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STRONG COINCIDENCE AND OVERLAP COINCIDENCE

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ABSTRACT. We show that strong coincidences of a certain many choices of control points are equivalent to overlap coincidence for the suspension tiling of Pisot substitution. The result is valid for degree ≥ 2 as well, under certain topological conditions. This result gives a converse of the paper [2] and elucidates the tight relationship between two coincidences.

1. INTRODUCTION

Self-affine tiling dynamical system in \mathbb{R}^d is a generalization of *substitution* dynamical system on letters, which gives a nice model of self-inducing structures appear in dynamical systems, number theory and the mathematics of aperiodic order. Pure discreteness of self-affine tiling dynamics is long studied from many points of views. The idea of *coincidence*¹ appeared firstly in Kamae [8], and then in a comprehensive form in Dekking [6] for constant length substitution (see also [16]). Generalizing a pioneer work of Rauzy [17], Arnoux-Ito [3] gave a geometric realization of irreducible Pisot unit substitution of degree d . They defined *strong coincidence*, which ensures that their geometric substitution gives rise to a domain exchange of \mathbb{R}^{d-1} , which is also semi-conjugate to the toral rotation of \mathbb{T}^d . It is remarkable that in many cases, it is even conjugate to the total rotation, which immediately implies that the system in pure discrete (see [4, 7, 5, 18] for further developments). On the other hand, *overlap coincidence* introduced by Solomyak [19] is an equivalent condition for pure discreteness of a given self-affine tiling dynamical system. This is also described as a geometric/combinatorial condition which guarantees that the tiling and its translation by return vectors become exponentially close if we iteratively enlarge return vectors by substitution. Lee [11] showed deeper characterizations that overlap coincidence is equivalent to *algebraic coincidence*, and the fact that the corresponding point set is an inter-model set.

Until now, the relation between strong coincidence and overlap coincidence is not fully understood. Motivated by the claim of Nakaiishi [15], Akiyama-Lee [2] generalized the notion of strong coincidence to \mathbb{R}^d and showed that overlap coincidence implies strong coincidence, and moreover *simultaneous coincidence*, provided that the associated point set is admissible and its height group is trivial. In this paper we shall give a converse statement for the suspension tiling of Pisot substitution at the expense of assuming many strong coincidences at a time, that is, strong coincidences on a certain many choices of control points imply overlap coincidence and vice versa. If every tile is connected and the tiling is not a collection of unbounded connected identical colored patches, then the same result holds for $d \geq 2$ (Theorem 3.2). This result elucidates the tight relationship between two coincidences.

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¹In their notation, the *column number* is one.

2. TERMINOLOGIES

2.1. Tiles and tilings. We shall briefly recall basic definitions used in this paper. A *tile* in \mathbb{R}^d is defined as a pair $T = (A, i)$ where A is a compact set in \mathbb{R}^d which is the closure of its interior, and $i = \ell(T) \in \{1, \dots, m\}$ is the *color* of T . We call A the *support* of T and denote $\text{supp}(T) = A$. The *translate* of T is defined by $g + T = (g + A, i)$ for $g \in \mathbb{R}^d$. Let $\mathcal{A} = \{T_1, \dots, T_m\}$ be a finite set of tiles in \mathbb{R}^d such that $T_i = (A_i, i)$; we will call them *prototiles*. A tiling \mathcal{T} is a collection of translates of prototiles which covers \mathbb{R}^d without interior overlaps. A finite collection of tiles which appear in \mathcal{T} is called a *patch*. A *generalized patch* is a collection of tiles in \mathcal{T} whose cardinality is not necessarily finite. Its support is defined to be the union of the supports of tiles. The diameter of a generalized patch is the supremum of Euclidean distance of two points lie within the support of the patch. A map Ω from \mathcal{A} to the set of patches is called a *substitution* with a $d \times d$ expansive matrix Q if there exist finite sets $\mathcal{D}_{ij} \subset \mathbb{R}^d$ for $i, j \leq m$ such that

$$(2.1) \quad \Omega(T_j) = \{T_i + u : u \in \mathcal{D}_{ij}, i = 1, \dots, m\}$$

with

$$(2.2) \quad Q\mathcal{A}_j = \bigcup_{i=1}^m (A_i + \mathcal{D}_{ij}) = \bigcup_{i=1}^m \bigcup_{u \in \mathcal{D}_{ij}} (A_i + u) \quad \text{for } j \leq m,$$

and the last union has mutually disjoint interiors. The substitution (2.1) extends to all translates of prototiles and patches in a natural way. A *substitution tiling* of Ω is a tiling \mathcal{T} that all the patches of \mathcal{T} is a sub-patch of $\Omega^n(T)$ for some $n \in \mathbb{N}$ and $T \in \mathcal{T}$. A substitution tiling \mathcal{T} is a *fixed point* of Ω if $\Omega(\mathcal{T}) = \mathcal{T}$ holds. We say that a substitution tiling is *primitive* if the corresponding substitution matrix $M = (\#\mathcal{D}_{ij})$ is primitive, and *irreducible* if the characteristic polynomial of M is irreducible. We say that \mathcal{T} has *finite local complexity* (FLC) if for any R there are only finitely many patches of diameter less than R up to translation. A tiling \mathcal{T} is *repetitive* if every patch is relatively dense in \mathcal{T} . A FLC substitution tiling of a primitive substitution is called a *self-affine tiling*. Every self-affine tiling is repetitive, which follows from the primitivity of substitution. Let $\lambda > 1$ be the Perron-Frobenius eigenvalue of the substitution matrix M and D be the set of eigenvalues of Q . By the tiling criterion of Lagarias-Wang [9], λ is the element of D of maximum modulus. We say that Q fulfills *Pisot family condition* if every algebraic conjugate μ of an element of D with $|\mu| \geq 1$ is contained in D .

The set of all substitution tilings of Ω forms a *tiling space*. By using a fixed point \mathcal{T} of Ω , we can describe this space as the orbit closure of \mathcal{T} under the translation action: $X_{\mathcal{T}} = \overline{\{\mathcal{T} - g : g \in \mathbb{R}^d\}}$, the closure is taken by ‘local topology’. The FLC assumption implies $X_{\mathcal{T}}$ is compact and we get a topological dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d)$ where \mathbb{R}^d acts by translations. This system is minimal and uniquely ergodic ([19, 12]), and we are interested in the spectra of self-affine tiling dynamical systems. Tiling dynamical system $X_{\mathcal{T}}$ has *pure discrete spectrum* if the eigenfunctions for the \mathbb{R}^d -action forms a complete orthonormal basis of $L^2(X_{\mathcal{T}}, \mu)$ [19].

2.2. Control points. A Delone set is a relatively dense and uniformly discrete subset of \mathbb{R}^d . We say that $\Lambda = (\Lambda_i)_{i \leq m}$ is a *Delone multi-color set* in \mathbb{R}^d if each Λ_i is Delone and $\cup_{i=1}^m \Lambda_i \subset \mathbb{R}^d$ is Delone. We say that $\Lambda \subset \mathbb{R}^d$ is a *Meyer set* if it is a Delone set and $\Lambda - \Lambda$ is uniformly discrete in \mathbb{R}^d [10]. $\Lambda = (\Lambda_i)_{i \leq m}$ is called a *substitution Delone multi-color set* if Λ is a Delone multi-color set

and there exist an expansive matrix Q and finite sets \mathcal{D}_{ij} for $i, j \leq m$ such that

$$(2.3) \quad \Lambda_i = \bigcup_{j=1}^m (Q\Lambda_j + \mathcal{D}_{ij}), \quad i \leq m,$$

where the union on the right side is disjoint.

Given a fixed point \mathcal{T} of Ω , we can associate a substitution Delone multi-color set $\mathbf{\Lambda}_{\mathcal{T}} = (\Lambda_i)_{i \leq m}$ of \mathcal{T} by taking representative points of tiles in the relatively same positions for the same color tiles in the tiling. There is a canonical way to choose representative points, called *control points*. A tile map $\gamma = \gamma_{\Omega}$ is a map from \mathcal{T} to itself which sends a tile T to the one in $\Omega(T)$ such that $\gamma(T_1)$ and $\gamma(T_2)$ are located in the same relative position in $\Omega(T_1)$ and $\Omega(T_2)$ whenever $\ell(T_1) = \ell(T_2)$. A control point $c(T)$ of $T \in \mathcal{T}$ is defined by

$$c(T) = \bigcap_{n=1}^{\infty} Q^{-n}(\gamma^n(T)).$$

Control points are representative points, i.e., $U - c(U) = V - c(V)$ holds if $\ell(U) = \ell(V)$ with $U, V \in \mathcal{T}$. Let Λ_i be the set of control points of color i . Clearly, $j = \ell(\gamma(T_i))$ implies $Q\Lambda_i \subset \Lambda_j$ and the set of control points $\mathcal{C} = \bigcup_{i=1}^m \Lambda_i$ is invariant under the expansion by Q , that is, $Q\mathcal{C} \subset \mathcal{C}$. We obtain an associated substitution Delone multi-color set $\mathbf{\Lambda} = \mathbf{\Lambda}_{\mathcal{T}} = (\Lambda_i)_{i \leq m}$.

In section 3, we have to assume a lot of strong coincidences by changing control points for a given tiling \mathcal{T} . If we change control points of tiles of \mathcal{T} by $\Lambda'_i = \Lambda_i - g_i$, then the set equation will be shifted like

$$\Lambda'_i = \bigcup_{j=1}^m Q\Lambda'_j + \mathcal{D}'_{ij}$$

with $\mathcal{D}'_{ij} = \{d_{ij} + Qg_j - g_i : 1 \leq i, j \leq m\}$. The corresponding tile equation becomes

$$QA'_j = \bigcup A'_i + \mathcal{D}'_{ij}$$

which is satisfied by $A'_j = A_j + g_j$. So we set $\text{supp}(T'_j) = A'_j$ and $\ell(T'_j) = \ell(T_j)$. To avoid heavy notation, we do not distinguish such changes of control points and use the same symbols Λ_i and T_i .

2.3. Coincidences. The set of *return vectors* is defined by $\Xi(\mathcal{T}) = \{y \in \mathbb{R}^d : U = V - y, \text{ where } U, V \in \mathcal{T}\}$. A triple (U, y, V) , with $U, V \in \mathcal{T}$ and $y \in \Xi(\mathcal{T})$, is called an *overlap* if

$$(\text{supp}(U))^{\circ} \cap (\text{supp}(V) - y)^{\circ} \neq \emptyset.$$

An overlap (U, y, V) is a *coincidence* if $U = V - y$. Let $\mathcal{O} = (U, y, V)$ be an overlap in \mathcal{T} , we define ℓ -th *inflated overlap*

$$\Omega^{\ell} \mathcal{O} = \{(U', Q^{\ell}y, V') : U' \in \Omega^{\ell}(U), V' \in \Omega^{\ell}(V), \text{ and } (U', Q^{\ell}y, V') \text{ is an overlap}\}.$$

We say that a self-affine tiling \mathcal{T} admits *overlap coincidence* if there exists $\ell \in \mathbb{Z}_+$ such that for each overlap \mathcal{O} in \mathcal{T} , $\Omega^{\ell} \mathcal{O}$ contains a coincidence. Two overlaps (U, y, V) and (U_1, y_1, V_1) are equivalent, if there is $x \in \mathbb{R}^d$ that both $U_1 = U - x$ and $V_1 - y_1 = V - y - x$ hold. The equivalence class is denoted by $(\widetilde{U, y, V})$. Hereafter we assume an important condition that $\Xi(\mathcal{T})$ forms a Meyer set. This condition is equivalent to the Pisot family condition for Q , if Q is diagonalizable and all its eigenvalues are algebraic conjugate with the same multiplicity [13]. The number of equivalence classes of overlaps is finite, by the Meyer property of $\Xi(\mathcal{T})$. The action of Ω is well-defined on equivalence classes of overlaps. An *overlap graph with multiplicity* is a finite directed graph whose vertices are the equivalence classes of overlaps. Multiplicities of the edge from $(\widetilde{U, y, V})$ to $(\widetilde{A, z, B})$ is given by the

number of overlaps in $\Omega((U, y, V))$ equivalent to (A, z, B) (c.f. [1]). Overlap coincidence is confirmed by checking whether from each vertex of this graph there is a path leading to a coincidence. Overlap coincidence is equivalent to pure discreteness of self-affine tiling dynamical system $X_{\mathcal{T}}$ [19].

Strong coincidence on letter substitution is naturally generalized to self-similar tiling in \mathbb{R}^d in [2]. We adapt this definition to control points. Let \mathcal{T} be a self-affine tiling in \mathbb{R}^d and $\mathcal{A} = \{T_1, \dots, T_m\}$ be the prototile set of \mathcal{T} . We say that the set of the control points is *admissible* if $\cap_{i \leq m} (\text{supp}(T_i) - c(T_i))$ has non-empty interior.

Let \mathcal{T} be the fixed point of Ω . Let $c(T_i)$ ($i = 1, \dots, m$) be the admissible control points and $\mathbf{\Lambda}$ be an associated substitution Delone multi-color set for which $\mathcal{T} = \{T_i - c(T_i) + u_i \mid u_i \in \Lambda_i, i \leq m\}$. If for any $1 \leq i, j \leq m$, there is a positive integer L that

$$(2.4) \quad \Omega^L(T_i - c(T_i)) \cap \Omega^L(T_j - c(T_j)) \neq \emptyset,$$

then we say that $\mathbf{\Lambda}$ admits *strong coincidence*. In other words, strong coincidence means that for every pair of tiles $(U, V) \in \mathcal{T}^2$, $\Omega^L(U - c(U))$ and $\Omega^L(V - c(V))$ share a common tile in the same position for some L .

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The set of *eventually return vectors* is defined by

$$\mathcal{G} := \bigcup_{k=0}^{\infty} Q^{-k}(\Lambda_i - \Lambda_i), \quad \text{for some } i \leq m$$

which is independent of the choice of i , by primitivity of Ω . The tiling dynamical system is invariant under replacement of the substitution rule Ω by Ω^n . We consider control points of Ω^n as well. Hereafter we put $\Lambda = \bigcup_{i=1}^m \Lambda_i$ for $\mathbf{\Lambda} = \mathbf{\Lambda}_{\mathcal{T}} = (\Lambda_i)$ to distinguish the multi-color set and its union. Let $\langle \mathcal{G} \rangle$ be the additive subgroup of \mathbb{R}^d generated by \mathcal{G} . We say that \mathcal{T} satisfies *multiple strong coincidence of level n* if all multi-color Delone set $\mathbf{\Lambda}$'s generated by admissible control points of Ω^n with $\Lambda - \Lambda \subset \langle \mathcal{G} \rangle$ admit strong coincidence.

Hereafter when we speak about a topological/metrical property (connected, bounded, diameter) of a generalized patch, it refers to the corresponding property of its support. A *rod* is an unbounded connected generalized patch of \mathcal{T} whose tiles have an identical color. A *rod tiling* is a tiling that every tile belongs to a rod. For ease of negation, a *non-rod* tiling is a tiling which is not a rod tiling. A tiling is called *non-periodic* if there are no non-trivial period, i.e., $\{p \in \mathbb{R}^d \mid \mathcal{T} + p = \mathcal{T}\} = \{0\}$.

Remark 3.1. There are many examples of periodic self-affine rod tiling. Consider a tiling of \mathbb{R}^2 by squares $[0, 1]^2 + (x, y)$ with $(x, y) \in \mathbb{Z}^2$ and their colors are defined by $y \pmod{2}$ or $x + y \pmod{2}$. However we do not know an example of non-periodic self-affine rod tiling.

Theorem 3.2. *Let \mathcal{T} be a non-rod self affine tiling by connected tiles such that $\Xi(\mathcal{T})$ is a Meyer set. Then there is a constant n depending only on \mathcal{T} that \mathcal{T} satisfies multiple strong coincidence of level n if and only if \mathcal{T} satisfies overlap coincidence.*

Consider a substitution σ over m letters $\{1, 2, \dots, m\}$ whose substitution matrix is $M_{\sigma} = (|\sigma(j)|_i)$, where $|w|_i$ is the number of letter i in a word w . We say that σ is a *Pisot* substitution, if the Perron Frobenius root β of M_{σ} is a Pisot number. The canonical *suspension tiling* \mathcal{T} in \mathbb{R} of σ with an expansion factor β is defined by associating to the letters the intervals whose lengths are given by a left eigenvector of M_{σ} corresponding to β .

Corollary 3.3. *The statement is valid for the suspension tiling of a Pisot substitution.*

Indeed, 1×1 matrix $Q = (\beta)$ satisfies Pisot family condition, tiles are intervals and the suspension tiling can not be a rod tiling, since it has at least two translationally inequivalent tiles in \mathbb{R} .

Remark 3.4. Multiple strong coincidence of level n requires many strong coincidences at a time for a fixed tiling \mathcal{T} even when $n = 1$. In dimension one, the claim of Nakaishi [15] reads a single strong coincidence implies overlap coincidence. Theorem 3.2 covers general cases but the requirement is much stronger. It would be interesting is to make smaller the constant n in Theorem 3.2. For e.g., can we take $n = 1$?

We prepare a lemma.

Lemma 3.5. *Let G be a strongly connected finite directed graph and C be a set of cycles of G . Then there is a subgraph $G(C)$ of G with the following property.*

- *The set of vertices of $G(C)$ is equal to that of G .*
- *Every vertex has exactly one outgoing edge.*
- *The set of cycles of $G(C)$ is equal to C .*

Proof. Put $H_0 = C$. We inductively construct H_i for $i = 0, 1, \dots$ which satisfies:

- Every vertex has exactly one outgoing edge.
- The set of cycles of H_i is equal to C .

Assume that the induced graph $G \setminus H_i$ is non empty and take a vertex v from $G \setminus H_i$. Since G is strongly connected, there is a path from v leading to H_i . So there is a vertex $u \in G \setminus H_i$ and an edge from u to a vertex of H_i . We define H_{i+1} by adding this u and the outgoing edge. Then H_{i+1} clearly satisfies above two conditions. Since G is finite, we find m that $G \setminus H_m$ is empty, i.e., the set of vertices of G and H_m are the same. We finish the proof by taking $G(C) = H_m$. \square

Proof of Theorem 3.2. Theorem 4.3 of [2] shows that overlap coincidence of \mathcal{T} implies multiple strong coincidence of level n for any $n \geq 1$. We prove that there is a constant n such that multiple strong coincidence of level n implies overlap coincidence.

Assume that \mathcal{T} does not admit overlap coincidence. Construct the overlap graph G of \mathcal{T} with multiplicity. Since \mathcal{T} does not admit overlap coincidence, there is a strongly connected component² S of G such that its spectral radius is equal to $|\det(Q)|$ and from each overlap of S there is no path leading to a coincidence in G . Without loss of generality, we may assume that the incidence matrix of S is primitive³. Thus we can find a positive integer n_0 such that for every overlap (U, y, V) , $\Omega^{n_0}(U, y, V)$ contains an overlap equivalent to (U, y, V) . Since $\Xi(\mathcal{T})$ is a Meyer set, number of equivalence classes of overlaps is finite and bounded by a constant which depends only on \mathcal{T} . Thus there is an upper bound of n_0 which depends only on \mathcal{T} . We further assume multiple strong coincidence of level $n = n_0$ on \mathcal{T} and derive a contradiction.

We claim that in the component S there is an overlap (U, y, V) with $\ell(U) \neq \ell(V)$ for any non-rod self-affine tiling by connected tiles. Assume on the contrary that all overlaps in S are of the

²In this assertion, one can take either usual overlaps or potential overlaps as we like.

³If the incidence matrix of S is irreducible but not primitive, then take a suitable power of Ω by Perron-Frobenius theory.

form (A, z, B) with $\ell(A) = \ell(B)$. Since S does not contain a coincidence, $z \neq 0$ for these overlaps. Taking k -th inflated overlap of (A, z, B) , we obtain of patches P and Q , both contain large balls, say $B_p(r)$ and $B_q(r)$, that the tiles of P close to p and the tiles of Q close to q are in multiple correspondence in the following sense. Putting $x = Q^k z$, for a tile $U \in P$ close to p there are several (at least two) tiles $V \in Q$ that (U, x, V) are overlaps in⁴ S and $\text{supp}(U)$ is contained in the union of $\text{supp}(V - x)$, and the same statements hold after interchanging the role of U and V . Take a tile U with $p \in \text{supp}(U) \subset B_p(r)$. Then overlaps (U, x, V) with $V \in Q$ give rise to a patch $\mathcal{V}_1 = \bigcup V$ that $\text{supp}(U) \subsetneq \text{supp}(\mathcal{V}_1) - x$. By assumption, $\ell(\mathcal{V}_1) = \ell(U)$ for every $V_1 \in \mathcal{V}_1$. By using path connectedness of tiles⁵, the patch \mathcal{V}_1 is path connected. If $\text{supp}(\mathcal{V}_1) \subset B_q(r)$, then there is a patch $\mathcal{U}_1 = \bigcup U_1$ where $U_1 \in P$ are taken from all overlaps of the form (U_1, x, V_1) with some $V_1 \in \mathcal{V}_1$. This patch is also path connected and satisfies $\text{supp}(\mathcal{V}_1) - x \subsetneq \text{supp}(\mathcal{U}_1)$ and each tile of \mathcal{U}_1 has the same color as U . In this manner, by taking large r , we obtain a long sequence of path connected patches

$$\text{supp}(U) \subsetneq \text{supp}(\mathcal{V}_1 - x) \subsetneq \text{supp}(\mathcal{U}_1) \subsetneq \text{supp}(\mathcal{V}_2 - x) \subsetneq \text{supp}(\mathcal{U}_2) \subsetneq \dots$$

The number of tiles strictly increases and all tiles appear in this sequence has the same color $\ell(U)$. This shows for any $M > 0$, there exists a ball of radius R that each tile U in the ball belongs to a connected patch in \mathcal{T} having diameter greater than M , whose tiles have an identical color $\ell(U)$. Therefore by using FLC, among $X_{\mathcal{T}}$ we can choose a rod tiling. Being a rod tiling is invariant under translation and closure operation, using minimality of $X_{\mathcal{T}}$ we see that every tiling in $X_{\mathcal{T}}$ is a rod tiling. This gives a contradiction, which finishes the proof of the claim.

Consider a directed graph \mathcal{V} over $\{1, \dots, m\}$ whose edge $i \rightarrow j$ is given if there are $U, V \in S$ that $V \in \Omega^n(U)$ with $i = \ell(U)$ and $j = \ell(V)$. Clearly \mathcal{V} is strongly connected as well. Pick one overlap (U, y, V) from S that $\ell(U) \neq \ell(V)$ and select one of the overlaps equivalent to (U, y, V) in $\Omega^n(U, y, V)$. We select a tile map $\gamma = \gamma_{\Omega^n}$ which sends $\gamma(U)$ to this U in (U, y, V) , and $\gamma(V)$ to the V in (U, y, V) , which correspond to two cycles $\ell(U) \rightarrow \ell(U)$ and $\ell(V) \rightarrow \ell(V)$ on \mathcal{V} . Let C be the set of these two cycles and take $\mathcal{V}(C)$ by Lemma 3.5. The tile map $\gamma = \gamma_{\Omega^n}$ is chosen so that $\ell(U) \rightarrow \ell(\gamma(U))$ for $U \in \{T_1, \dots, T_m\}$ forms the set of edges of $\mathcal{V}(C)$. By the choice of the subgraph, every path of length m on this subgraph must fall into one of the two cycles. Note that by this choice of γ , the control points of U and $V - y$ are exactly matching, because both of them are equal to a common point $\bigcap_{k=1}^{\infty} Q^{-nk} (\gamma^k(U) \cap \gamma^k(V - y))$.

We claim that by this γ , we have $\Lambda - \Lambda \subset \langle \mathcal{G} \rangle$. In fact, since every overlap in the overlap graph is of the form (A, z, B) with $z \in \bigcup_{i=1}^m (\Lambda_i - \Lambda_i)$, and control points of U and $V - y$ are matching on (U, y, V) , i.e., $c(U) = c(V) - y$, we have $c(U) - c(V) \in \mathcal{G}$. By construction of \mathcal{V} for any $x, y \in \Lambda$, we have $Q^m x, Q^m y \in \Lambda_{\ell(U)} \cup \Lambda_{\ell(V)}$. For instance, if $Q^m x \in \Lambda_{\ell(U)}$ and $Q^m y \in \Lambda_{\ell(V)}$, then $Q^m x = c(U) + f, Q^m y = c(V) + g$ hold with $f \in \Lambda_{\ell(U)} - \Lambda_{\ell(U)}, g \in \Lambda_{\ell(V)} - \Lambda_{\ell(V)}$. Therefore we have $\Lambda - \Lambda \subset \langle \mathcal{G} \rangle$.

We also see that the set of control points $\Lambda = (\Lambda_i)$ associated to γ is admissible. In fact, since (U, y, V) is an overlap, $\text{supp}(U) \cap \text{supp}(V - y)$ has an inner point. Since $y = c(V) - c(U)$, we have $(\text{supp}(U - c(U)))^\circ \cap (\text{supp}(V - c(V)))^\circ \neq \emptyset$. The admissibility follows from $Q^m x \in \Lambda_{\ell(U)} \cup \Lambda_{\ell(V)}$ for any $x \in \Lambda$.

⁴We say that an overlap belongs to S if its equivalence class does.

⁵Connectedness and path connectedness are equivalent for self-affine tiles [14].

Summing up, from $(\widetilde{U}, y, V) \in S$, we have chosen a tile map γ_{Ω^n} which produces a substitution Delone multi-color set of admissible control points with $\Lambda - \Lambda \subset \langle \mathcal{G} \rangle$. By the assumption of multiple strong coincidence of level n , we know $\Omega^k(U - c(U)) \cap \Omega^k(V - c(V))$ is non empty for some k , which shows that (U, y, V) leads to a coincidence, giving a desired contradiction. \square

Remark 3.6. We use the assumptions that each tile is connected and \mathcal{T} is a non-rod tiling only to show that there is an overlap $(U, y, V) \in S$ that $\ell(U) \neq \ell(V)$, which allows us to define a tile map. It is likely that these assumptions are not necessary, i.e., every non-periodic self-affine tiling that $\Xi(\mathcal{T})$ is a Meyer set, then such overlap must appear in S .

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