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THE QUANDLE COLORING INVARIANT OF A REDUCIBLE HANDLEBODY-KNOT

By

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Abstract. A handlebody-knot is a handlebody embedded in the 3-sphere. We provide methods to detect the irreducibility of a handlebody-knot by using the quandle coloring invariant.

1. Introduction

A *genus g handlebody-knot* [2] is a genus g handlebody embedded in the 3-sphere S^3. Two handlebody-knots are *equivalent* if one can be transformed into the other by an isotopy of S^3. A handlebody-knot H is *reducible* if there exists a 2-sphere in S^3 such that the intersection of H and the 2-sphere is an essential disk properly embedded in H. A handlebody-knot is *irreducible* if it is not reducible. In [4], Moriuchi, Suzuki and the authors gave a table of genus two handlebody-knots up to six crossings, and classified them according to the crossing number and the irreducibility. In this paper, we provide methods to detect the irreducibility, which were used in [4].

Let B_1, B_2 be 3-balls in S^3 such that $B_1 \cup B_2 = S^3$ and $B_1 \cap B_2 = \partial B_1 = \partial B_2$. Let H_i be a genus g_i handlebody-knot in B_i for $i = 1, 2$. When $H_1 \cap H_2$ is one disk, $H_1 \cup H_2$ is a genus $g_1 + g_2$ handlebody-knot in S^3. We denote it by $H_1 \# H_2$, where we remark that the handlebody-knot $H_1 \# H_2$ depends only on the handlebody-knots H_1, H_2. If a handlebody-knot H is reducible, then there exist handlebody-knots H_1, H_2 such that $H = H_1 \# H_2$. Since the exterior of H is the boundary connected sum of those of H_1, H_2, the fundamental group of the exterior of H is the free product of those of H_1, H_2. In general, it is not easy to determine whether or not a given group is the free product of two nontrivial groups, although some results for particular groups were known (see, for example, [6, 7]).

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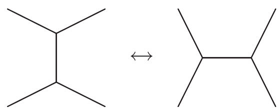

Figure 1: An IH-move

In this paper, we use the quandle coloring invariant defined in [2, 3] to detect the irreducibility. A quandle coloring for knots is a generalization of the Fox coloring. To define a quandle coloring on a handlebody-knot diagram, we need to overcome an obstacle arising from a trivalent vertex. In [2, 3], the quandle coloring invariant and the quandle cocycle invariant are defined for handlebody-knots by introducing the notion of a flow. The flow is the key to the methods we provide in this paper.

A *spatial trivalent graph* is a finite trivalent graph embedded in S^3. Two spatial trivalent graphs are *equivalent* if one can be transformed into the other by an isotopy of S^3. When a handlebody-knot H is a regular neighborhood of a spatial trivalent graph K, we say that H is represented by K. In this paper, a circle is regarded as a trivalent graph. Then any handlebody-knot can be represented by some spatial trivalent graph. Suzuki [10] introduced the notion of the neighborhood equivalence for spatial graphs. The neighborhood equivalence class of a spatial connected trivalent graph is a handlebody-knot. The ∂-irreducibility of the exterior of a spatial graph is investigated in [9, 11], where we note that a handlebody-knot is irreducible if its exterior is ∂-irreducible.

A *diagram* of a handlebody-knot is a diagram of a spatial trivalent graph which represents the handlebody-knot. An *IH-move* is a local spatial move on spatial trivalent graphs as described in Figure 1, where the replacement is applied in a disk embedded in S^3. The following enables us to study handlebody-knots through their diagrams.

Theorem 1 ([2]). *For spatial trivalent graphs K_1 and K_2, the following are equivalent.*

- *K_1 and K_2 represent an equivalent handlebody-knot.*
- *K_1 and K_2 are related by a finite sequence of IH-moves.*
- *Diagrams of K_1 and K_2 are related by a finite sequence of the moves depicted in Figure 2.*

In Section 2, we recall the definition of the quandle coloring invariant for a handlebody-knot. In Section 3, we show some properties of the quandle coloring

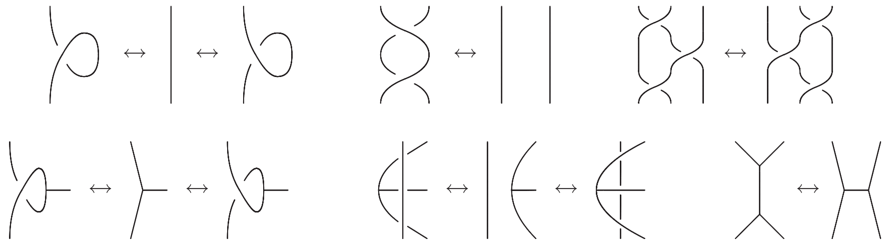

Figure 2

Figure 3

invariant of a reducible handlebody-knot. In Section 4, we show that some handlebody-knots are irreducible by using our results.

2. Quandle Colorings

In this section, we introduce a flow of a handlebody-knot and recall the definition of the quandle coloring invariant defined in [3].

We denote by $G(H)$ the fundamental group of the exterior of a handlebody-knot H. We denote by $\mathrm{Flow}(H; A)$ the set of homomorphisms from $G(H)$ to an abelian group A. We call an element of $\mathrm{Flow}(H; A)$ an *A-flow* of H.

Let D be a diagram of a handlebody-knot H. We denote by $\mathscr{A}(D)$ the set of arcs of D, where an arc is a piece of a curve such that its endpoint is an undercrossing or a vertex. We denote by $\mathcal{O}_\alpha$ the set of orientations of an arc $\alpha \in \mathscr{A}(D)$, which consists of two orientations. We represent an orientation $o \in \mathcal{O}_\alpha$ by a co-orientation, which is obtained by rotating a usual orientation $\pi/2$ counterclockwise on the diagram D. An A-flow φ of H assigns a pair $(o, s) \in \mathcal{O}_\alpha \times A$ to each arc $\alpha \in \mathscr{A}(D)$ up to the equivalence relation $(o, s) \sim (-o, -s)$ like the Wirtinger presentation, where $-o$ is the inverse of an orientation o. Then the assignment satisfies the conditions shown in Figure 3 at every crossing and vertex, where an element of A is represented with an underline. Conversely, an assignment satisfying the conditions gives an A-flow. For $\varphi \in \mathrm{Flow}(H; A)$ and $\alpha \in \mathscr{A}(D)$, we define a map $\varphi_\alpha : \mathcal{O}_\alpha \to A$ so that $(o, \varphi_\alpha(o)) \in \mathcal{O}_\alpha \times A$ is assigned to

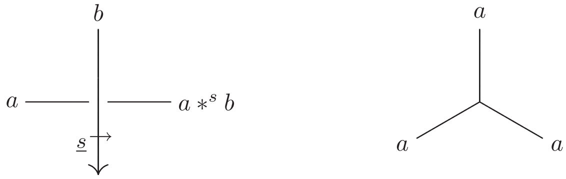

Figure 4

the arc α by the A-flow φ. Then we have $\varphi_\alpha(-o) = -\varphi_\alpha(o)$. We define the A-flow 0 so that 0_α is the zero map for any arc $\alpha \in \mathscr{A}(D)$.

An *A-flowed handlebody-knot* (H, φ) is a pair of a handlebody-knot H and an A-flow $\varphi \in \mathrm{Flow}(H; A)$. Two A-flowed handlebody-knots (H, φ) and (H', φ') are *equivalent* if H and H' are equivalent with an isotopy f of S^3 satisfying $\varphi = \varphi' \circ f_*$, where $f_* : G(H) \to G(H')$ is the isomorphism induced by f. Let D be a diagram of a handlebody-knot H. We denote by (D, φ) the diagram of an A-flowed handlebody-knot (H, φ), which is obtained by adding the information of the A-flow φ on the diagram D.

A *quandle* [5, 8] is a non-empty set X with a binary operation $* : X \times X \to X$ satisfying the following axioms.

- For any $a \in X$, $a * a = a$.
- For any $a \in X$, the map $S_a : X \to X$ defined by $S_a(x) = x * a$ is a bijection.
- For any $a, b, c \in X$, $(a * b) * c = (a * c) * (b * c)$.

The *type* of a quandle X is defined by

$$\text{type } X := \min\{i \in \mathbf{Z}_{>0} \mid S_a^i = \mathrm{id}_X \text{ for any } a \in X\}.$$

We set type $X := \infty$ if we do not have such a positive integer i. We note that the type of a finite quandle is finite. For $i \in \mathbf{Z}$ and $a, b \in X$, we define $a *^i b := S_b^i(a)$. For a quandle X, we set

$$\mathbf{Z}_X := \begin{cases} \mathbf{Z}/(\text{type } X)\mathbf{Z} & \text{if type } X \text{ is finite,} \\ \mathbf{Z} & \text{otherwise.} \end{cases}$$

Then $a *^i b$ is well-defined for $i \in \mathbf{Z}_X$.

Let X be a quandle. Let (D, φ) be a diagram of a $\mathbf{Z}_X$-flowed handlebody-knot. An *X-coloring* of (D, φ) is a map $C : \mathscr{A}(D) \to X$ satisfying the following conditions (Figure 4):

C_1. For a crossing χ, we have

$$C(\chi_1) *^{\varphi_{\chi_0}(o)} C(\chi_0) = C(\chi_2),$$

where χ_1, χ_2 are the under-arcs at the crossing χ such that an orientation o of the over-arc χ_0 at χ points to χ_2.

C_2. For a vertex ω, we have

$$C(\omega_1) = C(\omega_2) = C(\omega_3),$$

where ω_1, ω_2 and ω_3 are the arcs incident to ω.

We note that an X-coloring C does not depend on the choice of the orientation o of each arc, since the equality in the condition C_1 is equivalent to the following equality

$$C(\chi_2) *^{\varphi_{\chi_0}(-o)} C(\chi_0) = C(\chi_1).$$

We denote by $Col_X(D, \varphi)$ the set of all X-colorings of (D, φ), and denote by $\#Col_X(D, \varphi)$ the number of X-colorings in $Col_X(D, \varphi)$.

Theorem 2 ([3]). *Let X be a quandle. Let H be a handlebody-knot represented by a diagram D. For diagrams (D_1, φ) and (D_2, φ) of a $\mathbf{Z}_X$-flowed handlebody-knot (H, φ), there exists a one-to-one correspondence between $Col_X(D_1, \varphi)$ and $Col_X(D_2, \varphi)$. Then $\#Col_X(D, \varphi)$ is an invariant of a $\mathbf{Z}_X$-flowed handlebody-knot (H, φ). Furthermore, $\#Col_X^{\Sigma}(H)$ is an invariant of a handlebody-knot H, where $\#Col_X^{\Sigma}(H)$ is the multiset defined by*

$$\#Col_X^{\Sigma}(H) := \{\#Col_X(D, \varphi) \mid \varphi \in \mathrm{Flow}(H; \mathbf{Z}_X)\}.$$

An X-coloring C is *trivial* if the map C is a constant map. A handlebody-knot is *trivial* if it is equivalent to a handlebody standardly embedded in S^3. If D is a diagram of a trivial handlebody-knot, then any X-coloring $C \in Col_X(D, \varphi)$ is trivial for any $\mathbf{Z}_X$-flow φ. When there exist a $\mathbf{Z}_X$-flow $\varphi \in \mathrm{Flow}(H; \mathbf{Z}_X)$ and a nontrivial X-coloring $C \in Col_X(D, \varphi)$, we say that H has a nontrivial X-coloring. The property "H has a nontrivial X-coloring" is preserved under the moves depicted in Figure 2.

3. Colorings for Reducible Handlebody-knots

In this section, we investigate the quandle coloring invariant of a reducible handlebody-knot.

Let X be a quandle, and $x \in X$. Let D be a diagram of a handlebody-knot H, and $\alpha \in \mathscr{A}(D)$. For $\varphi \in \mathrm{Flow}(H; \mathbf{Z}_X)$, we define

$$Col_X(D, \varphi)_{\alpha}^{x} := \{C \in Col_X(D, \varphi) \mid C(\alpha) = x\}.$$

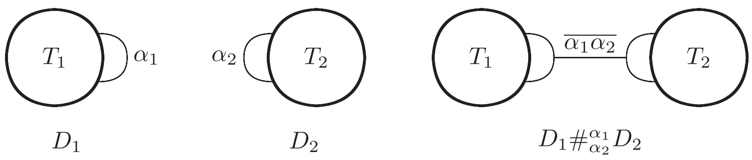

Figure 5

Then we have $\#Col_X(D,\varphi)^x_\alpha = 1$ if $\varphi = 0$ or if H is a trivial handlebody-knot, where $\#S$ denotes the number of elements in a set S. The subgroup of $\mathrm{Aut}(X)$ generated by $\{S_a \mid a \in X\}$ acts on a quandle X in a natural way. If the action is transitive, then X is said to be *connected*.

Lemma 3. *Let X be a connected quandle. Let D be a diagram of a handlebody-knot H, and $\alpha \in \mathscr{A}(D)$. For $\varphi \in \mathrm{Flow}(H;\mathbf{Z}_X)$ and $x, y \in X$, we have*

$$\#Col_X(D,\varphi)^x_\alpha = \#Col_X(D,\varphi)^y_\alpha.$$

Proof. Since X is a connected quandle, there exist $a_1,\dots,a_n \in X$ such that $(((x * a_1) * a_2) * \cdots) * a_n = y$. Set $x_1 := x$ and $x_{i+1} := x_i * a_i$ for $i \in \{1,\dots,n\}$. Then $x_{n+1} = y$. We define $f_i : Col_X(D,\varphi)^{x_i}_\alpha \to Col_X(D,\varphi)^{x_{i+1}}_\alpha$ by the equality $(f_i(C))(\beta) = C(\beta) * a_i$ for $\beta \in \mathscr{A}(D)$. The map f_i is well-defined, since we have $(a *^s b) *^t c = (a *^t c) *^s (b *^t c)$ for $a, b, c \in X$ and $s, t \in \mathbf{Z}_X$. Since the map $S_{a_i} : X \to X$ is bijective, the map $f_i : Col_X(D,\varphi)^{x_i}_\alpha \to Col_X(D,\varphi)^{x_{i+1}}_\alpha$ is injective, which implies $\#Col_X(D,\varphi)^{x_i}_\alpha \le \#Col_X(D,\varphi)^{x_{i+1}}_\alpha$. Thus we have $\#Col_X(D,\varphi)^x_\alpha \le \#Col_X(D,\varphi)^y_\alpha$. In the same way, we have $\#Col_X(D,\varphi)^y_\alpha \le \#Col_X(D,\varphi)^x_\alpha$. Therefore $\#Col_X(D,\varphi)^x_\alpha = \#Col_X(D,\varphi)^y_\alpha$. □

Let X be a quandle. Let D_i be a diagram of a handlebody-knot H_i, and α_i one of the outermost arcs of D_i for $i = 1, 2$. We denote by $D_1\#^{\alpha_1}_{\alpha_2}D_2$ the diagram obtained from D_1 and D_2 by attaching an arc $\overline{\alpha_1\alpha_2}$ between α_1 and α_2 as shown in Figure 5. Then $D_1\#^{\alpha_1}_{\alpha_2}D_2$ represents the reducible handlebody-knot $H_1\#H_2$. For $\varphi_1 \in \mathrm{Flow}(H_1;\mathbf{Z}_X)$ and $\varphi_2 \in \mathrm{Flow}(H_2;\mathbf{Z}_X)$, we define the $\mathbf{Z}_X$-flow $\varphi_1\#\varphi_2$ of $H_1\#H_2$ by the equalities

$$(\varphi_1\#\varphi_2)_{\overline{\alpha_1\alpha_2}} = 0 \quad \text{and} \quad (\varphi_1\#\varphi_2)_\alpha = (\varphi_i)_\alpha$$

for $\alpha \in \mathscr{A}(D_i)$ and $i \in \{1, 2\}$. Then we have the bijection

$$\mathrm{Flow}(H_1;\mathbf{Z}_X) \times \mathrm{Flow}(H_2;\mathbf{Z}_X) \to \mathrm{Flow}(H_1\#H_2;\mathbf{Z}_X)$$

which sends (φ_1, φ_2) to $\varphi_1 \# \varphi_2$. Let $x \in X$. For $C_1 \in Col_X(D_1, \varphi_1)^x_{\alpha_1}$ and $C_2 \in Col_X(D_2, \varphi_2)^x_{\alpha_2}$, we define the X-coloring $C_1 \# C_2$ of $(D_1 \#^{\alpha_1}_{\alpha_2} D_2, \varphi_1 \# \varphi_2)$ by the equalities

$$(C_1 \# C_2)(\overline{\alpha_1 \alpha_2}) = x \quad \text{and} \quad (C_1 \# C_2)(\alpha) = C_i(\alpha)$$

for $\alpha \in \mathscr{A}(D_i)$ and $i \in \{1, 2\}$. Then we have the bijection

$$Col_X(D_1, \varphi_1)^x_{\alpha_1} \times Col_X(D_2, \varphi_2)^x_{\alpha_2} \to Col_X(D_1 \# D_2, \varphi_1 \# \varphi_2)^x_{\overline{\alpha_1 \alpha_2}}$$

which sends (C_1, C_2) to $C_1 \# C_2$. By the equalities

$$\begin{aligned} \#Col_X^{\Sigma}(H_1 \# H_2) &= \{\#Col_X(D_1 \#^{\alpha_1}_{\alpha_2} D_2, \varphi) \mid \varphi \in \mathrm{Flow}(H_1 \# H_2; \mathbf{Z}_X)\} \\ &= \left\{ \sum_{x \in X} \#Col_X(D_1 \#^{\alpha_1}_{\alpha_2} D_2, \varphi)^x_{\overline{\alpha_1 \alpha_2}} \,\middle|\, \varphi \in \mathrm{Flow}(H_1 \# H_2; \mathbf{Z}_X) \right\} \\ &= \left\{ \sum_{x \in X} \#Col_X(D_1, \varphi_1)^x_{\alpha_1} \#Col_X(D_2, \varphi_2)^x_{\alpha_2} \,\middle|\, \varphi_i \in \mathrm{Flow}(H_i; \mathbf{Z}_X) \right\}, \end{aligned}$$

we have the following lemma.

Lemma 4. *Let X be a quandle. Let D_i be a diagram of a handlebody-knot H_i, and α_i one of the outermost arcs of D_i for $i = 1, 2$. We have*

$$\#Col_X^{\Sigma}(H_1 \# H_2) = \left\{ \sum_{x \in X} \prod_{i=1}^{2} \#Col_X(D_i, \varphi_i)^x_{\alpha_i} \,\middle|\, \varphi_i \in \mathrm{Flow}(H_i; \mathbf{Z}_X) \right\}.$$

Theorem 5. *Let $H = H_1 \# H_2$ be a reducible handlebody-knot, where H_i is a genus g_i handlebody-knot. Let X be a finite quandle. Put $n_i := (\text{type } X)^{g_i}$ for $i = 1, 2$.*

(1) *If X is a connected quandle, then we have*

$$\#Col_X^{\Sigma}(H) = \{a \cdot b \cdot \#X \mid a \in A_1, b \in A_2\}$$

for some multisets A_1, A_2 such that A_i consists of n_i positive integers including 1 for $i = 1, 2$.

(2) *If H_1 is a trivial handlebody-knot, then the multiplicity of every element of the multiset $\#Col_X^{\Sigma}(H)$ is divisible by n_1.*

Proof. Let D_i be a diagram of the handlebody-knot H_i, and α_i one of the outermost arcs of D_i for $i = 1, 2$. Then $D_1 \#^{\alpha_1}_{\alpha_2} D_2$ is a diagram of the handlebody-knot H. We note that $\#\mathrm{Flow}(H_i; \mathbf{Z}_X) = (\#\mathbf{Z}_X)^{g_i} = (\text{type } X)^{g_i} = n_i$ for $i = 1, 2$.

(1) Fix $x_0 \in X$. Put $A_i := \{\#Col_X(D_i, \varphi_i)_{\alpha_i}^{x_0} \mid \varphi_i \in \mathrm{Flow}(H_i; \mathbf{Z}_X)\}$ for $i = 1, 2$. By the equality $\#Col_X(D_i, 0)_{\alpha_i}^{x_0} = 1$, we have $1 \in A_i$ for $i = 1, 2$. By Lemmas 3 and 4, we have

$$\begin{aligned}
\#Col_X^{\Sigma}(H) &= \left\{ \sum_{x \in X} \prod_{i=1}^{2} \#Col_X(D_i, \varphi_i)_{\alpha_i}^{x} \,\middle|\, \varphi_i \in \mathrm{Flow}(H_i; \mathbf{Z}_X) \right\} \\
&= \left\{ \sum_{x \in X} \prod_{i=1}^{2} \#Col_X(D_i, \varphi_i)_{\alpha_i}^{x_0} \,\middle|\, \varphi_i \in \mathrm{Flow}(H_i; \mathbf{Z}_X) \right\} \\
&= \left\{ \sum_{x \in X} a \cdot b \,\middle|\, a \in A_1, b \in A_2 \right\} \\
&= \{a \cdot b \cdot \#X \mid a \in A_1, b \in A_2\}.
\end{aligned}$$

(2) Since H_1 is a trivial handlebody-knot, we have $\#Col_X(D_1, \varphi_1)_{\alpha_1}^{x} = 1$ for any $\varphi_1 \in \mathrm{Flow}(H_1; \mathbf{Z}_X)$ and $x \in X$. By Lemma 4, we have

$$\begin{aligned}
\#Col_X^{\Sigma}(H) &= \left\{ \sum_{x \in X} \prod_{i=1}^{2} \#Col_X(D_i, \varphi_i)_{\alpha_i}^{x} \,\middle|\, \varphi_i \in \mathrm{Flow}(H_i; \mathbf{Z}_X) \right\} \\
&= \left\{ \sum_{x \in X} \#Col_X(D_2, \varphi_2)_{\alpha_2}^{x} \,\middle|\, \begin{array}{l} \varphi_1 \in \mathrm{Flow}(H_1; \mathbf{Z}_X), \\ \varphi_2 \in \mathrm{Flow}(H_2; \mathbf{Z}_X) \end{array} \right\} \\
&= \left\{ \#Col_X(D_2, \varphi_2) \,\middle|\, \begin{array}{l} \varphi_1 \in \mathrm{Flow}(H_1; \mathbf{Z}_X), \\ \varphi_2 \in \mathrm{Flow}(H_2; \mathbf{Z}_X) \end{array} \right\}.
\end{aligned}$$

Hence the multiplicity of every element of the multiset $\#Col_X^{\Sigma}(H)$ is divisible by n_1. □

The *tunnel number* $t(H)$ of a handlebody-knot H is the minimum number of mutually disjoint 1-handles embedded in the exterior of H such that a trivial handlebody-knot is obtained from H by attaching the 1-handles. The tunnel number $t(H)$ of a genus g handlebody-knot H coincides with the Heegaard genus of the exterior of H minus g. Let $H = H_1 \# H_2$ be a reducible genus g handlebody-knot, where H_i is a genus g_i handlebody-knot for $i = 1, 2$. Then $g = g_1 + g_2$. The exterior of the handlebody-knot H is the boundary connected sum of those of H_1, H_2. Since the Heegaard genus is additive under the boundary connected sum [1], we have $t(H) = t(H_1) + t(H_2)$.

Proposition 6. *Let H be a reducible handlebody-knot with $t(H) = 1$. Let X be a finite quandle. Then the multiplicity of every element of the multiset $\#Col_X^{\Sigma}(H)$ is divisible by* type X.

Proof. Let H_1, H_2 be handlebody-knots such that $H = H_1 \# H_2$. By the equalities $t(H_1) + t(H_2) = t(H) = 1$, we have $t(H_1) = 0$ or $t(H_2) = 0$, which implies that H_1 or H_2 is a trivial genus g handlebody-knot. Since g is a positive integer, by Theorem 5 (2), the multiplicity of every element of the multiset $\#Col_X^{\Sigma}(H)$ is divisible by type X. □

Lemma 7. *Let $H = H_1 \# H_2$ be a reducible handlebody-knot. Let X be a quandle. If H_1 has a nontrivial X-coloring, then H has a nontrivial X-coloring.*

Proof. Let D_i be a diagram of the handlebody-knot H_i, and α_i one of the outermost arcs of D_i for $i = 1, 2$. Then $D_1 \#_{\alpha_2}^{\alpha_1} D_2$ is a diagram of the handlebody-knot H. Since H_1 has a nontrivial X-coloring, there exists a nontrivial X-coloring $C_1 \in Col_X(D_1, \varphi_1)$ for some $\varphi_1 \in \mathrm{Flow}(H_1; \mathbf{Z}_X)$. Let $C_2 \in Col_X(D_2, 0)$ be the trivial X-coloring defined by $C_2(\alpha_2) = C_1(\alpha_1)$. Since $C_1 \# C_2$ is a nontrivial X-coloring of $(D_1 \#_{\alpha_2}^{\alpha_1} D_2, \varphi_1 \# 0)$, H has a nontrivial X-coloring. □

Theorem 8. *Let $H = H_1 \# H_2$ be a reducible handlebody-knot. If H_1 is a nontrivial genus one handlebody-knot, then there exists a quandle X such that H has a nontrivial X-coloring.*

Proof. Let K be the nontrivial knot which represents the nontrivial genus one handlebody-knot H_1. Let X be the fundamental quandle of K. Since every nontrivial knot has a nontrivial coloring by its fundamental quandle [5, 8], H_1 has a nontrivial X-coloring. Hence, by Lemma 7, H has a nontrivial X-coloring. □

If $H = H_1 \# H_2$ is a nontrivial reducible genus two handlebody-knot, then H_1 or H_2 is a nontrivial genus one handlebody-knot. Hence, by Theorem 8, we have the following corollary.

Corollary 9. *Let H be a nontrivial reducible genus two handlebody-knot. Then there exists a quandle X such that H has a nontrivial X-coloring.*

4. Applications

In this section, we show that some handlebody-knots are irreducible by using the results in the previous section.

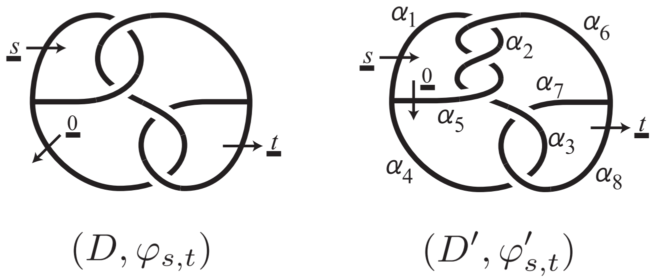

Figure 6

Example 10. Let R_p be the dihedral quandle, which is the quandle consisting of the set $\mathbf{Z}/p\mathbf{Z}$ with the binary operation defined by $a * b = 2b - a$. The type of the dihedral quandle R_p is 2. Let $(H, \varphi_{s,t})$ be the $\mathbf{Z}/2\mathbf{Z}$-flowed handlebody-knot represented by the diagram $(D, \varphi_{s,t})$ depicted in Figure 6. Then we have

$$\begin{aligned}\#Col_{R_3}^{\Sigma}(H) &= \{\#Col_{R_3}(D, \varphi) \mid \varphi \in \mathrm{Flow}(H; \mathbf{Z}/2\mathbf{Z})\} \\ &= \{\#Col_{R_3}(D, \varphi_{s,t}) \mid s, t \in \mathbf{Z}/2\mathbf{Z}\} \\ &= \{9, 3, 3, 3\},\end{aligned}$$

where we note that

$$\#Col_{R_3}(D, \varphi_{s,t}) = \begin{cases} 9 & \text{if } s = t = 1, \\ 3 & \text{otherwise.} \end{cases}$$

By Theorem 5 (1), if H is reducible, then $\#Col_{R_3}^{\Sigma}(H) = \{3, 3a, 3b, 3ab\}$ for some positive integers a, b. Thus the handlebody-knot H is irreducible. Proposition 6 also implies that H is irreducible, since the tunnel number of the handlebody-knot H is 1.

Example 11. Let X be a quandle. Let $(H', \varphi'_{s,t})$ be the $\mathbf{Z}_X$-flowed handlebody-knot represented by the diagram $(D', \varphi'_{s,t})$ depicted in Figure 6, where $\alpha_1, \ldots, \alpha_8$ indicate the arcs of D'. For an X-coloring C of $(D', \varphi'_{s,t})$, we set $c_i := C(\alpha_i) \in X$ for $i = 1, \ldots, 8$. From the coloring conditions for $(D', \varphi'_{s,t})$, we have the following equalities.

$$\begin{aligned}&c_1 *^0 c_6 = c_2, \quad c_5 *^s c_2 = c_6, \quad c_2 *^0 c_5 = c_3, \quad c_3 *^t c_8 = c_4, \\ &c_7 *^s c_3 = c_8, \quad c_1 = c_4 = c_5, \quad c_6 = c_7 = c_8.\end{aligned}$$

By the first and third equalities, we have $c_1 = c_2 = c_3$. Then, by the second equality, we have $c_1 = c_6$. Thus we have $c_1 = \cdots = c_8$, which implies that H' has no nontrivial X-coloring. Therefore, by Corollary 9, H' is irreducible.

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