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Paper

On the embedding constant of the Sobolev type inequality for fractional derivatives

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Abstract: This paper is concerned with the embedding constant of the Sobolev type inequality for fractional derivatives on $\Omega \subset \mathbb{R}^N$ $(N \in \mathbb{N})$. The constant is explicitly described using the analytic semigroup over $L^2(\Omega)$ generated by the Laplace operator. Some numerical examples of estimating the embedding constant are also provided.

Key Words: embedding constant, Sobolev inequality, fractional derivative, analytic semigroup

1. Introduction

Let Ω be a domain in $\mathbb{R}^N$ $(N \in \mathbb{N})$. For $p \geq 1$, we denote the usual Lebesgue space by

$$L^p(\Omega) := \begin{cases} \{f : \Omega \to \mathbb{R} \mid \int_\Omega |f(x)|^p dx < \infty\} & (1 \leq p < \infty), \\ \{f : \Omega \to \mathbb{R} \mid \operatorname{ess\,sup}_{x\in\Omega} |f(x)| < \infty\} & (p = \infty) \end{cases}$$

with the norm

$$\|f\|_{L^p(\Omega)} := \begin{cases} \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}} & (1 \leq p < \infty), \\ \operatorname{ess\,sup}_{x\in\Omega} |f(x)| & (p = \infty), \end{cases}$$

respectively. Let a function space

$$H_0^1(\Omega) := \left\{u \in L^2(\Omega) \mid \nabla u \in (L^2(\Omega))^2 \text{ and } u = 0 \text{ on } x \in \partial\Omega \text{ in the trace sense.}\right\},$$

where the L^2 inner product is denoted by $(u, v)_{L^2(\Omega)} = \int_\Omega u(x)v(x)dx$. Let $A : \mathcal{D}(A) \to L^2(\Omega)$ be an operator defined by

$$(Au, v)_{L^2(\Omega)} = (\nabla u, \nabla v)_{L^2(\Omega)}, \ \forall v \in H_0^1(\Omega), \tag{1}$$

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where $\mathcal{D}(A) := \{u \in H_0^1(\Omega) \mid Au \in L^2(\Omega)\}$ denotes the domain of A. For $i \in \mathbb{N}$, let λ_i be an eigenvalue[1] of A satisfying $0 < \lambda_1 \leq \lambda_2 \leq \cdots$. A function $\psi_i \in \mathcal{D}(A)$ denotes an eigenfunction of A corresponding to λ_i satisfying $(\psi_i, \psi_j)_{L^2(\Omega)} = \delta_{i,j}$, where $\delta_{i,j}$ is Kronecker's delta[2]. For $u \in L^2(\Omega)$, we express $u = \sum_{j=1}^{\infty} c_j \psi_j$ using the spectral decomposition, where $c_i = (u, \psi_i)_{L^2(\Omega)}$. Then, since $A : \mathcal{D}(A) \to L^2(\Omega)$ is a positive definite and self-adjoint operator, the fractional power of A is defined by

$$A^\alpha u = \sum_{i=1}^{\infty} \lambda_i^\alpha c_i \psi_i \in L^2(\Omega) \tag{2}$$

for $0 \leq \alpha \leq 1$, where $\mathcal{D}(A^\alpha) = \{u = \sum_{i=1}^{\infty} c_i \psi_i \in L^2(\Omega) \mid \sum_{i=1}^{\infty} \lambda_i^{2\alpha} c_i^2 < \infty\}$ denotes the domain of A^α. Let us define a function space[3] X_α as $X_\alpha = \mathcal{D}(A^\alpha)$ endowed with the norm $\|u\|_{X_\alpha} = \|A^\alpha u\|_{L^2(\Omega)}$. We note that $X_0 = L^2(\Omega)$ and $X_1 = \mathcal{D}(A)$. For two function spaces Y and Z satisfying $Y \subset Z$ endowed with the norm $\|\cdot\|_Y$ and $\|\cdot\|_Z$, the constant $C > 0$ is referred to as the embedding constant from Y to Z if the following inequality holds:

$$\sup_{u \in Y \setminus \{0\}} \frac{\|u\|_Z}{\|u\|_Y} \leq C < \infty. \tag{3}$$

Note that C is independent of all functions in Y. Furthermore, the value of $\sup_{u \in Y \setminus \{0\}} \|u\|_Z / \|u\|_Y$ is referred to as the best constant of C.

The main aim of this paper is to obtain the embedding constant $C_{p,\alpha}$ from X_α to $L^p(\Omega)$ such that

$$\|u\|_{L^p(\Omega)} \leq C_{p,\alpha} \|u\|_{X_\alpha}, \ \forall u \in X_\alpha \tag{4}$$

for $\alpha > N(1/2 - 1/p)/2$.

The inequality (4) is known as a Sobolev type inequality for fractional derivatives. The existence[4] of the embedding constant for (4) has been studied and applied to a branch of partial differential equations ([2–7], etc.). The embedding constant can be used in many different ways to show the existence of solutions to partial differential equations. For example, the explicit value of the embedding constant from $H_0^1(\Omega)$ to $L^p(\Omega)$ plays an essential role in numerical verification of the existence of solutions to partial differential equations [8, 9].

If we consider (4) in $\mathbb{R}^N$, the best constant of the embedding constant from X_α to $L^p(\mathbb{R}^N)$ has been shown. The best constant for $p = 2N/(N-2)$, $\alpha = 1/2$, and $N \geq 3$ was given by Aubin [10] and Talenti [11]. Later, the best constant for $p = 2N/(N - 4\alpha)$ and $0 < \alpha < N/4$ was also obtained by Lieb [12].

Some embedding constants were obtained for the inequality (4) on the bounded domain, for example, Nakao and Yamamoto [8] derived the embedding constant for $p \in (2, \infty)$ and $\alpha = 1/2$ using the best constant given by [10, 11]. Xiao and Zhai [13] provided a formula for the embedding constant for $2 \leq p < \infty$ and $\alpha = N/4$ imposing some assumptions on the function $u \in X_\alpha$ by using the Riesz kernel and the classical Lorentz space.

For a bounded or unbounded domain with a Lipschitz boundary, Plum [9] has proposed a formula that also provides the embedding constant for $\alpha = 1/2$. The details of these embedding constants are sketched in Section 2.

In this paper, we investigate the embedding constant on a bounded domain for $\alpha > N(1/2-1/p)/2$ using two lemmas, which are presented as in Lemma 1 and Lemma 2, with respect to the analytic semigroup[5] e^{-tA} over $L^2(\Omega)$ generated by $-A$. Our main theorem provides a formula for obtaining the embedding constant.

[1] As the inverse of the operator A is a compact and self-adjoint operator, the spectral theorem shows that the operator A has positive discrete spectrum (cf. [1]).

[2] Namely, λ_i and ψ_i satisfy $(\nabla\psi_i, \nabla v)_{L^2(\Omega)} = \lambda_i(\psi_i, v)_{L^2(\Omega)}, \forall v \in H_0^1(\Omega)$.

[3] The operator A^α is a closed and invertible operator. The closeness of A^α implies that X_α endowed with the graph norm: $\|u\|_{L^2(\Omega)} + \|u\|_{X_\alpha}$ is a Banach space. Because A^α is invertible, the graph norm is equivalent to the norm $\|u\|_{X_\alpha}$ (c.f. [4]).

[4] The existence of $C_{p,\alpha}$ for $\alpha > N(1/2 - 1/p)/2$ has been shown (e.g., [4]).

[5] We note that the operator $-A$ generates the analytic semigroup e^{-tA} (c.f. [4]).

Theorem 1. *Let $\Omega \subset \mathbb{R}^N$ $(N \in \mathbb{N})$ be a bounded domain. The minimum eigenvalue of A is denoted by $\lambda_{\min}$. For $2 < p \leq \infty$, let r and α be real values such that $1/r = 1/2 - 1/p$ and $N/(2r) < \alpha \leq 1$, where $1/p = 0$ if $p = \infty$. Then,*

$$\|u\|_{L^p(\Omega)} \leq C_{p,\alpha}\|u\|_{X_\alpha}, \forall u \in \mathcal{D}(A^\alpha), \tag{5}$$

holds for

$$C_{p,\alpha} = \frac{\alpha^\alpha \Gamma\left(\alpha - \frac{N}{2r}\right)}{(4\pi)^{\frac{N}{2r}}\left(\frac{N}{2r}\right)^{\frac{N}{2r}}\left(\alpha - \frac{N}{2r}\right)^{\alpha - \frac{N}{2r}}\Gamma(\alpha)}\lambda_{\min}^{-\left(\alpha - \frac{N}{2r}\right)}, \tag{6}$$

where $\Gamma(x)$ is the Gamma function defined by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ for $x > 0$. Furthermore, if $p = 2$ and $0 \leq \alpha \leq 1$, the inequality (5) also holds for

$$C_{p,\alpha} = \lambda_{\min}^{-\alpha}. \tag{7}$$

Moreover, the following corollary is obtained by combining the best constant described in [12]:

Corollary 1. *Let $\Omega \subset \mathbb{R}^N$ $(N \in \mathbb{N})$ be a bounded domain. The minimum eigenvalue of A is denoted by $\lambda_{\min}$. For $2 < p < \infty$, let r and α be real values such that $1/r = 1/2 - 1/p$ and $N/(2r) < \alpha \leq 1$. We impose $u = 0$ on $\partial\Omega$ in the trace sense on $D(A^\alpha)$. Then,*

$$\|u\|_{L^p(\Omega)} \leq \tilde{C}_{p,\alpha}\|u\|_{X_\alpha}, \ \forall u \in \mathcal{D}(A^\alpha),$$

holds for

$$\tilde{C}_{p,\alpha} = \frac{\Gamma\left(\frac{N}{p}\right)^{\frac{1}{2}}\Gamma(N)^{\frac{1}{r}}}{(4\pi)^{\frac{N}{2r}}\Gamma\left(\frac{N(p-1)}{p}\right)^{\frac{1}{2}}\Gamma\left(\frac{N}{2}\right)^{\frac{1}{r}}}\lambda_{\min}^{-\left(\alpha - \frac{N}{2r}\right)},$$

where $\Gamma(x)$ is the Gamma function defined by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ for $x > 0$.

This paper is organized as follows: we provide some results of previous studies about embedding constants in Section 2. We prove Theorem 1 and Corollary 1 in Section 3. We present numerical examples of estimating the embedding constants including some results of previous studies in Section 4.

2. Some previous studies related to $C_{p,\alpha}$

Here we briefly describe previous studies of embedding constants from X_α to $L^p(\Omega)$. We note that $X_{1/2} = H_0^1(\Omega)$ and $\|u\|_{X_{1/2}} = \sqrt{(\nabla u, \nabla u)_{L^2(\Omega)}}$ hold (cf. [14]). If the domain is $\mathbb{R}^N$ $(N \in \mathbb{N})$, the best constant of the Sobolev type inequality was that given by Aubin [10] and Talenti [11]. They independently derived the following estimate:

Theorem 2 ([10, 11])**.** *For $N \geq 2$, let q be a real number satisfying $1 < q < N$. Let $p = Nq/(N-q)$. For any point $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$, we define $|x|_2 := \sqrt{|x_1|^2 + \cdots + |x_N|^2}$. Then,*

$$\left(\int_{\mathbb{R}^N} |u(x)|^p dx\right)^{\frac{1}{p}} \leq T_p \left(\int_{\mathbb{R}^N} |\nabla u(x)|_2^q dx\right)^{\frac{1}{q}} \tag{8}$$

holds for

$$T_p = \pi^{-\frac{1}{2}} N^{-\frac{1}{q}} \left(\frac{q-1}{N-q}\right)^{1-\frac{1}{q}} \left(\frac{\Gamma(1+\frac{N}{2})\Gamma(N)}{\Gamma(\frac{N}{q})\Gamma(1+N-\frac{N}{q})}\right)^{\frac{1}{N}} \tag{9}$$

and T_p is the best constant of (8)*.*

Lieb [12] also obtained the best constant as follows:

Theorem 3 ([12])**.** *For $N \in \mathbb{N}$, let α be a real number satisfying $0 < \alpha < N/4$. Then,*

$$\|u\|_{L^{\frac{2N}{N-4\alpha}}(\mathbb{R}^N)} \leq E_\alpha \|A^\alpha u\|_{L^2(\mathbb{R}^N)}, \ \forall u \in \mathcal{D}(A^\alpha) \tag{10}$$

holds for

$$E_\alpha = 2^{-2\alpha}\pi^{-\alpha}\sqrt{\frac{\Gamma(\frac{N-4\alpha}{2})}{\Gamma(\frac{N+4\alpha}{2})}}\left(\frac{\Gamma(N)}{\Gamma(N/2)}\right)^{\frac{2\alpha}{N}} \tag{11}$$

and E_α is the best constant of (10).

If Ω is any bounded domain, some embedding constants were obtained. By using a zero-extension and Theorem 2, the embedding constant from $X_{1/2}$ $(= H_0^1(\Omega))$ to $L^p(\Omega)$ can be given as follows:

Theorem 4 ([8, 15])**.** *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain. Let p be a real number such that $p \in (N/(N-1), 2N/(N-2))$ if $N \geq 3$ and $p \in (2, \infty)$ if $N = 2$. Moreover, let $q = Np/(N+p)$. Then,*

$$\|u\|_{L^p(\Omega)} \leq M_p \left\|A^{\frac{1}{2}} u\right\|_{L^2(\Omega)}, \ \forall u \in \mathcal{D}(A^{\frac{1}{2}})$$

holds for

$$M_p = |\Omega|^{\frac{2-q}{2q}} T_p,$$

where $|\Omega|$ is the measure of Ω and T_p is a constant in (9).

Xiao and Zhai [13] obtained the following embedding constant:

Theorem 5 ([13])**.** *Let $\Omega \subset \mathbb{R}^N$ $(N \in \mathbb{N})$ be a bounded domain and $2 \leq p < \infty$. Let real numbers r and γ satisfying $1/r = 1/2 + 1/p$ and $(1-\gamma)/r = 1/2$. For $u \in \mathcal{D}(A^{N/4})$ satisfying* $\operatorname{supp}(A^{N/4}u) \subset \Omega$,

$$\|u\|_{L^p(\Omega)} \leq J_p \left\|A^{\frac{N}{4}} u\right\|_{L^2(\Omega)} \tag{12}$$

holds for

$$J_p = \frac{N^{\frac{\gamma-1}{r}} \omega_{N-1}^{\frac{1-\gamma}{r}} |\Omega|^{\frac{\gamma}{r}}}{2^{\frac{N}{2}} \pi^{\frac{N}{2}} \gamma^{\frac{1}{r}}}, \tag{13}$$

where ω_{N-1} is the surface area of the unit sphere in $\mathbb{R}^N$.

Remark 1. *Theorem 5 is obtained by substituting $q = p$ and $p = 2$ into (2) of Theorem 2.1 in [13].*

For cases in which the bounded or unbounded domain Ω have a Lipschitz boundary, Plum [9] provided the embedding constant using the minimum eigenvalue of A.

Theorem 6 ([9])**.** *Let $\lambda_{\min}$ be the minimum eigenvalue of the Laplace operator for $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ with a Lipschitz boundary. Specify $p \in [2, 2N/(N-2)))$ and $s = N(1/p - 1/2 + 1/N)$, where $2N/(N-2) = \infty$ if $N = 2$. Then,*

$$\|u\|_{L^p(\Omega)} \leq L_p \left\|A^{\frac{1}{2}} u\right\|_{L^2(\Omega)}, \ \forall u \in \mathcal{D}(A^{\frac{1}{2}}) \tag{14}$$

holds for

$$L_p = \begin{cases} \left(\frac{1}{2}\right)^{\frac{1}{2}+\frac{2\nu-3}{p}} \left[\frac{p}{2}\left(\frac{p}{2}-1\right)\cdots\left(\frac{p}{2}-\nu+2\right)\right]^{\frac{2}{p}} \lambda_{\min}^{-\frac{1}{p}} & (N=2), \\ \left(\frac{N-1}{\sqrt{N}(N-2)}\right)^{1-s} \lambda_{\min}^{-\frac{s}{2}} & (N \geq 3), \end{cases} \tag{15}$$

respectively, where ν is the maximum integer such that $\nu \leq p/2$ and the term in brackets is 1 if $\nu = 1$ and $N = 2$.

3. Proofs of Theorem 1 and Corollary 1

We introduce two fundamental lemmas in order to prove Theorem 1 and Corollary 1. The following lemma holds by using the fundamental theory of a semigroup:

Lemma 1 (cf. [4,14])**.** *Let $\Omega \subset \mathbb{R}^N$ $(N \in \mathbb{N})$ be a bounded domain. For $0 \le \beta \le 1$, $A^\beta : \mathcal{D}(A^\beta) \to L^2(\Omega)$ is invertible and*

$$(A^\beta)^{-1}u = \Gamma(\beta)^{-1}\int_0^\infty t^{\beta-1}e^{-tA}u\ dt \tag{16}$$

is expressed for $u \in L^2(\Omega)$[6]*.*

Moreover, some properties of the Dirichlet heat kernel give the following lemma:

Lemma 2 (cf. [16])**.** *Let $\Omega \subset \mathbb{R}^N$ $(N \in \mathbb{N})$ be a bounded domain. For $1 \le p < q \le \infty$, put $1/r = 1/p - 1/q$, where $1/q = 0$ if $q = \infty$. For all $t \in (0,\infty)$,*

$$\|e^{-tA}u\|_{L^q(\Omega)} \le (4\pi t)^{-\frac{N}{2r}}\|u\|_{L^p(\Omega)},\ \forall u \in L^p(\Omega)$$

holds.

For $0 \le \alpha < 1$, $A^{-\alpha}$ denotes $(A^\alpha)^{-1}$. For any bounded operator $T : L^p(\Omega) \to L^q(\Omega)$ $(1 \le p, q \le \infty)$, let

$$\|T\|_{L^p,L^q} = \sup_{u\in L^p(\Omega)\setminus\{0\}}\frac{\|Tu\|_{L^q(\Omega)}}{\|u\|_{L^p(\Omega)}}.$$

First, we prove Theorem 1.

Proof of Theorem 1. First, we show that Theorem 1 holds for $2 < p \le \infty$. Let r and α be real values such that $1/r = 1/2 - 1/p$ and $N/(2r) < \alpha \le 1$, where $1/p = 0$ if $p = \infty$. Put $u \in \mathcal{D}(A^\alpha)$. From Lemma 1,

$$\begin{aligned}
\|u\|_{L^p(\Omega)} &= \|A^{-\alpha}A^\alpha u\|_{L^p(\Omega)}\\
&\le \Gamma(\alpha)^{-1}\int_0^\infty t^{\alpha-1}\|e^{-tA}A^\alpha u\|_{L^p(\Omega)}dt\\
&\le \Gamma(\alpha)^{-1}\int_0^\infty t^{\alpha-1}\|e^{-tA}\|_{L^2,L^p}\|A^\alpha u\|_{L^2(\Omega)}dt\\
&\le \Gamma(\alpha)^{-1}\int_0^\infty t^{\alpha-1}\|e^{-\beta tA}\|_{L^2,L^p}\|e^{-(1-\beta)tA}\|_{L^2,L^2}\|A^\alpha u\|_{L^2(\Omega)}dt
\end{aligned}$$

holds for $0 < \beta < 1$. The spectral mapping theorem and Lemma 2 state that

$$\begin{aligned}
\|u\|_{L^p(\Omega)} &\le \Gamma(\alpha)^{-1}\int_0^\infty t^{\alpha-1}(4\pi\beta t)^{-\frac{N}{2r}}e^{-t(1-\beta)\lambda_{\min}}\|A^\alpha u\|_{L^2(\Omega)}dt\\
&= (4\pi\beta)^{-\frac{N}{2r}}\Gamma(\alpha)^{-1}\int_0^\infty t^{\alpha-1-\frac{N}{2r}}e^{-t(1-\beta)\lambda_{\min}}dt\|A^\alpha u\|_{L^2(\Omega)}\\
&= (4\pi\beta)^{-\frac{N}{2r}}\Gamma(\alpha)^{-1}\left(\frac{1}{(1-\beta)\lambda_{\min}}\right)^{\alpha-1-\frac{N}{2r}}\int_0^\infty s^{\alpha-1-\frac{N}{2r}}e^{-s}\left(\frac{1}{(1-\beta)\lambda_{\min}}\right)ds\|A^\alpha u\|_{L^2(\Omega)}\\
&= \frac{\Gamma\left(\alpha-\frac{N}{2r}\right)}{(4\pi)^{\frac{N}{2r}}g(\beta)\Gamma(\alpha)}\lambda_{\min}^{-\left(\alpha-\frac{N}{2r}\right)}\|A^\alpha u\|_{L^2(\Omega)}
\end{aligned} \tag{17}$$

holds, where $g(\beta) := \beta^{\frac{N}{2r}}(1-\beta)^{\alpha-\frac{N}{2r}}$ $(0 < \beta < 1)$ and $\Gamma(\alpha - N/2r) < \infty$ from $\alpha > N/(2r)$. Because the function g admits the maximal value at $\beta = \frac{N}{2r\alpha}(< 1)$, it follows that

$$\|u\|_{L^p(\Omega)} \le \frac{\alpha^\alpha\Gamma\left(\alpha-\frac{N}{2r}\right)}{(4\pi)^{\frac{N}{2r}}\left(\frac{N}{2r}\right)^{\frac{N}{2r}}\left(\alpha-\frac{N}{2r}\right)^{\alpha-\frac{N}{2r}}\Gamma(\alpha)}\lambda_{\min}^{-\left(\alpha-\frac{N}{2r}\right)}\|A^\alpha u\|_{L^2(\Omega)}. \tag{18}$$

[6]The function $(A^\beta)^{-1}u$ can be expressed by using the Dunford integral (e.g., [14]). The resulting expression corresponds with the right hand of (16) (e.g., [4]).

Next, we prove Theorem 1 for $p = 2$. For $0 \leq \alpha \leq 1$ and $u \in \mathcal{D}(A^\alpha)$, the spectral mapping theorem and Lemma 1 yield

$$\begin{aligned}\|u\|_{L^2(\Omega)} &= \|A^{-\alpha}A^\alpha u\|_{L^2(\Omega)} \\ &\leq \Gamma(\alpha)^{-1}\int_0^\infty t^{\alpha-1}\|e^{-tA}A^\alpha u\|_{L^2(\Omega)}dt \\ &\leq \Gamma(\alpha)^{-1}\int_0^\infty t^{\alpha-1}\|e^{-tA}\|_{L^2,L^2}dt\ \|A^\alpha u\|_{L^2(\Omega)} \\ &\leq \Gamma(\alpha)^{-1}\int_0^\infty t^{\alpha-1}e^{-t\lambda_{\min}}dt\ \|A^\alpha u\|_{L^2(\Omega)} \\ &= \lambda_{\min}^{-\alpha}\|A^\alpha u\|_{L^2(\Omega)}. \end{aligned} \tag{19}$$

From (19), $\|A^{-\alpha}\|_{L^2,L^2} \leq \lambda_{\min}^{-\alpha}$ and $C_{2,\alpha} = \lambda_{\min}^{-\alpha}$ hold for $0 \leq \alpha \leq 1$. □

Next, we provide the proof of Corollary 1 by using Theorem 3 and (19).

Proof of Corollary 1. For $\beta = N(p-2)/4p$ $(< N/4)$, let E_β be defined by (11). As a results of extension by zero and Theorem 3, it follows for $u \in \mathcal{D}(A^\alpha)$

$$\begin{aligned}\|u\|_{L^p(\Omega)} &= \|u\|_{L^p(\mathbb{R}^N)} \\ &\leq E_\beta\|A^\beta u\|_{L^2(\mathbb{R}^N)} \\ &= E_\beta\|A^\beta u\|_{L^2(\Omega)},\end{aligned}$$

where $\beta = N(p-2)/4p = N/2r < \alpha$ and $\mathcal{D}(A^\alpha) \subset \mathcal{D}(A^\beta)$ for $\beta < \alpha$ (cf. [4]). Moreover, (19) gives

$$\begin{aligned}\|u\|_{L^p(\Omega)} &\leq E_\beta\|A^{\beta-\alpha}\|_{L^2,L^2}\|A^\alpha u\|_{L^2(\Omega)} \\ &\leq E_\beta\lambda_{\min}^{\beta-\alpha}\|A^\alpha u\|_{L^2(\Omega)}.\end{aligned} \tag{20}$$

The inequality (20) implies that Corollary 1 holds. □

Remark 2. *All elements $u \in X_\alpha$ do not always satisfy $u = 0$ on $\partial\Omega$ in the trace sense. For example, we consider the regularity of functions in X_α if Ω is bounded and convex. Then, it is well known that the function space X_α is equivalent to the fractional Sobolev space $H^{2\alpha}(\Omega)$ for $0 \leq \alpha < 1/4$ [17]. Note that all elements $u \in X_\alpha$ satisfy $u = 0$ on $\partial\Omega$ in the trace sense for $1/4 < \alpha \leq 1$ and $\alpha \neq 3/4$ [17]. Moreover, if $\Omega \subset \mathbb{R}^2$ is a convex polygon, it is proved that all elements $u \in X_\alpha$ satisfy $u = 0$ on $\partial\Omega$ even for $\alpha = 3/4$ [18].*

4. Numerical examples

In this section, we provide some numerical examples to estimate the embedding constant $C_{p,\alpha}$ in Theorem 1 and $\tilde{C}_{p,\alpha}$ in Corollary 1. All computations were carried out on computer running Windows 7 Professional with an Intel (R) Core (TM) i7-5600U CPU and 16GB RAM. We used MATLAB R2012a with INTLAB ver. 7.1 [19]. Let $\Omega := (0,1) \times (0,1)$ and $\alpha = 1/2$. We note $\lambda_{\min} = 2\pi^2$. We computed $C_{p,1/2}$ in Theorem 1, $\tilde{C}_{p,1/2}$ in Corollary 1, M_p in Theorem 4, J_p in Theorem 5, and L_p in Theorem 6, respectively. The values of these constants are displayed in Table I.

In Table I, $C_{p,1/2}$ is a rough estimate compared with the other estimates. However, $\tilde{C}_{p,1/2}$ is tighter than the other values except for M_p.

Varying $p = 3,\ 4\ ,5\ ,6$, and α such that $1/2 - 1/p < \alpha \leq 1$, $C_{p,\alpha}$ in Theorem 1 and $\tilde{C}_{p,\alpha}$ in Corollary 1 are plotted on the domain $\Omega = (0,1) \times (0,1)$ in Figs. 1, 2, 3, and 4, respectively. The plots in the four figures indicate that the estimate in Corollary 1 is sharper than that in Theorem 1. However, as can be seen in Fig. 1, $\tilde{C}_{p,\alpha}$ is not plotted for $1/6 < \alpha < 1/4$ because the estimate in Corollary 1 does not hold for $\alpha < 1/4$ (e.g., Remark 2 of this paper).

Let $\Omega = (0,2) \times (0,2) \setminus [1,2] \times [1,2]$ and $\alpha = 1/2$. Then, the minimum eigenvalue over the domain Ω is included in $[9.639717, 9.639724]$ [20]. We compute $C_{p,1/2}$, $\tilde{C}_{p,1/2}$, M_p, J_p, and L_p, respectively. The value of these constants are displayed in Table II. Similar to the results in Table I, $C_{p,\alpha}$ is the

Table I. Comparison of each value on the domain $\Omega = (0,1) \times (0,1)$.

p	$C_{p,1/2}$	$\tilde{C}_{p,1/2}$	M_p	J_p	L_p
3	0.504227914	0.298833496	0.279911047	0.605357242	0.329648994
4	0.728930690	0.356352736	0.318309887	0.643037069	0.398942281
5	0.934611867	0.406084557	0.357803885	0.678020304	0.489090310
6	1.129584278	0.450720364	0.395853999	0.710834333	0.552669458

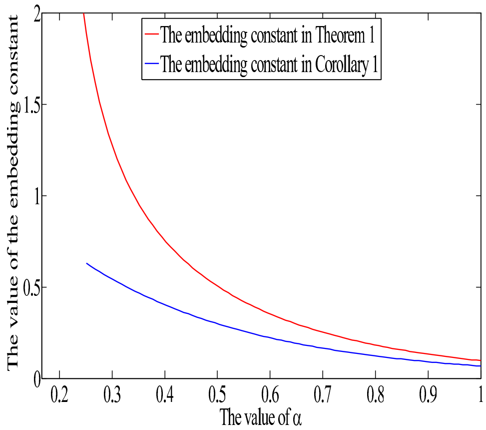

Fig. 1. Values of $C_{3,\alpha}$ and $\tilde{C}_{3,\alpha}$ on the domain $\Omega = (0,1) \times (0,1)$.

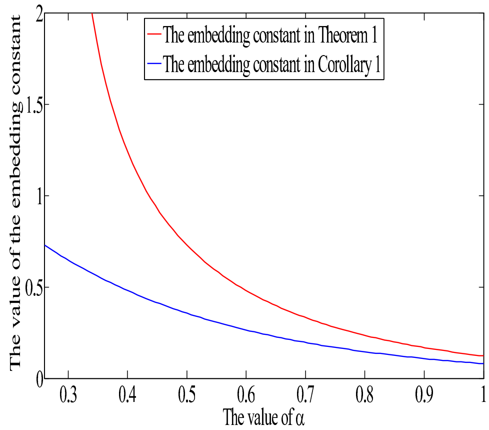

Fig. 2. Values of $C_{4,\alpha}$ and $\tilde{C}_{4,\alpha}$ on the domain $\Omega = (0,1) \times (0,1)$.

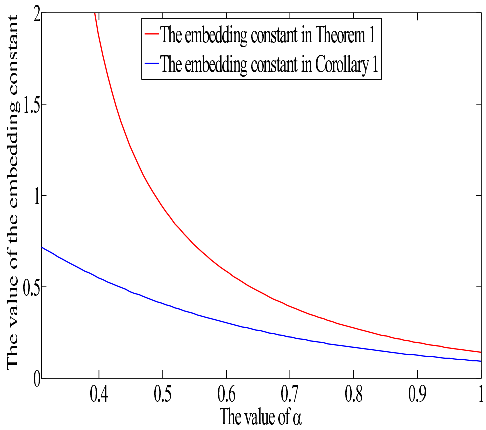

Fig. 3. Values of $C_{5,\alpha}$ and $\tilde{C}_{5,\alpha}$ on the domain $\Omega = (0,1) \times (0,1)$.

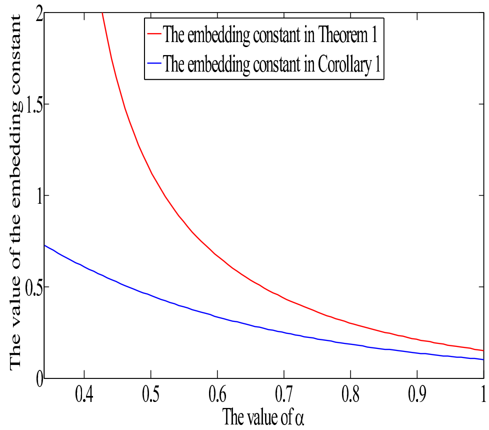

Fig. 4. Values of $C_{6,\alpha}$ and $\tilde{C}_{6,\alpha}$ on the domain $\Omega = (0,1) \times (0,1)$.

Table II. Comparison of each value on the domain $\Omega = (0,2) \times (0,2) \setminus [1,2] \times [1,2]$.

p	$C_{p,1/2}$	$\tilde{C}_{p,1/2}$	M_p	J_p	L_p
3	0.640297840	0.379476099	0.403701587	0.8730762213	0.418607405
4	0.871972121	0.426281476	0.418919370	0.8462843754	0.477228565
5	1.078659524	0.468672602	0.445727370	0.8446308696	0.564471670
6	1.272905652	0.507907653	0.475395696	0.8536672189	0.622792022

rough estimate and the value of $\tilde{C}_{p,\alpha}$ is tighter than the other constants except for M_p. Varying $p = 3,\ 4\ ,5\ ,6$ and α such that $1/2 - 1/p < \alpha \leq 1$, $C_{p,\alpha}$ in Theorem 1 are plotted on the domain $\Omega = (0,2) \times (0,2) \setminus [1,2] \times [1,2]$ in Figs. 5, 6, 7, and 8, respectively. On the other hand, the values of $\tilde{C}_{p,\alpha}$ are not plotted. This is because the domain Ω is a non-convex domain; therefore, it is difficult for us to judge the range of α in which all elements $u \in \mathcal{D}(A^\alpha)$ satisfy $u = 0$ on $\partial\Omega$ in the trace sense.

Moreover, we recall that our main theorem enables us to obtain the embedding constant from X_α to $L^\infty(\Omega)$ for $\alpha > 1/2$ by Theorem 1. Figures 9 and 10 show the embedding constant $C_{\infty,\alpha}$ in Theorem 1

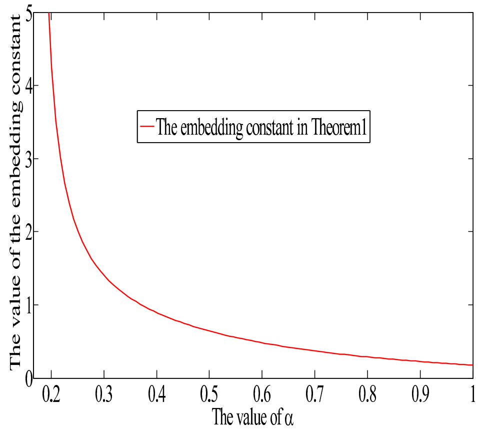

Fig. 5. Values of $C_{3,\alpha}$ on the domain $\Omega = (0,2) \times (0,2) \setminus [1,2] \times [1,2]$.

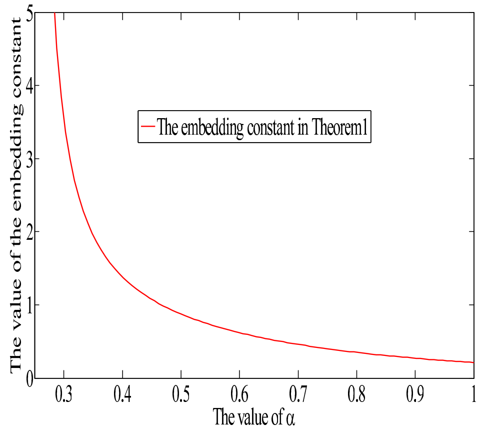

Fig. 6. Values of $C_{4,\alpha}$ on the domain $\Omega = (0,2) \times (0,2) \setminus [1,2] \times [1,2]$.

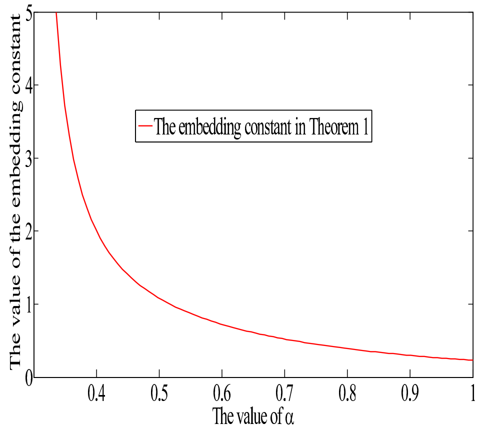

Fig. 7. Values of $C_{5,\alpha}$ on the domain $\Omega = (0,2) \times (0,2) \setminus [1,2] \times [1,2]$.

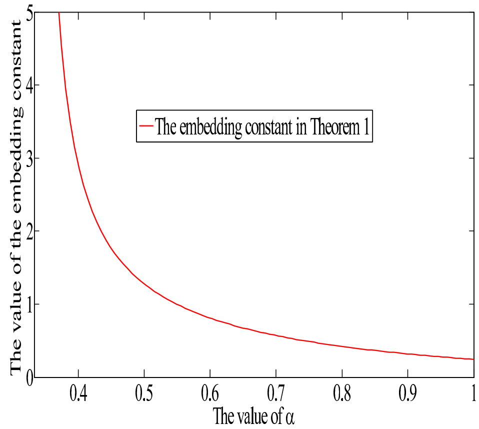

Fig. 8. Values of $C_{6,\alpha}$ on the domain $\Omega = (0,2) \times (0,2) \setminus [1,2] \times [1,2]$.

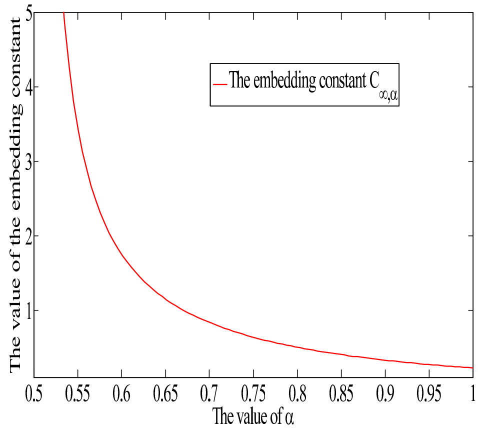

Fig. 9. Values of $C_{\infty,\alpha}$ on the domain $\Omega = (0,1) \times (0,1)$.

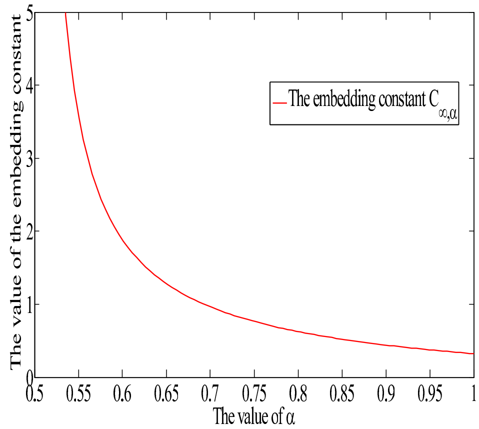

Fig. 10. Values of $C_{\infty,\alpha}$ on the domain $\Omega = (0,2) \times (0,2) \setminus [1,2] \times [1,2]$.

for $1/2 < \alpha \leq 1$ on $\Omega = (0,1) \times (0,1)$ and $\Omega = (0,2) \times (0,2) \setminus [1,2] \times [1,2]$, respectively. The results in these two figures indicate that each embedding constant $C_{\infty,\alpha}$ seems to grow up if α tends to $1/2$, respectively. Note that we cannot obtain the embedding constant X_α to $L^\infty(\Omega)$ using Corollary 1.

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References

[1] R. Dautray, J. Lions, M. Artola, and M. Cessenat, *Mathematical Analysis and Numerical Methods for Science and Technology: Spectral Theory and Applications*, Springer-Verlag, New York, 2000.

[2] J.L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications*, Springer Berlin Heidelberg, 1972.

[3] R.A. Adams, *Sobolev spaces*, Second ed., Academic Press, New York, 1975.

[4] A. Pazy, *Semigroup of linear operators and applications to partial differential equations*, Springer, 1983.

[5] A. Cotsiolis and N.K. Tavoularis, "Best constants for Sobolev inequalities for higher order fractional derivatives," *J. Math. Anal. Appl.*, vol. 295, pp. 225–236, July 2004.

[6] G.A. Anastassiou, "Fractional Sobolev type inequalities," *Applicable Analysis*, vol. 87, no. 5, pp. 607–624, May 2008.

[7] A. Aibeche, W. Chikouche, and Y. Daikh, "Real interpolation spaces between the domain of the laplace operator with transmission conditions and L^p on a polygonal domain," *Electronic Journal of Differential Equations*, vol. 2012, no. 10, pp. 1–14, January 2012.

[8] M.T. Nakao and N. Yamamoto, *Numerical verification*, Nihonhyouron-sya, 1988. (in Japanese)

[9] M. Plum, "Existence and multiplicity proofs for semilinear elliptic boundary value problems by computer assistance," *DMV Jahresbericht:* JB 110, pp. 19–54, 2008.

[10] T. Aubin, "Problémes isopérimetriques et spaces de Sobolev," *Journal of Differential Geometry*, vol. 11, pp. 573–598, 1976. (in French)

[11] G. Talenti, "Best constant in Sobolev inequality," *Annali di Matematica pura ed Applicata*, vol. 110, pp. 353–372, 1976.

[12] E.H. Lieb, "Sharp constants in the Hardy-Littlewood-Sobolev inequalities and related inequalities," *Annals of Mathematics.*, vol. 118, pp. 349–374, 1983.

[13] J. Xiao and Zh. Zhai, "Fractional Sobolev, Moser-Trudinger, Morrey-Sobolev inequalities under lorentz norms," *Journal of Mathematical Sciences*, vol. 166, no. 3, pp. 357–376, 2010.

[14] A. Yagi, *Abstract parabolic evolution equations and their applications*, Springer-Verlag, Berlin Heidelberg, 2010.

[15] K. Tanaka, K. Sekine, M. Mizuguchi, and S. Oishi, "Numerical method for deriving sharp inclusion of the Sobolev embedding constant on bounded convex domain," arXiv:1503.05468, http://arxiv.org/abs/1503.05468.

[16] P. Quittner and P. Souplet, *Superlinear parabolic problems blow-up, global existence and steady states*, Birkhäuser-Verlag, Basel, 2007.

[17] A. Yagi, "H_∞ functional calculus and characterization of domains of fractional powers," *Operator Theory: Advances and Applications*, vol. 187, pp. 217–235, 2008.

[18] N. Saito, "On interpolation spaces in a polygon," *Memories of the Faculty of Human development University of Toyama*, vol. 2, no. 1, pp. 7–12, 2007.

[19] S.M. Rump, "INTLAB-INTerval LABoratory," in *Developments in Reliable Computing*, ed. Tibor Csendes, Kluwer, pp. 77–104, Kluwer Academic Publisher, Dordrecht, 1999.

[20] X. Liu, T. Okayama, and S. Oishi, "High-Precision Eigenvalue Bound for the Laplacian with Singularities," *Computer Mathematics*, pp. 311–323, Springer-Verlag Berlin Heidelberg, 2014.