

## Boundary Value Problems of Nonlinear Elastostatics

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# **Boundary Value Problems of Nonlinear Elastostatics**

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# Domain with Corner Singularities

$$\{ \alpha = 1 \}$$

$\Omega$



$$\{ \alpha = 0 \}$$

## Abstract

- This talk is devoted to a **semigroup approach** to an initial-boundary value problem of **nonlinear elastodynamics** in the case where the boundary condition is a **regularization** of the **genuine mixed displacement-traction** boundary condition.

## My Works

- **Taira**: On boundary value problems of nonlinear elastostatics, *Osaka Journal of Mathematics*, 33 (1996), 555-585.  
<http://projecteuclid.org/euclid.ojm/1200786927>
- **Taira**: Introduction to boundary value problems of nonlinear elastostatics, *Tsukuba Journal of Mathematics*, 32 (2008), 67-138
- **Taira** : A mixed problem of linear elastodynamics, *Journal of Evolution Equations*, 13 (2013), 481-507  
DOI: 10.1007/s00028-013-0187-1

## References (1)

- **Ciarlet**: Mathematical elasticity, North-Holland, 1988
- **Duvaut et Lions**: Les inéquations en mécanique et en physique, Dunod, 1972
- **Marsden and Hughes**: Mathematical foundations of elasticity, Prentice-Hall, 1983
- **Valent**: Boundary value problems of finite elasticity, Springer-Verlag, 1988

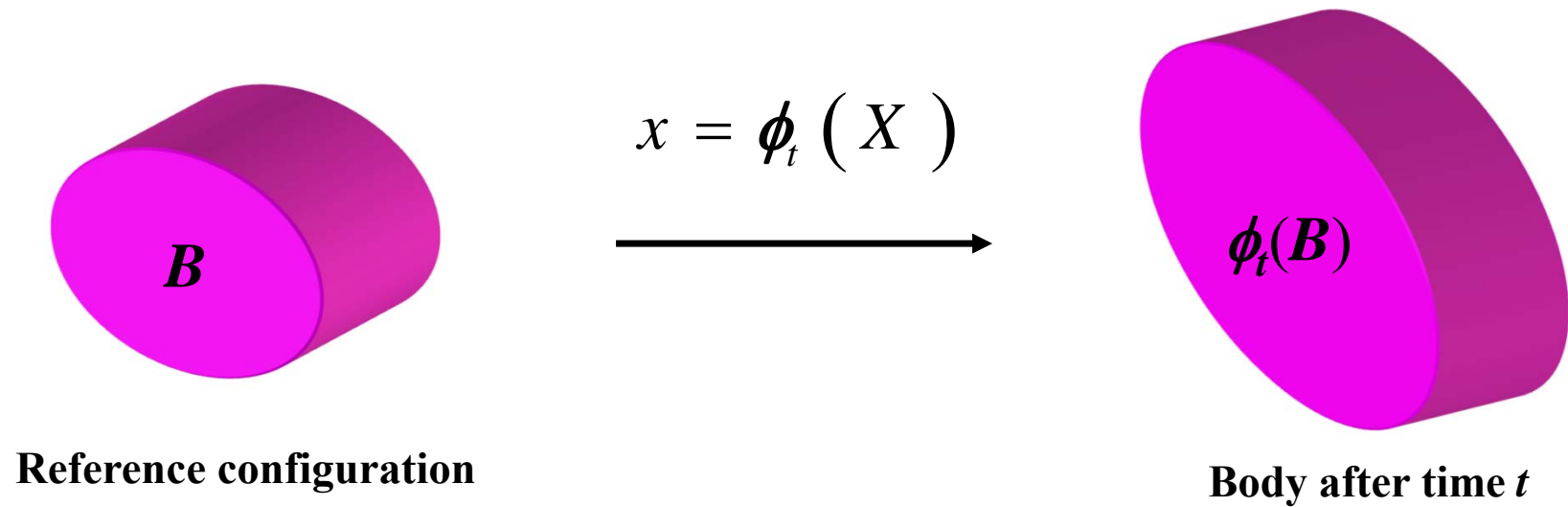
## References (2)

- **Goldstein**: Semigroups of linear operators and applications, Oxford University Press, 1985
- **Weiss**: Abstract vibrating systems, J. Math. Mech. 17 (1967), 241—255.

**Formulation  
of  
the Problem**



# Nonlinear Elasticity

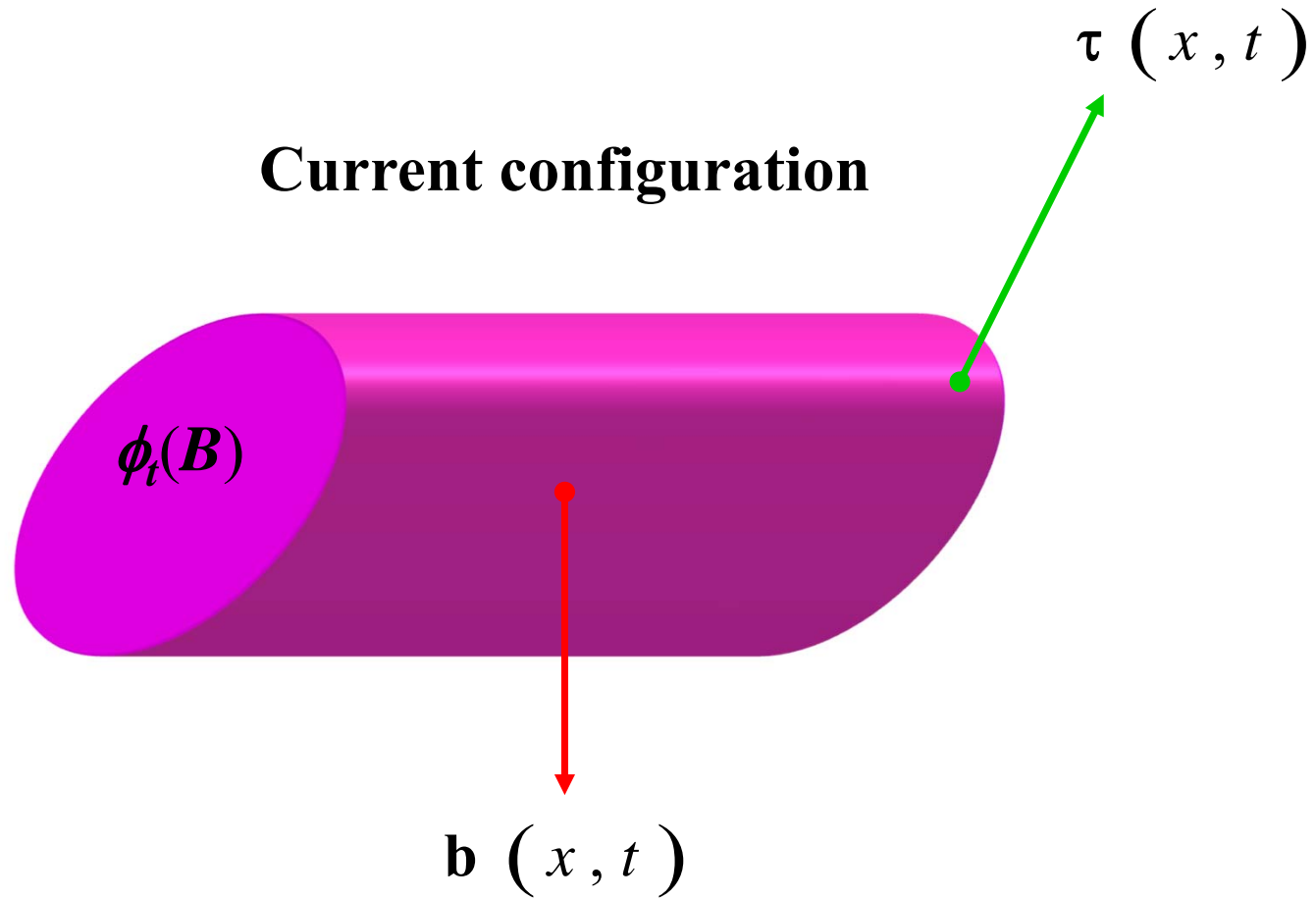


## Notation

Notation	Meaning
$\Omega$	Domain in $\mathbb{R}^3$
$B = \overline{\Omega}$	Reference configuration
$\phi_t$	Deformation of $B$
$X = (X_1, X_2, X_3)$	Material point
$x = (x_1, x_2, x_3)$	Spatial point

# Eulerian Description (1)

**Current configuration**



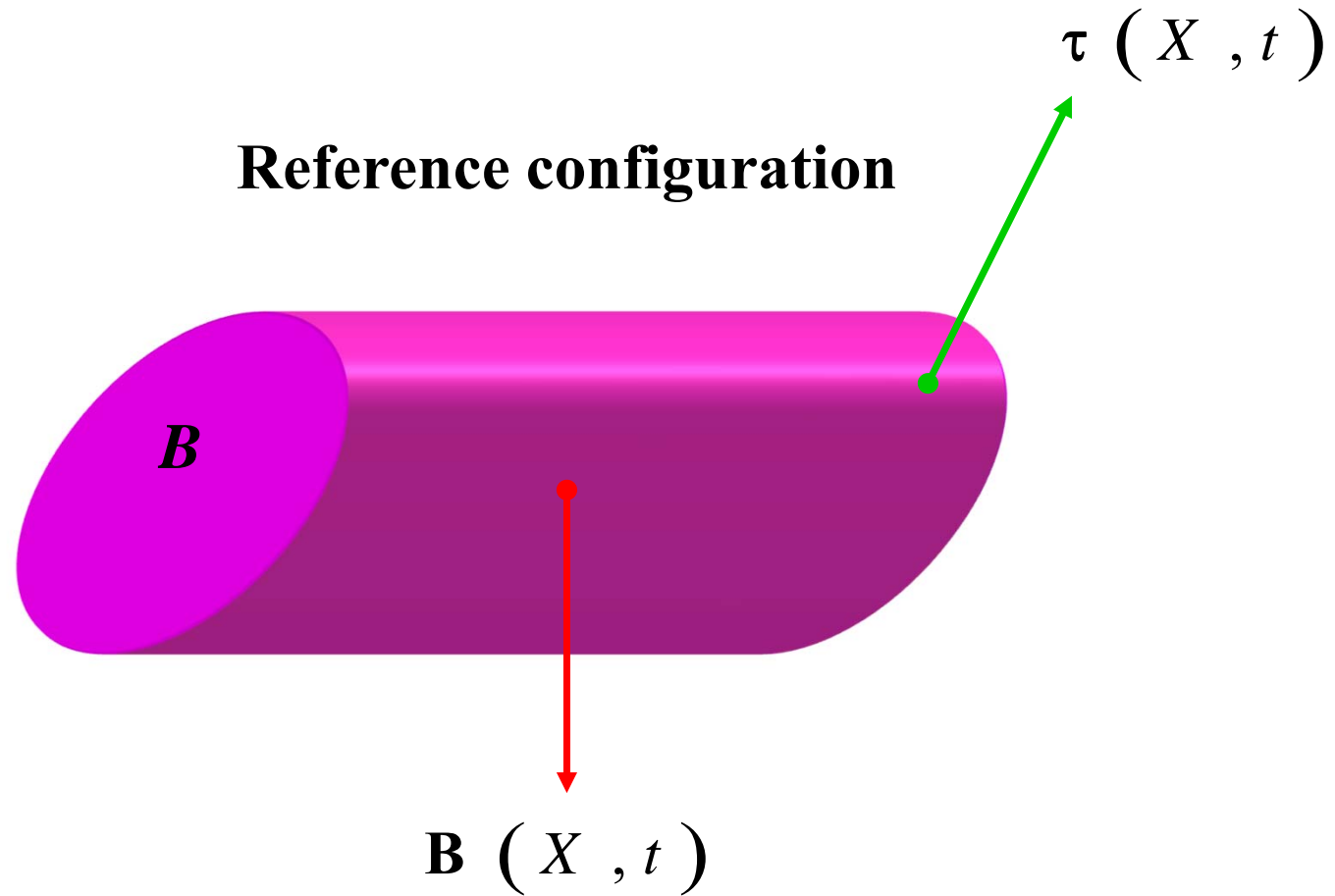
# Leonhard Euler

**Leonhard Euler (1707-1783)**

**Swiss Mathematician, Physicist, Astronomer  
and Engineer**

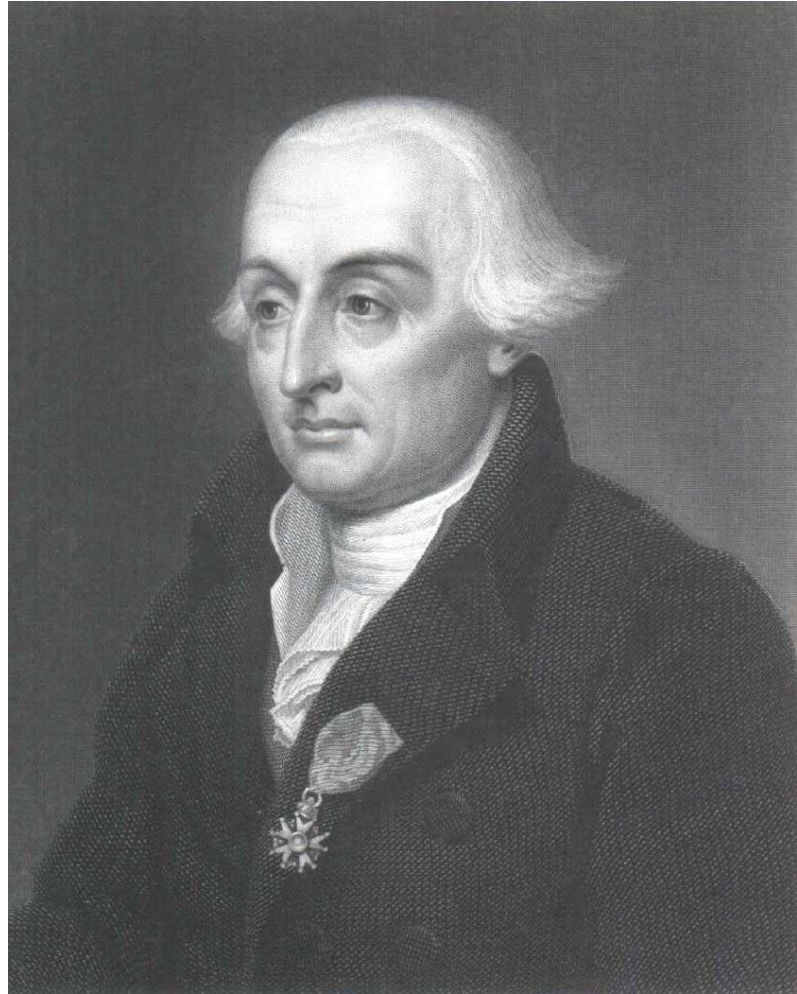


# Lagrangian Description



# Joseph-Louis Lagrange

**Joseph-Louis Lagrange (1736-1813)**  
**Italian Mathematician and Astronomer**





## Continuum Mechanics (1)

<b>Form</b>	<b>Conservation of mass</b>	<b>Balance of linear momentum</b>
<b>Eulerian form</b>	$\dot{\rho} + \rho \operatorname{div} \mathbf{v} = 0$	$\rho \dot{\mathbf{v}} = \operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{b}$
<b>Lagrangian form</b>	$\rho_0 (X)$ $= \rho (\phi_t (X), t) J (X, t)$	$\rho_0 \frac{\partial \mathbf{V}}{\partial t} = \operatorname{Div} \mathbf{P} + \rho_0 \mathbf{B}$

## Divergence of a Tensor Field

$$\mathbf{T} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}$$

$\Rightarrow$

$$\text{div } \mathbf{T} = \begin{pmatrix} \partial_1 T_{11} + \partial_2 T_{12} + \partial_3 T_{13} \\ \partial_1 T_{21} + \partial_2 T_{22} + \partial_3 T_{23} \\ \partial_1 T_{31} + \partial_2 T_{32} + \partial_3 T_{33} \end{pmatrix}$$

## The divergence theorem for a tensor field

$$\int_{\Omega} \operatorname{div} \mathbf{T} \, dx = \int_{\partial \Omega} \mathbf{T} \cdot \mathbf{n} \, d\sigma$$

## Notation (1)

<b>Notation</b>	<b>Meaning</b>
$\rho_0 ( X ), \rho ( x, t )$	<b>Density</b>
$\mathbf{V}(X, t), \mathbf{v}(x, t)$	<b>Velocity</b>
$\mathbf{B}(X, t), \mathbf{b}(x, t)$	<b>Body force</b>
$\boldsymbol{\sigma} ( x, t )$	<b>Cauchy stress tensor</b>
$\mathbf{P}(X, t)$	<b>First Piola-Kirchhoff stress tensor</b>

# Louis Augustin Cauchy

**Louis Augustin Cauchy (1789-1857)**  
**French Mathematician and Physicist**



## Notation (2)

Notation	Meaning
$\phi_t ( X )$	Deformation
$\mathbf{F} ( X , t ) = D \phi_t ( X )$	Deformation gradient
$J ( X , t ) = \det \mathbf{F} ( X , t )$	Jacobian
$\dot{\bullet} = \frac{D}{D t} = \frac{\partial}{\partial t} + \mathbf{v} \bullet \nabla$	Material derivative

# Deformation Gradient

$$\boldsymbol{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} : \overline{\Omega} \rightarrow \mathbf{R}^3$$

$\Rightarrow$

$$\nabla \boldsymbol{\varphi} = \begin{pmatrix} \partial_1 \varphi_1 & \partial_2 \varphi_1 & \partial_3 \varphi_1 \\ \partial_1 \varphi_2 & \partial_2 \varphi_2 & \partial_3 \varphi_2 \\ \partial_1 \varphi_3 & \partial_2 \varphi_3 & \partial_3 \varphi_3 \end{pmatrix}$$



## Displacement gradient

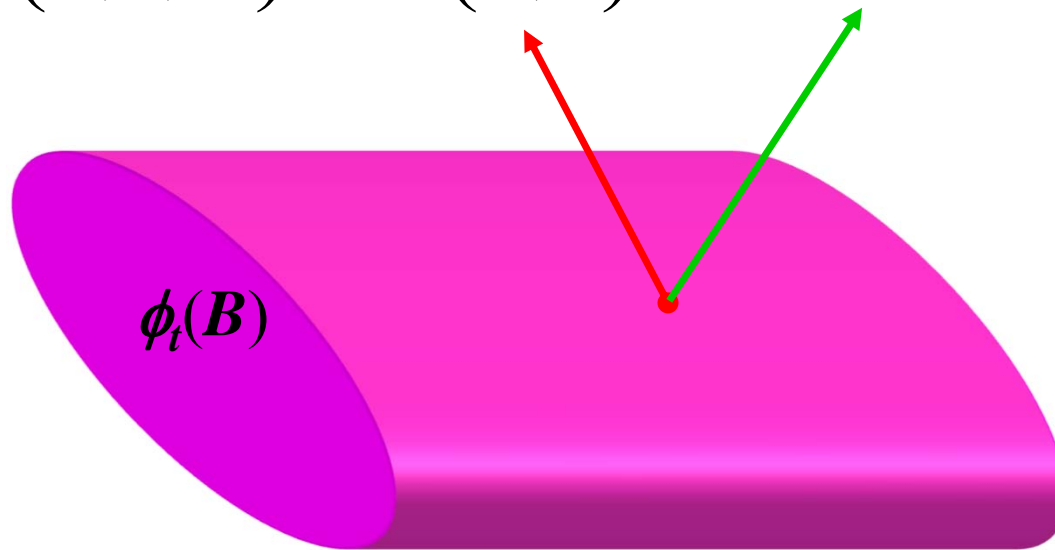
$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \varphi_1 - x_1 \\ \varphi_2 - x_2 \\ \varphi_3 - x_3 \end{pmatrix} : \overline{\Omega} \rightarrow \mathbf{R}^3$$

$\Rightarrow$

$$\nabla \mathbf{u} = \begin{pmatrix} \partial_1 u_1 & \partial_2 u_1 & \partial_3 u_1 \\ \partial_1 u_2 & \partial_2 u_2 & \partial_3 u_2 \\ \partial_1 u_3 & \partial_2 u_3 & \partial_3 u_3 \end{pmatrix} = \nabla \boldsymbol{\varphi} - \mathbf{I}$$

## Eulerian Description (2)

$$\mathbf{t}(x, t, \mathbf{n}) = \boldsymbol{\sigma}(x, t) \cdot \mathbf{n}$$



**Current configuration**

## Continuum Mechanics (2)

<b>Form</b>	<b>Cauchy's Theorem</b>
<b>Eulerian form</b>	$\mathbf{t}(x, t, \mathbf{n}) = \boldsymbol{\sigma}(x, t) \cdot \mathbf{n}$
<b>Lagrangian form</b>	$\mathbf{T}(X, t, \mathbf{N}) = \mathbf{P}(X, t) \cdot \mathbf{N}$

## Notation (3)

<b>Notation</b>	<b>Meaning</b>
$\mathbf{t} ( x , t , \mathbf{n} )$	<b>Cauchy stress vector</b>
$\boldsymbol{\sigma} ( x , t )$	<b>Cauchy stress tensor</b>
$\mathbf{T} ( X , t , \mathbf{N} )$	<b>First Piola-Kirchhoff stress vector</b>
$\mathbf{P} ( X , t )$	<b>First Piola-Kirchhoff stress tensor</b>
$\mathbf{n} , \mathbf{N}$	<b>Outward unit normal vector</b>

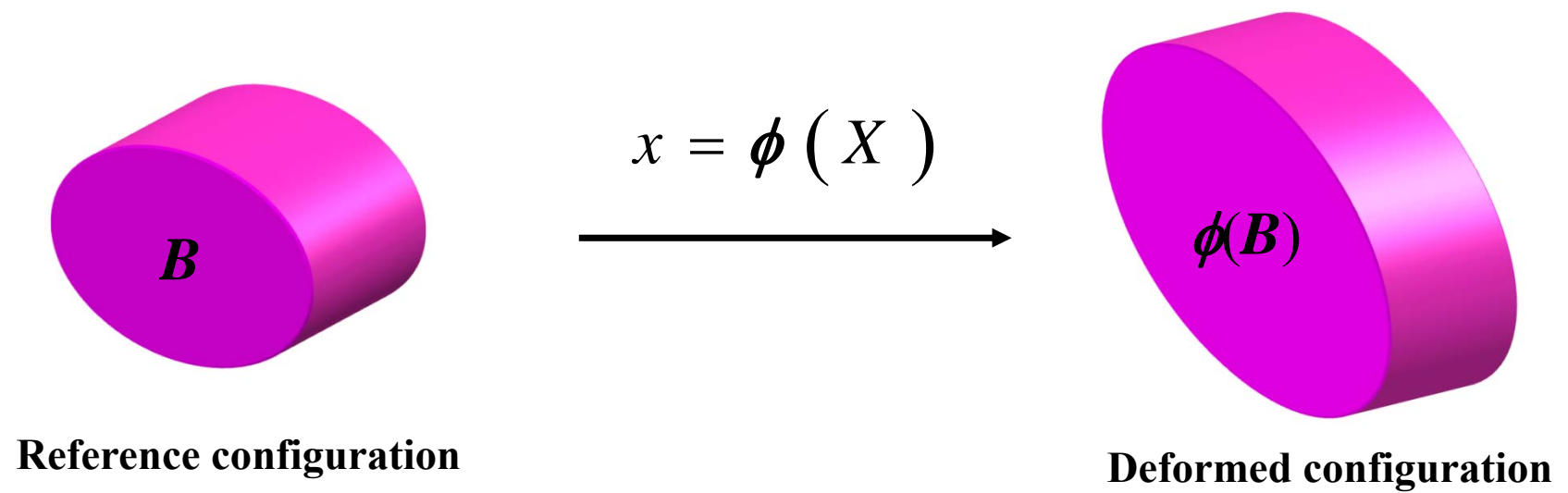
## Continuum Mechanics (3)

Form	Balance of angular momentum	Conservation of energy
Eulerian form	$\sigma = {}^t \sigma$	$\rho \dot{e} + \operatorname{div} \mathbf{q} = \operatorname{tr}(\sigma \mathbf{d}) + \rho r$
Lagrangian form	$\mathbf{S} = {}^t \mathbf{S}$	$\rho_0 \frac{\partial E}{\partial t} + \operatorname{D} \operatorname{iv} \mathbf{Q} = \operatorname{tr}(\mathbf{S} \mathbf{D}) + \rho_0 R$

## Notation (4)

Notation	Meaning
$\mathbf{S}(X, t)$	Second Piola-Kirchhoff stress tensor
$E(X, t), e(x, t)$	Internal energy function
$\mathbf{D}(X, t), \mathbf{d}(x, t)$	Material rate of deformation tensor
$R(X, t), r(x, t)$	Heat supply
$\mathbf{Q}(X, t), \mathbf{q}(x, t)$	Heat flux vector

# Nonlinear Elastostatics



## Notation

Notation	Meaning
$\Omega$	Bounded domain in $\mathbb{R}^3$ with <b>smooth</b> boundary
$B = \overline{\Omega}$	Reference configuration
$\phi$	Deformation of $B$
$X = (X_1, X_2, X_3)$	Material point
$x = (x_1, x_2, x_3)$	Spatial point



# Elasticity

A material is said to be **elastic**

$\Leftrightarrow$

$\exists$  A function  $\hat{\mathbf{P}}(X, \mathbf{F})$  of points  $X \in B$   
and  $3 \times 3$  matrices  $\mathbf{F} = (F_{ij})$  with  $\det \mathbf{F} > 0$   
such that

$$\mathbf{P}(X) = \hat{\mathbf{P}}(X, D\phi(X))$$

## Constitutive Function

$\hat{P}(X, F)$  is called a **constitutive function**

# Hyper-Elasticity

A material is said to be **hyperelastic**

$\Leftrightarrow$

$\exists$  A function  $W(X, \mathbf{F})$  of points  $X \in B$  and  $3 \times 3$  matrices  $\mathbf{F} = (F_{ij})$  with  $\det \mathbf{F} > 0$  such that

$$\hat{\mathbf{P}}(X, \mathbf{F}) = \frac{\partial W}{\partial \mathbf{F}}(X, \mathbf{F})$$

$$\hat{P}_{ij}(X, \mathbf{F}) = \frac{\partial W}{\partial F_{ij}}(X, \mathbf{F})$$

## Stored Energy Function

$W(X, \mathbf{F})$  is called a **stored energy function**

## First Elasticity Tensor

The **four - index tensor**  
defined by the formula

$$\mathbf{A} = \frac{\partial \widehat{\mathbf{P}}}{\partial \mathbf{F}} = \frac{\partial^2 W}{\partial \mathbf{F} \partial \mathbf{F}},$$

$$\begin{aligned} A_{ij\ell m} (X, \mathbf{F}) &= \frac{\partial \widehat{P}_{ij}}{\partial F_{\ell m}} (X, \mathbf{F}) \\ &= \frac{\partial^2 W}{\partial F_{ij} \partial F_{\ell m}} (X, \mathbf{F}) \end{aligned}$$

is called the **first elasticity tensor**.

# Typical Examples

## Hencky-Nadai Elasto-Plastic Material

The **stored energy function** has the form

$$W ( X , \mathbf{F} ) = \frac{3}{4} \int_0^{\Gamma(\mathbf{F})} g ( \xi ) d \xi + \frac{K}{2} \left( \sum_{k=1}^3 F_{kk} - 3 \right)^2$$

$$g ( \xi ) \in C^\infty ( [0, \infty ), \mathbf{R} )$$

$$\Gamma ( \mathbf{F} ) = \frac{4}{3} \sum_{i,j=1}^3 \left( \frac{1}{2} ( F_{ij} + F_{ji} ) - \frac{1}{3} \left( \sum_{k=1}^3 F_{kk} F_{kk} \right) \delta_{ij} \right)^2$$

## The Modulus of Compression

$K$  is called the **modulus of compression**



## Saint Venant-Kirchhoff Isotropic Material

The **stored energy function** has the form

$$W(X, \mathbf{F}) = \frac{\lambda(X)}{8} \left( \sum_{k=1}^3 C_{kk}(\mathbf{F}) - 3 \right)^2 + \frac{\mu(X)}{4} \sum_{i,j=1}^3 \left( C_{ij}(\mathbf{F}) - \delta_{ij} \right)^2$$
$$C_{ij}(\mathbf{F}) = \sum_{k=1}^3 F_{ki} F_{kj}$$

# Saint Venant

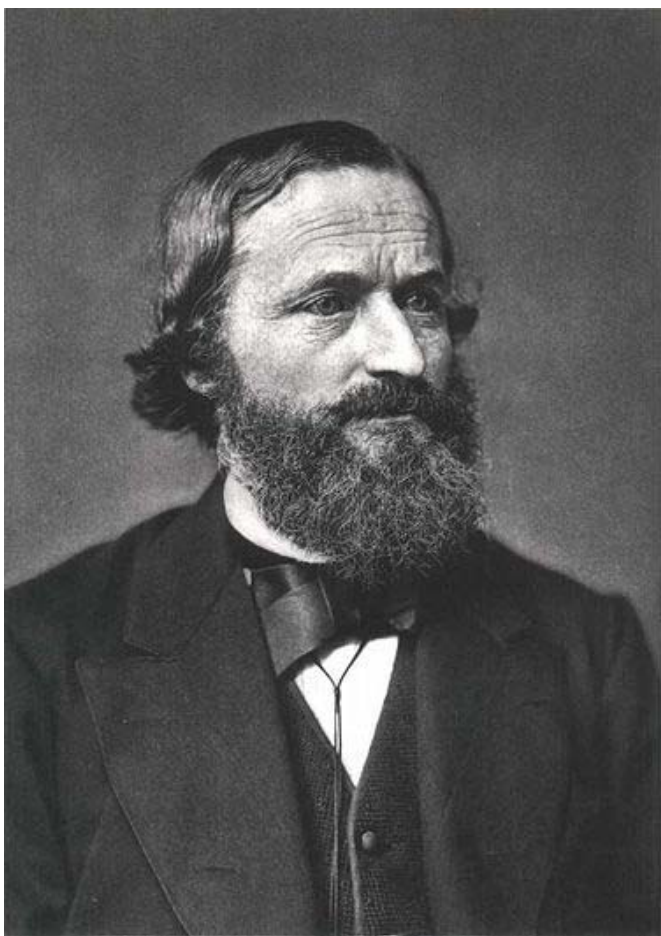
**Adhémar Jean Claude Barré de Saint Venant  
(1797-1886)**

**French Mechanician and Mathematician**



# Gustav Robert Kirchhoff

**Gustav Robert Kirchhoff (1824-1887)**  
**Prussian/German Physicist and Chemist**



## Lamé functions

$\lambda(X)$ ,  $\mu(X)$  are called **Lame functions**

# The Right Cauchy-Green tensor

$$C_{ij}(\mathbf{F}) = \sum_{k=1}^3 F_{ki} F_{kj}$$

## Green-Saint Venant strain tensor

$$\mathbf{E} = \frac{1}{2} (\mathbf{C} - \mathbf{I})$$

$$\mathbf{E} = (E_{ij})$$

$$= \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_{k=1}^3 \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_i} \right)$$



## Saint Venant-Kirchhoff Isotropic Material

$$W(X, \mathbf{F}) = \frac{\lambda(X)}{2} (\text{tr } \mathbf{E})^2 + \mu(X) \text{tr}(\mathbf{E}^2)$$

**Formulation  
of  
the Results**

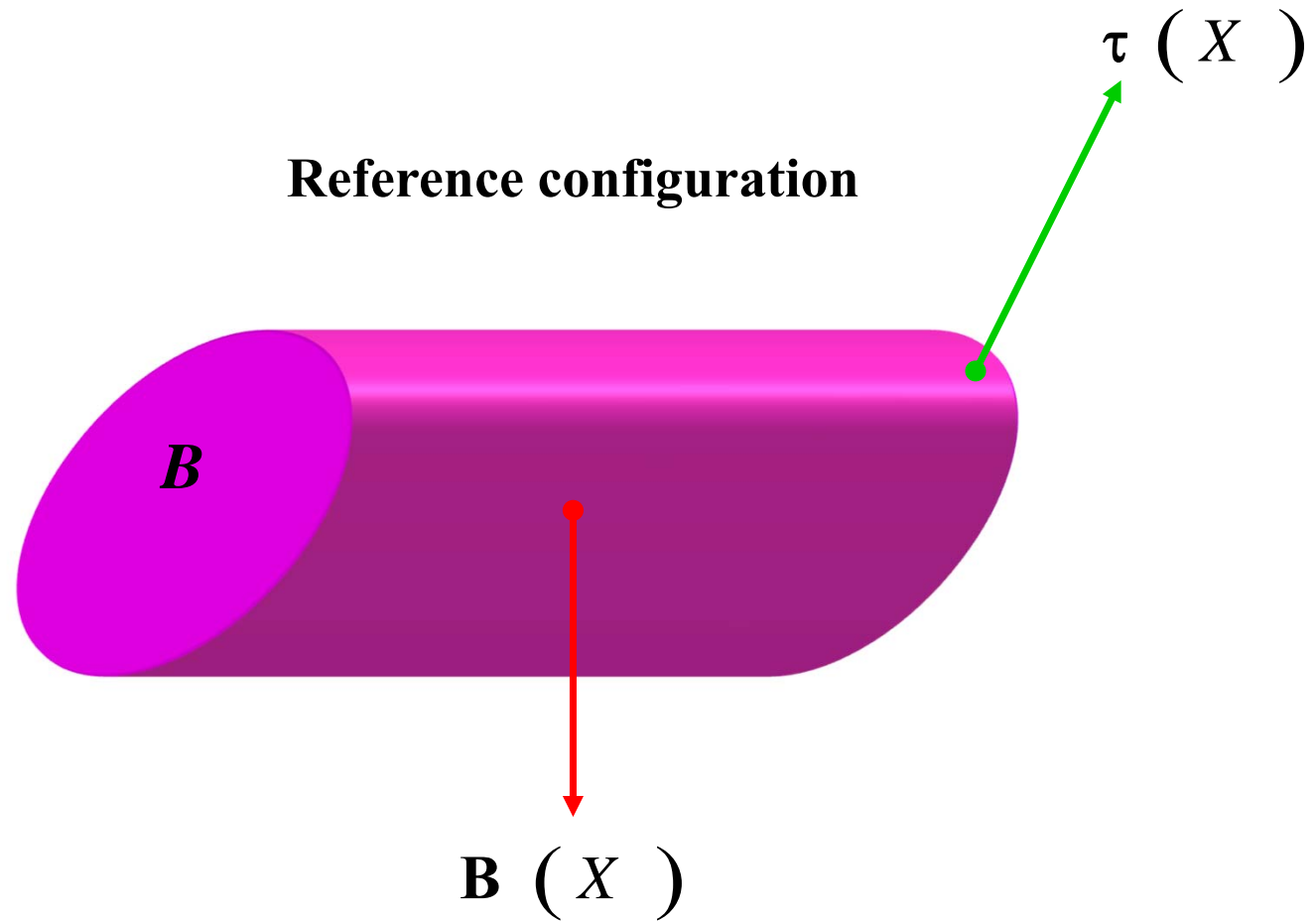
## Nonlinear Boundary Value Problem (Lagrange)

This talk is devoted to an  $L^p$  approach to the following equilibrium equations for the unknown deformation  $\phi(X)$

$$\text{Div } \hat{\mathbf{P}}(X, D\phi(X)) + \mathbf{B}(X) = \mathbf{0} \quad \text{in } \Omega,$$

$$\alpha(X) \hat{\mathbf{P}}(X, D\phi(X)) \cdot \mathbf{N}(X) + (1 - \alpha(X)) \phi(X) = \boldsymbol{\tau}(X) \quad \text{on } \partial\Omega.$$

# Lagrangian Description



## Notation

Notation	Meaning
$\hat{\mathbf{P}}(X, \mathbf{F})$	Constitutive function
$\mathbf{B}(X)$	Body force
$\boldsymbol{\tau}(X)$	Surface force
$\mathbf{N}(X)$	Outward unit normal
$\alpha(X)$	$\alpha(X) \in C^\infty(\partial\Omega)$
	$0 \leq \alpha(X) \leq 1$ on $\partial\Omega$

## Boundary Condition

$\{ \alpha = 1 \}$  (traction)



The diagram shows a domain  $\Omega$  represented by a cyan rectangle with a thick blue border. This rectangle is positioned on top of a magenta rectangular base. The magenta base is labeled with the boundary condition  $\{ \alpha = 0 \}$  (displacement). The cyan rectangle is labeled with the boundary condition  $\{ \alpha = 1 \}$  (traction).

$\Omega$

$\{ \alpha = 0 \}$  (displacement)

## Example of Regularization (Real axis)

$$\alpha(x_1) = \begin{cases} e^{-\frac{1}{x_1}} & \text{for } x_1 > 0, \\ 0 & \text{for } x_1 \leq 0 \end{cases}$$

## Example of Regularization (Unit Circle)

$$a(x_1, x_2) = a(\cos \theta, \sin \theta) =$$

$$\left\{ \begin{array}{l} \exp[2/\pi + 1/(\theta + \pi/2)](1 - \exp[2/\pi - 1/(\theta + \pi)]), \\ \theta \in (-\pi, -\pi/2) \\ 0, \theta \in [-\pi/2, 0] \\ \exp[2/\pi - 1/\theta](1 - \exp[2/\pi + 1/(\theta - \pi/2)]), \\ \theta \in (0, \pi/2), \\ 1, \theta \in [\pi/2, \pi] \end{array} \right.$$



## Main Results

1. If the linearized problem has unique solutions, then so does the nonlinear one, nearby. This is done by using the **linear  $L^p$  theory** and the **inverse mapping theorem**.
2. Our results can be applied to the **Saint Venant-Kirchhoff** elastic material and the **Hencky-Nadai** elasto-plastic material.

## Crucial Point

- The crucial point is how to find a **function space** associated with the **degenerate** boundary condition in which the linearized problem has unique solutions.

## Function Spaces (1)

For all real  $s$ , we define **Sobolev spaces**

$$\mathbf{H}^{s,p}(\Omega, \mathbf{R}^3) = \mathbf{H}^{s,p}(\mathbf{R}^3, \mathbf{R}^3)|_{\Omega}$$

For all  $s > 1/p$ , we define **Besov spaces**

$$\mathbf{B}^{s-1/p,p}(\partial\Omega, \mathbf{R}^3) = \mathbf{H}^{s,p}(\Omega, \mathbf{R}^3)|_{\partial\Omega}.$$

## Function Spaces (2)

$$s > 1 + 1/p$$

$$\mathbf{B}_{(\alpha)}^{s-1-1/p,p}(\partial\Omega, \mathbf{R}^3) \\ = \alpha(X) \mathbf{B}^{s-1-1/p,p}(\partial\Omega, \mathbf{R}^3) + (1 - \alpha(X)) \mathbf{B}^{s-1/p,p}(\partial\Omega, \mathbf{R}^3)$$

$$(A) \quad 0 \leq \alpha(X) \leq 1 \text{ and } \alpha(X) \neq 1 \text{ on } \partial\Omega.$$

## Definition of a norm

$$\begin{aligned} & \|\varphi\|_{\mathbf{B}_{(\alpha)}^{s-1-1/p,p}(\partial\Omega, \mathbf{R}^3)} \\ &= \inf \left\{ \|\varphi_1\|_{s-1-1/p,p} + \|\varphi_2\|_{s-1/p,p} : \varphi = \alpha(X)\varphi_1 + (1 - \alpha(X))\varphi_2 \right\} \end{aligned}$$

(A)  $0 \leq \alpha(X) \leq 1$  and  $\alpha(X) \not\equiv 1$  on  $\partial\Omega$ .

## Function Spaces (3)

$$\mathbf{B}_{(\alpha)}^{s-1-1/p,p}(\partial\Omega, \mathbf{R}^3) = \mathbf{B}^{s-1/p,p}(\partial\Omega, \mathbf{R}^3)$$

if  $\alpha(X) \equiv 0$  on  $\partial\Omega$  (**D i r i c h l e t**)

$$\mathbf{B}_{(\alpha)}^{s-1-1/p,p}(\partial\Omega, \mathbf{R}^3) = \mathbf{B}^{s-1-1/p,p}(\partial\Omega, \mathbf{R}^3)$$

if  $\alpha(X) \equiv 1$  on  $\partial\Omega$  (**N e u m a n n**)

## Function Spaces (4)

$$X = \mathbf{H}^{s,p} \left( \Omega, \mathbf{R}^3 \right),$$

$$Y = \mathbf{H}^{s-2,p} \left( \Omega, \mathbf{R}^3 \right) \times \mathbf{B}_{(\alpha)}^{s-1-1/p,p} \left( \partial\Omega, \mathbf{R}^3 \right).$$

# Nonlinear Map

$$F : X \rightarrow Y$$

$$F(\phi) = \left( -\text{Div} \hat{\mathbf{P}}(D\phi), \alpha \hat{\mathbf{P}}(D\phi) \cdot \mathbf{N} + (1 - \alpha) \phi|_{\partial\Omega} \right)$$



## Fundamental Assumptions (1)

(A)  $0 \leq \alpha(X) \leq 1$  and  $\alpha(X) \neq 1$  on  $\partial\Omega$ .

(P)  $\mathbf{P} \left( D \overset{\circ}{\phi} \right) = 0$  when  $\overset{\circ}{\phi} = \mathbf{I}_{\Omega}$ .

## Remarks

1. Condition (P) implies that the reference configuration is a **natural state**.
2. Condition (A) implies that our boundary condition is not equal to the **pure traction** boundary condition

## Fundamental Assumptions (2)

$$(H) \quad \overset{\circ}{\mathbb{A}} = \frac{\partial^2 W}{\partial \mathbf{F} \partial \mathbf{F}} \quad \text{at } \overset{\circ}{\phi} = \mathbf{I}_\Omega$$

enjoys the property of **symmetry**

$$\boxed{\overset{\circ}{\mathbb{A}}_{ijlm} = \overset{\circ}{\mathbb{A}}_{lmij} = \overset{\circ}{\mathbb{A}}_{jilm}}$$

and is **uniformly pointwise stable**

$$\boxed{\frac{1}{2} \mathbf{e} \cdot \overset{\circ}{\mathbb{A}} \cdot \mathbf{e} \geq \exists \eta \|\mathbf{e}\|}$$

## Main Theorem (1)

**Main Theorem states that if the linearized problem is uniformly pointwise stable, then, for **slight perturbations** of the load or boundary conditions from their values at the natural state, then the nonlinear problem has a **unique solution**.**

## Main Theorem (2)

$$1 < p < \infty, \quad s > 3 / p + 1.$$

$\exists$  A neighborhood  $U$  of  $\phi^\circ = \mathbf{I}_\Omega$   
and a neighborhood  $V$  of the point

$$\left( -\text{Div } \widehat{\mathbf{P}} \left( D \phi^\circ \right), \alpha \widehat{\mathbf{P}} \left( D \phi^\circ \right) \cdot \mathbf{N} + (1 - \alpha) \phi^\circ \Big|_{\partial\Omega} \right)$$

such that the **nonlinear map**

$$\boxed{F : U \rightarrow V}$$

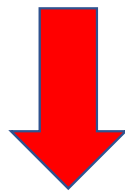
is **bijective**.

# Typical Examples

## Hencky-Nadai Elasto-Plastic Material

(A)  $0 \leq \alpha(X) \leq 1$  and  $\alpha(X) \neq 1$  on  $\partial\Omega$ .

(G)  $g(0) > 0$  and  $K > 0$



**Condition (H) is satisfied and  
so Main Theorem applies.**

## Saint Venant-Kirchhoff Isotropic Material

(A)  $0 \leq \alpha(X) \leq 1$  and  $\alpha(X) \neq 1$  on  $\partial\Omega$ .

(M)  $\exists c_1 > 0, \exists c_2 > 0$  such that

$$\mu(X) \geq c_1 \text{ on } \Omega,$$

$$\lambda(X) + \frac{2}{3}\mu(X) \geq c_2 \text{ on } \Omega$$



**Condition (H) is satisfied and  
so Main Theorem applies.**



# Linearized Problems (Frechet Derivatives)

## Mixed Problem of Linear Elastodynamics

We study the following mixed problem of **linear elastodynamics**:

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} - \operatorname{div} (\mathbf{a} \cdot \nabla \mathbf{u}) = \mathbf{f} \quad \text{in } \Omega \times (0, \infty),$$

$$\mathbf{u} \Big|_{t=0} = \mathbf{u}_0 \quad \text{in } \Omega,$$

$$\frac{\partial \mathbf{u}}{\partial t} \Big|_{t=0} = \mathbf{u}_1 \quad \text{in } \Omega,$$

$$\alpha(x')(\mathbf{a} \cdot \nabla \mathbf{u} \cdot \mathbf{n}) + (1 - \alpha(x'))\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \times (0, \infty).$$

## Example (I)

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} - \operatorname{div} (\lambda \operatorname{tr}(\mathbf{e}(\mathbf{u}))\mathbf{I} + 2\mu \mathbf{e}(\mathbf{u})) = \mathbf{f} \quad \text{in } \Omega \times (0, \infty),$$

$$\mathbf{u} \Big|_{t=0} = \mathbf{u}_0 \quad \text{in } \Omega,$$

$$\frac{\partial \mathbf{u}}{\partial t} \Big|_{t=0} = \mathbf{u}_1 \quad \text{in } \Omega,$$

$$\alpha(x')(\boldsymbol{\tau}(\mathbf{u}) \cdot \mathbf{n}) + (1 - \alpha(x'))\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \times (0, \infty).$$

## Example (I)

$$\mathbf{e}(\mathbf{u}) = (e_{ij}) = \left( \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right), \text{ strain tensor}$$

$$\boldsymbol{\tau}(\mathbf{u}) = (\tau_{ij}) = \left( \lambda e_{kk} \delta_{ij} + 2\mu e_{ij} \right), \text{ stress tensor}$$

## Example (Ia)

$$(a_{ijlm}) = \left( \lambda \delta_{ij} \delta_{lm} + \mu (\delta_{il} \delta_{jm} + \delta_{im} \delta_{jl}) \right)$$

$\lambda, \mu$  : **Lame moduli**

$$(H) \quad \mu > 0, \quad k = \frac{3\lambda + 2\mu}{3} > 0.$$

$k$  : **m o d u l u s o f c o m p r e s s i o n**

## Example (Ib)

$$(a_{ijlm}) = \left( \left( K - \frac{2}{3} g(0) \right) \delta_{ij} \delta_{lm} + g(0) (\delta_{il} \delta_{jm} + \delta_{im} \delta_{jl}) \right)$$

$K$  is the **modulus of compression**

$$g(\xi) \in C^\infty([0, \infty), \mathbf{R})$$

$$(G) \quad K > 0, \quad g(0) > 0.$$

## Example (II)

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = f \quad \text{in } \Omega \times (0, \infty),$$

$$u \Big|_{t=0} = u_0 \quad \text{in } \Omega,$$

$$\frac{\partial u}{\partial t} \Big|_{t=0} = u_1 \quad \text{in } \Omega,$$

$$\alpha(x') \frac{\partial u}{\partial \mathbf{n}} + (1 - \alpha(x'))u = 0 \quad \text{on } \partial \Omega \times (0, \infty).$$

## Example (II)

$$(a_{ijlm}) = (\delta_{il} \delta_{jm}).$$



## Linear Boundary Value Problem

We are reduced to the study of a problem of **linear elastostatics** for the unknown vector function  $\mathbf{v}$ :

$$\mathbf{A}\mathbf{v} = \operatorname{div}(\mathbf{a} \cdot \nabla \mathbf{v}) = \mathbf{f} \quad \text{in } \Omega,$$

$$\mathbf{B}_\alpha \mathbf{v} = \alpha(x')(\mathbf{a} \cdot \nabla \mathbf{v} \cdot \mathbf{n}) + (1 - \alpha(x'))\mathbf{v} = \varphi \quad \text{on } \partial\Omega$$

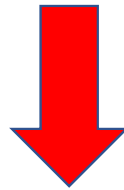
$\mathbf{a} = (a_{ijlm})$  : smooth elasticity tensor.

$\mathbf{n}$  : the outward unit normal

## Fundamental Existence and Uniqueness Theorem

$$(A) \quad 0 \leq \alpha(X) \leq 1 \text{ and } \alpha(X) \not\equiv 1 \text{ on } \partial\Omega.$$

$$(H) \quad \begin{array}{l} a_{ijlm} = a_{lmij} = a_{jilm} \\ \frac{1}{2} \mathbf{e} \cdot \mathbf{a} \cdot \mathbf{e} \geq \exists \eta \|\mathbf{e}\|^2 \end{array}$$



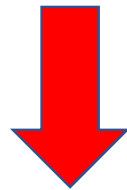
$$(\mathbf{A}, \mathbf{B}_\alpha): \mathbf{H}^{s,p}(\Omega, \mathbf{R}^3) \rightarrow \mathbf{H}^{s-2,p}(\Omega, \mathbf{R}^3) \times \mathbf{B}_{(\alpha)}^{s-1-1/p,p}(\partial\Omega, \mathbf{R}^3)$$

is an algebraic and topological **isomorphism**

# Regularity Theorem

## Regularity Theorem

$$(H) \quad \begin{cases} a_{ijlm} = a_{lmij} = a_{jilm} \\ \frac{1}{2} \mathbf{e} \cdot \mathbf{a} \cdot \mathbf{e} \geq \exists \eta \|\mathbf{e}\|^2 \end{cases}$$



$$1 < p < \infty, \quad s > 1/p + 1$$

$$\mathbf{u} \in L^p(\Omega, \mathbf{R}^3), \quad \mathbf{A} \mathbf{u} \in \mathbf{H}^{s-2, p}(\Omega, \mathbf{R}^3),$$

$$\mathbf{B}_{(\alpha)} \mathbf{u} \in \mathbf{B}_{(\alpha)}^{s-1-1/p, p}(\partial\Omega, \mathbf{R}^3) \Rightarrow \mathbf{u} \in \mathbf{H}^{s, p}(\Omega, \mathbf{R}^3).$$

# **Construction of a Parametrix**

## Symbol of a Pseudo-Differential Operator

$$\begin{aligned} & \mathbf{t}(x', \xi') \\ &= \alpha(x') \mathbf{p}_1(x', \xi') \\ &+ \left[ (1 - \alpha(x')) \mathbf{I} + \alpha(x') \mathbf{p}_0(x', \xi') \right] \\ &+ \dots \end{aligned}$$

Here:

$$\mathbf{p}_1(x', \xi') \geq \exists c_0 |\xi'| \mathbf{I} \text{ on } T^*(\partial\Omega)$$

## Elementary Lemma

$$\begin{aligned} f(x) &\in C^2(\mathbf{R}), \\ f(x) &\geq 0 \text{ on } \mathbf{R}, \\ \sup_{x \in \mathbf{R}} |f''(x)| &\leq \exists c \end{aligned}$$

$\Rightarrow$

$$|f'(x)| \leq \sqrt{2c} (f(x))^{1/2} \text{ on } \mathbf{R}.$$

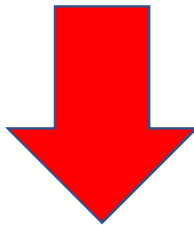
## Criteria for Parametrixes (1)

$$A = p(x, D) \in L_{1,0}^m(\Omega).$$

Assume that:

$$\left| D_{\xi}^{\alpha} D_x^{\beta} p(x, \xi) \right| \leq \exists C_{K, \alpha, \beta} \left| p(x, \xi) \right| (1 + |\xi|)^{-|\alpha| + (1/2)|\beta|},$$

$$\left| p(x, \xi)^{-1} \right| \leq \exists C_K, \quad \forall x \in K \subset \Omega, \quad \forall |\xi| \geq C_K.$$





## Criteria for Parametrixes (2)

$\exists B \in L_{1,1/2}^0(\Omega)$  such that

$$AB \equiv I \pmod{L^{-\infty}(\Omega)},$$
$$BA \equiv I \pmod{L^{-\infty}(\Omega)}.$$

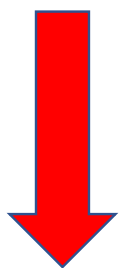
# Uniqueness Theorem

## Uniqueness Theorem

$$\mathbf{v} \in \mathbf{H}^{s,p}(\Omega, \mathbf{R}^3)$$

$$\mathbf{A} \mathbf{v} = \operatorname{div}(\mathbf{a} \cdot \nabla \mathbf{v}) = \mathbf{0} \quad \text{in } \Omega,$$

$$\mathbf{B}_\alpha \mathbf{v} = \alpha(x')(\mathbf{a} \cdot \nabla \mathbf{v} \cdot \mathbf{n}) + (1 - \alpha(x'))\mathbf{v} = \mathbf{0} \quad \text{on } \partial\Omega,$$



$$(A) \quad 0 \leq \alpha(X) \leq 1 \quad \text{and} \quad \alpha(X) \neq 1 \quad \text{on } \partial\Omega.$$

$$(H) \quad \left[ \begin{array}{l} a_{ijlm} = a_{lmij} = a_{jilm} \\ \frac{1}{2} \mathbf{e} \cdot \mathbf{a} \cdot \mathbf{e} \geq \exists \eta \|\mathbf{e}\|^2 \end{array} \right]$$

$$\mathbf{v} = \mathbf{0} \quad \text{in } \Omega$$

## Korn's Inequality

A non-empty set :  $\gamma \subset \partial\Omega$

$$\mathbf{u} \in \mathbf{H}^{1,2}(\Omega, \mathbf{R}^3)$$

$$\mathbf{u} = \mathbf{0} \text{ on } \gamma$$



$$\int_{\Omega} \|\mathbf{e}\|^2 dx \geq \exists c(\gamma) \left( \int_{\Omega} \|\mathbf{u}\|^2 dx + \int_{\partial\Omega} \|\nabla \mathbf{u}\|^2 dx \right)$$

$$e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) : \text{the } \mathbf{strain\ tensor}$$

# **Existence Theorem**

# Existence Theorem

$$\mathbf{f} \in \mathbf{H}^{s-2,p}(\Omega, \mathbf{R}^3), \quad \varphi \in \mathbf{B}_{(\alpha)}^{s-1-1/p,p}(\partial\Omega, \mathbf{R}^3),$$



$$(A) \quad 0 \leq \alpha(x) \leq 1 \text{ and } \alpha(x) \neq 1 \text{ on } \partial\Omega.$$

$$(H) \quad \begin{array}{l} a_{ijlm} = a_{lmij} = a_{jilm} \\ \frac{1}{2} \mathbf{e} \cdot \mathbf{a} \cdot \mathbf{e} \geq \exists \eta \|\mathbf{e}\|^2 \end{array}$$

$$\exists \mathbf{v} \in \mathbf{H}^{s,p}(\Omega, \mathbf{R}^3)$$

$$\mathbf{A} \mathbf{v} = \operatorname{div}(\mathbf{a} \cdot \nabla \mathbf{v}) = \mathbf{f} \text{ in } \Omega,$$

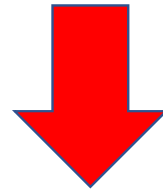
$$\mathbf{B}_{\alpha} \mathbf{v} = \alpha(x')(\mathbf{a} \cdot \nabla \mathbf{v} \cdot \mathbf{n}) + (1 - \alpha(x')) \mathbf{v} = \varphi \text{ on } \partial\Omega$$

## The Operator A

$$\mathbf{A} : L^2(\Omega, \mathbf{R}^3) \rightarrow L^2(\Omega, \mathbf{R}^3)$$

$$(a) \quad D(\mathbf{A}) = \{ \mathbf{u} \in H^2(\Omega, \mathbf{R}^3) : \mathbf{B}_\alpha \mathbf{u} = \mathbf{0} \}.$$

$$(b) \quad \mathbf{A} \mathbf{u} = \operatorname{div}(\mathbf{a} \cdot \nabla \mathbf{u}), \quad \forall \mathbf{u} \in D(\mathbf{A}).$$



–  $A$  : positive definite, self - adjoint

## Function Space (1)

$\mathbf{B} = \sqrt{-\mathbf{A}}$  : the **square root** of  $-\mathbf{A}$   
 $H_{\mathbf{A}}$  = the **domain**  $D(\mathbf{B})$  with the inner  
product  $(\mathbf{u}, \mathbf{v})_{\mathbf{A}} = (\mathbf{B}\mathbf{u}, \mathbf{B}\mathbf{v})$ .

(A)  $0 \leq \alpha(X) \leq 1$  and  $\alpha(X) \neq 1$  on  $\partial\Omega$ .



## Function Space (2)

$H_{\mathbf{A}}$  = the **completion** of  $D(\mathbf{A})$  with respect to the inner product

$$(\mathbf{u}, \mathbf{v})_{\mathbf{A}} = -(\mathbf{A}\mathbf{u}, \mathbf{v})$$

$$= \int_{\Omega} \nabla \mathbf{u} \cdot \mathbf{a} \cdot \overline{\nabla \mathbf{v}} \, dx + \int_{\{\alpha \neq 0\}} \frac{1 - \alpha}{\alpha} \mathbf{u} \cdot \overline{\mathbf{v}} \, d\sigma$$

## Function Spaces (4)

$$\mathbf{H}_A = H_0^1(\Omega, \mathbf{R}^3)$$

if  $\alpha(X) \equiv 0$  on  $\partial\Omega$  (**Dirichlet**)

$$\mathbf{H}_A = H^1(\Omega, \mathbf{R}^3)$$

if  $\alpha(X) \equiv 1$  on  $\partial\Omega$  (**Neumann**)

## Embedding Properties

$$D(\mathbf{A}) \subset H_{\mathbf{A}} \subset H^1(\Omega, \mathbf{R}^3)$$

# Existence and Uniqueness Theorem

$$\mathbf{u}_0 \in D(\mathbf{A}), \mathbf{u}_1 \in H_{\mathbf{A}}, \mathbf{f} \in C^1([0, \infty), L^2(\Omega, \mathbf{R}^3)),$$



$$\frac{\partial^2 \mathbf{u}}{\partial t^2} - \operatorname{div}(\mathbf{a} \cdot \nabla \mathbf{u}) = \mathbf{f} \quad \text{in } \Omega \times (0, \infty),$$

$$\mathbf{u} \Big|_{t=0} = \mathbf{u}_0 \quad \text{in } \Omega,$$

$$\frac{\partial \mathbf{u}}{\partial t} \Big|_{t=0} = \mathbf{u}_1 \quad \text{in } \Omega,$$

$$\alpha(x')(\mathbf{a} \cdot \nabla \mathbf{u} \cdot \mathbf{n}) + (1 - \alpha(x'))\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \times (0, \infty)$$

has a **unique solution**

$$\mathbf{u} \in C^2([0, \infty), L^2(\Omega, \mathbf{R}^3)) \cap C^1([0, \infty), H_{\mathbf{A}}) \cap C([0, \infty), D(\mathbf{A}))$$

## Energy Estimate

$$\begin{aligned} & \|\mathbf{u}(t)\|_{H^2}^2 + \|\mathbf{u}'(t)\|_{H_A}^2 + \|\mathbf{u}''(t)\|_{L^2}^2 \\ & \leq \exists C \left( \|\mathbf{u}_0\|_{H^2}^2 + \|\mathbf{u}_1\|_{H_A}^2 + \|\mathbf{f}(0)\|_{L^2}^2 + \int_0^t \|\mathbf{f}'(s)\|_{L^2}^2 ds \right). \end{aligned}$$

# Regularity Theorem

$$\forall \mathbf{u}_0 \in D(\mathbf{A}), \forall \mathbf{u}_1 \in H_{\mathbf{A}}, \forall \mathbf{f} \in C^1([0, \infty), L^2(\Omega, \mathbf{R}^3))$$

$\Rightarrow$

$$\exists \mathbf{u} \in C^2([0, \infty), L^2(\Omega, \mathbf{R}^3)) \cap C^1([0, \infty), H_{\mathbf{A}}) \cap C([0, \infty), D(\mathbf{A}))$$

such that

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} - \operatorname{div}(\mathbf{a} \bullet \nabla \mathbf{u}) = \mathbf{f} \quad \text{in } \Omega \times (0, \infty),$$

$$\mathbf{u}|_{t=0} = \mathbf{u}_0 \quad \text{in } \Omega,$$

$$\left. \frac{\partial \mathbf{u}}{\partial t} \right|_{t=0} = \mathbf{u}_1 \quad \text{in } \Omega,$$

$$\alpha(x')(\mathbf{a} \bullet \nabla \mathbf{u} \bullet \mathbf{n}) + (1 - \alpha(x'))\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \times (0, \infty)$$

# Semigroup Approach

## Semigroup Approach (1)

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} - \operatorname{div} (\mathbf{a} \cdot \nabla \mathbf{u}) = \mathbf{f} \quad \text{in } \Omega \times (0, \infty),$$

$$\mathbf{u} \Big|_{t=0} = \mathbf{u}_0 \quad \text{in } \Omega,$$

$$\frac{\partial \mathbf{u}}{\partial t} \Big|_{t=0} = \mathbf{u}_1 \quad \text{in } \Omega,$$

$$\alpha(x') (\mathbf{a} \cdot \nabla \mathbf{u} \cdot \mathbf{n}) + (1 - \alpha(x')) \mathbf{u} = \mathbf{0} \quad \text{on } \partial \Omega \times (0, \infty).$$

$$\mathbf{u}''(t) = \mathbf{A} \mathbf{u}(t) + \mathbf{f}(t),$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{u}'(0) = \mathbf{u}_1$$



## Semigroup Approach (2)

$$\begin{aligned}\mathbf{u}''(t) &= \mathbf{A} \mathbf{u}(t) + \mathbf{f}(t), \\ \mathbf{u}(0) &= \mathbf{u}_0, \quad \mathbf{u}'(0) = \mathbf{u}_1\end{aligned}$$

$$\begin{aligned}\mathbf{U}(t) &= \begin{pmatrix} \mathbf{u}(t) \\ \mathbf{u}'(t) \end{pmatrix}, \quad \mathbf{U}(0) = \begin{pmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \end{pmatrix}, \\ \mathbf{F}(t) &= \begin{pmatrix} 0 \\ \mathbf{f}(t) \end{pmatrix}, \quad \mathfrak{A} = \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{A} & 0 \end{pmatrix}\end{aligned}$$

## Semigroup Approach (3)

$$\begin{aligned} \mathbf{U}'(t) &= \mathfrak{A}\mathbf{U}(t) + \mathbf{F}(t), \\ \mathbf{U}(0) &= \mathbf{U}_0. \end{aligned}$$

$$\begin{aligned} X &= H_{\mathbf{A}} \times L^2(\Omega, \mathbf{R}^3), \\ D(\mathfrak{A}) &= D(\mathbf{A}) \times H_{\mathbf{A}}. \end{aligned}$$

## Semigroup Approach (4)

$$e^{t\mathbf{A}} = \cos(t\mathbf{B}) \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} + \mathbf{B}^{-1} \sin(t\mathbf{B}) \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{A} & \mathbf{0} \end{pmatrix}$$

$$\mathbf{B} = \sqrt{-\mathbf{A}}, \quad \mathbf{B}^2 = -\mathbf{A}.$$

## Representation Formula

$$\mathbf{u}(t) = \cos(t\mathbf{B})\mathbf{u}_0 + \mathbf{B}^{-1}\sin(t\mathbf{B})\mathbf{u}_1 + \int_0^t \mathbf{B}^{-1}\sin((t-s)\mathbf{B})\mathbf{f}(s)ds.$$

$$\mathbf{B} = \sqrt{-\mathbf{A}}, \quad \mathbf{B}^2 = -\mathbf{A}.$$

# Open Problems

## Open Problems

1. The first problem is to generalize Main Theorem to the case where the domain has **corner singularities**.
2. The second problem is to study the case where the function  $\alpha(X)$  is the **characteristic function** of a subset of the boundary.

## References (4)

- **Ito**: On a mixed problem of linear elastodynamics with time-dependent discontinuous boundary condition, *Osaka J. Math.* 27 (1990), 667—707.

## Nonlinear Boundary Value Problem (Lagrange)

This talk is devoted to an  $L^p$  approach to the following equilibrium equations for the unknown deformation  $\phi(X)$

$$\text{Div } \hat{\mathbf{P}}(X, D\phi(X)) + \mathbf{B}(X) = \mathbf{0} \quad \text{in } \Omega,$$

$$\alpha(X) \hat{\mathbf{P}}(X, D\phi(X)) \cdot \mathbf{N}(X) + (1 - \alpha(X)) \phi(X) = \boldsymbol{\tau}(X) \quad \text{on } \partial\Omega.$$

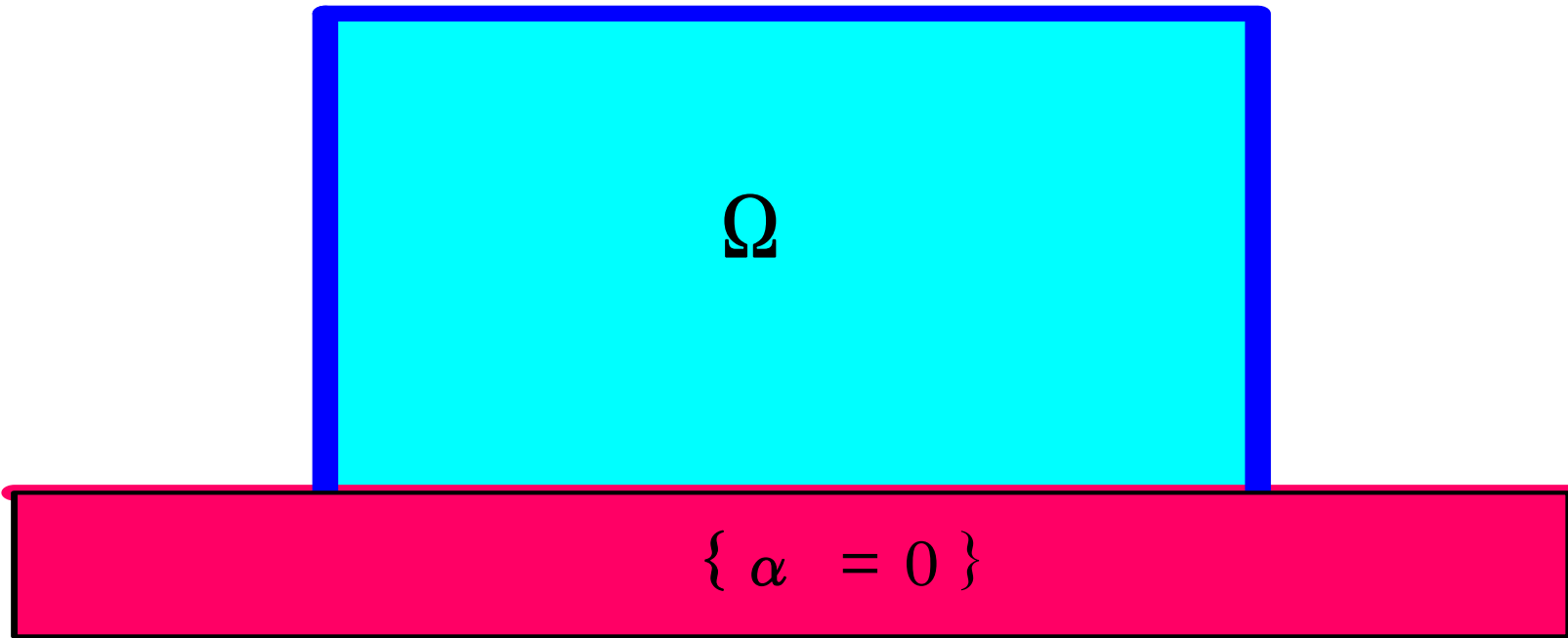


# Domain with Corner Singularities

$$\{ \alpha = 1 \}$$

$\Omega$

$$\{ \alpha = 0 \}$$




## Boundary Condition (1)

$$\alpha(x) = \begin{cases} 1 & \text{on } \partial_N \Omega, \\ 0 & \text{on } \partial_D \Omega \end{cases}$$

## Boundary Condition (2)

$$\alpha(x')(\mathbf{a} \cdot \nabla \mathbf{u} \cdot \mathbf{n}) + (1 - \alpha(x'))\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \times (0, \infty)$$



A diagram showing a horizontal line representing a boundary. The left portion of the line is blue, and the right portion is pink. A vertical dotted red line connects the two segments, indicating a jump or discontinuity at that point. Above the line, a solid black horizontal line is drawn, which is also discontinuous at the jump point, with a vertical dotted red line connecting the two segments.

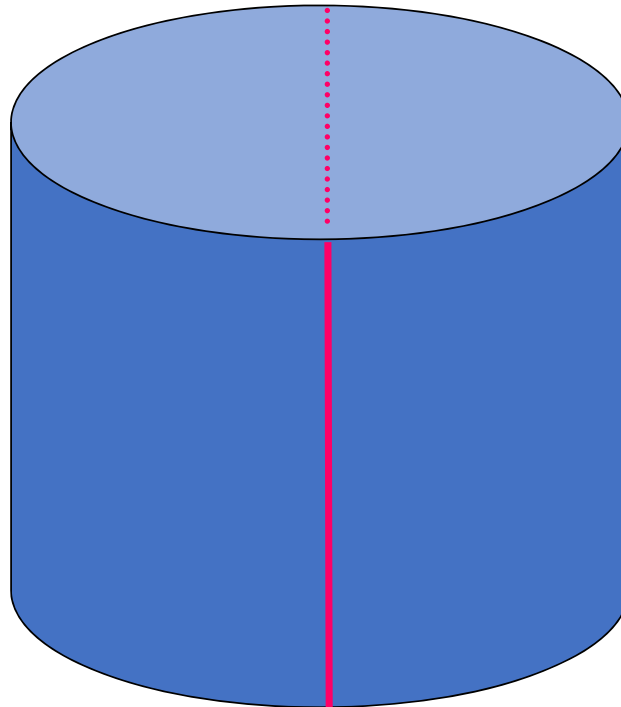
$$\partial_D \Omega = \{\alpha = 0\}$$

$$\partial_N \Omega = \{\alpha = 1\}$$

## Geometry of the Second Problem

$$\Omega \times (0, T)$$

$$\partial_D \Omega \times (0, T)$$



$$\partial_N \Omega \times (0, T)$$

## Fundamental Function Space

$$\mathbf{V}_D(0) = \left\{ \mathbf{u} \in H^1(\Omega, \mathbf{R}^3) : \mathbf{u} = \mathbf{0} \text{ on } \partial_D \Omega \right\}$$

## Definition of a Weak Solution (1)

$$\mathbf{u}_0 \in \mathbf{V}_D(0), \mathbf{u}_1 \in L^2(\Omega, \mathbf{R}^3), \mathbf{f} \in L^2(\Omega \times (0, T), \mathbf{R}^3)$$

A function

$$\mathbf{u} \in H^1(\Omega \times (0, T), \mathbf{R}^3)$$

is called a **weak solution** if it satisfies the conditions

$$\mathbf{u}(0, \bullet) = \mathbf{u}_0,$$

$$\mathbf{u} = \mathbf{0} \text{ on } \partial_D \Omega \times (0, T)$$

## Definition of a Weak Solution(2)

$$\begin{aligned} & -\int_0^T \left( \frac{\partial \mathbf{u}}{\partial t}, \frac{\partial \eta}{\partial t} \right)_{L^2} dt + \int_0^T (\mathbf{a} \cdot \nabla \mathbf{u}, \nabla \eta)_{L^2} dt \\ & = (\mathbf{u}_1, \eta(0, \cdot))_{L^2} + \int_0^T (\mathbf{f}, \eta)_{L^2} dt \end{aligned}$$

$$\forall \eta \in H^1(\Omega \times (0, T), \mathbf{R}^3), \eta = 0 \text{ on } \partial_D \Omega \times (0, T), \eta(T, \cdot) = 0.$$

## Existence and Uniqueness Theorem

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} - \operatorname{div} (\mathbf{a} \cdot \nabla \mathbf{u}) = \mathbf{f} \quad \text{in } \Omega \times (0, T),$$

$$\mathbf{u} \Big|_{t=0} = \mathbf{u}_0 \quad \text{in } \Omega,$$

$$\frac{\partial \mathbf{u}}{\partial t} \Big|_{t=0} = \mathbf{u}_1 \quad \text{in } \Omega,$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial_D \Omega \times (0, T),$$

$$\mathbf{a} \cdot \nabla \mathbf{u} \cdot \mathbf{n} = \mathbf{0} \quad \text{on } \partial_N \Omega \times (0, T)$$

has a **unique weak solution**

$$\mathbf{u} \in H^1(\Omega \times (0, T), \mathbf{R}^3)$$

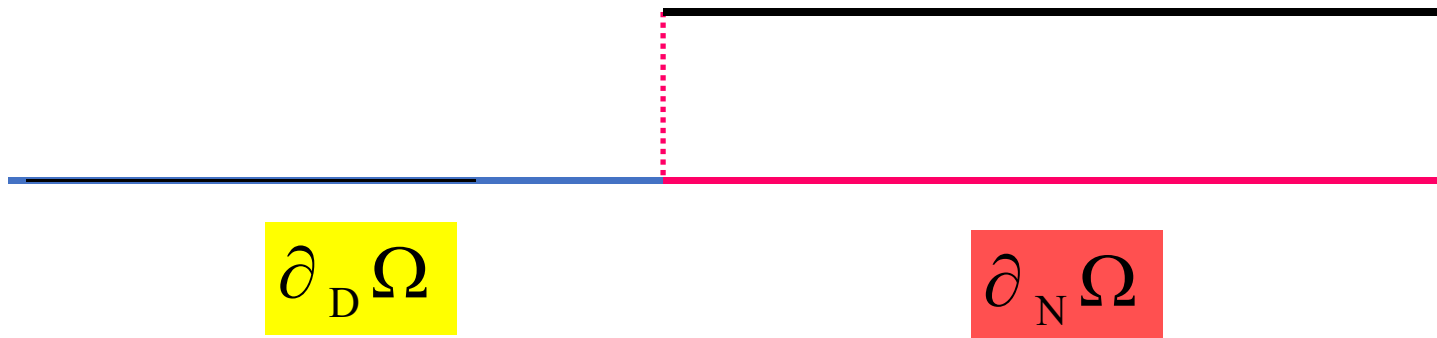


## Energy Estimate for Weak Solutions

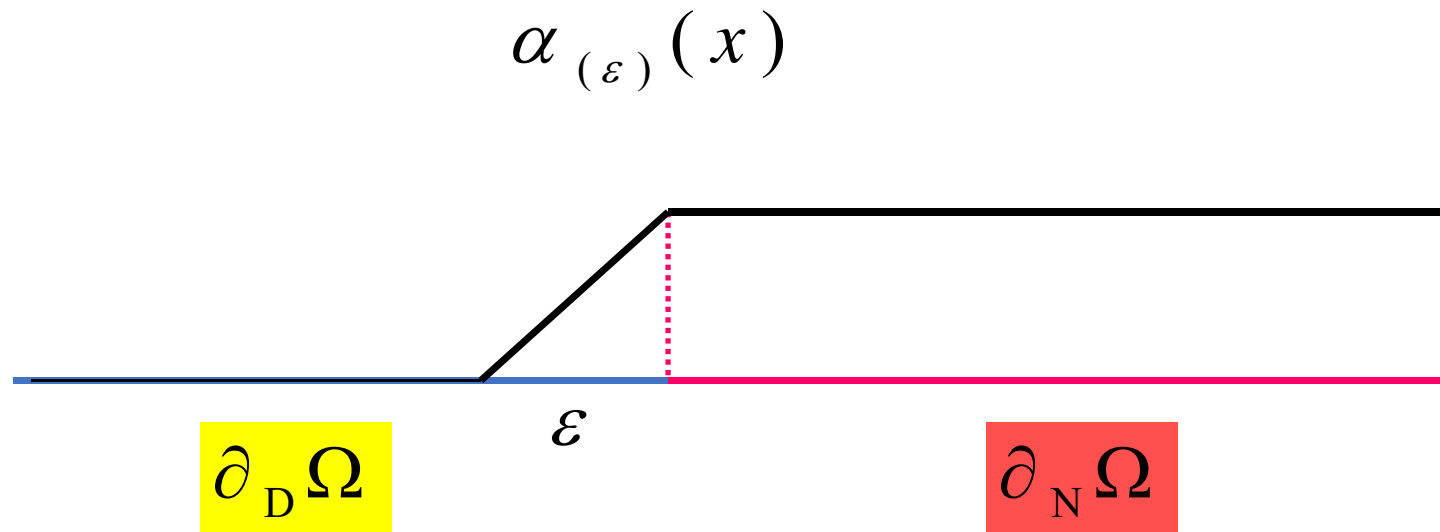
$$\begin{aligned} & \|\mathbf{u}(t)\|_{H^1}^2 + \|\mathbf{u}'(t)\|_{L^2}^2 \\ & \leq \exists C(T) \left( \|\mathbf{u}_0\|_{H^1}^2 + \|\mathbf{u}_1\|_{L^2}^2 + \int_0^t \|\mathbf{f}(s)\|_{L^2}^2 ds \right). \end{aligned}$$

# The Original Boundary Condition

$\alpha(x)$



# Approximations to the Boundary Condition



## Regularized Mixed Problems

We study the following **regularized** mixed problem:

$$\frac{\partial^2 \mathbf{u}_\varepsilon}{\partial t^2} - \operatorname{div}(\mathbf{a} \cdot \nabla \mathbf{u}_\varepsilon) = \mathbf{f}^{(\varepsilon)} \quad \text{in } \Omega \times (0, \infty),$$

$$\mathbf{u}_\varepsilon \Big|_{t=0} = \mathbf{u}_0^{(\varepsilon)} \quad \text{in } \Omega,$$

$$\frac{\partial \mathbf{u}_\varepsilon}{\partial t} \Big|_{t=0} = \mathbf{u}_1^{(\varepsilon)} \quad \text{in } \Omega,$$

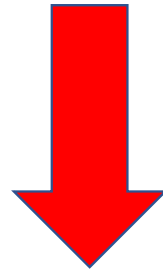
$$\alpha_{(\varepsilon)} (\mathbf{a} \cdot \nabla \mathbf{u}_\varepsilon \cdot \mathbf{n}) + (1 - \alpha_{(\varepsilon)}) \mathbf{u}_\varepsilon = \mathbf{0} \quad \text{on } \partial\Omega \times (0, \infty).$$

## Energy Estimate for Regularized Solutions

$$\forall \varepsilon > 0$$

$$\begin{aligned} & \|\mathbf{u}_\varepsilon(t)\|_{H^1}^2 + \|\mathbf{u}'_\varepsilon(t)\|_{L^2}^2 \\ & \leq \exists C(T) \left( \|\mathbf{u}_0^{(\varepsilon)}\|_{H^1}^2 + \|\mathbf{u}_1^{(\varepsilon)}\|_{L^2}^2 + \int_0^t \|\mathbf{f}^{(\varepsilon)}(s)\|_{L^2}^2 ds \right). \end{aligned}$$

## Energy Estimate for Weak Solutions



$$\varepsilon \downarrow 0$$

$$\begin{aligned} & \|\mathbf{u}(t)\|_{H^1}^2 + \|\mathbf{u}'(t)\|_{L^2}^2 \\ & \leq \exists C(T) \left( \|\mathbf{u}_0\|_{H^1}^2 + \|\mathbf{u}_1\|_{L^2}^2 + \int_0^t \|\mathbf{f}(s)\|_{L^2}^2 ds \right). \end{aligned}$$

END