2017年度 数理科学III

11/7 (火)

第4回

monoidal category

$B = \langle B, \square, e, \alpha, \lambda, \rho \rangle$

B : category

$\square : B \times B \to B$ bifunctor

$e \in B$ object

$\alpha = \alpha_{a,b,c} : a \square (b \square c) \cong (a \square b) \square c$ natural isomorphism

$\lambda_a : e \square a \cong a$

$\rho_a : a \square e \cong a$

$$\begin{tikzcd}
& (a\square b)\square(c\square d) \arrow[dr, "\alpha"] & \\
a\square(b\square(c\square d)) \arrow[ur, "\alpha"] \arrow[d, "1\square\alpha"'] & & ((a\square b)\square c)\square d \\
a\square((b\square c)\square d) \arrow[rr, "\alpha"] & & (a\square(b\square c))\square d \arrow[u, "\alpha\square 1"']
\end{tikzcd}$$

pentagonal diagram

triangular diagram

$$\begin{tikzcd}
a\square(e\square c) \arrow[rr] \arrow[dr, "1_a\square\lambda_c"'] & & (a\square e)\square c \arrow[dl, "\rho_a\square 1_c"] \\
& a\square c &
\end{tikzcd}$$

$\lambda_e = \rho_e : e \square e \to e$

ex) Any category with finite products

Ab abelian groups $A \otimes B$ (tensor product)

$\mathbb{Z} \times A \cong A$

Coherence Theorem

Every diagram commutes

The class of diagrams at issue are the diagrams in a monoidal category built up from instances of α, λ and ρ by multi cation.

binary word of length 0 e_0 (the empty word)

of — 1 (_)

V : binary word of length m
w : 〃 n
$\Rightarrow V \square W = (V) \square (W)$

$((\_ \square \_) \square e_0) \square \_$ length 3

V, W of the same length

$V \to W$ arrow

category W : monoidal category

Wについては、任意の diagram が commute

$V, W \mapsto V \square W$

unit e_0

α, λ, ρ

morphism of monoidal categories

$T : \langle B, \square, e, \alpha, \lambda, \rho \rangle \to \langle B', \square', e', \alpha', \lambda', \rho' \rangle$

$T : B \to B'$ functor

$T(a \square b) = Ta \square' Tb$

$T(f \square g) = Tf \square' Tg$ $\quad T\lambda_a = \lambda'_{Ta}$

$Te = e'$ $\quad T\rho_a = \rho'_{Ta}$

$T\alpha_{a,b,c} = \alpha'_{Ta,Tb,Tc}$

Mon Cat

Theorem 1

任意の monoidal category B 任意の $b \in B$

ただ1個の monoidal categories の morphism

$W \to B$ ~~such that~~

with $(\_) \to b$

coherence Theorem

W edge α, λ, ρ

$B^n = \underbrace{B \times \cdots \times B}_{n個} \to B$

$(\_)$ $B \to B$ identity functor

formal

$(e_0)_B = 1 \to B$ is the constant functor $e \in B$

$(\_)_B$: the identity functor of B

w of length n
w' of length n'

W_B
W'_B

$(w \square w')_B : B^{n+n'} = B^n \times B^{n'} \xrightarrow{W_B \times W'_B} B \times B \xrightarrow{\square} B$

Corollary

B : monoidal category

$\exists$ function assigning to each pair of words V, W of the same length

n a unique natural isomorphism

$can_B(V,W) : V_B \to W_B : B^n \to B$

called the canonical map from V_B to W_B

in such a way that the identity arrow $e \to e$ is canonical (between functors of 0 variables) the identity transformation.

$id_B : I_B \to I_B$ is canonical, $\alpha, \alpha^{-1}, \lambda, \lambda^{-1}, \rho, \rho^{-1}$ are canonical and the composite as well as the $\square$-product of two canonical map is canonical.

Corollary の証明

the given monoidal category B

$\Downarrow$

monoidal category $It(B)$: category

objects $\langle n, T \rangle$ with T any functor $B^n \to B$ morphisms.

$f : \langle n, T \rangle \to \langle n, T' \rangle$ natural transformation

$\langle m, S \rangle \square \langle n, T \rangle = \langle m+n, S \square T \rangle$

$S \square T : B^{m+n} \cong B^m \times B^n \xrightarrow{S \times T} B \times B \xrightarrow{\square} B$

e $1 \to B$ constant functor of e

$\lambda : e \square T \to T$

$\lambda_{Ta} : e \square Ta \to Ta$

λは natural

ρ, α

The identity functor

$I : B \to B$ an object of $It(B)$

From the Theorem.

$\exists !$ morphism of monoidal categories with $(\_) \mapsto I$

$V \to W$ for V, W of the same length

$V_B \to W_B$ natural transformation

$can_B(V,W)$

preserves α, λ, ρ.

$can_B(e_0, e_0) = 1_e : e \to e$

$can_B((\_),(\_)) = id_B : B \to B$

$can_B(\_ \square (\_ \square \_), (\_ \square \_) \square \_)$

$= \alpha : B \square (B \square B) \to (B \square B) \square B$

$can_B(e_0 \square \_, (\_)) = \lambda$

$can_B(\_ \square e_0, (\_)) = \rho$

$can_B(v \square v', w \square w')$

$= can_B(v,w) \square can_B(v', w')$

次の形の diagram が commute

Vertices : words w of length n representing functors $w_B : B^n \to B$

edges : natural transformation $1_e, id_0, \alpha, \lambda, \rho$ and their $\square$ products