello, G. M. (1987). *Elliptic differential equations and obstacle problems*, The University Series in Mathematics (Plenum Press, New York London).
- [94] Trudinger, N. S. (1980). Local estimates for subsolutions and supersolutions of general second order elliptic quasilinear equations, *Invent. Math.* **61**, 67–79.
- [95] Vitanza, C. (1992).  $W^{2,p}$ -regularity for a class of elliptic second order equations with discontinuous coefficients, *Le Matematiche* **47**, 177–186.
- [96] von Waldenfels, W. (1964). Positive Halbgruppen auf einem  $n$ -dimensionalen Torus, *Arch. Math.* **15**, 191–203.
- [97] Wentzell (Ventcel'), A. D. (1959). On boundary conditions for multidimensional diffusion processes (in Russian), *Teoriya Veroyat. i ee Primen.* **4**, 172–185. English translation in *Theory Prob. and its Appl.* **4**, 164–177.
- [98] Wloka, J. (1980). *Partial differential equations* (Cambridge University

- Press, Cambridge).
- [99] Yosida, K. (1995). *Functional analysis*, reprint of the sixth (1980) edition, Classics in Mathematics (Springer-Verlag, Berlin Heidelberg New York).
- [100] Ziemer, W. P. (1989). *Weakly differentiable functions*, Graduate Texts in Mathematics, Vol. 120 (Springer-Verlag, New York).

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