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Higher-Dimensional General Jacobi Identities I

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Abstract

It was shown by the author [International Journal of Theoretical Physics 36 (1997), 1099-1131] in synthetic differential geometry that what is called the general Jacobi identity obtaining in microcubes underlies the Jacobi identity of vector fields. It is well known in the theory of Lie algebras that a plethora of higher-dimensional generalizations of the Jacobi identity hold, though it is usually established not as a direct derivation from the axioms of Lie algebras but by making an appeal to the so-called Poincaré-Birkhoff-Witt theorem. The general Jacobi identity was rediscovered by Kirill Mackenzie in the second decade of this century [Geometric Methods in Physics, 357-366, Birkhäuser/Springer 2013]. The principal objective in this paper is to investigate a four-dimensional generalization of the general Jacobi identity in detail. In a subsequent paper we will propose a uniform method for establishing a bevy of higher-dimensional generalizations of the Jacobi identity under a single umbrella.

1 Introduction

It is known in synthetic differential geometry (cf. [2] and [4]) that vector fields on a microlinear space M forms a Lie algebra, for which the following antisymmetry holds:

$$[X_1, X_2] + [X_2, X_1] = 0 \quad (1)$$

It was shown in [3] that a bit deeper theorem in the following underlies the above identity.

Theorem 1 *Let M be a microlinear space. Given microsquares $\gamma_{12}, \gamma_{21} : D^2 \rightarrow M$ with $\gamma_{12} \mid D(2) = \gamma_{21} \mid D(2)$, we have*

$$\left(\gamma_{12} \dot{-} \gamma_{21} \right) + \left(\gamma_{21} \dot{-} \gamma_{12} \right) = 0 \quad (2)$$

Now we consider the famous Jacobi identity.

$$[X_1, [X_2, X_3]] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] = 0 \quad (3)$$

It claims that the sum of $[X_1, [X_2, X_3]]$'s with the three cyclic permutations of $\{1, 2, 3\}$ applied vanishes. We note in passing that the three permutations of $\{1, 2, 3\}$ are no other than the three even permutations of $\{1, 2, 3\}$. It has been demonstrated in [6], [7], [8] and [10] that the following deeper theorem underlies the above identity.

Theorem 2 (General Jacobi Identity) *Let M be a microlinear space. Given microcubes $\gamma_{123}, \gamma_{132}, \gamma_{213}, \gamma_{231}, \gamma_{312}, \gamma_{321} : D^3 \rightarrow M$ with*

$$\begin{aligned} \gamma_{123} \mid \{(d_1, d_2, d_3) \in D^3 \mid d_2 d_3 = 0\} &= \gamma_{132} \mid \{(d_1, d_2, d_3) \in D^3 \mid d_2 d_3 = 0\} \\ \gamma_{231} \mid \{(d_1, d_2, d_3) \in D^3 \mid d_2 d_3 = 0\} &= \gamma_{321} \mid \{(d_1, d_2, d_3) \in D^3 \mid d_2 d_3 = 0\} \\ \gamma_{231} \mid \{(d_1, d_2, d_3) \in D^3 \mid d_1 d_3 = 0\} &= \gamma_{213} \mid \{(d_1, d_2, d_3) \in D^3 \mid d_1 d_3 = 0\} \\ \gamma_{312} \mid \{(d_1, d_2, d_3) \in D^3 \mid d_1 d_3 = 0\} &= \gamma_{132} \mid \{(d_1, d_2, d_3) \in D^3 \mid d_1 d_3 = 0\} \\ \gamma_{312} \mid \{(d_1, d_2, d_3) \in D^3 \mid d_1 d_2 = 0\} &= \gamma_{321} \mid \{(d_1, d_2, d_3) \in D^3 \mid d_1 d_2 = 0\} \\ \gamma_{123} \mid \{(d_1, d_2, d_3) \in D^3 \mid d_1 d_2 = 0\} &= \gamma_{213} \mid \{(d_1, d_2, d_3) \in D^3 \mid d_1 d_2 = 0\} \end{aligned}$$

we have

$$\begin{aligned} &\left(\left(\gamma_{123} \begin{array}{c} \cdot \\ 1 \end{array} \gamma_{132} \right) \dot{-} \left(\gamma_{231} \begin{array}{c} \cdot \\ 1 \end{array} \gamma_{321} \right) \right) + \\ &\left(\left(\gamma_{231} \begin{array}{c} \cdot \\ 2 \end{array} \gamma_{213} \right) \dot{-} \left(\gamma_{312} \begin{array}{c} \cdot \\ 2 \end{array} \gamma_{132} \right) \right) + \\ &\left(\left(\gamma_{312} \begin{array}{c} \cdot \\ 3 \end{array} \gamma_{321} \right) \dot{-} \left(\gamma_{123} \begin{array}{c} \cdot \\ 3 \end{array} \gamma_{213} \right) \right) \\ &= 0 \end{aligned} \quad (4)$$

The general Jacobi identity was rediscovered by Kirill Mackenzie [5] in a somewhat different context. We add that the general Jacobi identity plays a fundamental role in a combinatorial or geometric proof of Jacobi-like identities in so-called Frölicher-Nijenhuis calculus (cf. [9]).

Now we consider the following four-dimensional analogue of the Jacobi identity.

$$\begin{aligned} &[X_1, [X_2, [X_3, X_4]]] + [X_1, [X_3, [X_4, X_2]]] + [X_1, [X_4, [X_2, X_3]]] + \\ &[X_2, [X_1, [X_4, X_3]]] + [X_2, [X_3, [X_1, X_4]]] + [X_2, [X_4, [X_3, X_1]]] + \\ &[X_3, [X_1, [X_2, X_4]]] + [X_3, [X_2, [X_4, X_1]]] + [X_3, [X_4, [X_1, X_2]]] + \\ &[X_4, [X_1, [X_3, X_2]]] + [X_4, [X_2, [X_1, X_3]]] + [X_4, [X_3, [X_2, X_1]]] \\ &= 0 \end{aligned} \quad (5)$$

It claims that the sum of $[X_1, [X_2, [X_3, X_4]]$'s with the twelve even permutations of $\{1, 2, 3, 4\}$ applied vanishes.

The principal objective in this paper is to establish a four-dimensional version of the general Jacobi identity underpinning the above identity (5). In a subsequent paper we will discuss a slew of higher-dimensional general Jacobi identities underlying the higher-dimensional Jacobi identities discussed in [1] and [12] (the former called them *generalized Jacobi identities*) from a coherent standpoint. For a good introduction to generalized Jacobi identities, the reader is referred to Chapter 8 of [11]. We know well that various higher-dimensional Jacobi identities are logical consequences of the three-dimensional Jacobi identity, but we guess that higher-dimensional general Jacobi identities are by no means logical consequences of the three-dimensional general Jacobi identity. We assume the reader to be familiar with [4] up to Chapter 3.

2 Strong Differences

First we introduce the notion of a simplicial small object after [6], though in a somewhat generalized form.

Notation 3 (*Simplicial small objects*) *Let n be a natural number. Given a subset \mathfrak{p} of*

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i \leq n, 1 \leq j \leq n, i \neq j\}$$

and a subset ξ of

$$\{i \in \mathbb{N} \mid 1 \leq i \leq n\}$$

$D^n \{\mathfrak{p}, \xi\}$ denotes the set

$$\{(d_1, \dots, d_n) : D^n \mid d_i d_j = 0 \text{ for any } (i, j) \in \mathfrak{p}, d_i = 0 \text{ for any } i \in \xi\}$$

which is surely a small object. By way of example, we have

$$\begin{aligned} D(2) &= D^2 \{(1, 2)\} \\ D(3) &= D^3 \{(1, 2), (1, 3), (2, 3)\} \end{aligned}$$

and $D^3 \{1, 3\}$ can be identified with D via the canonical isomorphism

$$d \in D \mapsto (0, d, 0) \in D^3 \{1, 3\}$$

The notion of strong difference in synthetic differential geometry is based upon the following lemma.

Lemma 4 (*cf. the first Lemma in §3.4 of [4]*) *The diagram*

$$\begin{array}{ccc} & D^3 \{(1, 3), (2, 3)\} & \\ & \nearrow & \nwarrow \\ D^2 & & D^2 \\ & \nwarrow & \nearrow \\ & D^2 \{(1, 2)\} & \end{array} \quad (6)$$

with the lower two arrows being the canonical injections and the upper two arrows being

$$\begin{aligned} j_1^2 : (d_1, d_2) \in D^2 &\mapsto (d_1, d_2, d_1 d_2) \in D^3 \{(1, 3), (2, 3)\} \\ j_2^2 : (d_1, d_2) \in D^2 &\mapsto (d_1, d_2, 0) \in D^3 \{(1, 3), (2, 3)\} \end{aligned}$$

from left to right is a quasi-colimit diagram.

Corollary 5 Let M be a microlinear space with two microsquares $\gamma_1, \gamma_2 : D^2 \rightarrow M$ abiding by

$$\gamma_1 \mid D^2 \{(1, 2)\} = \gamma_2 \mid D^2 \{(1, 2)\}$$

Then there exists a unique mapping

$$\mathbf{n}_{(\gamma_1, \gamma_2)}^2 : D^3 \{(1, 3), (2, 3)\} \rightarrow M$$

such that $\mathbf{n}_{(\gamma_1, \gamma_2)}^2 \circ j_1^2 = \gamma_1$ and $\mathbf{n}_{(\gamma_1, \gamma_2)}^2 \circ j_2^2 = \gamma_2$.

Notation 6 In the above notation in Corollary 5 we write $\gamma_1 \dot{-} \gamma_2$ for the mapping

$$d \in D \mapsto \mathbf{n}_{(\gamma_1, \gamma_2)}^2(0, 0, d)$$

The notion of strong difference can easily be relativized.

Lemma 7 Let n be a natural number. The diagram

$$\begin{array}{ccc} & D^{n+3} \{(n+1, n+3), (n+2, n+3)\} & \\ D^{n+2} & \nearrow & \nwarrow D^{n+2} \\ & D^{n+2} \{(n+1, n+2)\} & \nearrow \end{array} \quad (7)$$

with the lower two arrows being the canonical injections and the upper two arrows being

$$\begin{aligned} j_1^{n+2} : (d_1, \dots, d_n, d_{n+1}, d_{n+2}) \in D^{n+2} &\mapsto (d_1, \dots, d_n, d_{n+1}, d_{n+2}, d_{n+1}d_{n+2}) \in D^{n+3} \{(n+1, n+3), (n+2, n+3)\} \\ j_2^{n+2} : (d_1, \dots, d_n, d_{n+1}, d_{n+2}) \in D^{n+2} &\mapsto (d_1, \dots, d_n, d_{n+1}, d_{n+2}, 0) \in D^{n+3} \{(n+1, n+3), (n+2, n+3)\} \end{aligned}$$

from left to right is a quasi-colimit diagram.

Corollary 8 Let n be a natural number. Let M be a microlinear space with two mappings $\gamma_1, \gamma_2 : D^{n+2} \rightarrow M$ abiding by

$$\gamma_1 \mid D^{n+2} \{(n+1, n+2)\} = \gamma_2 \mid D^{n+2} \{(n+1, n+2)\}$$

Then there exists a unique mapping

$$\mathbf{n}_{(\gamma_1, \gamma_2)}^{n+2} : D^{n+3} \{(n+1, n+3), (n+2, n+3)\} \rightarrow M$$

such that $\mathbf{n}_{(\gamma_1, \gamma_2)}^{n+2} \circ j_1^{n+2} = \gamma_1$ and $\mathbf{n}_{(\gamma_1, \gamma_2)}^{n+2} \circ j_2^{n+2} = \gamma_2$.

Notation 9 Let n be a natural number. Let M be a microlinear space. Given $\gamma : D^n \rightarrow M$ and a permutation σ of $\{1, \dots, n\}$, we write γ^σ for the mapping

$$(d_1, \dots, d_n) \in D^n \mapsto \gamma(d_{\sigma^{-1}(1)}, \dots, d_{\sigma^{-1}(n)}) \in M$$

Notation 10 Let n be a natural number. Let M be a microlinear space.

1. Given $\gamma_1, \gamma_2 : D^{n+2} \rightarrow M$ with

$$\gamma_1 | D^{n+2} \{(n+1, n+2)\} = \gamma_2 | D^{n+2} \{(n+1, n+2)\}$$

we write $\gamma_1 \underset{1\dots n}{\dot{=}} \gamma_2$ for the mapping

$$(d_1, \dots, d_n, d_{n+1}) \in D^{n+1} \mapsto \mathbf{n}_{(\gamma_1, \gamma_2)}^{n+2} (d_1, \dots, d_n, 0, 0, d_{n+1}) \in M$$

2. Given a permutation σ of $\{1, \dots, n, n+1, n+2\}$ and $\gamma_1, \gamma_2 : D^{n+2} \rightarrow M$ with

$$\gamma_1 | D^{n+2} \{(\sigma(n+1), \sigma(n+2))\} = \gamma_2 | D^{n+2} \{(\sigma(n+1), \sigma(n+2))\}$$

we write $\gamma_1 \underset{\sigma(1)\dots\sigma(n)}{\dot{=}} \gamma_2$ for $(\gamma_1)^\sigma \underset{1\dots n}{\dot{=}} (\gamma_2)^\sigma$.

The following result is well known.

Lemma 11 (cf. Proposition 6 in §2.2 of [4]) The diagram

$$\begin{array}{ccc} & D^2 \{(1, 2)\} & \\ & \nearrow & \nwarrow \\ D^2 \{1\} & & D^2 \{2\} \\ & \nwarrow & \nearrow \\ & 1 & \end{array} \quad (8)$$

with the four arrows being the canonical injections is a quasi-colimit diagram.

Corollary 12 Let M be a microlinear space. Given $\gamma_1, \gamma_2 : D^2 \rightarrow M$,

$$\gamma_1 | D^2 \{(1, 2)\} = \gamma_2 | D^2 \{(1, 2)\}$$

obtains iff both

$$\gamma_1 | D^2 \{1\} = \gamma_2 | D^2 \{1\}$$

and

$$\gamma_1 | D^2 \{2\} = \gamma_2 | D^2 \{2\}$$

obtain.

It can readily be relativized and generalized.

Lemma 13 *The diagram*

$$\begin{array}{ccc}
& & D^{n+m_1+m_2} \left\{ \begin{array}{l} (n+i, n+m_1+j) \mid \\ 1 \leq i \leq m_1, 1 \leq j \leq m_2 \end{array} \right\} \\
& \nearrow & \nwarrow \\
D^{n+m_1+m_2} \left\{ \begin{array}{l} n+1, \dots, \\ n+m_1 \end{array} \right\} & & D^{n+m_1+m_2} \left\{ \begin{array}{l} n+m_1+1, \dots \\ n+m_1+m_2 \end{array} \right\} \\
& \nwarrow & \nearrow \\
& D^n &
\end{array} \tag{9}$$

with the four arrows being the canonical injections is a quasi-colimit diagram.

Corollary 14 *Let M be a microlinear space. Given $\gamma_1, \gamma_2 : D^{n+m_1+m_2} \rightarrow M$,*

$$\begin{aligned}
& \gamma_1 \mid D^{n+m_1+m_2} \{(n+i, n+m_1+j) \mid 1 \leq i \leq m_1, 1 \leq j \leq m_2\} \\
& = \gamma_2 \mid D^{n+m_1+m_2} \{(n+i, n+m_1+j) \mid 1 \leq i \leq m_1, 1 \leq j \leq m_2\}
\end{aligned}$$

obtains iff both

$$\gamma_1 \mid D^{n+m_1+m_2} \{n+1, \dots, n+m_1\} = \gamma_2 \mid D^{n+m_1+m_2} \{n+1, \dots, n+m_1\}$$

and

$$\gamma_1 \mid D^{n+m_1+m_2} \{n+m_1+1, \dots, n+m_1+m_2\} = \gamma_2 \mid D^{n+m_1+m_2} \{n+m_1+1, \dots, n+m_1+m_2\}$$

obtain.

Proposition 15 *Let M be a microlinear space. Then we have the following two statements:*

1. *Given*

$$\gamma_1 : D^4 \rightarrow M, \gamma_2 : D^4 \rightarrow M, \gamma_3 : D^4 \rightarrow M, \gamma_4 : D^4 \rightarrow M$$

if it holds that

$$\begin{aligned}
& \gamma_1 \mid D^4 \{(3, 4)\} = \gamma_2 \mid D^4 \{(3, 4)\} \\
& \gamma_3 \mid D^4 \{(3, 4)\} = \gamma_4 \mid D^4 \{(3, 4)\} \\
& \gamma_1 \mid D^4 \{(2, 3), (2, 4)\} = \gamma_3 \mid D^4 \{(2, 3), (2, 4)\} \tag{10}
\end{aligned}$$

$$\gamma_2 \mid D^4 \{(2, 3), (2, 4)\} = \gamma_4 \mid D^4 \{(2, 3), (2, 4)\} \tag{11}$$

then all of

$$\begin{aligned}
& \gamma_1 \overset{\cdot}{\underset{12}{\dashv}} \gamma_2 \\
& \gamma_3 \overset{\cdot}{\underset{12}{\dashv}} \gamma_4 \\
& \left(\gamma_1 \overset{\cdot}{\underset{12}{\dashv}} \gamma_2 \right) \overset{\cdot}{\underset{1}{\dashv}} \left(\gamma_3 \overset{\cdot}{\underset{12}{\dashv}} \gamma_4 \right)
\end{aligned}$$

are well defined.

2. Given

$$\begin{aligned}\gamma_1 : D^4 \rightarrow M, \gamma_2 : D^4 \rightarrow M, \gamma_3 : D^4 \rightarrow M, \gamma_4 : D^4 \rightarrow M, \\ \gamma_5 : D^4 \rightarrow M, \gamma_6 : D^4 \rightarrow M, \gamma_7 : D^4 \rightarrow M, \gamma_8 : D^4 \rightarrow M\end{aligned}$$

if it holds that

$$\begin{aligned}\gamma_1 \mid D^4 \{(3, 4)\} &= \gamma_2 \mid D^4 \{(3, 4)\} \\ \gamma_3 \mid D^4 \{(3, 4)\} &= \gamma_4 \mid D^4 \{(3, 4)\} \\ \gamma_5 \mid D^4 \{(3, 4)\} &= \gamma_6 \mid D^4 \{(3, 4)\} \\ \gamma_7 \mid D^4 \{(3, 4)\} &= \gamma_8 \mid D^4 \{(3, 4)\} \\ \gamma_1 \mid D^4 \{(2, 3), (2, 4)\} &= \gamma_3 \mid D^4 \{(2, 3), (2, 4)\} \\ \gamma_2 \mid D^4 \{(2, 3), (2, 4)\} &= \gamma_4 \mid D^4 \{(2, 3), (2, 4)\} \\ \gamma_5 \mid D^4 \{(2, 3), (2, 4)\} &= \gamma_7 \mid D^4 \{(2, 3), (2, 4)\} \\ \gamma_6 \mid D^4 \{(2, 3), (2, 4)\} &= \gamma_8 \mid D^4 \{(2, 3), (2, 4)\} \\ \gamma_1 \mid D^4 \{(1, 2), (1, 3), (1, 4)\} &= \gamma_5 \mid D^4 \{(1, 2), (1, 3), (1, 4)\} \quad (12) \\ \gamma_2 \mid D^4 \{(1, 2), (1, 3), (1, 4)\} &= \gamma_6 \mid D^4 \{(1, 2), (1, 3), (1, 4)\} \quad (13) \\ \gamma_3 \mid D^4 \{(1, 2), (1, 3), (1, 4)\} &= \gamma_7 \mid D^4 \{(1, 2), (1, 3), (1, 4)\} \quad (14) \\ \gamma_4 \mid D^4 \{(1, 2), (1, 3), (1, 4)\} &= \gamma_8 \mid D^4 \{(1, 2), (1, 3), (1, 4)\} \quad (15)\end{aligned}$$

then all of

$$\begin{aligned}\dot{\gamma}_1 \dot{\gamma}_2 \\ \dot{\gamma}_3 \dot{\gamma}_4 \\ \dot{\gamma}_5 \dot{\gamma}_6 \\ \dot{\gamma}_7 \dot{\gamma}_8 \\ \left(\dot{\gamma}_1 \dot{\gamma}_2 \right) \dot{\gamma}_3 \dot{\gamma}_4 \\ \left(\dot{\gamma}_5 \dot{\gamma}_6 \right) \dot{\gamma}_7 \dot{\gamma}_8 \\ \left(\left(\dot{\gamma}_1 \dot{\gamma}_2 \right) \dot{\gamma}_3 \dot{\gamma}_4 \right) \dot{\gamma}_5 \dot{\gamma}_6 \left(\dot{\gamma}_7 \dot{\gamma}_8 \right)\end{aligned}$$

are well defined.

Proof. We deal with the above two statements in order.

1. For the first statement, we have to show that

$$\left(\dot{\gamma}_1 \dot{\gamma}_2 \right) \mid D^3 \{(2, 3)\} = \left(\dot{\gamma}_3 \dot{\gamma}_4 \right) \mid D^3 \{(2, 3)\}$$

which is, by dint of Corollary 14, tantamount to showing that

$$\left(\gamma_1 \begin{smallmatrix} \cdot \\ \hline \gamma_2 \end{smallmatrix} \right) | D^3 \{2\} = \left(\gamma_3 \begin{smallmatrix} \cdot \\ \hline \gamma_4 \end{smallmatrix} \right) | D^3 \{2\} \quad (16)$$

$$\left(\gamma_1 \begin{smallmatrix} \cdot \\ \hline \gamma_2 \end{smallmatrix} \right) | D^3 \{3\} = \left(\gamma_3 \begin{smallmatrix} \cdot \\ \hline \gamma_4 \end{smallmatrix} \right) | D^3 \{3\} \quad (17)$$

because of the quasi-colimit diagram

$$\begin{array}{ccc} & D^3 \{(2, 3)\} & \\ \nearrow & & \nwarrow \\ D^3 \{2\} & & D^3 \{3\} \\ \nwarrow & & \nearrow \\ & D^3 \{2, 3\} & \end{array}$$

with the four arrows being the canonical injections (Lemma 13 with $n = m_1 = m_2 = 1$). Due to the quasi-colimit diagram

$$\begin{array}{ccc} & D^4 \{(2, 3), (2, 4)\} & \\ \nearrow & & \nwarrow \\ D^4 \{2\} & & D^4 \{3, 4\} \\ \nwarrow & & \nearrow \\ & D^4 \{2, 3, 4\} & \end{array}$$

with the four arrows being the canonical injections (Lemma 13 with $n = m_1 = 1$ and $m_2 = 2$), the condition (10) is equivalent to the conditions

$$\gamma_1 | D^4 \{2\} = \gamma_3 | D^4 \{2\} \quad (18)$$

$$\gamma_1 | D^4 \{3, 4\} = \gamma_3 | D^4 \{3, 4\} \quad (19)$$

while the condition (11) is equivalent to the conditions

$$\gamma_2 | D^4 \{2\} = \gamma_4 | D^4 \{2\} \quad (20)$$

$$\gamma_2 | D^4 \{3, 4\} = \gamma_4 | D^4 \{3, 4\} \quad (21)$$

In order to show that (16) obtains, we note that the quasi-colimit diagram in (7) with $n = 2$ is to be restricted to the quasi-colimit diagram

$$\begin{array}{ccc} & D^5 \{2, (3, 5), (4, 5)\} & \\ \nearrow & & \nwarrow \\ D^4 \{2\} & & D^4 \{2\} \\ \nwarrow & & \nearrow \\ & D^4 \{2, (3, 4)\} & \end{array}$$

so that the conditions (18) and (20) imply (16). It is easy to see that

$$\begin{aligned} \left(\left(\gamma_1 \begin{smallmatrix} \cdot \\ \hline \gamma_2 \end{smallmatrix} \right) | D^3 \{3\} \right) \circ i_2^3 &= (\gamma_1 | D^4 \{3, 4\}) \circ i_2^4 = (\gamma_2 | D^4 \{3, 4\}) \circ i_2^4 \\ \left(\left(\gamma_3 \begin{smallmatrix} \cdot \\ \hline \gamma_4 \end{smallmatrix} \right) | D^3 \{3\} \right) \circ i_2^3 &= (\gamma_3 | D^4 \{3, 4\}) \circ i_2^4 = (\gamma_4 | D^4 \{3, 4\}) \circ i_2^4 \end{aligned}$$

obtain with

$$\begin{aligned} i_2^3 &: (d_1, d_2) \in D^2 \mapsto (d_1, d_2, 0) \in D^3 \\ i_2^4 &: (d_1, d_2) \in D^2 \mapsto (d_1, d_2, 0, 0) \in D^4 \end{aligned}$$

so that (19) and (21) imply (17).

2. For the second statement, we have to show that

$$\left(\left(\gamma_1 \begin{smallmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{smallmatrix} \gamma_2 \right) \begin{smallmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{smallmatrix} \left(\gamma_3 \begin{smallmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{smallmatrix} \gamma_4 \right) \right) | D^2 \{(1, 2)\} = \left(\left(\gamma_5 \begin{smallmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{smallmatrix} \gamma_6 \right) \begin{smallmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{smallmatrix} \left(\gamma_7 \begin{smallmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{smallmatrix} \gamma_8 \right) \right) | D^2 \{(1, 2)\}$$

which is tantamount to showing that

$$\left(\left(\gamma_1 \begin{smallmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{smallmatrix} \gamma_2 \right) \begin{smallmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{smallmatrix} \left(\gamma_3 \begin{smallmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{smallmatrix} \gamma_4 \right) \right) | D^2 \{1\} = \left(\left(\gamma_5 \begin{smallmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{smallmatrix} \gamma_6 \right) \begin{smallmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{smallmatrix} \left(\gamma_7 \begin{smallmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{smallmatrix} \gamma_8 \right) \right) | D^2 \{1\} \quad (22)$$

$$\left(\left(\gamma_1 \begin{smallmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{smallmatrix} \gamma_2 \right) \begin{smallmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{smallmatrix} \left(\gamma_3 \begin{smallmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{smallmatrix} \gamma_4 \right) \right) | D^2 \{2\} = \left(\left(\gamma_5 \begin{smallmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{smallmatrix} \gamma_6 \right) \begin{smallmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{smallmatrix} \left(\gamma_7 \begin{smallmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{smallmatrix} \gamma_8 \right) \right) | D^2 \{2\} \quad (23)$$

because of the quasi-colimit diagram

$$\begin{array}{ccc} & D^2 \{(1, 2)\} & \\ & \nearrow & \nwarrow \\ D^2 \{1\} & & D^2 \{2\} \\ & \nwarrow & \nearrow \\ & D^2 \{1, 2\} & \end{array}$$

with the four arrows being the canonical injections (Lemma 11). Due to the quasi-colimit diagram

$$\begin{array}{ccc} & D^4 \{(1, 2), (1, 3), (1, 4)\} & \\ & \nearrow & \nwarrow \\ D^4 \{1\} & & D^4 \{2, 3, 4\} \\ & \nwarrow & \nearrow \\ & D^4 \{1, 2, 3, 4\} & \end{array}$$

with the four arrows being the canonical injections (Lemma 13 with $n = 0$, $m_1 = 1$ and $m_2 = 3$), we have

- the condition (12) is equivalent to the conditions

$$\gamma_1 | D^4 \{1\} = \gamma_5 | D^4 \{1\} \quad (24)$$

$$\gamma_1 | D^4 \{2, 3, 4\} = \gamma_5 | D^4 \{2, 3, 4\} \quad (25)$$

- the condition (13) is equivalent to the conditions

$$\gamma_2 | D^4 \{1\} = \gamma_6 | D^4 \{1\} \quad (26)$$

$$\gamma_2 | D^4 \{2, 3, 4\} = \gamma_6 | D^4 \{2, 3, 4\} \quad (27)$$

- the condition (14) is equivalent to the conditions

$$\gamma_3 | D^4 \{1\} = \gamma_7 | D^4 \{1\} \quad (28)$$

$$\gamma_3 | D^4 \{2, 3, 4\} = \gamma_7 | D^4 \{2, 3, 4\} \quad (29)$$

- the condition (15) is equivalent to the conditions

$$\gamma_4 | D^4 \{1\} = \gamma_8 | D^4 \{1\} \quad (30)$$

$$\gamma_4 | D^4 \{2, 3, 4\} = \gamma_8 | D^4 \{2, 3, 4\} \quad (31)$$

In order to show that (22) obtains, we note first that the quasi-colimit diagram in 7 with $n = 2$ is to be restricted to the quasi-colimit diagram

$$\begin{array}{ccc} & D^5 \{1, (3, 5), (4, 5)\} & \\ & \nearrow & \nwarrow \\ D^4 \{1\} & & D^4 \{1\} \\ & \nwarrow & \nearrow \\ & D^4 \{1, (3, 4)\} & \end{array}$$

so that we have

- the conditions (24) and (26) imply

$$\left(\gamma_1 \begin{smallmatrix} \cdot \\ \hline \cdot \end{smallmatrix} \gamma_2 \right) | D^3 \{1\} = \left(\gamma_5 \begin{smallmatrix} \cdot \\ \hline \cdot \end{smallmatrix} \gamma_6 \right) | D^3 \{1\} \quad (32)$$

- the conditions (28) and (30) imply

$$\left(\gamma_3 \begin{smallmatrix} \cdot \\ \hline \cdot \end{smallmatrix} \gamma_4 \right) | D^3 \{1\} = \left(\gamma_7 \begin{smallmatrix} \cdot \\ \hline \cdot \end{smallmatrix} \gamma_8 \right) | D^3 \{1\} \quad (33)$$

We note also that the quasi-colimit diagram in (7) with $n = 1$ is to be restricted to the quasi-colimit diagram

$$\begin{array}{ccc} & D^4 \{1, (2, 4), (3, 4)\} & \\ & \nearrow & \nwarrow \\ D^3 \{1\} & & D^3 \{1\} \\ & \nwarrow & \nearrow \\ & D^3 \{1, (2, 3)\} & \end{array}$$

so that the conditions (32) and (??) imply the condition (22). It is easy to see that

$$\begin{aligned} & \left(\left(\left(\gamma_1 \begin{smallmatrix} \cdot \\ \hline \cdot \end{smallmatrix} \gamma_2 \right) \begin{smallmatrix} \cdot \\ \hline \cdot \end{smallmatrix} \left(\gamma_3 \begin{smallmatrix} \cdot \\ \hline \cdot \end{smallmatrix} \gamma_4 \right) \right) | D^2 \{2\} \right) \circ i^2 \\ &= \left(\left(\gamma_1 \begin{smallmatrix} \cdot \\ \hline \cdot \end{smallmatrix} \gamma_2 \right) | D^3 \{2, 3\} \right) \circ i^3 = \left(\left(\gamma_3 \begin{smallmatrix} \cdot \\ \hline \cdot \end{smallmatrix} \gamma_4 \right) | D^3 \{2, 3\} \right) \circ i^3 \\ &= (\gamma_1 | D^4 \{2, 3, 4\}) \circ i^4 = (\gamma_2 | D^4 \{2, 3, 4\}) \circ i^4 \\ &= (\gamma_3 | D^4 \{2, 3, 4\}) \circ i^4 = (\gamma_4 | D^4 \{2, 3, 4\}) \circ i^4 \end{aligned}$$

and

$$\begin{aligned}
& \left(\left(\left(\gamma_5 \begin{smallmatrix} \cdot \\ 12 \end{smallmatrix} \gamma_6 \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_7 \begin{smallmatrix} \cdot \\ 12 \end{smallmatrix} \gamma_8 \right) \right) \mid D^2 \{2\} \right) \circ i^2 \\
&= \left(\left(\gamma_5 \begin{smallmatrix} \cdot \\ 12 \end{smallmatrix} \gamma_6 \right) \mid D^3 \{2, 3\} \right) \circ i^3 = \left(\left(\gamma_7 \begin{smallmatrix} \cdot \\ 12 \end{smallmatrix} \gamma_8 \right) \mid D^3 \{2, 3\} \right) \circ i^3 \\
&= (\gamma_5 \mid D^4 \{2, 3, 4\}) \circ i^4 = (\gamma_6 \mid D^4 \{2, 3, 4\}) \circ i^4 \\
&= (\gamma_7 \mid D^4 \{2, 3, 4\}) \circ i^4 = (\gamma_8 \mid D^4 \{2, 3, 4\}) \circ i^4
\end{aligned}$$

obtain with

$$\begin{aligned}
i^2 &: d \in D \mapsto (d, 0) \in D^2 \{2\} \\
i^3 &: d \in D \mapsto (d, 0, 0) \in D^3 \{2, 3\} \\
i^4 &: d \in D \mapsto (d, 0, 0, 0) \in D^4 \{2, 3, 4\}
\end{aligned}$$

so that (25), (27), (29) and (31) imply (23).

■

Notation 16 *Let M be a microlinear space.*

1. We denote by $\mathfrak{X}(M)$ the totality of vector fields on M . It forms a Lie algebra. We take the third viewpoint of a vector field in the essentially equivalent three discussed in §3.2 of [4]. Namely, a vector field X on M is a mapping $d \in D \mapsto X_d \in M^M$ with $X_0 = \text{id}_M$.
2. Given $X, \dots, X_n \in \mathfrak{X}(M)$, we denote by $X_n * \dots * X_1$ the mapping $(d_1, \dots, d_n) \in D^n \mapsto (X_n)_{d_n} \circ \dots \circ (X_1)_{d_1} \in M^M$.
3. We recall (cf. Proposition 8 in §3.4 of [4]) that

$$[X_1, X_2] = X_2 * X_1 - (X_1 * X_2)^{\sigma_{21}}$$

with

$$\sigma_{21} = \begin{pmatrix} 12 \\ 21 \end{pmatrix}$$

We recall (cf. Proposition 2.7 of [6]) that

$$\begin{aligned}
& [X_1, [X_2, X_3]] \\
&= \left(X_3 * X_2 * X_1 \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} (X_2 * X_3 * X_1)^{\sigma_{132}} \right) - \left((X_1 * X_3 * X_2)^{\sigma_{231}} \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} (X_1 * X_2 * X_3)^{\sigma_{321}} \right)
\end{aligned}$$

with

$$\sigma_{132} = \begin{pmatrix} 123 \\ 132 \end{pmatrix}, \sigma_{231} = \begin{pmatrix} 123 \\ 312 \end{pmatrix}, \sigma_{321} = \begin{pmatrix} 123 \\ 321 \end{pmatrix}$$

We note that

$$\begin{aligned}
& [X_1, [X_2, [X_3, X_4]]] \\
&= \left(\begin{array}{c} \left(X_4 * X_3 * X_2 * X_1 \frac{\cdot}{12} (X_3 * X_4 * X_2 * X_1)^{\sigma_{1243}} \right) \frac{\cdot}{1} \\ \left((X_2 * X_4 * X_3 * X_1)^{\sigma_{1342}} \frac{\cdot}{12} (X_2 * X_3 * X_4 * X_1)^{\sigma_{1432}} \right) \end{array} \right) \frac{\cdot}{1} \\
& \left(\begin{array}{c} \left((X_1 * X_4 * X_3 * X_2)^{\sigma_{2341}} \frac{\cdot}{12} (X_1 * X_3 * X_4 * X_2)^{\sigma_{2431}} \right) \frac{\cdot}{1} \\ \left((X_1 * X_2 * X_4 * X_3)^{\sigma_{3421}} \frac{\cdot}{12} (X_1 * X_2 * X_3 * X_4)^{\sigma_{4321}} \right) \end{array} \right) \frac{\cdot}{1}
\end{aligned}$$

with

$$\begin{aligned}
\sigma_{1243} &= \begin{pmatrix} 1234 \\ 1243 \end{pmatrix}, \sigma_{1342} = \begin{pmatrix} 1234 \\ 1423 \end{pmatrix}, \sigma_{1432} = \begin{pmatrix} 1234 \\ 1432 \end{pmatrix}, \sigma_{2341} = \begin{pmatrix} 1234 \\ 4123 \end{pmatrix}, \\
\sigma_{2431} &= \begin{pmatrix} 1234 \\ 4132 \end{pmatrix}, \sigma_{3421} = \begin{pmatrix} 1234 \\ 4312 \end{pmatrix}, \sigma_{4321} = \begin{pmatrix} 1234 \\ 4321 \end{pmatrix}
\end{aligned}$$

3 A Four-Dimensional General Jacobi Identity

Theorem 17 *The diagram consisting of objects*

$$\begin{aligned}
& P, Q^{1234}, Q^{1243}, Q^{1324}, Q^{1342}, Q^{1423}, Q^{1432}, Q^{2134}, Q^{2143}, Q^{2314}, Q^{2341}, Q^{2413}, Q^{2431}, \\
& Q^{3124}, Q^{3142}, Q^{3214}, Q^{3241}, Q^{3412}, Q^{3421}, Q^{4123}, Q^{4132}, Q^{4213}, Q^{4231}, Q^{4312}, Q^{4321}, \\
& R_{12}^{1234,1243}, R_{12}^{1342,1432}, R_{12}^{2341,2431}, R_{12}^{3421,4321}, R_{12}^{2134,2143}, R_{12}^{3412,4312}, R_{13}^{1324,1342}, R_{13}^{1243,1423}, \\
& R_{13}^{3241,3421}, R_{13}^{2431,4231}, R_{13}^{3124,3142}, R_{13}^{2413,4213}, R_{14}^{1423,1432}, R_{14}^{1234,1324}, R_{14}^{4231,4321}, R_{14}^{2341,3241}, \\
& R_{14}^{4123,4132}, R_{14}^{2314,3214}, R_{23}^{2314,2341}, R_{23}^{2143,2413}, R_{23}^{3142,3412}, R_{23}^{1432,4132}, R_{23}^{3214,3241}, R_{23}^{1423,4123}, \\
& R_{24}^{2413,2431}, R_{24}^{2134,2314}, R_{24}^{4132,4312}, R_{24}^{1342,3142}, R_{24}^{4213,4231}, R_{24}^{1324,3124}, R_{34}^{3412,3421}, R_{34}^{3124,3214}, \\
& R_{34}^{4123,4213}, R_{34}^{1243,2143}, R_{34}^{4312,4321}, R_{34}^{1234,2134}
\end{aligned}$$

with

$$P = D^{53} \left\{ \begin{array}{l} (1, 5), (2, 5), (1, 6), (3, 6), (1, 7), (4, 7), (2, 8), (3, 8), (2, 9), (4, 9), \\ (3, 10), (4, 10), (5, 6), (5, 7), (5, 8), (5, 9), (6, 7), (6, 8), (6, 10), (7, 9), \\ (7, 10), (8, 9), (8, 10), (9, 10), (1, i_{11,15}), (2, i_{11,15}), (3, i_{11,15}), \\ (5, i_{11,15}), (6, i_{11,15}), (7, i_{11,15}), (8, i_{11,15}), (9, i_{11,15}), (10, i_{11,15}), \\ (1, i_{16,20}), (2, i_{16,20}), (4, i_{16,20}), (5, i_{16,20}), (6, i_{16,20}), (7, i_{16,20}), \\ (8, i_{16,20}), (9, i_{16,20}), (10, i_{16,20}), (1, i_{21,25}), (3, i_{21,25}), (4, i_{21,25}), \\ (5, i_{21,25}), (6, i_{21,25}), (7, i_{21,25}), (8, i_{21,25}), (9, i_{21,25}), (10, i_{21,25}), \\ (2, i_{26,30}), (3, i_{26,30}), (4, i_{26,30}), (5, i_{26,30}), (6, i_{26,30}), (7, i_{26,30}), \\ (8, i_{26,30}), (9, i_{26,30}), (10, i_{26,30}), (i_{11,15}, i'_{11,15}), (i_{11,15}, i_{16,20}), \\ (i_{11,15}, i_{21,25}), (i_{11,15}, i_{26,30}), (i_{16,20}, i'_{16,20}), (i_{16,20}, i_{21,25}), \\ (i_{16,20}, i_{26,30}), (i_{21,25}, i'_{21,25}), (i_{21,25}, i_{26,30}), (i_{26,30}, i'_{26,30}), \\ (1, i_{31,53}), (2, i_{31,53}), (3, i_{31,53}), (4, i_{31,53}), (5, i_{31,53}), (6, i_{31,53}), \\ (7, i_{31,53}), (8, i_{31,53}), (9, i_{31,53}), (10, i_{31,53}), (i_{11,15}, i_{31,53}), \\ (i_{16,20}, i_{31,53}), (i_{21,25}, i_{31,53}), (i_{26,30}, i_{31,53}), (i_{31,53}, i'_{31,53}) \\ | i_{11,15}, i'_{11,15}, i_{16,20}, i'_{16,20}, i_{21,25}, i'_{21,25}, i_{26,30}, i'_{26,30}, i_{31,53}, i'_{31,53} \in \mathbb{N}, \\ 1 \leq i_{11,15} \leq 15, 11 \leq i'_{11,15} \leq 15, 16 \leq i_{16,20} \leq 20, \\ 16 \leq i'_{16,20} \leq 20, 21 \leq i_{21,25} \leq 25, 21 \leq i'_{21,25} \leq 25, \\ 26 \leq i_{26,30} \leq 30, 26 \leq i'_{26,30} \leq 30, 31 \leq i_{31,53} \leq 53, \\ 31 \leq i'_{31,53} \leq 53 \end{array} \right.$$

$$\begin{aligned} Q^{1234} &= Q^{1243} = Q^{1324} = Q^{1342} = Q^{1423} = Q^{1432} = \\ Q^{2134} &= Q^{2143} = Q^{2314} = Q^{2341} = Q^{2413} = Q^{2431} = \\ Q^{3124} &= Q^{3142} = Q^{3214} = Q^{3241} = Q^{3412} = Q^{3421} = \\ Q^{4123} &= Q^{4132} = Q^{4213} = Q^{4231} = Q^{4312} = Q^{4321} = D^4 \end{aligned}$$

$$\begin{aligned} R_{12}^{1234,1243} &= R_{12}^{1342,1432} = R_{12}^{2341,2431} = R_{12}^{3421,4321} = \\ R_{12}^{2134,2143} &= R_{12}^{3412,4312} = D^4 \{(3, 4)\} \end{aligned}$$

$$\begin{aligned} R_{13}^{1324,1342} &= R_{13}^{1243,1423} = R_{13}^{3241,3421} = R_{13}^{2431,4231} = \\ R_{13}^{3124,3142} &= R_{13}^{2413,4213} = D^4 \{(2, 4)\} \end{aligned}$$

$$\begin{aligned} R_{14}^{1423,1432} &= R_{14}^{1234,1324} = R_{14}^{4231,4321} = R_{14}^{2341,3241} = \\ R_{14}^{4123,4132} &= R_{14}^{2314,3214} = D^4 \{(2, 3)\} \end{aligned}$$

$$\begin{aligned} R_{23}^{2314,2341} &= R_{23}^{2143,2413} = R_{23}^{3142,3412} = R_{23}^{1432,4132} = \\ R_{23}^{3214,3241} &= R_{23}^{1423,4123} = D^4 \{(1, 4)\} \end{aligned}$$

$$\begin{aligned} R_{24}^{2413,2431} &= R_{24}^{2134,2314} = R_{24}^{4132,4312} = R_{24}^{1342,3142} = \\ R_{24}^{4213,4231} &= R_{24}^{1324,3124} = D^4 \{(1, 3)\} \end{aligned}$$

$$R_{34}^{3412,3421} = R_{34}^{3124,3214} = R_{34}^{4123,4213} = R_{34}^{1243,2143} =$$

$$R_{34}^{4312,4321} = R_{34}^{1234,2134} = D^4 \{(1, 2)\}$$

and of morphisms

$$f_{1234} : Q^{1234} \rightarrow P, f_{1243} : Q^{1243} \rightarrow P, f_{1324} : Q^{1324} \rightarrow P, f_{1342} : Q^{1342} \rightarrow P,$$

$$f_{1423} : Q^{1423} \rightarrow P, f_{1432} : Q^{1432} \rightarrow P, f_{2134} : Q^{2134} \rightarrow P, f_{2143} : Q^{2143} \rightarrow P,$$

$$f_{2314} : Q^{2314} \rightarrow P, f_{2341} : Q^{2341} \rightarrow P, f_{2413} : Q^{2413} \rightarrow P, f_{2431} : Q^{2431} \rightarrow P,$$

$$f_{3124} : Q^{3124} \rightarrow P, f_{3142} : Q^{3142} \rightarrow P, f_{3214} : Q^{3214} \rightarrow P, f_{3241} : Q^{3241} \rightarrow P,$$

$$f_{3412} : Q^{3412} \rightarrow P, f_{3421} : Q^{3421} \rightarrow P, f_{4123} : Q^{4123} \rightarrow P, f_{4132} : Q^{4132} \rightarrow P,$$

$$f_{4213} : Q^{4213} \rightarrow P, f_{4231} : Q^{4231} \rightarrow P, f_{4312} : Q^{4312} \rightarrow P, f_{4321} : Q^{4321} \rightarrow P,$$

$$g_{12}^{1234,1243} : R_{12}^{1234,1243} \rightarrow Q^{1234}, h_{12}^{1234,1243} : R_{12}^{1234,1243} \rightarrow Q^{1243},$$

$$g_{12}^{1342,1432} : R_{12}^{1342,1432} \rightarrow Q^{1342}, h_{12}^{1342,1432} : R_{12}^{1342,1432} \rightarrow Q^{1432},$$

$$g_{12}^{2341,2431} : R_{12}^{2341,2431} \rightarrow Q^{2341}, h_{12}^{2341,2431} : R_{12}^{2341,2431} \rightarrow Q^{2431},$$

$$g_{12}^{3421,4321} : R_{12}^{3421,4321} \rightarrow Q^{3421}, h_{12}^{3421,4321} : R_{12}^{3421,4321} \rightarrow Q^{4321},$$

$$g_{12}^{2134,2143} : R_{12}^{2134,2143} \rightarrow Q^{2134}, h_{12}^{2134,2143} : R_{12}^{2134,2143} \rightarrow Q^{2143},$$

$$g_{12}^{3412,4312} : R_{12}^{3412,4312} \rightarrow Q^{3412}, h_{12}^{3412,4312} : R_{12}^{3412,4312} \rightarrow Q^{4312},$$

$$g_{13}^{1324,1342} : R_{13}^{1324,1342} \rightarrow Q^{1324}, h_{13}^{1324,1342} : R_{13}^{1324,1342} \rightarrow Q^{1342},$$

$$g_{13}^{1243,1423} : R_{13}^{1243,1423} \rightarrow Q^{1243}, h_{13}^{1243,1423} : R_{13}^{1243,1423} \rightarrow Q^{1423},$$

$$g_{13}^{3241,3421} : R_{13}^{3241,3421} \rightarrow Q^{3241}, h_{13}^{3241,3421} : R_{13}^{3241,3421} \rightarrow Q^{3421},$$

$$g_{13}^{2431,4231} : R_{13}^{2431,4231} \rightarrow Q^{2431}, h_{13}^{2431,4231} : R_{13}^{2431,4231} \rightarrow Q^{4231},$$

$$g_{13}^{3124,3142} : R_{13}^{3124,3142} \rightarrow Q^{3124}, h_{13}^{3124,3142} : R_{13}^{3124,3142} \rightarrow Q^{3142},$$

$$g_{13}^{2413,4213} : R_{13}^{2413,4213} \rightarrow Q^{2413}, h_{13}^{2413,4213} : R_{13}^{2413,4213} \rightarrow Q^{4213},$$

$$g_{14}^{1423,1432} : R_{14}^{1423,1432} \rightarrow Q^{1423}, h_{14}^{1423,1432} : R_{14}^{1423,1432} \rightarrow Q^{1432},$$

$$g_{14}^{1234,1324} : R_{14}^{1234,1324} \rightarrow Q^{1234}, h_{14}^{1234,1324} : R_{14}^{1234,1324} \rightarrow Q^{1324},$$

$$g_{14}^{4231,4321} : R_{14}^{4231,4321} \rightarrow Q^{4231}, h_{14}^{4231,4321} : R_{14}^{4231,4321} \rightarrow Q^{4321},$$

$$g_{14}^{2341,3241} : R_{14}^{2341,3241} \rightarrow Q^{2341}, h_{14}^{2341,3241} : R_{14}^{2341,3241} \rightarrow Q^{3241},$$

$$g_{14}^{4123,4132} : R_{14}^{4123,4132} \rightarrow Q^{4123}, h_{14}^{4123,4132} : R_{14}^{4123,4132} \rightarrow Q^{4132},$$

$$g_{14}^{2314,3214} : R_{14}^{2314,3214} \rightarrow Q^{2314}, h_{14}^{2314,3214} : R_{14}^{2314,3214} \rightarrow Q^{3214},$$

$$\begin{aligned}
g_{23}^{2314,2341} : R_{23}^{2314,2341} &\rightarrow Q^{2314}, h_{23}^{2314,2341} : R_{23}^{2314,2341} \rightarrow Q^{2341}, \\
g_{23}^{2143,2413} : R_{23}^{2143,2413} &\rightarrow Q^{2143}, h_{23}^{2143,2413} : R_{23}^{2143,2413} \rightarrow Q^{2413}, \\
g_{23}^{3142,3412} : R_{23}^{3142,3412} &\rightarrow Q^{3142}, h_{23}^{3142,3412} : R_{23}^{3142,3412} \rightarrow Q^{3412}, \\
g_{23}^{1432,4132} : R_{23}^{1432,4132} &\rightarrow Q^{1432}, h_{23}^{1432,4132} : R_{23}^{1432,4132} \rightarrow Q^{4132}, \\
g_{23}^{3214,3241} : R_{23}^{3214,3241} &\rightarrow Q^{3214}, h_{23}^{3214,3241} : R_{23}^{3214,3241} \rightarrow Q^{3241}, \\
g_{23}^{1423,4123} : R_{23}^{1423,4123} &\rightarrow Q^{1423}, h_{23}^{1423,4123} : R_{23}^{1423,4123} \rightarrow Q^{4123},
\end{aligned}$$

$$\begin{aligned}
g_{24}^{2413,2431} : R_{24}^{2413,2431} &\rightarrow Q^{2413}, h_{24}^{2413,2431} : R_{24}^{2413,2431} \rightarrow Q^{2431}, \\
g_{24}^{2134,2314} : R_{24}^{2134,2314} &\rightarrow Q^{2134}, h_{24}^{2134,2314} : R_{24}^{2134,2314} \rightarrow Q^{2314}, \\
g_{24}^{4132,4312} : R_{24}^{4132,4312} &\rightarrow Q^{4132}, h_{24}^{4132,4312} : R_{24}^{4132,4312} \rightarrow Q^{4312}, \\
g_{24}^{1342,3142} : R_{24}^{1342,3142} &\rightarrow Q^{1342}, h_{24}^{1342,3142} : R_{24}^{1342,3142} \rightarrow Q^{3142}, \\
g_{24}^{4213,4231} : R_{24}^{4213,4231} &\rightarrow Q^{4213}, h_{24}^{4213,4231} : R_{24}^{4213,4231} \rightarrow Q^{4231}, \\
g_{24}^{1324,3124} : R_{24}^{1324,3124} &\rightarrow Q^{1324}, h_{24}^{1324,3124} : R_{24}^{1324,3124} \rightarrow Q^{3124},
\end{aligned}$$

$$\begin{aligned}
g_{34}^{3412,3421} : R_{34}^{3412,3421} &\rightarrow Q^{3412}, h_{34}^{3412,3421} : R_{34}^{3412,3421} \rightarrow Q^{3421}, \\
g_{34}^{3124,3214} : R_{34}^{3124,3214} &\rightarrow Q^{3124}, h_{34}^{3124,3214} : R_{34}^{3124,3214} \rightarrow Q^{3214}, \\
g_{34}^{4123,4213} : R_{34}^{4123,4213} &\rightarrow Q^{4123}, h_{34}^{4123,4213} : R_{34}^{4123,4213} \rightarrow Q^{4213}, \\
g_{34}^{1243,2143} : R_{34}^{1243,2143} &\rightarrow Q^{1243}, h_{34}^{1243,2143} : R_{34}^{1243,2143} \rightarrow Q^{2143}, \\
g_{34}^{4312,4321} : R_{34}^{4312,4321} &\rightarrow Q^{4312}, h_{34}^{4312,4321} : R_{34}^{4312,4321} \rightarrow Q^{4321}, \\
g_{34}^{1234,2134} : R_{34}^{1234,2134} &\rightarrow Q^{1234}, h_{34}^{1234,2134} : R_{34}^{1234,2134} \rightarrow Q^{2134}
\end{aligned}$$

with

$$\begin{aligned}
&f_{1234}(d_1, d_2, d_3, d_4) \\
&= (d_1, d_2, d_3, d_4, \underset{5}{0}, \dots, \underset{53}{0})
\end{aligned}$$

$$\begin{aligned}
&f_{1243}(d_1, d_2, d_3, d_4) \\
&= \left(d_1, d_2, d_3, d_4, \underset{5}{0}, \dots, \underset{9}{0}, \underset{10}{d_3 d_4}, \underset{11}{0}, \dots, \underset{20}{0}, \underset{22}{d_1 d_3 d_4}, \dots, \underset{25}{0}, \underset{27}{d_2 d_3 d_4}, \dots, \underset{30}{0}, \right. \\
&\quad \left. d_1 d_2 d_3 d_4, \underset{32}{0}, \dots, \underset{53}{0} \right)
\end{aligned}$$

$$\begin{aligned}
&f_{1324}(d_1, d_2, d_3, d_4) \\
&= \left(d_1, d_2, d_3, d_4, \underset{5}{0}, \dots, \underset{7}{0}, \underset{9}{d_2 d_3}, \underset{10}{0}, \dots, \underset{12}{0}, \underset{26}{d_1 d_2 d_3}, \dots, \underset{28}{0}, \dots, \underset{31}{0}, \right. \\
&\quad \left. d_1 d_2 d_3 d_4, \underset{33}{0}, \dots, \underset{53}{0} \right)
\end{aligned}$$

$$\begin{aligned}
& f_{1342}(d_1, d_2, d_3, d_4) \\
&= \begin{pmatrix} d_1, d_2, d_3, d_4, \underset{5}{0}, \dots, \underset{7}{0}, d_2d_3, d_2d_4, \underset{10}{0}, d_1d_2d_3, \underset{12}{0}, \dots, \underset{15}{0}, d_1d_2d_4, \underset{17}{0}, \dots, \underset{27}{0}, \\ d_2d_3d_4, \underset{29}{0}, \dots, \underset{32}{0}, d_1d_2d_3d_4, \underset{34}{0}, \dots, \underset{53}{0} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
& f_{1423}(d_1, d_2, d_3, d_4) \\
&= \begin{pmatrix} d_1, d_2, d_3, d_4, \underset{5}{0}, \dots, \underset{8}{0}, d_2d_4, d_3d_4, \underset{11}{0}, \dots, \underset{15}{0}, d_1d_2d_4, \underset{17}{0}, \dots, \underset{20}{0}, d_1d_3d_4, \\ \underset{22}{0}, \dots, \underset{28}{0}, d_2d_3d_4, \underset{30}{0}, \dots, \underset{33}{0}, d_1d_2d_3d_4, \underset{35}{0}, \dots, \underset{53}{0} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
& f_{1432}(d_1, d_2, d_3, d_4) \\
&= \begin{pmatrix} d_1, d_2, d_3, d_4, \underset{5}{0}, \dots, \underset{7}{0}, d_2d_3, d_2d_4, d_3d_4, d_1d_2d_3, \underset{12}{0}, \dots, \underset{15}{0}, d_1d_2d_4, \underset{16}{0}, \dots, \underset{20}{0}, \\ d_1d_3d_4, \underset{22}{0}, \dots, \underset{29}{0}, d_2d_3d_4, \underset{31}{0}, \dots, \underset{34}{0}, d_1d_2d_3d_4, \underset{36}{0}, \dots, \underset{53}{0} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
& f_{2134}(d_1, d_2, d_3, d_4) \\
&= \left(d_1, d_2, d_3, d_4, d_1d_2, \underset{6}{0}, \dots, \underset{11}{0}, d_1d_2d_3, \underset{13}{0}, \dots, \underset{16}{0}, d_1d_2d_4, \underset{18}{0}, \dots, \underset{35}{0}, d_1d_2d_3d_4, \underset{37}{0}, \dots, \underset{53}{0} \right)
\end{aligned}$$

$$\begin{aligned}
& f_{2143}(d_1, d_2, d_3, d_4) \\
&= \begin{pmatrix} d_1, d_2, d_3, d_4, d_1d_2, \underset{6}{0}, \dots, \underset{9}{0}, d_3d_4, \underset{11}{0}, d_1d_2d_3, \underset{13}{0}, \dots, \underset{16}{0}, d_1d_2d_4, \underset{18}{0}, \dots, \underset{20}{0}, d_1d_3d_4, \\ \underset{22}{0}, \dots, \underset{25}{0}, d_2d_3d_4, \underset{27}{0}, \dots, \underset{36}{0}, d_1d_2d_3d_4, \underset{38}{0}, \dots, \underset{53}{0} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
& f_{2314}(d_1, d_2, d_3, d_4) \\
&= \begin{pmatrix} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, \underset{7}{0}, \dots, \underset{12}{0}, d_1d_2d_3, \underset{14}{0}, \dots, \underset{16}{0}, d_1d_2d_4, \underset{18}{0}, \dots, \underset{21}{0}, d_1d_3d_4, \\ \underset{23}{0}, \dots, \underset{37}{0}, d_1d_2d_3d_4, \underset{39}{0}, \dots, \underset{53}{0} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
& f_{2341}(d_1, d_2, d_3, d_4) \\
&= \begin{pmatrix} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, d_1d_4, \underset{8}{0}, \dots, \underset{12}{0}, d_1d_2d_3, \underset{14}{0}, \dots, \underset{17}{0}, d_1d_2d_4, \underset{19}{0}, \dots, \underset{22}{0}, \\ d_1d_3d_4, \underset{24}{0}, \dots, \underset{38}{0}, d_1d_2d_3d_4, \underset{40}{0}, \dots, \underset{53}{0} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
& f_{2413}(d_1, d_2, d_3, d_4) \\
&= \begin{pmatrix} d_1, d_2, d_3, d_4, d_1d_2, \underset{6}{0}, d_1d_4, \underset{8}{0}, \underset{9}{0}, d_3d_4, \underset{11}{0}, d_1d_2d_3, \underset{13}{0}, \dots, \underset{17}{0}, d_1d_2d_4, \underset{19}{0}, \dots, \underset{23}{0}, \\ d_1d_3d_4, \underset{25}{0}, d_2d_3d_4, \underset{27}{0}, \dots, \underset{39}{0}, d_1d_2d_3d_4, \underset{41}{0}, \dots, \underset{53}{0} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
& f_{2431}(d_1, d_2, d_3, d_4) \\
&= \begin{pmatrix} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, d_1d_4, \underset{8}{0}, \underset{9}{0}, d_3d_4, \underset{11}{0}, \underset{12}{0}, d_1d_2d_3, \underset{14}{0}, \dots, \underset{17}{0}, d_1d_2d_4, \\ \underset{19}{0}, \dots, \underset{24}{0}, d_1d_3d_4, d_2d_3d_4, \underset{27}{0}, \dots, \underset{40}{0}, d_1d_2d_3d_4, \underset{42}{0}, \dots, \underset{53}{0} \end{pmatrix}
\end{aligned}$$

$$f_{3124}(d_1, d_2, d_3, d_4) = \begin{pmatrix} d_1, d_2, d_3, d_4, 0, d_1d_3, 0, d_2d_3, 0, \dots, 0, d_1d_2d_3, 0, \dots, 0, d_1d_3d_4, 0, \dots, 0, \\ d_2d_3d_4, 0, \dots, 0, d_1d_2d_3d_4, 0, \dots, 0 \end{pmatrix}$$

$$f_{3142}(d_1, d_2, d_3, d_4) = \begin{pmatrix} d_1, d_2, d_3, d_4, 0, d_1d_3, 0, d_2d_3, d_2d_4, 0, \dots, 0, d_1d_2d_3, 0, d_1d_2d_4, 0, \dots, 0, d_1d_3d_4, \\ 0, \dots, 0, d_2d_3d_4, 0, \dots, 0, d_1d_2d_3d_4, 0, \dots, 0 \end{pmatrix}$$

$$f_{3214}(d_1, d_2, d_3, d_4) = \begin{pmatrix} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, 0, d_2d_3, 0, \dots, 0, d_1d_2d_3, 0, d_1d_2d_4, 0, \dots, 0, d_1d_3d_4, \\ 0, \dots, 0, d_2d_3d_4, 0, \dots, 0, d_1d_2d_3d_4, 0, \dots, 0 \end{pmatrix}$$

$$f_{3241}(d_1, d_2, d_3, d_4) = \begin{pmatrix} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, d_1d_4, d_2d_3, 0, \dots, 0, d_1d_2d_3, 0, 0, d_1d_2d_4, 0, \dots, 0, \\ d_1d_3d_4, 0, \dots, 0, d_2d_3d_4, 0, \dots, 0, d_1d_2d_3d_4, 0, \dots, 0 \end{pmatrix}$$

$$f_{3412}(d_1, d_2, d_3, d_4) = \begin{pmatrix} d_1, d_2, d_3, d_4, 0, d_1d_3, d_1d_4, d_2d_3, d_2d_4, 0, \dots, 0, d_1d_2d_3, 0, \dots, 0, d_1d_2d_4, \\ 0, \dots, 0, d_1d_3d_4, 0, \dots, 0, d_2d_3d_4, 0, \dots, 0, d_1d_2d_3d_4, 0, \dots, 0 \end{pmatrix}$$

$$f_{3421}(d_1, d_2, d_3, d_4) = \begin{pmatrix} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, d_1d_4, d_2d_3, d_2d_4, 0, \dots, 0, d_1d_2d_3, 0, \dots, 0, d_1d_2d_4, \\ 0, 0, d_1d_3d_4, 0, \dots, 0, d_2d_3d_4, 0, \dots, 0, d_1d_2d_3d_4, 0, \dots, 0 \end{pmatrix}$$

$$f_{4123}(d_1, d_2, d_3, d_4) = \begin{pmatrix} d_1, d_2, d_3, d_4, 0, 0, d_1d_4, 0, 0, d_2d_4, d_3d_4, 0, \dots, 0, d_1d_2d_4, 0, \dots, 0, d_1d_3d_4, \\ 0, \dots, 0, d_2d_3d_4, 0, \dots, 0, d_1d_2d_3d_4, 0, \dots, 0 \end{pmatrix}$$

$$f_{4132}(d_1, d_2, d_3, d_4) = \begin{pmatrix} d_1, d_2, d_3, d_4, 0, 0, d_1d_4, d_2d_3, d_2d_4, d_3d_4, d_1d_2d_3, 0, \dots, 0, d_1d_2d_4, 0, \dots, 0, \\ d_1d_3d_4, 0, \dots, 0, d_2d_3d_4, 0, \dots, 0, d_1d_2d_3d_4, 0, \dots, 0 \end{pmatrix}$$

$$f_{4213}(d_1, d_2, d_3, d_4) = \begin{pmatrix} d_1, d_2, d_3, d_4, d_1d_2, 0, d_1d_4, 0, d_2d_4, d_3d_4, 0, d_1d_2d_3, 0, \dots, 0, d_1d_2d_4, \\ 0, \dots, 0, d_1d_3d_4, 0, \dots, 0, d_2d_3d_4, 0, \dots, 0, d_1d_2d_3d_4, 0, \dots, 0 \end{pmatrix}$$

$$\begin{aligned}
& f_{4231}(d_1, d_2, d_3, d_4) \\
&= \begin{pmatrix} d_1, d_2, d_3, d_4, d_1 d_2, d_1 d_3, d_1 d_4, 0, d_2 d_4, d_3 d_4, 0, 0, 0, d_1 d_2 d_3, 0, \dots, 0, \\ d_1 d_2 d_4, 0, \dots, 0, d_1 d_3 d_4, 0, \dots, 0, d_2 d_3 d_4, 0, \dots, 0, d_1 d_2 d_3 d_4, 0, 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
& f_{4312}(d_1, d_2, d_3, d_4) \\
&= \begin{pmatrix} d_1, d_2, d_3, d_4, 0, d_1 d_3, d_1 d_4, d_2 d_3, d_2 d_4, d_3 d_4, 0, \dots, 0, d_1 d_2 d_3, 0, \dots, 0, \\ d_1 d_2 d_4, 0, \dots, 0, d_1 d_3 d_4, 0, \dots, 0, d_2 d_3 d_4, 0, \dots, 0, d_1 d_2 d_3 d_4, 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
& f_{4321}(d_1, d_2, d_3, d_4) \\
&= \begin{pmatrix} d_1, d_2, d_3, d_4, d_1 d_2, d_1 d_3, d_1 d_4, d_2 d_3, d_2 d_4, d_3 d_4, 0, \dots, 0, d_1 d_2 d_3, 0, \dots, 0, \\ d_1 d_2 d_4, 0, \dots, 0, d_1 d_3 d_4, 0, \dots, 0, d_2 d_3 d_4, 0, \dots, 0, d_1 d_2 d_3 d_4 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
g_{12}^{1234,1243} &= g_{12}^{1342,1432} = g_{12}^{2341,2431} = g_{12}^{3421,4321} = g_{12}^{2134,2143} = g_{12}^{3412,4312} = \\
h_{12}^{1234,1243} &= h_{12}^{1342,1432} = h_{12}^{2341,2431} = h_{12}^{3421,4321} = h_{12}^{2134,2143} = h_{12}^{3412,4312}
\end{aligned}$$

$$g_{12}^{1234,1243}(d_1, d_2, d_3, d_4) = h_{12}^{1234,1243}(d_1, d_2, d_3, d_4) = (d_1, d_2, d_3, d_4)$$

$$\begin{aligned}
g_{13}^{1324,1342} &= g_{13}^{1243,1423} = g_{13}^{3241,3421} = g_{13}^{2431,4231} = g_{13}^{3124,3142} = g_{13}^{2413,4213} = \\
h_{13}^{1324,1342} &= h_{13}^{1243,1423} = h_{13}^{3241,3421} = h_{13}^{2431,4231} = h_{13}^{3124,3142} = h_{13}^{2413,4213}
\end{aligned}$$

$$g_{13}^{1324,1342}(d_1, d_2, d_3, d_4) = h_{13}^{1324,1342}(d_1, d_2, d_3, d_4) = (d_1, d_2, d_3, d_4)$$

$$\begin{aligned}
g_{14}^{1423,1432} &= g_{14}^{1234,1324} = g_{14}^{4231,4321} = g_{14}^{2341,3241} = g_{14}^{4123,4132} = g_{14}^{2314,3214} = \\
h_{14}^{1423,1432} &= h_{14}^{1234,1324} = h_{14}^{4231,4321} = h_{14}^{2341,3241} = h_{14}^{4123,4132} = h_{14}^{2314,3214}
\end{aligned}$$

$$g_{14}^{1423,1432}(d_1, d_2, d_3, d_4) = h_{14}^{1423,1432}(d_1, d_2, d_3, d_4) = (d_1, d_2, d_3, d_4)$$

$$\begin{aligned}
g_{23}^{2314,2341} &= g_{23}^{2143,2413} = g_{23}^{3142,3412} = g_{23}^{1432,4132} = g_{23}^{3214,3241} = g_{23}^{1423,4123} = \\
h_{23}^{2314,2341} &= h_{23}^{2143,2413} = h_{23}^{3142,3412} = h_{23}^{1432,4132} = h_{23}^{3214,3241} = h_{23}^{1423,4123}
\end{aligned}$$

$$g_{23}^{2314,2341}(d_1, d_2, d_3, d_4) = h_{23}^{2314,2341}(d_1, d_2, d_3, d_4) = (d_1, d_2, d_3, d_4)$$

$$\begin{aligned}
g_{24}^{2413,2431} &= g_{24}^{2134,2314} = g_{24}^{4132,4312} = g_{24}^{1342,3142} = g_{24}^{4213,4231} = g_{24}^{1324,3124} = \\
h_{24}^{2413,2431} &= h_{24}^{2134,2314} = h_{24}^{4132,4312} = h_{24}^{1342,3142} = h_{24}^{4213,4231} = h_{24}^{1324,3124}
\end{aligned}$$

$$g_{24}^{2413,2431}(d_1, d_2, d_3, d_4) = h_{24}^{2413,2431}(d_1, d_2, d_3, d_4) = (d_1, d_2, d_3, d_4)$$

$$\begin{aligned}
g_{34}^{3412,3421} &= g_{34}^{3124,3214} = g_{34}^{4123,4213} = g_{34}^{1243,2143} = g_{34}^{4312,4321} = g_{34}^{1234,2134} = \\
h_{34}^{3412,3421} &= h_{34}^{3124,3214} = h_{34}^{4123,4213} = h_{34}^{1243,2143} = h_{34}^{4312,4321} = h_{34}^{1234,2134}
\end{aligned}$$

$$g_{34}^{3412,3421}(d_1, d_2, d_3, d_4) = h_{34}^{3412,3421}(d_1, d_2, d_3, d_4) = (d_1, d_2, d_3, d_4)$$

is a quasi-colimit diagram.

Proof. Let $\theta^{1234} : Q^{1234} \rightarrow \mathbb{R}$, $\theta^{1243} : Q^{1243} \rightarrow \mathbb{R}$, $\theta^{1324} : Q^{1324} \rightarrow \mathbb{R}$, $\theta^{1342} : Q^{1342} \rightarrow \mathbb{R}$, $\theta^{1423} : Q^{1423} \rightarrow \mathbb{R}$, $\theta^{1432} : Q^{1432} \rightarrow \mathbb{R}$, $\theta^{2134} : Q^{2134} \rightarrow \mathbb{R}$, $\theta^{2143} : Q^{2143} \rightarrow \mathbb{R}$, $\theta^{2314} : Q^{2314} \rightarrow \mathbb{R}$, $\theta^{2341} : Q^{2341} \rightarrow \mathbb{R}$, $\theta^{2413} : Q^{2413} \rightarrow \mathbb{R}$, $\theta^{2431} : Q^{2431} \rightarrow \mathbb{R}$, $\theta^{3124} : Q^{3124} \rightarrow \mathbb{R}$, $\theta^{3142} : Q^{3142} \rightarrow \mathbb{R}$, $\theta^{3214} : Q^{3214} \rightarrow \mathbb{R}$, $\theta^{3241} : Q^{3241} \rightarrow \mathbb{R}$, $\theta^{3412} : Q^{3412} \rightarrow \mathbb{R}$, $\theta^{3421} : Q^{3421} \rightarrow \mathbb{R}$, $\theta^{4123} : Q^{4123} \rightarrow \mathbb{R}$, $\theta^{4132} : Q^{4132} \rightarrow \mathbb{R}$, $\theta^{4213} : Q^{4213} \rightarrow \mathbb{R}$, $\theta^{4231} : Q^{4231} \rightarrow \mathbb{R}$, $\theta^{4312} : Q^{4312} \rightarrow \mathbb{R}$ and $\theta^{4321} : Q^{4321} \rightarrow \mathbb{R}$ be mappings, which are to be of the following forms by dint of the general Kock-Lawvere axiom (cf. §2.1.3 of [4]):

$$\begin{aligned} & \theta^{1234} (d_1, d_2, d_3, d_4) \\ &= a^{1234} + a_1^{1234} d_1 + a_2^{1234} d_2 + a_3^{1234} d_3 + a_4^{1234} d_4 + a_{12}^{1234} d_1 d_2 + a_{13}^{1234} d_1 d_3 + a_{14}^{1234} d_1 d_4 + \\ & a_{23}^{1234} d_2 d_3 + a_{24}^{1234} d_2 d_4 + a_{34}^{1234} d_3 d_4 + a_{123}^{1234} d_1 d_2 d_3 + a_{124}^{1234} d_1 d_2 d_4 + a_{134}^{1234} d_1 d_3 d_4 + \\ & a_{234}^{1234} d_2 d_3 d_4 + a_{1234}^{1234} d_1 d_2 d_3 d_4 \end{aligned}$$

$$\begin{aligned} & \theta^{1243} (d_1, d_2, d_3, d_4) \\ &= a^{1243} + a_1^{1243} d_1 + a_2^{1243} d_2 + a_3^{1243} d_3 + a_4^{1243} d_4 + a_{12}^{1243} d_1 d_2 + a_{13}^{1243} d_1 d_3 + a_{14}^{1243} d_1 d_4 + \\ & a_{23}^{1243} d_2 d_3 + a_{24}^{1243} d_2 d_4 + a_{34}^{1243} d_3 d_4 + a_{123}^{1243} d_1 d_2 d_3 + a_{124}^{1243} d_1 d_2 d_4 + a_{134}^{1243} d_1 d_3 d_4 + \\ & a_{234}^{1243} d_2 d_3 d_4 + a_{1234}^{1243} d_1 d_2 d_3 d_4 \end{aligned}$$

$$\begin{aligned} & \theta^{1324} (d_1, d_2, d_3, d_4) \\ &= a^{1324} + a_1^{1324} d_1 + a_2^{1324} d_2 + a_3^{1324} d_3 + a_4^{1324} d_4 + a_{12}^{1324} d_1 d_2 + a_{13}^{1324} d_1 d_3 + a_{14}^{1324} d_1 d_4 + \\ & a_{23}^{1324} d_2 d_3 + a_{24}^{1324} d_2 d_4 + a_{34}^{1324} d_3 d_4 + a_{123}^{1324} d_1 d_2 d_3 + a_{124}^{1324} d_1 d_2 d_4 + a_{134}^{1324} d_1 d_3 d_4 + \\ & a_{234}^{1324} d_2 d_3 d_4 + a_{1234}^{1324} d_1 d_2 d_3 d_4 \end{aligned}$$

$$\begin{aligned} & \theta^{1342} (d_1, d_2, d_3, d_4) \\ &= a^{1342} + a_1^{1342} d_1 + a_2^{1342} d_2 + a_3^{1342} d_3 + a_4^{1342} d_4 + a_{12}^{1342} d_1 d_2 + a_{13}^{1342} d_1 d_3 + a_{14}^{1342} d_1 d_4 + \\ & a_{23}^{1342} d_2 d_3 + a_{24}^{1342} d_2 d_4 + a_{34}^{1342} d_3 d_4 + a_{123}^{1342} d_1 d_2 d_3 + a_{124}^{1342} d_1 d_2 d_4 + a_{134}^{1342} d_1 d_3 d_4 + \\ & a_{234}^{1342} d_2 d_3 d_4 + a_{1234}^{1342} d_1 d_2 d_3 d_4 \end{aligned}$$

$$\begin{aligned} & \theta^{1423} (d_1, d_2, d_3, d_4) \\ &= a^{1423} + a_1^{1423} d_1 + a_2^{1423} d_2 + a_3^{1423} d_3 + a_4^{1423} d_4 + a_{12}^{1423} d_1 d_2 + a_{13}^{1423} d_1 d_3 + a_{14}^{1423} d_1 d_4 + \\ & a_{23}^{1423} d_2 d_3 + a_{24}^{1423} d_2 d_4 + a_{34}^{1423} d_3 d_4 + a_{123}^{1423} d_1 d_2 d_3 + a_{124}^{1423} d_1 d_2 d_4 + a_{134}^{1423} d_1 d_3 d_4 + \\ & a_{234}^{1423} d_2 d_3 d_4 + a_{1234}^{1423} d_1 d_2 d_3 d_4 \end{aligned}$$

$$\begin{aligned} & \theta^{1432} (d_1, d_2, d_3, d_4) \\ &= a^{1432} + a_1^{1432} d_1 + a_2^{1432} d_2 + a_3^{1432} d_3 + a_4^{1432} d_4 + a_{12}^{1432} d_1 d_2 + a_{13}^{1432} d_1 d_3 + a_{14}^{1432} d_1 d_4 + \\ & a_{23}^{1432} d_2 d_3 + a_{24}^{1432} d_2 d_4 + a_{34}^{1432} d_3 d_4 + a_{123}^{1432} d_1 d_2 d_3 + a_{124}^{1432} d_1 d_2 d_4 + a_{134}^{1432} d_1 d_3 d_4 + \\ & a_{234}^{1432} d_2 d_3 d_4 + a_{1234}^{1432} d_1 d_2 d_3 d_4 \end{aligned}$$

$$\begin{aligned}
& \theta^{2134}(d_1, d_2, d_3, d_4) \\
&= a^{2134} + a_1^{2134}d_1 + a_2^{2134}d_2 + a_3^{2134}d_3 + a_4^{2134}d_1 + a_{12}^{2134}d_1d_2 + a_{13}^{2134}d_1d_3 + a_{14}^{2134}d_1d_4 + \\
& a_{23}^{2134}d_2d_3 + a_{24}^{2134}d_2d_4 + a_{34}^{2134}d_3d_4 + a_{123}^{2134}d_1d_2d_3 + a_{124}^{2134}d_1d_2d_4 + a_{134}^{2134}d_1d_3d_4 + \\
& a_{234}^{2134}d_2d_3d_4 + a_{1234}^{2134}d_1d_2d_3d_4
\end{aligned}$$

$$\begin{aligned}
& \theta^{2143}(d_1, d_2, d_3, d_4) \\
&= a^{2143} + a_1^{2143}d_1 + a_2^{2143}d_2 + a_3^{2143}d_3 + a_4^{2143}d_1 + a_{12}^{2143}d_1d_2 + a_{13}^{2143}d_1d_3 + a_{14}^{2143}d_1d_4 + \\
& a_{23}^{2143}d_2d_3 + a_{24}^{2143}d_2d_4 + a_{34}^{2143}d_3d_4 + a_{123}^{2143}d_1d_2d_3 + a_{124}^{2143}d_1d_2d_4 + a_{134}^{2143}d_1d_3d_4 + \\
& a_{234}^{2143}d_2d_3d_4 + a_{1234}^{2143}d_1d_2d_3d_4
\end{aligned}$$

$$\begin{aligned}
& \theta^{2314}(d_1, d_2, d_3, d_4) \\
&= a^{2314} + a_1^{2314}d_1 + a_2^{2314}d_2 + a_3^{2314}d_3 + a_4^{2314}d_1 + a_{12}^{2314}d_1d_2 + a_{13}^{2314}d_1d_3 + a_{14}^{2314}d_1d_4 + \\
& a_{23}^{2314}d_2d_3 + a_{24}^{2314}d_2d_4 + a_{34}^{2314}d_3d_4 + a_{123}^{2314}d_1d_2d_3 + a_{124}^{2314}d_1d_2d_4 + a_{134}^{2314}d_1d_3d_4 + \\
& a_{234}^{2314}d_2d_3d_4 + a_{1234}^{2314}d_1d_2d_3d_4
\end{aligned}$$

$$\begin{aligned}
& \theta^{2341}(d_1, d_2, d_3, d_4) \\
&= a^{2341} + a_1^{2341}d_1 + a_2^{2341}d_2 + a_3^{2341}d_3 + a_4^{2341}d_1 + a_{12}^{2341}d_1d_2 + a_{13}^{2341}d_1d_3 + a_{14}^{2341}d_1d_4 + \\
& a_{23}^{2341}d_2d_3 + a_{24}^{2341}d_2d_4 + a_{34}^{2341}d_3d_4 + a_{123}^{2341}d_1d_2d_3 + a_{124}^{2341}d_1d_2d_4 + a_{134}^{2341}d_1d_3d_4 + \\
& a_{234}^{2341}d_2d_3d_4 + a_{1234}^{2341}d_1d_2d_3d_4
\end{aligned}$$

$$\begin{aligned}
& \theta^{2413}(d_1, d_2, d_3, d_4) \\
&= a^{2413} + a_1^{2413}d_1 + a_2^{2413}d_2 + a_3^{2413}d_3 + a_4^{2413}d_1 + a_{12}^{2413}d_1d_2 + a_{13}^{2413}d_1d_3 + a_{14}^{2413}d_1d_4 + \\
& a_{23}^{2413}d_2d_3 + a_{24}^{2413}d_2d_4 + a_{34}^{2413}d_3d_4 + a_{123}^{2413}d_1d_2d_3 + a_{124}^{2413}d_1d_2d_4 + a_{134}^{2413}d_1d_3d_4 + \\
& a_{234}^{2413}d_2d_3d_4 + a_{1234}^{2413}d_1d_2d_3d_4
\end{aligned}$$

$$\begin{aligned}
& \theta^{2431}(d_1, d_2, d_3, d_4) \\
&= a^{2431} + a_1^{2431}d_1 + a_2^{2431}d_2 + a_3^{2431}d_3 + a_4^{2431}d_1 + a_{12}^{2431}d_1d_2 + a_{13}^{2431}d_1d_3 + a_{14}^{2431}d_1d_4 + \\
& a_{23}^{2431}d_2d_3 + a_{24}^{2431}d_2d_4 + a_{34}^{2431}d_3d_4 + a_{123}^{2431}d_1d_2d_3 + a_{124}^{2431}d_1d_2d_4 + a_{134}^{2431}d_1d_3d_4 + \\
& a_{234}^{2431}d_2d_3d_4 + a_{1234}^{2431}d_1d_2d_3d_4
\end{aligned}$$

$$\begin{aligned}
& \theta^{3124}(d_1, d_2, d_3, d_4) \\
&= a^{3124} + a_1^{3124}d_1 + a_2^{3124}d_2 + a_3^{3124}d_3 + a_4^{3124}d_1 + a_{12}^{3124}d_1d_2 + a_{13}^{3124}d_1d_3 + a_{14}^{3124}d_1d_4 + \\
& a_{23}^{3124}d_2d_3 + a_{24}^{3124}d_2d_4 + a_{34}^{3124}d_3d_4 + a_{123}^{3124}d_1d_2d_3 + a_{124}^{3124}d_1d_2d_4 + a_{134}^{3124}d_1d_3d_4 + \\
& a_{234}^{3124}d_2d_3d_4 + a_{1234}^{3124}d_1d_2d_3d_4
\end{aligned}$$

$$\begin{aligned}
& \theta^{3142}(d_1, d_2, d_3, d_4) \\
&= a^{3142} + a_1^{3142}d_1 + a_2^{3142}d_2 + a_3^{3142}d_3 + a_4^{3142}d_1 + a_{12}^{3142}d_1d_2 + a_{13}^{3142}d_1d_3 + a_{14}^{3142}d_1d_4 + \\
& a_{23}^{3142}d_2d_3 + a_{24}^{3142}d_2d_4 + a_{34}^{3142}d_3d_4 + a_{123}^{3142}d_1d_2d_3 + a_{124}^{3142}d_1d_2d_4 + a_{134}^{3142}d_1d_3d_4 + \\
& a_{234}^{3142}d_2d_3d_4 + a_{1234}^{3142}d_1d_2d_3d_4
\end{aligned}$$

$$\begin{aligned}
& \theta^{3214}(d_1, d_2, d_3, d_4) \\
&= a^{3214} + a_1^{3214}d_1 + a_2^{3214}d_2 + a_3^{3214}d_3 + a_4^{3214}d_1 + a_{12}^{3214}d_1d_2 + a_{13}^{3214}d_1d_3 + a_{14}^{3214}d_1d_4 + \\
& a_{23}^{3214}d_2d_3 + a_{24}^{3214}d_2d_4 + a_{34}^{3214}d_3d_4 + a_{123}^{3214}d_1d_2d_3 + a_{124}^{3214}d_1d_2d_4 + a_{134}^{3214}d_1d_3d_4 + \\
& a_{234}^{3214}d_2d_3d_4 + a_{1234}^{3214}d_1d_2d_3d_4
\end{aligned}$$

$$\begin{aligned}
& \theta^{3241}(d_1, d_2, d_3, d_4) \\
&= a^{3241} + a_1^{3241}d_1 + a_2^{3241}d_2 + a_3^{3241}d_3 + a_4^{3241}d_1 + a_{12}^{3241}d_1d_2 + a_{13}^{3241}d_1d_3 + a_{14}^{3241}d_1d_4 + \\
& a_{23}^{3241}d_2d_3 + a_{24}^{3241}d_2d_4 + a_{34}^{3241}d_3d_4 + a_{123}^{3241}d_1d_2d_3 + a_{124}^{3241}d_1d_2d_4 + a_{134}^{3241}d_1d_3d_4 + \\
& a_{234}^{3241}d_2d_3d_4 + a_{1234}^{3241}d_1d_2d_3d_4
\end{aligned}$$

$$\begin{aligned}
& \theta^{3412}(d_1, d_2, d_3, d_4) \\
&= a^{3412} + a_1^{3412}d_1 + a_2^{3412}d_2 + a_3^{3412}d_3 + a_4^{3412}d_1 + a_{12}^{3412}d_1d_2 + a_{13}^{3412}d_1d_3 + a_{14}^{3412}d_1d_4 + \\
& a_{23}^{3412}d_2d_3 + a_{24}^{3412}d_2d_4 + a_{34}^{3412}d_3d_4 + a_{123}^{3412}d_1d_2d_3 + a_{124}^{3412}d_1d_2d_4 + a_{134}^{3412}d_1d_3d_4 + \\
& a_{234}^{3412}d_2d_3d_4 + a_{1234}^{3412}d_1d_2d_3d_4
\end{aligned}$$

$$\begin{aligned}
& \theta^{3421}(d_1, d_2, d_3, d_4) \\
&= a^{3421} + a_1^{3421}d_1 + a_2^{3421}d_2 + a_3^{3421}d_3 + a_4^{3421}d_1 + a_{12}^{3421}d_1d_2 + a_{13}^{3421}d_1d_3 + a_{14}^{3421}d_1d_4 + \\
& a_{23}^{3421}d_2d_3 + a_{24}^{3421}d_2d_4 + a_{34}^{3421}d_3d_4 + a_{123}^{3421}d_1d_2d_3 + a_{124}^{3421}d_1d_2d_4 + a_{134}^{3421}d_1d_3d_4 + \\
& a_{234}^{3421}d_2d_3d_4 + a_{1234}^{3421}d_1d_2d_3d_4
\end{aligned}$$

$$\begin{aligned}
& \theta^{4123}(d_1, d_2, d_3, d_4) \\
&= a^{4123} + a_1^{4123}d_1 + a_2^{4123}d_2 + a_3^{4123}d_3 + a_4^{4123}d_1 + a_{12}^{4123}d_1d_2 + a_{13}^{4123}d_1d_3 + a_{14}^{4123}d_1d_4 + \\
& a_{23}^{4123}d_2d_3 + a_{24}^{4123}d_2d_4 + a_{34}^{4123}d_3d_4 + a_{123}^{4123}d_1d_2d_3 + a_{124}^{4123}d_1d_2d_4 + a_{134}^{4123}d_1d_3d_4 + \\
& a_{234}^{4123}d_2d_3d_4 + a_{1234}^{4123}d_1d_2d_3d_4
\end{aligned}$$

$$\begin{aligned}
& \theta^{4132}(d_1, d_2, d_3, d_4) \\
&= a^{4132} + a_1^{4132}d_1 + a_2^{4132}d_2 + a_3^{4132}d_3 + a_4^{4132}d_1 + a_{12}^{4132}d_1d_2 + a_{13}^{4132}d_1d_3 + a_{14}^{4132}d_1d_4 + \\
& a_{23}^{4132}d_2d_3 + a_{24}^{4132}d_2d_4 + a_{34}^{4132}d_3d_4 + a_{123}^{4132}d_1d_2d_3 + a_{124}^{4132}d_1d_2d_4 + a_{134}^{4132}d_1d_3d_4 + \\
& a_{234}^{4132}d_2d_3d_4 + a_{1234}^{4132}d_1d_2d_3d_4
\end{aligned}$$

$$\begin{aligned}
& \theta^{4213}(d_1, d_2, d_3, d_4) \\
&= a^{4213} + a_1^{4213}d_1 + a_2^{4213}d_2 + a_3^{4213}d_3 + a_4^{4213}d_1 + a_{12}^{4213}d_1d_2 + a_{13}^{4213}d_1d_3 + a_{14}^{4213}d_1d_4 + \\
& a_{23}^{4213}d_2d_3 + a_{24}^{4213}d_2d_4 + a_{34}^{4213}d_3d_4 + a_{123}^{4213}d_1d_2d_3 + a_{124}^{4213}d_1d_2d_4 + a_{134}^{4213}d_1d_3d_4 + \\
& a_{234}^{4213}d_2d_3d_4 + a_{1234}^{4213}d_1d_2d_3d_4
\end{aligned}$$

$$\begin{aligned}
& \theta^{4231}(d_1, d_2, d_3, d_4) \\
&= a^{4231} + a_1^{4231}d_1 + a_2^{4231}d_2 + a_3^{4231}d_3 + a_4^{4231}d_1 + a_{12}^{4231}d_1d_2 + a_{13}^{4231}d_1d_3 + a_{14}^{4231}d_1d_4 + \\
& a_{23}^{4231}d_2d_3 + a_{24}^{4231}d_2d_4 + a_{34}^{4231}d_3d_4 + a_{123}^{4231}d_1d_2d_3 + a_{124}^{4231}d_1d_2d_4 + a_{134}^{4231}d_1d_3d_4 + \\
& a_{234}^{4231}d_2d_3d_4 + a_{1234}^{4231}d_1d_2d_3d_4
\end{aligned}$$

$$\begin{aligned}
& \theta^{4312}(d_1, d_2, d_3, d_4) \\
&= a^{4312} + a_1^{4312}d_1 + a_2^{4312}d_2 + a_3^{4312}d_3 + a_4^{4312}d_1 + a_{12}^{4312}d_1d_2 + a_{13}^{4312}d_1d_3 + a_{14}^{4312}d_1d_4 + \\
& a_{23}^{4312}d_2d_3 + a_{24}^{4312}d_2d_4 + a_{34}^{4312}d_3d_4 + a_{123}^{4312}d_1d_2d_3 + a_{124}^{4312}d_1d_2d_4 + a_{134}^{4312}d_1d_3d_4 + \\
& a_{234}^{4312}d_2d_3d_4 + a_{1234}^{4312}d_1d_2d_3d_4
\end{aligned}$$

$$\begin{aligned}
& \theta^{4321}(d_1, d_2, d_3, d_4) \\
&= a^{4321} + a_1^{4321}d_1 + a_2^{4321}d_2 + a_3^{4321}d_3 + a_4^{4321}d_1 + a_{12}^{4321}d_1d_2 + a_{13}^{4321}d_1d_3 + a_{14}^{4321}d_1d_4 + \\
& a_{23}^{4321}d_2d_3 + a_{24}^{4321}d_2d_4 + a_{34}^{4321}d_3d_4 + a_{123}^{4321}d_1d_2d_3 + a_{124}^{4321}d_1d_2d_4 + a_{134}^{4321}d_1d_3d_4 + \\
& a_{234}^{4321}d_2d_3d_4 + a_{1234}^{4321}d_1d_2d_3d_4
\end{aligned}$$

The conditions

$$\begin{aligned}
\theta^{1234} \circ g_{12}^{1234,1243} &= \theta^{1243} \circ h_{12}^{1234,1243}, \theta^{1342} \circ g_{12}^{1342,1432} = \theta^{1432} \circ h_{12}^{1342,1432}, \\
\theta^{2341} \circ g_{12}^{2341,2431} &= \theta^{2431} \circ h_{12}^{2341,2431}, \theta^{3421} \circ g_{12}^{3421,4321} = \theta^{4321} \circ h_{12}^{3421,4321}, \\
\theta^{2134} \circ g_{12}^{2134,2143} &= \theta^{2143} \circ h_{12}^{2134,2143}, \theta^{3412} \circ g_{12}^{3412,4312} = \theta^{4312} \circ h_{12}^{3412,4312}, \\
\theta^{1324} \circ g_{13}^{1324,1342} &= \theta^{1342} \circ h_{13}^{1324,1342}, \theta^{1243} \circ g_{13}^{1243,1423} = \theta^{1423} \circ h_{13}^{1243,1423}, \\
\theta^{3241} \circ g_{13}^{3241,3421} &= \theta^{3421} \circ h_{13}^{3241,3421}, \theta^{2431} \circ g_{13}^{2431,4231} = \theta^{4231} \circ h_{13}^{2431,4231}, \\
\theta^{3124} \circ g_{13}^{3124,3142} &= \theta^{3142} \circ h_{13}^{3124,3142}, \theta^{2413} \circ g_{13}^{2413,4213} = \theta^{4213} \circ h_{13}^{2413,4213}, \\
\theta^{1423} \circ g_{14}^{1423,1432} &= \theta^{1432} \circ h_{14}^{1423,1432}, \theta^{1234} \circ g_{14}^{1234,1324} = \theta^{1324} \circ h_{14}^{1234,1324}, \\
\theta^{4231} \circ g_{14}^{4231,4321} &= \theta^{4321} \circ h_{14}^{4231,4321}, \theta^{2341} \circ g_{14}^{2341,3241} = \theta^{3241} \circ h_{14}^{2341,3241}, \\
\theta^{4123} \circ g_{14}^{4123,4132} &= \theta^{4132} \circ h_{14}^{4123,4132}, \theta^{2314} \circ g_{14}^{2314,3214} = \theta^{3214} \circ h_{14}^{2314,3214}, \\
\theta^{2314} \circ g_{23}^{2314,2341} &= \theta^{2341} \circ h_{23}^{2314,2341}, \theta^{2143} \circ g_{23}^{2143,2413} = \theta^{2413} \circ h_{23}^{2143,2413}, \\
\theta^{3142} \circ g_{23}^{3142,3412} &= \theta^{3412} \circ h_{23}^{3142,3412}, \theta^{1432} \circ g_{23}^{1432,4132} = \theta^{4132} \circ h_{23}^{1432,4132}, \\
\theta^{3214} \circ g_{23}^{3214,3241} &= \theta^{3241} \circ h_{23}^{3214,3241}, \theta^{1423} \circ g_{23}^{1423,4123} = \theta^{4123} \circ h_{23}^{1423,4123}, \\
\theta^{2413} \circ g_{24}^{2413,2431} &= \theta^{2431} \circ h_{24}^{2413,2431}, \theta^{2134} \circ g_{24}^{2134,2314} = \theta^{2314} \circ h_{24}^{2134,2314}, \\
\theta^{4132} \circ g_{24}^{4132,4312} &= \theta^{4312} \circ h_{24}^{4132,4312}, \theta^{1342} \circ g_{24}^{1342,3142} = \theta^{3142} \circ h_{24}^{1342,3142}, \\
\theta^{4213} \circ g_{24}^{4213,4231} &= \theta^{4231} \circ h_{24}^{4213,4231}, \theta^{1324} \circ g_{24}^{1324,3124} = \theta^{3124} \circ h_{24}^{1324,3124}, \\
\theta^{3412} \circ g_{34}^{3412,3421} &= \theta^{3421} \circ h_{34}^{3412,3421}, \theta^{3124} \circ g_{34}^{3124,3214} = \theta^{3214} \circ h_{34}^{3124,3214}, \\
\theta^{4123} \circ g_{34}^{4123,4213} &= \theta^{4213} \circ h_{34}^{4123,4213}, \theta^{1243} \circ g_{34}^{1243,2143} = \theta^{2143} \circ h_{34}^{1243,2143}, \\
\theta^{4312} \circ g_{34}^{4312,4321} &= \theta^{4321} \circ h_{34}^{4312,4321}, \theta^{1234} \circ g_{34}^{1234,2134} = \theta^{2134} \circ h_{34}^{1234,2134}
\end{aligned}$$

are tantamount to the following conditions in terms of coefficients of the polynomials

$$\begin{aligned}
a^{1342} &= a^{1432}, a_1^{1342} = a_1^{1432}, a_2^{1342} = a_2^{1432}, a_3^{1342} = a_3^{1432}, a_4^{1342} = a_4^{1432}, \\
a_{12}^{1342} &= a_{12}^{1432}, a_{13}^{1342} = a_{13}^{1432}, a_{14}^{1342} = a_{14}^{1432}, a_{23}^{1342} = a_{23}^{1432}, a_{24}^{1342} = a_{24}^{1432}, \\
a_{123}^{1342} &= a_{123}^{1432}, a_{124}^{1342} = a_{124}^{1432}
\end{aligned}$$

$$\begin{aligned}
a^{2341} &= a^{2431}, a_1^{2341} = a_1^{2431}, a_2^{2341} = a_2^{2431}, a_3^{2341} = a_3^{2431}, a_4^{2341} = a_4^{2431}, \\
a_{12}^{2341} &= a_{12}^{2431}, a_{13}^{2341} = a_{13}^{2431}, a_{14}^{2341} = a_{14}^{2431}, a_{23}^{2341} = a_{23}^{2431}, a_{24}^{2341} = a_{24}^{2431}, \\
a_{123}^{2341} &= a_{123}^{2431}, a_{124}^{2341} = a_{124}^{2431}
\end{aligned}$$

$$\begin{aligned}
a^{3421} &= a^{4321}, a_1^{3421} = a_1^{4321}, a_2^{3421} = a_2^{4321}, a_3^{3421} = a_3^{4321}, a_4^{3421} = a_4^{4321}, \\
a_{12}^{3421} &= a_{12}^{4321}, a_{13}^{3421} = a_{13}^{4321}, a_{14}^{3421} = a_{14}^{4321}, a_{23}^{3421} = a_{23}^{4321}, a_{24}^{3421} = a_{24}^{4321}, \\
a_{123}^{3421} &= a_{123}^{4321}, a_{124}^{3421} = a_{124}^{4321}
\end{aligned}$$

$$\begin{aligned}
a^{2341} &= a^{2314}, a_1^{2341} = a_1^{2314}, a_2^{2341} = a_2^{2314}, a_3^{2341} = a_3^{2314}, a_4^{2341} = a_4^{2314}, \\
a_{12}^{2341} &= a_{12}^{2314}, a_{13}^{2341} = a_{13}^{2314}, a_{14}^{2341} = a_{14}^{2314}, a_{24}^{2341} = a_{24}^{2314}, a_{34}^{2341} = a_{34}^{2314}, \\
a_{124}^{2341} &= a_{124}^{2314}, a_{134}^{2341} = a_{134}^{2314}
\end{aligned}$$

$$\begin{aligned}
a^{2413} &= a^{2143}, a_1^{2413} = a_1^{2143}, a_2^{2413} = a_2^{2143}, a_3^{2413} = a_3^{2143}, a_4^{2413} = a_4^{2143}, \\
a_{12}^{2413} &= a_{12}^{2143}, a_{13}^{2413} = a_{13}^{2143}, a_{14}^{2413} = a_{14}^{2143}, a_{24}^{2413} = a_{24}^{2143}, a_{34}^{2413} = a_{34}^{2143}, \\
a_{124}^{2413} &= a_{124}^{2143}, a_{134}^{2413} = a_{134}^{2143}
\end{aligned}$$

$$\begin{aligned}
a^{3412} &= a^{3142}, a_1^{3412} = a_1^{3142}, a_2^{3412} = a_2^{3142}, a_3^{3412} = a_3^{3142}, a_4^{3412} = a_4^{3142}, \\
a_{12}^{3412} &= a_{12}^{3142}, a_{13}^{3412} = a_{13}^{3142}, a_{14}^{3412} = a_{14}^{3142}, a_{24}^{3412} = a_{24}^{3142}, a_{34}^{3412} = a_{34}^{3142}, \\
a_{124}^{3412} &= a_{124}^{3142}, a_{134}^{3412} = a_{134}^{3142}
\end{aligned}$$

$$\begin{aligned}
a^{4132} &= a^{1432}, a_1^{4132} = a_1^{1432}, a_2^{4132} = a_2^{1432}, a_3^{4132} = a_3^{1432}, a_4^{4132} = a_4^{1432}, \\
a_{12}^{4132} &= a_{12}^{1432}, a_{13}^{4132} = a_{13}^{1432}, a_{14}^{4132} = a_{14}^{1432}, a_{24}^{4132} = a_{24}^{1432}, a_{34}^{4132} = a_{34}^{1432}, \\
a_{124}^{4132} &= a_{124}^{1432}, a_{134}^{4132} = a_{134}^{1432}
\end{aligned}$$

$$\begin{aligned}
a^{3421} &= a^{3421}, a_1^{3421} = a_1^{3421}, a_2^{3421} = a_2^{3421}, a_3^{3421} = a_3^{3421}, a_4^{3421} = a_4^{3421}, \\
a_{12}^{3421} &= a_{12}^{3421}, a_{13}^{3421} = a_{13}^{3421}, a_{14}^{3421} = a_{14}^{3421}, a_{23}^{3421} = a_{23}^{3421}, a_{24}^{3421} = a_{24}^{3421}, \\
a_{123}^{3421} &= a_{123}^{3421}, a_{124}^{3421} = a_{124}^{3421}
\end{aligned}$$

$$\begin{aligned}
a^{3124} &= a^{3214}, a_1^{3124} = a_1^{3214}, a_2^{3124} = a_2^{3214}, a_3^{3124} = a_3^{3214}, a_4^{3124} = a_4^{3214}, \\
a_{12}^{3124} &= a_{12}^{3214}, a_{13}^{3124} = a_{13}^{3214}, a_{14}^{3124} = a_{14}^{3214}, a_{23}^{3124} = a_{23}^{3214}, a_{24}^{3124} = a_{24}^{3214}, \\
a_{123}^{3124} &= a_{123}^{3214}, a_{124}^{3124} = a_{124}^{3214}
\end{aligned}$$

$$\begin{aligned}
a^{4123} &= a^{4213}, a_1^{4123} = a_1^{4213}, a_2^{4123} = a_2^{4213}, a_3^{4123} = a_3^{4213}, a_4^{4123} = a_4^{4213}, \\
a_{12}^{4123} &= a_{12}^{4213}, a_{13}^{4123} = a_{13}^{4213}, a_{14}^{4123} = a_{14}^{4213}, a_{23}^{4123} = a_{23}^{4213}, a_{24}^{4123} = a_{24}^{4213}, \\
a_{123}^{4123} &= a_{123}^{4213}, a_{124}^{4123} = a_{124}^{4213}
\end{aligned}$$

$$\begin{aligned}
a^{1243} &= a^{2143}, a_1^{1243} = a_1^{2143}, a_2^{1243} = a_2^{2143}, a_3^{1243} = a_3^{2143}, a_4^{1243} = a_4^{2143}, \\
a_{12}^{1243} &= a_{12}^{2143}, a_{13}^{1243} = a_{13}^{2143}, a_{14}^{1243} = a_{14}^{2143}, a_{23}^{1243} = a_{23}^{2143}, a_{24}^{1243} = a_{24}^{2143}, \\
a_{123}^{1243} &= a_{123}^{2143}, a_{124}^{1243} = a_{124}^{2143}
\end{aligned}$$

$$\begin{aligned}
a^{4123} &= a^{4132}, a_1^{4123} = a_1^{4132}, a_2^{4123} = a_2^{4132}, a_3^{4123} = a_3^{4132}, a_4^{4123} = a_4^{4132}, \\
a_{12}^{4123} &= a_{12}^{4132}, a_{13}^{4123} = a_{13}^{4132}, a_{23}^{4123} = a_{23}^{4132}, a_{24}^{4123} = a_{24}^{4132}, a_{34}^{4123} = a_{34}^{4132}, \\
a_{123}^{4123} &= a_{123}^{4132}, a_{234}^{4123} = a_{234}^{4132}
\end{aligned}$$

$$\begin{aligned}
a^{4231} &= a^{4321}, a_1^{4231} = a_1^{4321}, a_2^{4231} = a_2^{4321}, a_3^{4231} = a_3^{4321}, a_4^{4231} = a_4^{4321}, \\
a_{12}^{4231} &= a_{12}^{4321}, a_{13}^{4231} = a_{13}^{4321}, a_{23}^{4231} = a_{23}^{4321}, a_{24}^{4231} = a_{24}^{4321}, a_{34}^{4231} = a_{34}^{4321}, \\
a_{123}^{4231} &= a_{123}^{4321}, a_{234}^{4231} = a_{234}^{4321}
\end{aligned}$$

$$\begin{aligned}
a^{1234} &= a^{1324}, a_1^{1234} = a_1^{1324}, a_2^{1234} = a_2^{1324}, a_3^{1234} = a_3^{1324}, a_4^{1234} = a_4^{1324}, \\
a_{12}^{1234} &= a_{12}^{1324}, a_{13}^{1234} = a_{13}^{1324}, a_{23}^{1234} = a_{23}^{1324}, a_{24}^{1234} = a_{24}^{1324}, a_{34}^{1234} = a_{34}^{1324}, \\
a_{123}^{1234} &= a_{123}^{1324}, a_{234}^{1234} = a_{234}^{1324}
\end{aligned}$$

$$\begin{aligned}
a^{2314} &= a^{3214}, a_1^{2314} = a_1^{3214}, a_2^{2314} = a_2^{3214}, a_3^{2314} = a_3^{3214}, a_4^{2314} = a_4^{3214}, \\
a_{12}^{2314} &= a_{12}^{3214}, a_{13}^{2314} = a_{13}^{3214}, a_{23}^{2314} = a_{23}^{3214}, a_{24}^{2314} = a_{24}^{3214}, a_{34}^{2314} = a_{34}^{3214}, \\
a_{123}^{2314} &= a_{123}^{3214}, a_{234}^{2314} = a_{234}^{3214}
\end{aligned}$$

which can succinctly be summarized as

$$\begin{aligned}
a^{1234} &= a^{1243} = a^{1324} = a^{1342} = a^{1423} = a^{1432} = \\
a^{2134} &= a^{2143} = a^{2314} = a^{2341} = a^{2413} = a^{2431} = \\
a^{3124} &= a^{3142} = a^{3214} = a^{3241} = a^{3412} = a^{3421} = \\
a^{4123} &= a^{4132} = a^{4213} = a^{4231} = a^{4312} = a^{4321}
\end{aligned}$$

$$\begin{aligned}
a_1^{1234} &= a_1^{1243} = a_1^{1324} = a_1^{1342} = a_1^{1423} = a_1^{1432} = \\
a_1^{2134} &= a_1^{2143} = a_1^{2314} = a_1^{2341} = a_1^{2413} = a_1^{2431} = \\
a_1^{3124} &= a_1^{3142} = a_1^{3214} = a_1^{3241} = a_1^{3412} = a_1^{3421} = \\
a_1^{4123} &= a_1^{4132} = a_1^{4213} = a_1^{4231} = a_1^{4312} = a_1^{4321}
\end{aligned}$$

$$\begin{aligned}
a_2^{1234} &= a_2^{1243} = a_2^{1324} = a_2^{1342} = a_2^{1423} = a_2^{1432} = \\
a_2^{2134} &= a_2^{2143} = a_2^{2314} = a_2^{2341} = a_2^{2413} = a_2^{2431} = \\
a_2^{3124} &= a_2^{3142} = a_2^{3214} = a_2^{3241} = a_2^{3412} = a_2^{3421} = \\
a_2^{4123} &= a_2^{4132} = a_2^{4213} = a_2^{4231} = a_2^{4312} = a_2^{4321}
\end{aligned}$$

$$\begin{aligned}
a_3^{1234} &= a_3^{1243} = a_3^{1324} = a_3^{1342} = a_3^{1423} = a_3^{1432} = \\
a_3^{2134} &= a_3^{2143} = a_3^{2314} = a_3^{2341} = a_3^{2413} = a_3^{2431} = \\
a_3^{3124} &= a_3^{3142} = a_3^{3214} = a_3^{3241} = a_3^{3412} = a_3^{3421} = \\
a_3^{4123} &= a_3^{4132} = a_3^{4213} = a_3^{4231} = a_3^{4312} = a_3^{4321}
\end{aligned}$$

$$\begin{aligned}
a_4^{1234} &= a_4^{1243} = a_4^{1324} = a_4^{1342} = a_4^{1423} = a_4^{1432} = \\
a_4^{2134} &= a_4^{2143} = a_4^{2314} = a_4^{2341} = a_4^{2413} = a_4^{2431} = \\
a_4^{3124} &= a_4^{3142} = a_4^{3214} = a_4^{3241} = a_4^{3412} = a_4^{3421} = \\
a_4^{4123} &= a_4^{4132} = a_4^{4213} = a_4^{4231} = a_4^{4312} = a_4^{4321}
\end{aligned}$$

$$a_{12}^{1234} = a_{12}^{1243} = a_{12}^{1324} = a_{12}^{1342} = a_{12}^{1423} = a_{12}^{1432} =$$

$$a_{12}^{3124} = a_{12}^{3142} = a_{12}^{3412} = a_{12}^{4123} = a_{12}^{4132} = a_{12}^{4312} =$$

$$a_{12}^{2134} = a_{12}^{2143} = a_{12}^{2314} = a_{12}^{2341} = a_{12}^{2413} = a_{12}^{2431} =$$

$$a_{12}^{3214} = a_{12}^{3241} = a_{12}^{3421} = a_{12}^{4213} = a_{12}^{4231} = a_{12}^{4321} =$$

$$a_{13}^{1342} = a_{13}^{1324} = a_{13}^{1432} = a_{13}^{1423} = a_{13}^{1234} = a_{13}^{1243} =$$

$$a_{13}^{4132} = a_{13}^{4123} = a_{13}^{4213} = a_{13}^{2134} = a_{13}^{2143} = a_{13}^{2413} =$$

$$a_{13}^{3142} = a_{13}^{3124} = a_{13}^{3412} = a_{13}^{3421} = a_{13}^{3214} = a_{13}^{3241} =$$

$$a_{13}^{4312} = a_{13}^{4321} = a_{13}^{4231} = a_{13}^{2314} = a_{13}^{2341} = a_{13}^{2431} =$$

$$a_{14}^{1423} = a_{14}^{1432} = a_{14}^{1243} = a_{14}^{1234} = a_{14}^{1342} = a_{14}^{1324} =$$

$$a_{14}^{2143} = a_{14}^{2134} = a_{14}^{2314} = a_{14}^{3142} = a_{14}^{3124} = a_{14}^{3214} =$$

$$a_{14}^{4123} = a_{14}^{4132} = a_{14}^{4213} = a_{14}^{4231} = a_{14}^{4312} = a_{14}^{4321} =$$

$$a_{14}^{2413} = a_{14}^{2431} = a_{14}^{2341} = a_{14}^{3412} = a_{14}^{3421} = a_{14}^{3241} =$$

$$a_{23}^{2314} = a_{23}^{2341} = a_{23}^{2134} = a_{23}^{2143} = a_{23}^{2431} = a_{23}^{2413} =$$

$$a_{23}^{1234} = a_{23}^{1243} = a_{23}^{1423} = a_{23}^{4231} = a_{23}^{4213} = a_{23}^{4123} =$$

$$a_{23}^{3214} = a_{23}^{3241} = a_{23}^{3124} = a_{23}^{3142} = a_{23}^{3421} = a_{23}^{3412} =$$

$$a_{23}^{1324} = a_{23}^{1342} = a_{23}^{1432} = a_{23}^{4321} = a_{23}^{4312} = a_{23}^{4132} =$$

$$a_{24}^{2413} = a_{24}^{2431} = a_{24}^{2143} = a_{24}^{2134} = a_{24}^{2341} = a_{24}^{2314} =$$

$$a_{24}^{1243} = a_{24}^{1234} = a_{24}^{1324} = a_{24}^{3241} = a_{24}^{3214} = a_{24}^{3124} =$$

$$a_{24}^{4213} = a_{24}^{4231} = a_{24}^{4123} = a_{24}^{4132} = a_{24}^{4321} = a_{24}^{4312} =$$

$$a_{24}^{1423} = a_{24}^{1432} = a_{24}^{1342} = a_{24}^{3421} = a_{24}^{3412} = a_{24}^{3142} =$$

$$a_{34}^{3412} = a_{34}^{3421} = a_{34}^{3142} = a_{34}^{3124} = a_{34}^{3241} = a_{34}^{3214} =$$

$$a_{34}^{1342} = a_{34}^{1324} = a_{34}^{1234} = a_{34}^{2341} = a_{34}^{2314} = a_{34}^{2134} =$$

$$a_{34}^{4312} = a_{34}^{4321} = a_{34}^{4132} = a_{34}^{4123} = a_{34}^{4231} = a_{34}^{4213} =$$

$$a_{34}^{1432} = a_{34}^{1423} = a_{34}^{1243} = a_{34}^{2431} = a_{34}^{2413} = a_{34}^{2143} =$$

$$\begin{aligned}
a_{123}^{1234} &= a_{123}^{1243} = a_{123}^{1423} = a_{123}^{4123} \\
a_{123}^{1324} &= a_{123}^{1342} = a_{123}^{1432} = a_{123}^{4132} \\
a_{123}^{2134} &= a_{123}^{2143} = a_{123}^{2413} = a_{123}^{4213} \\
a_{123}^{2314} &= a_{123}^{2341} = a_{123}^{2431} = a_{123}^{4231} \\
a_{123}^{3124} &= a_{123}^{3142} = a_{123}^{3412} = a_{123}^{4312} \\
a_{123}^{3214} &= a_{123}^{3241} = a_{123}^{3421} = a_{123}^{4321} \\
a_{124}^{1243} &= a_{124}^{1234} = a_{124}^{1324} = a_{124}^{3124} \\
a_{124}^{1423} &= a_{124}^{1432} = a_{124}^{1342} = a_{124}^{3142} \\
a_{124}^{2143} &= a_{124}^{2134} = a_{124}^{2314} = a_{124}^{3214} \\
a_{124}^{2413} &= a_{124}^{2431} = a_{124}^{2341} = a_{124}^{3241} \\
a_{124}^{4123} &= a_{124}^{4132} = a_{124}^{4312} = a_{124}^{3412} \\
a_{124}^{4213} &= a_{124}^{4231} = a_{124}^{4321} = a_{124}^{3421} \\
a_{134}^{1342} &= a_{134}^{1324} = a_{134}^{1234} = a_{134}^{2134} \\
a_{134}^{1432} &= a_{134}^{1423} = a_{134}^{1243} = a_{134}^{2143} \\
a_{134}^{3142} &= a_{134}^{3124} = a_{134}^{3214} = a_{134}^{2314} \\
a_{134}^{3412} &= a_{134}^{3421} = a_{134}^{3241} = a_{134}^{2341} \\
a_{134}^{4132} &= a_{134}^{4123} = a_{134}^{4213} = a_{134}^{2413} \\
a_{134}^{4312} &= a_{134}^{4321} = a_{134}^{4231} = a_{134}^{2431} \\
a_{234}^{2341} &= a_{234}^{2314} = a_{234}^{2134} = a_{234}^{1234} \\
a_{234}^{2431} &= a_{234}^{2413} = a_{234}^{2143} = a_{234}^{1243} \\
a_{234}^{3241} &= a_{234}^{3214} = a_{234}^{3124} = a_{234}^{1324} \\
a_{234}^{3421} &= a_{234}^{3412} = a_{234}^{3142} = a_{234}^{1342} \\
a_{234}^{4231} &= a_{234}^{4213} = a_{234}^{4123} = a_{234}^{1423} \\
a_{234}^{4321} &= a_{234}^{4312} = a_{234}^{4132} = a_{234}^{1432}
\end{aligned}$$

Therefore there exists a unique mapping $\theta : P \rightarrow \mathbb{R}$ such that

$$\begin{aligned}
\theta^{1234} &= \theta \circ f_{1234}, \theta^{1243} = \theta \circ f_{1243}, \theta^{1324} = \theta \circ f_{1324}, \theta^{1342} = \theta \circ f_{1342}, \theta^{1423} = \theta \circ f_{1423}, \theta^{1432} = \theta \circ f_{1432}, \\
\theta^{2134} &= \theta \circ f_{2134}, \theta^{2143} = \theta \circ f_{2143}, \theta^{2314} = \theta \circ f_{2314}, \theta^{2341} = \theta \circ f_{2341}, \theta^{2413} = \theta \circ f_{2413}, \theta^{2431} = \theta \circ f_{2431}, \\
\theta^{3124} &= \theta \circ f_{3124}, \theta^{3142} = \theta \circ f_{3142}, \theta^{3214} = \theta \circ f_{3214}, \theta^{3241} = \theta \circ f_{3241}, \theta^{3412} = \theta \circ f_{3412}, \theta^{3421} = \theta \circ f_{3421}, \\
\theta^{4123} &= \theta \circ f_{4123}, \theta^{4132} = \theta \circ f_{4132}, \theta^{4213} = \theta \circ f_{4213}, \theta^{4231} = \theta \circ f_{4231}, \theta^{4312} = \theta \circ f_{4312}, \theta^{4321} = \theta \circ f_{4321}
\end{aligned}$$

The proof is now complete. ■

Remark 18 For our convenience we display the positions 5–30 of $f_{ijkl}(d_1, d_2, d_3, d_4)$ as tables as follows:

	$5/d_1d_2$	$6/d_1d_3$	$7/d_1d_4$	$8/d_2d_3$	$9/d_2d_4$	$10/d_3d_4$	
1234	0	0	0	0	0	0	
1243	0	0	0	0	0	d_3d_4	
1324	0	0	0	d_2d_3	0	0	
1342	0	0	0	d_2d_3	d_2d_4	0	
1423	0	0	0	0	d_2d_4	d_3d_4	
1432	0	0	0	d_2d_3	d_2d_4	d_3d_4	
2134	d_1d_2	0	0	0	0	0	
2143	d_1d_2	0	0	0	0	d_3d_4	
2314	d_1d_2	d_1d_3	0	0	0	0	
2341	d_1d_2	d_1d_3	d_1d_4	0	0	0	
2413	d_1d_2	0	d_1d_4	0	0	d_3d_4	
2431	d_1d_2	d_1d_3	d_1d_4	0	0	d_3d_4	(34)
3124	0	d_1d_3	0	d_2d_3	0	0	
3142	0	d_1d_3	0	d_2d_3	d_2d_4	0	
3214	d_1d_2	d_1d_3	0	d_2d_3	0	0	
3241	d_1d_2	d_1d_3	d_1d_4	d_2d_3	0	0	
3412	0	d_1d_3	d_1d_4	d_2d_3	d_2d_4	0	
3421	d_1d_2	d_1d_3	d_1d_4	d_2d_3	d_2d_4	0	
4123	0	0	d_1d_4	0	d_2d_4	d_3d_4	
4132	0	0	d_1d_4	d_2d_3	d_2d_4	d_3d_4	
4213	d_1d_2	0	d_1d_4	0	d_2d_4	d_3d_4	
4231	d_1d_2	d_1d_3	d_1d_4	0	d_2d_4	d_3d_4	
4312	0	d_1d_3	d_1d_4	d_2d_3	d_2d_4	d_3d_4	
4321	d_1d_2	d_1d_3	d_1d_4	d_2d_3	d_2d_4	d_3d_4	

	11 – 15/ $d_1d_2d_3$	16 – 20/ $d_1d_2d_4$	21 – 25/ $d_1d_3d_4$	26 – 30/ $d_2d_3d_4$
1234				
1243			21	26
1324	11			27
1342	11	16		28
1423		16	21	29
1432	11	16	21	30
2134	12	17		
2143	12	17	21	26
2314	13	17	22	
2341	13	18	23	
2413	12	18	24	26
2431	13	18	25	26
3124	14		22	27
3142	14	16	22	28
3214	15	17	22	27
3241	15	18	23	27
3412	14	19	23	28
3421	15	20	23	28
4123		19	24	29
4132	11	19	24	30
4213	12	20	24	29
4231	13	20	25	29
4312	14	19	25	30
4321	15	20	25	30

(35)

Corollary 19 *Let M be a microlinear space with mappings*

$$\begin{aligned}
&\gamma_{1234} : Q^{1234} \rightarrow M, \gamma_{1243} : Q^{1243} \rightarrow M, \gamma_{1324} : Q^{1324} \rightarrow M, \\
&\gamma_{1342} : Q^{1342} \rightarrow M, \gamma_{1423} : Q^{1423} \rightarrow M, \gamma_{1432} : Q^{1432} \rightarrow M, \\
&\gamma_{2134} : Q^{2134} \rightarrow M, \gamma_{2143} : Q^{2143} \rightarrow M, \gamma_{2314} : Q^{2314} \rightarrow M, \\
&\gamma_{2341} : Q^{2341} \rightarrow M, \gamma_{2413} : Q^{2413} \rightarrow M, \gamma_{2431} : Q^{2431} \rightarrow M, \\
&\gamma_{3124} : Q^{3124} \rightarrow M, \gamma_{3142} : Q^{3142} \rightarrow M, \gamma_{3214} : Q^{3214} \rightarrow M, \\
&\gamma_{3241} : Q^{3241} \rightarrow M, \gamma_{3412} : Q^{3412} \rightarrow M, \gamma_{3421} : Q^{3421} \rightarrow M, \\
&\gamma_{4123} : Q^{4123} \rightarrow M, \gamma_{4132} : Q^{4132} \rightarrow M, \gamma_{4213} : Q^{4213} \rightarrow M, \\
&\gamma_{4231} : Q^{4231} \rightarrow M, \gamma_{4312} : Q^{4312} \rightarrow M, \gamma_{4321} : Q^{4321} \rightarrow M
\end{aligned}$$

abiding by

$$\begin{aligned}
\gamma_{1234} \circ g_{12}^{1234,1243} &= \gamma_{1243} \circ h_{12}^{1234,1243}, \gamma_{1342} \circ g_{12}^{1342,1432} = \gamma_{1432} \circ h_{12}^{1342,1432}, \\
\gamma_{2341} \circ g_{12}^{2341,2431} &= \gamma_{2431} \circ h_{12}^{2341,2431}, \gamma_{3421} \circ g_{12}^{3421,4321} = \gamma_{4321} \circ h_{12}^{3421,4321}, \\
\gamma_{2134} \circ g_{12}^{2134,2143} &= \gamma_{2143} \circ h_{12}^{2134,2143}, \gamma_{3412} \circ g_{12}^{3412,4312} = \gamma_{4312} \circ h_{12}^{3412,4312}, \\
\gamma_{1324} \circ g_{13}^{1324,1342} &= \gamma_{1342} \circ h_{13}^{1324,1342}, \gamma_{1243} \circ g_{13}^{1243,1423} = \gamma_{1423} \circ h_{13}^{1243,1423}, \\
\gamma_{3241} \circ g_{13}^{3241,3421} &= \gamma_{3421} \circ h_{13}^{3241,3421}, \gamma_{2431} \circ g_{13}^{2431,4231} = \gamma_{4231} \circ h_{13}^{2431,4231}, \\
\gamma_{3124} \circ g_{13}^{3124,3142} &= \gamma_{3142} \circ h_{13}^{3124,3142}, \gamma_{2413} \circ g_{13}^{2413,4213} = \gamma_{4213} \circ h_{13}^{2413,4213}, \\
\gamma_{1423} \circ g_{14}^{1423,1432} &= \gamma_{1432} \circ h_{14}^{1423,1432}, \gamma_{1234} \circ g_{14}^{1234,1324} = \gamma_{1324} \circ h_{14}^{1234,1324}, \\
\gamma_{4231} \circ g_{14}^{4231,4321} &= \gamma_{4321} \circ h_{14}^{4231,4321}, \gamma_{2341} \circ g_{14}^{2341,3241} = \gamma_{3241} \circ h_{14}^{2341,3241}, \\
\gamma_{4123} \circ g_{14}^{4123,4132} &= \gamma_{4132} \circ h_{14}^{4123,4132}, \gamma_{2314} \circ g_{14}^{2314,3214} = \gamma_{3214} \circ h_{14}^{2314,3214}, \\
\gamma_{2314} \circ g_{23}^{2314,2341} &= \gamma_{2341} \circ h_{23}^{2314,2341}, \gamma_{2143} \circ g_{23}^{2143,2413} = \gamma_{2413} \circ h_{23}^{2143,2413}, \\
\gamma_{3142} \circ g_{23}^{3142,3412} &= \gamma_{3412} \circ h_{23}^{3142,3412}, \gamma_{1432} \circ g_{23}^{1432,4132} = \gamma_{4132} \circ h_{23}^{1432,4132}, \\
\gamma_{3214} \circ g_{23}^{3214,3241} &= \gamma_{3241} \circ h_{23}^{3214,3241}, \gamma_{1423} \circ g_{23}^{1423,4123} = \gamma_{4123} \circ h_{23}^{1423,4123}, \\
\gamma_{2413} \circ g_{24}^{2413,2431} &= \gamma_{2431} \circ h_{24}^{2413,2431}, \gamma_{2134} \circ g_{24}^{2134,2314} = \gamma_{2314} \circ h_{24}^{2134,2314}, \\
\gamma_{4132} \circ g_{24}^{4132,4312} &= \gamma_{4312} \circ h_{24}^{4132,4312}, \gamma_{1342} \circ g_{24}^{1342,3142} = \gamma_{3142} \circ h_{24}^{1342,3142}, \\
\gamma_{4213} \circ g_{24}^{4213,4231} &= \gamma_{4231} \circ h_{24}^{4213,4231}, \gamma_{1324} \circ g_{24}^{1324,3124} = \gamma_{3124} \circ h_{24}^{1324,3124}, \\
\gamma_{3412} \circ g_{34}^{3412,3421} &= \gamma_{3421} \circ h_{34}^{3412,3421}, \gamma_{3124} \circ g_{34}^{3124,3214} = \gamma_{3214} \circ h_{34}^{3124,3214}, \\
\gamma_{4123} \circ g_{34}^{4123,4213} &= \gamma_{4213} \circ h_{34}^{4123,4213}, \gamma_{1243} \circ g_{34}^{1243,2143} = \gamma_{2143} \circ h_{34}^{1243,2143}, \\
\gamma_{4312} \circ g_{34}^{4312,4321} &= \gamma_{4321} \circ h_{34}^{4312,4321}, \gamma_{1234} \circ g_{34}^{1234,2134} = \gamma_{2134} \circ h_{34}^{1234,2134}
\end{aligned}$$

Then there exists a unique mapping

$$\mathbf{m} : P \rightarrow M$$

such that

$$\begin{aligned}
\mathbf{m} \circ f_{1234} &= \gamma_{1234}, \mathbf{m} \circ f_{1243} = \gamma_{1243}, \mathbf{m} \circ f_{1324} = \gamma_{1324}, \mathbf{m} \circ f_{1342} = \gamma_{1342}, \mathbf{m} \circ f_{1423} = \gamma_{1423}, \mathbf{m} \circ f_{1432} = \gamma_{1432}, \\
\mathbf{m} \circ f_{2134} &= \gamma_{2134}, \mathbf{m} \circ f_{2143} = \gamma_{2143}, \mathbf{m} \circ f_{2314} = \gamma_{2314}, \mathbf{m} \circ f_{2341} = \gamma_{2341}, \mathbf{m} \circ f_{2413} = \gamma_{2413}, \mathbf{m} \circ f_{2431} = \gamma_{2431}, \\
\mathbf{m} \circ f_{3124} &= \gamma_{3124}, \mathbf{m} \circ f_{3142} = \gamma_{3142}, \mathbf{m} \circ f_{3214} = \gamma_{3214}, \mathbf{m} \circ f_{3241} = \gamma_{3241}, \mathbf{m} \circ f_{3412} = \gamma_{3412}, \mathbf{m} \circ f_{3421} = \gamma_{3421}, \\
\mathbf{m} \circ f_{4123} &= \gamma_{4123}, \mathbf{m} \circ f_{4132} = \gamma_{4132}, \mathbf{m} \circ f_{4213} = \gamma_{4213}, \mathbf{m} \circ f_{4231} = \gamma_{4231}, \mathbf{m} \circ f_{4312} = \gamma_{4312}, \mathbf{m} \circ f_{4321} = \gamma_{4321}
\end{aligned}$$

Theorem 20 Let M be a microlinear space. Let

$$\begin{aligned}
&\gamma_{1234}, \gamma_{1243}, \gamma_{1324}, \gamma_{1342}, \gamma_{1423}, \gamma_{1432}, \gamma_{2134}, \gamma_{2143}, \gamma_{2314}, \gamma_{2341}, \gamma_{2413}, \gamma_{2431}, \\
&\gamma_{3124}, \gamma_{3142}, \gamma_{3214}, \gamma_{3241}, \gamma_{3124}, \gamma_{3142}, \gamma_{4123}, \gamma_{4132}, \gamma_{4213}, \gamma_{4231}, \gamma_{4312}, \gamma_{4321} : D^4 \rightarrow M
\end{aligned}$$

with

$$\begin{aligned}
&\gamma_{1234} | D^4 \{(3, 4)\} = \gamma_{1243} | D^4 \{(3, 4)\}, \gamma_{1342} | D^4 \{(3, 4)\} = \gamma_{1432} | D^4 \{(3, 4)\}, \\
&\gamma_{2341} | D^4 \{(3, 4)\} = \gamma_{2431} | D^4 \{(3, 4)\}, \gamma_{3421} | D^4 \{(3, 4)\} = \gamma_{4321} | D^4 \{(3, 4)\}, \\
&\gamma_{2143} | D^4 \{(3, 4)\} = \gamma_{2134} | D^4 \{(3, 4)\}, \gamma_{4312} | D^4 \{(3, 4)\} = \gamma_{3412} | D^4 \{(3, 4)\}, \\
&\gamma_{1342} | D^4 \{(2, 4)\} = \gamma_{1324} | D^4 \{(2, 4)\}, \gamma_{1423} | D^4 \{(2, 4)\} = \gamma_{1243} | D^4 \{(2, 4)\}, \\
&\gamma_{3421} | D^4 \{(2, 4)\} = \gamma_{3241} | D^4 \{(2, 4)\}, \gamma_{4231} | D^4 \{(2, 4)\} = \gamma_{2431} | D^4 \{(2, 4)\}, \\
&\gamma_{3124} | D^4 \{(2, 4)\} = \gamma_{3142} | D^4 \{(2, 4)\}, \gamma_{2413} | D^4 \{(2, 4)\} = \gamma_{4213} | D^4 \{(2, 4)\}, \\
&\gamma_{1423} | D^4 \{(2, 3)\} = \gamma_{1432} | D^4 \{(2, 3)\}, \gamma_{1234} | D^4 \{(2, 3)\} = \gamma_{1324} | D^4 \{(2, 3)\}, \\
&\gamma_{4231} | D^4 \{(2, 3)\} = \gamma_{4321} | D^4 \{(2, 3)\}, \gamma_{2341} | D^4 \{(2, 3)\} = \gamma_{3241} | D^4 \{(2, 3)\}, \\
&\gamma_{4132} | D^4 \{(2, 3)\} = \gamma_{4123} | D^4 \{(2, 3)\}, \gamma_{3214} | D^4 \{(2, 3)\} = \gamma_{2314} | D^4 \{(2, 3)\}, \\
&\gamma_{2314} | D^4 \{(1, 4)\} = \gamma_{2341} | D^4 \{(1, 4)\}, \gamma_{2143} | D^4 \{(1, 4)\} = \gamma_{2413} | D^4 \{(1, 4)\}, \\
&\gamma_{3142} | D^4 \{(1, 4)\} = \gamma_{3412} | D^4 \{(1, 4)\}, \gamma_{1432} | D^4 \{(1, 4)\} = \gamma_{4132} | D^4 \{(1, 4)\}, \\
&\gamma_{3241} | D^4 \{(1, 4)\} = \gamma_{3214} | D^4 \{(1, 4)\}, \gamma_{4123} | D^4 \{(1, 4)\} = \gamma_{1423} | D^4 \{(1, 4)\}, \\
&\gamma_{2431} | D^4 \{(1, 3)\} = \gamma_{2413} | D^4 \{(1, 3)\}, \gamma_{2314} | D^4 \{(1, 3)\} = \gamma_{2134} | D^4 \{(1, 3)\}, \\
&\gamma_{4312} | D^4 \{(1, 3)\} = \gamma_{4132} | D^4 \{(1, 3)\}, \gamma_{3142} | D^4 \{(1, 3)\} = \gamma_{1342} | D^4 \{(1, 3)\}, \\
&\gamma_{4213} | D^4 \{(1, 3)\} = \gamma_{4231} | D^4 \{(1, 3)\}, \gamma_{1324} | D^4 \{(1, 3)\} = \gamma_{3124} | D^4 \{(1, 3)\}, \\
&\gamma_{3412} | D^4 \{(1, 2)\} = \gamma_{3421} | D^4 \{(1, 2)\}, \gamma_{3124} | D^4 \{(1, 2)\} = \gamma_{3214} | D^4 \{(1, 2)\}, \\
&\gamma_{4123} | D^4 \{(1, 2)\} = \gamma_{4213} | D^4 \{(1, 2)\}, \gamma_{1243} | D^4 \{(1, 2)\} = \gamma_{2143} | D^4 \{(1, 2)\}, \\
&\gamma_{4321} | D^4 \{(1, 2)\} = \gamma_{4312} | D^4 \{(1, 2)\}, \gamma_{2134} | D^4 \{(1, 2)\} = \gamma_{1234} | D^4 \{(1, 2)\}
\end{aligned}$$

Then we have

$$\begin{aligned}
& \left(\left(\left(\gamma_{1234} \begin{smallmatrix} \cdot \\ 12 \end{smallmatrix} \gamma_{1243} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{1342} \begin{smallmatrix} \cdot \\ 12 \end{smallmatrix} \gamma_{1432} \right) \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\left(\gamma_{2341} \begin{smallmatrix} \cdot \\ 12 \end{smallmatrix} \gamma_{2431} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{3421} \begin{smallmatrix} \cdot \\ 12 \end{smallmatrix} \gamma_{4321} \right) \right) \right) + \\
& \left(\left(\left(\gamma_{1342} \begin{smallmatrix} \cdot \\ 13 \end{smallmatrix} \gamma_{1324} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{1423} \begin{smallmatrix} \cdot \\ 13 \end{smallmatrix} \gamma_{1243} \right) \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\left(\gamma_{3421} \begin{smallmatrix} \cdot \\ 13 \end{smallmatrix} \gamma_{3241} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{4231} \begin{smallmatrix} \cdot \\ 13 \end{smallmatrix} \gamma_{2431} \right) \right) \right) + \\
& \left(\left(\left(\gamma_{1423} \begin{smallmatrix} \cdot \\ 14 \end{smallmatrix} \gamma_{1432} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{1234} \begin{smallmatrix} \cdot \\ 14 \end{smallmatrix} \gamma_{1324} \right) \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\left(\gamma_{4231} \begin{smallmatrix} \cdot \\ 14 \end{smallmatrix} \gamma_{4321} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{2341} \begin{smallmatrix} \cdot \\ 14 \end{smallmatrix} \gamma_{3241} \right) \right) \right) + \\
& \left(\left(\left(\gamma_{2143} \begin{smallmatrix} \cdot \\ 21 \end{smallmatrix} \gamma_{2134} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{2431} \begin{smallmatrix} \cdot \\ 21 \end{smallmatrix} \gamma_{2341} \right) \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\left(\gamma_{1432} \begin{smallmatrix} \cdot \\ 21 \end{smallmatrix} \gamma_{1342} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{4312} \begin{smallmatrix} \cdot \\ 21 \end{smallmatrix} \gamma_{3412} \right) \right) \right) + \\
& \left(\left(\left(\gamma_{2314} \begin{smallmatrix} \cdot \\ 23 \end{smallmatrix} \gamma_{2341} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{2143} \begin{smallmatrix} \cdot \\ 23 \end{smallmatrix} \gamma_{2413} \right) \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\left(\gamma_{3142} \begin{smallmatrix} \cdot \\ 23 \end{smallmatrix} \gamma_{3412} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{1432} \begin{smallmatrix} \cdot \\ 23 \end{smallmatrix} \gamma_{4132} \right) \right) \right) + \\
& \left(\left(\left(\gamma_{2431} \begin{smallmatrix} \cdot \\ 24 \end{smallmatrix} \gamma_{2413} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{2314} \begin{smallmatrix} \cdot \\ 24 \end{smallmatrix} \gamma_{2134} \right) \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\left(\gamma_{4312} \begin{smallmatrix} \cdot \\ 24 \end{smallmatrix} \gamma_{4132} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{3142} \begin{smallmatrix} \cdot \\ 24 \end{smallmatrix} \gamma_{1342} \right) \right) \right) + \\
& \left(\left(\left(\gamma_{3124} \begin{smallmatrix} \cdot \\ 31 \end{smallmatrix} \gamma_{3142} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{3241} \begin{smallmatrix} \cdot \\ 31 \end{smallmatrix} \gamma_{3421} \right) \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\left(\gamma_{1243} \begin{smallmatrix} \cdot \\ 31 \end{smallmatrix} \gamma_{1423} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{2413} \begin{smallmatrix} \cdot \\ 31 \end{smallmatrix} \gamma_{4213} \right) \right) \right) + \\
& \left(\left(\left(\gamma_{3241} \begin{smallmatrix} \cdot \\ 32 \end{smallmatrix} \gamma_{3214} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{3412} \begin{smallmatrix} \cdot \\ 32 \end{smallmatrix} \gamma_{3142} \right) \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\left(\gamma_{2413} \begin{smallmatrix} \cdot \\ 32 \end{smallmatrix} \gamma_{2143} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{4123} \begin{smallmatrix} \cdot \\ 32 \end{smallmatrix} \gamma_{1423} \right) \right) \right) + \\
& \left(\left(\left(\gamma_{3412} \begin{smallmatrix} \cdot \\ 34 \end{smallmatrix} \gamma_{3421} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{3124} \begin{smallmatrix} \cdot \\ 34 \end{smallmatrix} \gamma_{3214} \right) \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\left(\gamma_{4123} \begin{smallmatrix} \cdot \\ 34 \end{smallmatrix} \gamma_{4213} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{1243} \begin{smallmatrix} \cdot \\ 34 \end{smallmatrix} \gamma_{2143} \right) \right) \right) + \\
& \left(\left(\left(\gamma_{4132} \begin{smallmatrix} \cdot \\ 41 \end{smallmatrix} \gamma_{4123} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{4321} \begin{smallmatrix} \cdot \\ 41 \end{smallmatrix} \gamma_{4231} \right) \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\left(\gamma_{1324} \begin{smallmatrix} \cdot \\ 41 \end{smallmatrix} \gamma_{1234} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{3214} \begin{smallmatrix} \cdot \\ 41 \end{smallmatrix} \gamma_{2314} \right) \right) \right) + \\
& \left(\left(\left(\gamma_{4213} \begin{smallmatrix} \cdot \\ 42 \end{smallmatrix} \gamma_{4231} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{4132} \begin{smallmatrix} \cdot \\ 42 \end{smallmatrix} \gamma_{4312} \right) \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\left(\gamma_{2134} \begin{smallmatrix} \cdot \\ 42 \end{smallmatrix} \gamma_{2314} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{1324} \begin{smallmatrix} \cdot \\ 42 \end{smallmatrix} \gamma_{3124} \right) \right) \right) + \\
& \left(\left(\gamma_{4321} \begin{smallmatrix} \cdot \\ 43 \end{smallmatrix} \gamma_{4312} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{4213} \begin{smallmatrix} \cdot \\ 43 \end{smallmatrix} \gamma_{4123} \right) \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\left(\gamma_{3214} \begin{smallmatrix} \cdot \\ 43 \end{smallmatrix} \gamma_{3124} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{2134} \begin{smallmatrix} \cdot \\ 43 \end{smallmatrix} \gamma_{1234} \right) \right) \\
& = 0 \tag{36}
\end{aligned}$$

Proof. The proof is divided into thirteen steps.

1. Since

$$\begin{aligned}
& \gamma_{1234}(d_1, d_2, d_3, d_4) \\
& = \mathbf{m} \left(d_1, d_2, d_3, d_4, \underset{5}{0}, \dots, \underset{53}{0} \right)
\end{aligned}$$

and

$$\begin{aligned}
& \gamma_{1243}(d_1, d_2, d_3, d_4) \\
& = \mathbf{m} \left(d_1, d_2, d_3, d_4, \underset{5}{0}, \dots, \underset{9}{0}, \underset{10}{d_3 d_4}, \underset{11}{0}, \dots, \underset{20}{0}, \underset{22}{d_1 d_3 d_4}, \dots, \underset{25}{0}, \underset{27}{d_2 d_3 d_4}, \dots, \underset{30}{0}, \right. \\
& \quad \left. d_1 d_2 d_3 d_4, \underset{32}{0}, \dots, \underset{53}{0} \right)
\end{aligned}$$

we have

$$\begin{aligned}
& \left(\gamma_{1234} \begin{smallmatrix} \cdot \\ 12 \end{smallmatrix} \gamma_{1243} \right) (d_1, d_2, d_3) \\
&= \mathbf{m} \left(d_1, d_2, \underset{3}{0}, \dots, \underset{9}{0}, -d_3, \underset{11}{0}, \dots, \underset{20}{0}, -d_1 d_3, \underset{22}{0}, \dots, \underset{25}{0}, -d_2 d_3, \underset{27}{0}, \dots, \underset{30}{0}, -d_1 d_2 d_3, \underset{32}{0}, \dots, \underset{53}{0} \right)
\end{aligned} \tag{37}$$

For we have

$$\begin{aligned}
& \mathbf{n}_{(\gamma_{1234}, \gamma_{1243})}^4 (d_1, d_2, d_3, d_4, d_5) \\
&= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, \dots, \underset{9}{0}, d_3 d_4 - d_5, \underset{11}{0}, \dots, \underset{20}{0}, d_1 d_3 d_4 - d_1 d_5, \underset{22}{0}, \dots, \underset{25}{0}, \\ d_2 d_3 d_4 - d_2 d_5, \underset{27}{0}, \dots, \underset{30}{0}, d_1 d_2 d_3 d_4 - d_1 d_2 d_5, \underset{32}{0}, \dots, \underset{53}{0} \end{array} \right)
\end{aligned}$$

Since

$$\begin{aligned}
& \gamma_{1342} (d_1, d_2, d_3, d_4) \\
&= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, \dots, \underset{7}{0}, d_2 d_3, d_2 d_4, \underset{10}{0}, d_1 d_2 d_3, \underset{12}{0}, \dots, \underset{15}{0}, d_1 d_2 d_4, \underset{17}{0}, \dots, \underset{27}{0}, \\ d_2 d_3 d_4, 0, 0, 0, 0, d_1 d_2 d_3 d_4, \underset{34}{0}, \dots, \underset{53}{0} \end{array} \right)
\end{aligned}$$

and

$$\begin{aligned}
& \gamma_{1432} (d_1, d_2, d_3, d_4) \\
&= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, \dots, \underset{7}{0}, d_2 d_3, d_2 d_4, d_3 d_4, d_1 d_2 d_3, \underset{12}{0}, \dots, \underset{15}{0}, d_1 d_2 d_4, \underset{17}{0}, \dots, \underset{20}{0}, \\ d_1 d_3 d_4, \underset{22}{0}, \dots, \underset{29}{0}, d_2 d_3 d_4, \underset{31}{0}, \dots, \underset{34}{0}, d_1 d_2 d_3 d_4, \underset{36}{0}, \dots, \underset{53}{0} \end{array} \right)
\end{aligned}$$

we have

$$\begin{aligned}
& \left(\gamma_{1342} \begin{smallmatrix} \cdot \\ 12 \end{smallmatrix} \gamma_{1432} \right) (d_1, d_2, d_3) \\
&= \mathbf{m} \left(\begin{array}{c} d_1, d_2, \underset{3}{0}, \dots, \underset{9}{0}, -d_3, \underset{11}{0}, \dots, \underset{20}{0}, -d_1 d_3, \underset{22}{0}, \dots, \underset{27}{0}, d_2 d_3, \underset{29}{0}, -d_2 d_3, \underset{31}{0}, \underset{32}{0}, d_1 d_2 d_3, \underset{34}{0}, \\ -d_1 d_2 d_3, \underset{36}{0}, \dots, \underset{53}{0} \end{array} \right)
\end{aligned} \tag{38}$$

For we have

$$\begin{aligned}
& \mathbf{n}_{(\gamma_{1342}, \gamma_{1432})}^4 (d_1, d_2, d_3, d_4, d_5) \\
&= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, \dots, \underset{9}{0}, d_3 d_4 - d_5, \underset{11}{0}, \dots, \underset{20}{0}, d_1 d_3 d_4 - d_1 d_5, \underset{22}{0}, \dots, \underset{27}{0}, \\ d_2 d_5, \underset{29}{0}, d_2 d_3 d_4 - d_2 d_5, \underset{31}{0}, \underset{32}{0}, d_1 d_2 d_5, \underset{34}{0}, d_1 d_2 d_3 d_4 - d_1 d_2 d_5, \underset{36}{0}, \dots, \underset{53}{0} \end{array} \right)
\end{aligned}$$

For the sake of the completeness of our proof, we have provided the reason why (37) and (38) are derivable, but we will omit such reasoning from now

(40) and (41) imply that

$$\begin{aligned} & \left(\left(\gamma_{2341} \begin{smallmatrix} \cdot \\ 12 \end{smallmatrix} \gamma_{2431} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{3421} \begin{smallmatrix} \cdot \\ 12 \end{smallmatrix} \gamma_{4321} \right) \right) (d_1, d_2) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, \underset{2}{0}, \dots, \underset{25}{0}, -d_2, \underset{27}{0}, -d_2, \underset{29}{0}, d_2, \underset{31}{0}, \dots, \underset{38}{0}, d_1 d_2, \underset{40}{0}, \\ -d_1 d_2, \underset{42}{0}, \dots, \underset{46}{0}, -d_1 d_2, \underset{48}{0}, \dots, \underset{52}{0}, d_1 d_2 \end{array} \right) \end{aligned} \quad (42)$$

(39) and (42) imply that

$$\begin{aligned} & \left(\left(\left(\gamma_{1234} \begin{smallmatrix} \cdot \\ 12 \end{smallmatrix} \gamma_{1243} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{1342} \begin{smallmatrix} \cdot \\ 12 \end{smallmatrix} \gamma_{1432} \right) \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \right. \\ & \left. \left(\left(\gamma_{2341} \begin{smallmatrix} \cdot \\ 12 \end{smallmatrix} \gamma_{2431} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{3421} \begin{smallmatrix} \cdot \\ 12 \end{smallmatrix} \gamma_{4321} \right) \right) \right) (d) \\ &= \mathbf{m} \left(\underset{1}{0}, \dots, \underset{30}{0}, -d, \underset{32}{0}, -d, \underset{34}{0}, d, \underset{36}{0}, \dots, \underset{38}{0}, -d, 0, d, \underset{42}{0}, \dots, \underset{46}{0}, d, \underset{48}{0}, \dots, \underset{52}{0}, -d \right) \end{aligned} \quad (43)$$

2. Since

$$\begin{aligned} & \gamma_{1342} (d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, \dots, \underset{7}{0}, d_2 d_3, d_2 d_4, \underset{10}{0}, d_1 d_2 d_3, \underset{12}{0}, \dots, \underset{15}{0}, d_1 d_2 d_4, \underset{17}{0}, \dots, \underset{27}{0}, \\ d_2 d_3 d_4, 0, 0, 0, 0, d_1 d_2 d_3 d_4, \underset{34}{0}, \dots, \underset{53}{0} \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{1324} (d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, \dots, \underset{7}{0}, d_2 d_3, \underset{9}{0}, \underset{10}{0}, d_1 d_2 d_3, \underset{12}{0}, \dots, \underset{26}{0}, d_2 d_3 d_4, \underset{28}{0}, \dots, \underset{31}{0}, \\ d_1 d_2 d_3 d_4, \underset{33}{0}, \dots, \underset{53}{0} \end{array} \right) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{1342} \begin{smallmatrix} \cdot \\ 13 \end{smallmatrix} \gamma_{1324} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, 0, d_2, \underset{4}{0}, \dots, \underset{8}{0}, d_3, \underset{10}{0}, \dots, \underset{15}{0}, d_1 d_3, \underset{17}{0}, \dots, \underset{26}{0}, -d_2 d_3, d_2 d_3, \underset{29}{0}, \dots, \underset{31}{0}, \\ -d_1 d_2 d_3, d_1 d_2 d_3, \underset{34}{0}, \dots, \underset{53}{0} \end{array} \right) \end{aligned} \quad (44)$$

Since

$$\begin{aligned} & \gamma_{1423} (d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, \dots, \underset{8}{0}, d_2 d_4, d_3 d_4, \underset{11}{0}, \dots, \underset{15}{0}, d_1 d_2 d_4, \underset{17}{0}, \dots, \underset{20}{0}, d_1 d_3 d_4, \\ \underset{22}{0}, \dots, \underset{28}{0}, d_2 d_3 d_4, \underset{30}{0}, \dots, \underset{33}{0}, d_1 d_2 d_3 d_4, \underset{35}{0}, \dots, \underset{53}{0} \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{1243}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, \dots, \underset{9}{0}, \underset{10}{d_3 d_4}, \underset{11}{0}, \dots, \underset{20}{0}, \underset{22}{d_1 d_3 d_4}, \dots, \underset{25}{0}, \underset{27}{d_2 d_3 d_4}, \dots, \underset{30}{0}, \\ d_1 d_2 d_3 d_4, \underset{32}{0}, \dots, \underset{53}{0} \end{array} \right) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{1423} \dot{-}_{13} \gamma_{1243} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, 0, d_2, \underset{4}{0}, \dots, \underset{8}{0}, \underset{10}{d_3}, \underset{15}{0}, \dots, \underset{17}{0}, \underset{25}{d_1 d_3}, \dots, \underset{27}{0}, \underset{28}{-d_2 d_3}, \underset{30}{0}, \underset{27}{0}, \underset{28}{d_2 d_3}, \underset{30}{0}, \\ -d_1 d_2 d_3, \underset{32}{0}, \underset{33}{0}, \underset{35}{d_1 d_2 d_3}, \underset{35}{0}, \dots, \underset{53}{0} \end{array} \right) \end{aligned} \quad (45)$$

(44) and (45) imply that

$$\begin{aligned} & \left(\left(\gamma_{1324} \dot{-}_{13} \gamma_{1342} \right) \dot{-}_1 \left(\gamma_{1243} \dot{-}_{13} \gamma_{1423} \right) \right) (d_1, d_2) \\ &= \mathbf{m} \left(d_1, \underset{2}{0}, \dots, \underset{25}{0}, d_2, -d_2, d_2, -d_2, \underset{30}{0}, d_1 d_2, -d_1 d_2, d_1 d_2, -d_1 d_2, \underset{35}{0}, \dots, \underset{53}{0} \right) \end{aligned} \quad (46)$$

Since

$$\begin{aligned} & \gamma_{3421}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1 d_2, d_1 d_3, d_1 d_4, d_2 d_3, d_2 d_4, \underset{10}{0}, \dots, \underset{14}{0}, \underset{16}{d_1 d_2 d_3}, \dots, \underset{19}{0}, \underset{19}{d_1 d_2 d_4}, \\ 0, 0, d_1 d_3 d_4, \underset{24}{0}, \dots, \underset{27}{0}, \underset{29}{d_2 d_3 d_4}, \dots, \underset{46}{0}, \underset{48}{d_1 d_2 d_3 d_4}, \underset{48}{0}, \dots, \underset{53}{0} \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{3241}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1 d_2, d_1 d_3, d_1 d_4, d_2 d_3, \underset{9}{0}, \dots, \underset{14}{0}, \underset{16}{d_1 d_2 d_3}, \underset{17}{0}, \dots, \underset{19}{0}, \underset{19}{d_1 d_2 d_4}, \dots, \underset{22}{0}, \\ d_1 d_3 d_4, \underset{24}{0}, \dots, \underset{26}{0}, \underset{28}{d_2 d_3 d_4}, \dots, \underset{44}{0}, \underset{46}{d_1 d_2 d_3 d_4}, \underset{46}{0}, \dots, \underset{53}{0} \end{array} \right) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{3421} \dot{-}_{13} \gamma_{3241} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, 0, d_2, \underset{4}{0}, \dots, \underset{8}{0}, \underset{10}{d_3}, \dots, \underset{17}{0}, \underset{19}{-d_1 d_3}, \underset{21}{d_1 d_3}, \dots, \underset{26}{0}, \underset{26}{-d_2 d_3}, \underset{26}{d_2 d_3}, \\ \underset{29}{0}, \dots, \underset{44}{0}, \underset{46}{-d_1 d_2 d_3}, \underset{46}{0}, \underset{48}{d_1 d_2 d_3}, \underset{48}{0}, \dots, \underset{53}{0} \end{array} \right) \end{aligned} \quad (47)$$

Since

$$\begin{aligned} & \gamma_{4231}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1 d_2, d_1 d_3, d_1 d_4, \underset{8}{0}, \underset{11}{d_2 d_4}, \underset{12}{d_3 d_4}, \dots, \underset{14}{0}, \underset{14}{d_1 d_2 d_3}, \dots, \underset{19}{0}, \\ d_1 d_2 d_4, \underset{21}{0}, \dots, \underset{24}{0}, \underset{26}{d_1 d_3 d_4}, \dots, \underset{28}{0}, \underset{30}{d_2 d_3 d_4}, \dots, \underset{50}{0}, \underset{53}{d_1 d_2 d_3 d_4}, \underset{53}{0} \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{2431}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1 d_2, d_1 d_3, d_1 d_4, \underset{8}{0}, \underset{9}{0}, d_3 d_4, \underset{11}{0}, \underset{11}{0}, d_1 d_2 d_3, \underset{14}{0}, \dots, \underset{17}{0}, d_1 d_2 d_4, \\ \underset{19}{0}, \dots, \underset{24}{0}, d_1 d_3 d_4, d_2 d_3 d_4, \underset{27}{0}, \dots, \underset{40}{0}, d_1 d_2 d_3 d_4, \underset{42}{0}, \dots, \underset{53}{0} \end{array} \right) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{4231} \overset{\cdot}{-} \gamma_{2431} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, \underset{2}{0}, d_2, \underset{4}{0}, \dots, \underset{8}{0}, d_3, \underset{10}{0}, \dots, \underset{17}{0}, -d_1 d_3, \underset{19}{0}, d_1 d_3, \underset{21}{0}, \dots, \underset{25}{0}, -d_2 d_3, \underset{27}{0}, \underset{28}{0}, d_2 d_3, \\ \underset{30}{0}, \dots, \underset{40}{0}, -d_1 d_2 d_3, \underset{42}{0}, \dots, \underset{50}{0}, d_1 d_2 d_3, \underset{52}{0}, \underset{53}{0} \end{array} \right) \quad (48) \end{aligned}$$

(47) and (48) imply that

$$\begin{aligned} & \left(\left(\gamma_{3421} \overset{\cdot}{-} \gamma_{3241} \right) \overset{\cdot}{-} \left(\gamma_{4231} \overset{\cdot}{-} \gamma_{2431} \right) \right) (d_1, d_2) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, \underset{2}{0}, \dots, \underset{25}{0}, d_2, -d_2, d_2, -d_2, \underset{30}{0}, \dots, \underset{40}{0}, d_1 d_2, \underset{42}{0}, \dots, \underset{44}{0}, \\ -d_1 d_2, \underset{46}{0}, d_1 d_2, \underset{48}{0}, \dots, \underset{50}{0}, -d_1 d_2, \underset{52}{0}, \underset{53}{0} \end{array} \right) \quad (49) \end{aligned}$$

(46) and (49) imply that

$$\begin{aligned} & \left(\left(\left(\gamma_{1324} \overset{\cdot}{-} \gamma_{1342} \right) \overset{\cdot}{-} \left(\gamma_{1243} \overset{\cdot}{-} \gamma_{1423} \right) \right) \overset{\cdot}{-} \right. \\ & \left. \left(\left(\gamma_{3241} \overset{\cdot}{-} \gamma_{3421} \right) \overset{\cdot}{-} \left(\gamma_{2431} \overset{\cdot}{-} \gamma_{4231} \right) \right) \right) (d) \\ &= \mathbf{m} \left(\underset{1}{0}, \dots, \underset{30}{0}, d, -d, d, -d, \underset{35}{0}, \dots, \underset{40}{0}, -d, \underset{42}{0}, \dots, \underset{44}{0}, d, \underset{46}{0}, -d, \underset{48}{0}, \dots, \underset{50}{0}, d, \underset{52}{0}, \underset{53}{0} \right) \quad (50) \end{aligned}$$

3. Since

$$\begin{aligned} & \gamma_{1423}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, \dots, \underset{8}{0}, d_2 d_4, d_3 d_4, \underset{11}{0}, \dots, \underset{15}{0}, d_1 d_2 d_4, \underset{17}{0}, \dots, \underset{20}{0}, d_1 d_3 d_4, \\ \underset{22}{0}, \dots, \underset{28}{0}, d_2 d_3 d_4, \underset{30}{0}, \dots, \underset{33}{0}, d_1 d_2 d_3 d_4, \underset{35}{0}, \dots, \underset{53}{0} \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{1432}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, \dots, \underset{7}{0}, d_2 d_3, d_2 d_4, d_3 d_4, d_1 d_2 d_3, \underset{12}{0}, \dots, \underset{15}{0}, d_1 d_2 d_4, \underset{17}{0}, \dots, \underset{20}{0}, \\ d_1 d_3 d_4, \underset{22}{0}, \dots, \underset{29}{0}, d_2 d_3 d_4, \underset{31}{0}, \dots, \underset{34}{0}, d_1 d_2 d_3 d_4, \underset{36}{0}, \dots, \underset{53}{0} \end{array} \right) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{1423} \overset{\cdot}{\underset{14}{-}} \gamma_{1432} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, \underset{2}{0}, \underset{3}{0}, d_2, \underset{5}{0}, \dots, \underset{7}{0}, -d_3, \underset{9}{0}, \underset{10}{0}, -d_1 d_3, \underset{12}{0}, \dots, \underset{28}{0}, d_2 d_3, -d_2 d_3, \\ \underset{31}{0}, \dots, \underset{33}{0}, d_1 d_2 d_3, -d_1 d_2 d_3, \underset{36}{0}, \dots, \underset{53}{0} \end{array} \right) \end{aligned} \quad (51)$$

Since

$$\begin{aligned} & \gamma_{1234} (d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(d_1, d_2, d_3, d_4, \underset{5}{0}, \dots, \underset{53}{0} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{1324} (d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, \dots, \underset{7}{0}, d_2 d_3, \underset{9}{0}, \underset{10}{0}, d_1 d_2 d_3, \underset{12}{0}, \dots, \underset{26}{0}, d_2 d_3 d_4, \underset{28}{0}, \dots, \underset{30}{0}, \\ 0, d_1 d_2 d_3 d_4, \underset{33}{0}, \dots, \underset{53}{0} \end{array} \right) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{1234} \overset{\cdot}{\underset{14}{-}} \gamma_{1324} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, \underset{2}{0}, \underset{3}{0}, d_2, \underset{5}{0}, \dots, \underset{7}{0}, -d_3, \underset{9}{0}, \underset{10}{0}, -d_1 d_3, \underset{12}{0}, \dots, \underset{26}{0}, -d_2 d_3, \\ \underset{28}{0}, \dots, \underset{31}{0}, -d_1 d_2 d_3, \underset{33}{0}, \dots, \underset{53}{0} \end{array} \right) \end{aligned} \quad (52)$$

(51) and (52) imply that

$$\begin{aligned} & \left(\left(\gamma_{1423} \overset{\cdot}{\underset{14}{-}} \gamma_{1432} \right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{1234} \overset{\cdot}{\underset{14}{-}} \gamma_{1324} \right) \right) (d_1, d_2) \\ &= \mathbf{m} \left(d_1, \underset{2}{0}, \dots, \underset{26}{0}, d_2, \underset{28}{0}, d_2, -d_2, \underset{31}{0}, d_1 d_2, \underset{33}{0}, d_1 d_2, -d_1 d_2, \underset{36}{0}, \dots, \underset{53}{0} \right) \end{aligned} \quad (53)$$

Since

$$\begin{aligned} & \gamma_{4231} (d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1 d_2, d_1 d_3, d_1 d_4, \underset{8}{0}, d_2 d_4, d_3 d_4, \underset{11}{0}, \underset{12}{0}, d_1 d_2 d_3, \underset{14}{0}, \dots, \underset{19}{0}, \\ d_1 d_2 d_4, \underset{21}{0}, \dots, \underset{24}{0}, d_1 d_3 d_4, \underset{26}{0}, \dots, \underset{28}{0}, d_2 d_3 d_4, \underset{30}{0}, \dots, \underset{50}{0}, d_1 d_2 d_3 d_4, \underset{52}{0}, \underset{53}{0} \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{4321} (d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1 d_2, d_1 d_3, d_1 d_4, d_2 d_3, d_2 d_4, d_3 d_4, \underset{11}{0}, \dots, \underset{14}{0}, d_1 d_2 d_3, \underset{16}{0}, \dots, \underset{19}{0}, \\ d_1 d_2 d_4, \underset{21}{0}, \dots, \underset{24}{0}, d_1 d_3 d_4, \underset{26}{0}, \dots, \underset{29}{0}, d_2 d_3 d_4, \underset{31}{0}, \dots, \underset{52}{0}, d_1 d_2 d_3 d_4 \end{array} \right) \end{aligned}$$

we have

$$\begin{aligned}
& \left(\gamma_{4231} \overset{\cdot}{\underset{14}{-}} \gamma_{4321} \right) (d_1, d_2, d_3) \\
&= \mathbf{m} \left(\begin{array}{c} d_1, \underset{2}{0}, \underset{3}{0}, d_2, \underset{5}{0}, \dots, \underset{7}{0}, -d_3, \underset{9}{0}, \dots, \underset{12}{0}, d_1 d_3, \underset{14}{0}, -d_1 d_3, \underset{16}{0}, \dots, \underset{28}{0}, d_2 d_3, -d_2 d_3, \\ 0, \dots, \underset{31}{0}, d_1 d_2 d_3, \underset{50}{0}, -d_1 d_2 d_3 \end{array} \right) \quad (54)
\end{aligned}$$

Since

$$\begin{aligned}
& \gamma_{2341} (d_1, d_2, d_3, d_4) \\
&= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1 d_2, d_1 d_3, d_1 d_4, \underset{8}{0}, \dots, \underset{12}{0}, d_1 d_2 d_3, \underset{14}{0}, \dots, \underset{17}{0}, d_1 d_2 d_4, \underset{19}{0}, \dots, \underset{22}{0}, \\ d_1 d_3 d_4, \underset{24}{0}, \dots, \underset{38}{0}, d_1 d_2 d_3 d_4, \underset{40}{0}, \dots, \underset{53}{0} \end{array} \right)
\end{aligned}$$

and

$$\begin{aligned}
& \gamma_{3241} (d_1, d_2, d_3, d_4) \\
&= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1 d_2, d_1 d_3, d_1 d_4, d_2 d_3, \underset{9}{0}, \dots, \underset{14}{0}, d_1 d_2 d_3, \underset{16}{0}, \underset{17}{0}, d_1 d_2 d_4, \underset{19}{0}, \dots, \underset{22}{0}, \\ d_1 d_3 d_4, \underset{24}{0}, \dots, \underset{26}{0}, d_2 d_3 d_4, \underset{28}{0}, \dots, \underset{44}{0}, d_1 d_2 d_3 d_4, \underset{46}{0}, \dots, \underset{53}{0} \end{array} \right)
\end{aligned}$$

we have

$$\begin{aligned}
& \left(\gamma_{2341} \overset{\cdot}{\underset{14}{-}} \gamma_{3241} \right) (d_1, d_2, d_3) \\
&= \mathbf{m} \left(\begin{array}{c} d_1, \underset{2}{0}, \underset{3}{0}, d_2, \underset{5}{0}, \dots, \underset{7}{0}, -d_3, \underset{9}{0}, \dots, \underset{12}{0}, d_1 d_3, \underset{14}{0}, -d_1 d_3, \underset{16}{0}, \dots, \underset{26}{0}, -d_2 d_3, \\ 0, \dots, \underset{28}{0}, \dots, \underset{38}{0}, d_1 d_2 d_3, \underset{40}{0}, \dots, \underset{44}{0}, -d_1 d_2 d_3, \underset{46}{0}, \dots, \underset{53}{0} \end{array} \right) \quad (55)
\end{aligned}$$

(54) and (55) imply that

$$\begin{aligned}
& \left(\left(\gamma_{4231} \overset{\cdot}{\underset{14}{-}} \gamma_{4321} \right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{2341} \overset{\cdot}{\underset{14}{-}} \gamma_{3241} \right) \right) (d_1, d_2) \\
&= \mathbf{m} \left(\begin{array}{c} d_1, \underset{2}{0}, \dots, \underset{26}{0}, d_2, \underset{28}{0}, d_2, -d_2, \underset{31}{0}, \dots, \underset{38}{0}, -d_1 d_2, \\ 0, \dots, \underset{40}{0}, \dots, \underset{44}{0}, d_1 d_2, \underset{46}{0}, \dots, \underset{50}{0}, d_1 d_2, \underset{52}{0}, -d_1 d_2 \end{array} \right) \quad (56)
\end{aligned}$$

(53) and (56) imply that

$$\begin{aligned}
& \left(\left(\left(\gamma_{1423} \overset{\cdot}{\underset{14}{-}} \gamma_{1432} \right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{1234} \overset{\cdot}{\underset{14}{-}} \gamma_{1324} \right) \right) \overset{\cdot}{\underset{1}{-}} \right) (d) \\
&= \mathbf{m} \left(\begin{array}{c} 0, \dots, \underset{1}{0}, \dots, \underset{31}{0}, d, \underset{33}{0}, d, -d, \underset{36}{0}, \dots, \underset{38}{0}, d, \underset{40}{0}, \dots, \underset{44}{0}, -d, \underset{46}{0}, \dots, \underset{50}{0}, -d, \underset{52}{0}, d \end{array} \right) \quad (57)
\end{aligned}$$

4. Since

$$\begin{aligned} & \gamma_{2143}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1 d_2, 0, \dots, 0, d_3 d_4, 0, d_1 d_2 d_3, 0, \dots, 0, d_1 d_2 d_4, 0, \dots, 0, d_1 d_3 d_4, \\ 0, \dots, 0, d_2 d_3 d_4, 0, \dots, 0, d_1 d_2 d_3 d_4, 0, \dots, 0 \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{2134}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(d_1, d_2, d_3, d_4, d_1 d_2, 0, \dots, 0, d_1 d_2 d_3, 0, \dots, 0, d_1 d_2 d_4, 0, \dots, 0, d_1 d_2 d_3 d_4, 0, \dots, 0 \right) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{2143} \dot{-}_{21} \gamma_{2134} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \left(\begin{array}{c} d_2, d_1, 0, 0, d_1 d_2, 0, \dots, 0, d_3, 0, \dots, 0, d_2 d_3, 0, \dots, 0, d_1 d_3, \\ 0, \dots, 0, -d_1 d_2 d_3, d_1 d_2 d_3, 0, \dots, 0 \end{array} \right) \quad (58) \end{aligned}$$

Since

$$\begin{aligned} & \gamma_{2431}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1 d_2, d_1 d_3, d_1 d_4, 0, 0, d_3 d_4, 0, 0, d_1 d_2 d_3, 0, \dots, 0, d_1 d_2 d_4, \\ 0, \dots, 0, d_1 d_3 d_4, d_2 d_3 d_4, 0, \dots, 0, d_1 d_2 d_3 d_4, 0, \dots, 0 \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{2341}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1 d_2, d_1 d_3, d_1 d_4, 0, \dots, 0, d_1 d_2 d_3, 0, \dots, 0, d_1 d_2 d_4, 0, \dots, 0, \\ d_1 d_3 d_4, 0, \dots, 0, d_1 d_2 d_3 d_4, 0, \dots, 0 \end{array} \right) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{2431} \dot{-}_{21} \gamma_{2341} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \left(\begin{array}{c} d_2, d_1, 0, 0, d_1 d_2, 0, \dots, 0, d_3, 0, \dots, 0, -d_2 d_3, 0, d_2 d_3, d_1 d_3, \\ 0, \dots, 0, -d_1 d_2 d_3, 0, d_1 d_2 d_3, 0, \dots, 0 \end{array} \right) \quad (59) \end{aligned}$$

(58) and (59) imply that

$$\begin{aligned} & \left(\left(\gamma_{2143} \dot{-}_{21} \gamma_{2134} \right) \dot{-}_1 \left(\gamma_{2431} \dot{-}_{21} \gamma_{2341} \right) \right) (d_1, d_2) \\ &= \mathbf{m} \left(\begin{array}{c} 0, d_1, 0, \dots, 0, d_2, 0, d_2, 0, -d_2, 0, \dots, 0, \\ -d_1 d_2, d_1 d_2, 0, d_1 d_2, 0, -d_1 d_2, 0, \dots, 0 \end{array} \right) \quad (60) \end{aligned}$$

Since

$$\begin{aligned} & \gamma_{1432}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, \dots, \underset{7}{0}, d_2d_3, d_2d_4, d_3d_4, d_1d_2d_3, \underset{12}{0}, \dots, \underset{15}{0}, d_1d_2d_4, \underset{17}{0}, \dots, \underset{20}{0}, \\ d_1d_3d_4, \underset{22}{0}, \dots, \underset{29}{0}, d_2d_3d_4, \underset{31}{0}, \dots, \underset{34}{0}, d_1d_2d_3d_4, \underset{36}{0}, \dots, \underset{53}{0} \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{1342}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, \dots, \underset{7}{0}, d_2d_3, d_2d_4, \underset{10}{0}, d_1d_2d_3, \underset{12}{0}, \dots, \underset{15}{0}, d_1d_2d_4, \underset{17}{0}, \dots, \underset{27}{0}, \\ d_2d_3d_4, 0, 0, 0, 0, d_1d_2d_3d_4, \underset{34}{0}, \dots, \underset{53}{0} \end{array} \right) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{1432} \overset{\cdot}{-} \gamma_{1342} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \left(\begin{array}{c} d_2, d_1, \underset{3}{0}, \underset{4}{0}, d_1d_2, \underset{6}{0}, \dots, \underset{9}{0}, d_3, \underset{11}{0}, \dots, \underset{20}{0}, d_1d_3, \underset{22}{0}, \dots, \underset{27}{0}, -d_2d_3, \underset{29}{0}, d_2d_3, \\ \underset{31}{0}, \underset{32}{0}, -d_1d_2d_3, \underset{34}{0}, d_1d_2d_3, \underset{36}{0}, \dots, \underset{53}{0} \end{array} \right) \quad (61) \end{aligned}$$

Since

$$\begin{aligned} & \gamma_{4312}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, d_1d_3, d_1d_4, d_2d_3, d_2d_4, d_3d_4, \underset{11}{0}, \dots, \underset{13}{0}, d_1d_2d_3, \underset{15}{0}, \dots, \underset{18}{0}, \\ d_1d_2d_4, \underset{20}{0}, \dots, \underset{24}{0}, d_1d_3d_4, \underset{26}{0}, \dots, \underset{29}{0}, d_2d_3d_4, \underset{31}{0}, \dots, \underset{51}{0}, d_1d_2d_3d_4, \underset{53}{0} \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{3412}(d_1, d_2, d_3, d_4) \\ &= \left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, d_1d_3, d_1d_4, d_2d_3, d_2d_4, \underset{10}{0}, \dots, \underset{13}{0}, d_1d_2d_3, \underset{15}{0}, \dots, \underset{18}{0}, d_1d_2d_4, \\ \underset{20}{0}, \dots, \underset{22}{0}, d_1d_3d_4, \underset{24}{0}, \dots, \underset{27}{0}, d_2d_3d_4, \underset{29}{0}, \dots, \underset{45}{0}, d_1d_2d_3d_4, \underset{47}{0}, \dots, \underset{53}{0} \end{array} \right) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{4312} \overset{\cdot}{-} \gamma_{3412} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \left(\begin{array}{c} d_2, d_1, \underset{3}{0}, \underset{4}{0}, d_1d_2, \underset{6}{0}, \dots, \underset{9}{0}, d_3, \underset{11}{0}, \dots, \underset{20}{0}, \underset{21}{0}, \underset{22}{0}, -d_2d_3, \underset{24}{0}, d_2d_3, \underset{26}{0}, \underset{27}{0}, \\ -d_1d_3, \underset{29}{0}, d_1d_3, \underset{31}{0}, \dots, \underset{45}{0}, -d_1d_2d_3, \underset{47}{0}, \dots, \underset{51}{0}, d_1d_2d_3, \underset{53}{0} \end{array} \right) \quad (62) \end{aligned}$$

(61) and (62) imply that

$$\begin{aligned} & \left(\left(\gamma_{1432} \overset{\cdot}{-} \gamma_{1342} \right) \overset{\cdot}{-} \left(\gamma_{4312} \overset{\cdot}{-} \gamma_{3412} \right) \right) (d_1, d_2) \\ &= \mathbf{m} \left(\begin{array}{c} \underset{1}{0}, d_1, \underset{3}{0}, \dots, \underset{20}{0}, d_2, \underset{22}{0}, d_2, \underset{24}{0}, -d_2, \underset{26}{0}, \dots, \underset{32}{0}, -d_1d_2, \underset{34}{0}, d_1d_2, \\ \underset{36}{0}, \dots, \underset{45}{0}, d_1d_2, \underset{47}{0}, \dots, \underset{51}{0}, -d_1d_2, \underset{53}{0} \end{array} \right) \quad (63) \end{aligned}$$

(60) and (63) imply that

$$\begin{aligned} & \left(\left(\left(\gamma_{2143} \begin{smallmatrix} \cdot \\ 21 \end{smallmatrix} \gamma_{2134} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{2431} \begin{smallmatrix} \cdot \\ 21 \end{smallmatrix} \gamma_{2341} \right) \right) \begin{smallmatrix} \cdot \\ \cdot \end{smallmatrix} \right) \\ & \left(\left(\left(\gamma_{1432} \begin{smallmatrix} \cdot \\ 21 \end{smallmatrix} \gamma_{1342} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{4312} \begin{smallmatrix} \cdot \\ 21 \end{smallmatrix} \gamma_{3412} \right) \right) \right) (d) \\ & = \mathbf{m} \left(\begin{array}{cccccccccccccccccccc} 0, & \dots, & 0, & d, & 0, & -d, & -d, & d, & 0, & d, & 0, & -d, & 0, & \dots, & 0, & -d, & 0, & \dots, & 0, & d, & 0 \end{array} \right) \end{aligned} \quad (64)$$

5. Since

$$\begin{aligned} & \gamma_{2314} (d_1, d_2, d_3, d_4) \\ & = \mathbf{m} \left(\begin{array}{cccccccccccccccccccc} d_1, & d_2, & d_3, & d_4, & d_1 d_2, & d_1 d_3, & 0, & \dots, & 0, & d_1 d_2 d_3, & 0, & 0, & 0, & d_1 d_2 d_4, & 0, & \dots, & 0, & d_1 d_3 d_4, & 0 \\ 0, & \dots, & 0, & d_1 d_2 d_3 d_4, & 0, & \dots, & 0 \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{2341} (d_1, d_2, d_3, d_4) \\ & = \mathbf{m} \left(\begin{array}{cccccccccccccccccccc} d_1, & d_2, & d_3, & d_4, & d_1 d_2, & d_1 d_3, & d_1 d_4, & 0, & \dots, & 0, & d_1 d_2 d_3, & 0, & \dots, & 0, & d_1 d_2 d_4, & 0, & \dots, & 0, & d_1 d_3 d_4, & 0 \\ d_1 d_3 d_4, & 0, & \dots, & 0, & d_1 d_2 d_3 d_4, & 0, & \dots, & 0 \end{array} \right) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{2314} \begin{smallmatrix} \cdot \\ 23 \end{smallmatrix} \gamma_{2341} \right) (d_1, d_2, d_3) \\ & = \mathbf{m} \left(\begin{array}{cccccccccccccccccccc} 0, & d_1, & d_2, & 0, & \dots, & 0, & -d_3, & 0, & \dots, & 0, & d_1 d_3, & -d_1 d_3, & 0, & \dots, & 0, & d_2 d_3, & -d_2 d_3, & 0 \\ 0, & \dots, & 0, & d_1 d_2 d_3, & -d_1 d_2 d_3, & 0, & \dots, & 0 \end{array} \right) \end{aligned} \quad (65)$$

Since

$$\begin{aligned} & \gamma_{2143} (d_1, d_2, d_3, d_4) \\ & = \mathbf{m} \left(\begin{array}{cccccccccccccccccccc} d_1, & d_2, & d_3, & d_4, & d_1 d_2, & 0, & \dots, & 0, & d_3 d_4, & 0, & d_1 d_2 d_3, & 0, & \dots, & 0, & d_1 d_2 d_4, & 0, & 0, & 0, & d_1 d_3 d_4, & 0 \\ 0, & \dots, & 0, & d_2 d_3 d_4, & 0, & \dots, & 0, & d_1 d_2 d_3 d_4, & 0, & \dots, & 0 \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{2413} (d_1, d_2, d_3, d_4) \\ & = \mathbf{m} \left(\begin{array}{cccccccccccccccccccc} d_1, & d_2, & d_3, & d_4, & d_1 d_2, & 0, & d_1 d_4, & 0, & 0, & d_3 d_4, & 0, & d_1 d_2 d_3, & 0, & \dots, & 0, & d_1 d_2 d_4, & 0, & \dots, & 0, & d_1 d_3 d_4, & 0 \\ d_1 d_3 d_4, & 0, & d_2 d_3 d_4, & 0, & \dots, & 0, & d_1 d_2 d_3 d_4, & 0, & \dots, & 0 \end{array} \right) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{1432} \overset{\cdot}{\underset{23}{-}} \gamma_{4132} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \left(\begin{array}{c} 0, d_1, d_2, 0, \dots, 0, -d_3, 0, \dots, 0, d_1 d_3, 0, 0, -d_1 d_3, 0, d_2 d_3, 0, 0, \\ -d_2 d_3, 0, \dots, 0, d_1 d_2 d_3, 0, \dots, 0, -d_1 d_2 d_3, 0, \dots, 0 \end{array} \right) \end{aligned} \quad (69)$$

(68) and (69) imply that

$$\begin{aligned} & \left(\left(\gamma_{3142} \overset{\cdot}{\underset{23}{-}} \gamma_{3412} \right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{1432} \overset{\cdot}{\underset{23}{-}} \gamma_{4132} \right) \right) (d_1, d_2) \\ &= \mathbf{m} \left(\begin{array}{c} 0, d_1, 0, \dots, 0, -d_2, d_2, -d_2, d_2, 0, \dots, 0, -d_1 d_2, 0, \dots, 0, \\ d_1 d_2, 0, 0, -d_1 d_2, 0, 0, d_1 d_2, 0, \dots, 0 \end{array} \right) \end{aligned} \quad (70)$$

(67) and (70) imply that

$$\begin{aligned} & \left(\left(\left(\gamma_{2314} \overset{\cdot}{\underset{23}{-}} \gamma_{2341} \right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{2143} \overset{\cdot}{\underset{23}{-}} \gamma_{2413} \right) \right) \overset{\cdot}{\underset{1}{-}} \right. \\ & \left. \left(\left(\gamma_{3142} \overset{\cdot}{\underset{23}{-}} \gamma_{3412} \right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{1432} \overset{\cdot}{\underset{23}{-}} \gamma_{4132} \right) \right) \right) (d) \\ &= \mathbf{m} \left(\begin{array}{c} 0, \dots, 0, d, 0, -d, d, -d, d, 0, 0, -d, 0, 0, d, 0, 0, -d, 0, \dots, 0 \end{array} \right) \end{aligned} \quad (71)$$

6. Since

$$\begin{aligned} & \gamma_{2431} (d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1 d_2, d_1 d_3, d_1 d_4, 0, 0, d_3 d_4, 0, 0, d_1 d_2 d_3, 0, \dots, 0, d_1 d_2 d_4, \\ 0, \dots, 0, d_1 d_3 d_4, d_2 d_3 d_4, 0, \dots, 0, d_1 d_2 d_3 d_4, 0, \dots, 0 \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{2413} (d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1 d_2, 0, d_1 d_4, 0, 0, d_3 d_4, 0, d_1 d_2 d_3, 0, \dots, 0, d_1 d_2 d_4, 0, \dots, 0, \\ d_1 d_3 d_4, 0, d_2 d_3 d_4, 0, \dots, 0, d_1 d_2 d_3 d_4, 0, \dots, 0 \end{array} \right) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{2431} \overset{\cdot}{\underset{24}{-}} \gamma_{2413} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \left(\begin{array}{c} 0, d_1, 0, d_2, 0, d_3, 0, \dots, 0, -d_1 d_3, d_1 d_3, 0, \dots, 0, \\ -d_2 d_3, d_2 d_3, 0, \dots, 0, -d_1 d_2 d_3, d_1 d_2 d_3, 0, \dots, 0 \end{array} \right) \end{aligned} \quad (72)$$

Since

$$\begin{aligned} & \gamma_{2314}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1 d_2, d_1 d_3, 0, \dots, 0, d_1 d_2 d_3, 0, 0, 0, d_1 d_2 d_4, 0, \dots, 0, d_1 d_3 d_4, \\ 0, \dots, 0, d_1 d_2 d_3 d_4, 0, \dots, 0 \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{2134}(d_1, d_2, d_3, d_4) \\ &= \left(d_1, d_2, d_3, d_4, d_1 d_2, 0, \dots, 0, d_1 d_2 d_3, 0, \dots, 0, d_1 d_2 d_4, 0, \dots, 0, d_1 d_2 d_3 d_4, 0, \dots, 0 \right) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{2314} \frac{\cdot}{24} \gamma_{2134} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \left(\begin{array}{c} 0, d_1, 0, d_2, 0, d_3, 0, \dots, 0, -d_1 d_3, d_1 d_3, 0, \dots, 0, d_2 d_3, \\ 0, \dots, 0, -d_1 d_2 d_3, 0, d_1 d_2 d_3, 0, \dots, 0 \end{array} \right) \quad (73) \end{aligned}$$

(72) and (73) imply that

$$\begin{aligned} & \left(\left(\gamma_{2431} \frac{\cdot}{24} \gamma_{2413} \right) \frac{\cdot}{1} \left(\gamma_{2314} \frac{\cdot}{24} \gamma_{2134} \right) \right) (d_1, d_2) \\ &= \mathbf{m} \left(\begin{array}{c} 0, d_1, 0, \dots, 0, -d_2, 0, -d_2, d_2, 0, \dots, 0, d_1 d_2, 0, -d_1 d_2, 0, -d_1 d_2, d_1 d_2, 0, \dots, 0 \\ \end{array} \right) \quad (74) \end{aligned}$$

Since

$$\begin{aligned} & \gamma_{4312}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, 0, d_1 d_3, d_1 d_4, d_2 d_3, d_2 d_4, d_3 d_4, 0, \dots, 0, d_1 d_2 d_3, 0, \dots, 0, \\ d_1 d_2 d_4, 0, \dots, 0, d_1 d_3 d_4, 0, \dots, 0, d_2 d_3 d_4, 0, \dots, 0, d_1 d_2 d_3 d_4, 0 \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{4132}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, 0, 0, d_1 d_4, d_2 d_3, d_2 d_4, d_3 d_4, d_1 d_2 d_3, 0, \dots, 0, d_1 d_2 d_4, 0, \dots, 0, \\ d_1 d_3 d_4, 0, \dots, 0, d_2 d_3 d_4, 0, \dots, 0, d_1 d_2 d_3 d_4, 0, \dots, 0 \end{array} \right) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{4312} \frac{\cdot}{24} \gamma_{4132} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \left(\begin{array}{c} 0, d_1, 0, d_2, 0, d_3, 0, \dots, 0, -d_1 d_3, 0, 0, d_1 d_3, 0, \dots, 0, \\ -d_2 d_3, d_2 d_3, 0, \dots, 0, -d_1 d_2 d_3, 0, 0, d_1 d_2 d_3, 0 \end{array} \right) \quad (75) \end{aligned}$$

Since

$$\begin{aligned} & \gamma_{3142}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, \dots, \underset{7}{0}, d_2d_3, d_2d_4, \underset{10}{0}, \dots, \underset{13}{0}, d_1d_2d_3, 0, d_1d_2d_4, \underset{17}{0}, \dots, \underset{21}{0}, d_1d_3d_4, \\ \underset{23}{0}, \dots, \underset{27}{0}, d_2d_3d_4, \underset{29}{0}, \dots, \underset{42}{0}, d_1d_2d_3d_4, \underset{44}{0}, \dots, \underset{53}{0} \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{1342}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, \dots, \underset{7}{0}, d_2d_3, d_2d_4, \underset{10}{0}, d_1d_2d_3, \underset{12}{0}, \dots, \underset{15}{0}, d_1d_2d_4, \underset{17}{0}, \dots, \underset{27}{0}, \\ d_2d_3d_4, 0, 0, 0, 0, d_1d_2d_3d_4, \underset{34}{0}, \dots, \underset{53}{0} \end{array} \right) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{3142} \overset{\cdot}{\underset{24}{-}} \gamma_{1342} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \left(\begin{array}{c} \underset{1}{0}, d_1, \underset{3}{0}, d_2, \underset{5}{0}, d_3, \underset{7}{0}, \dots, \underset{10}{0}, -d_1d_3, \underset{12}{0}, \underset{13}{0}, d_1d_3, \underset{15}{0}, \dots, \underset{21}{0}, d_2d_3, \\ \underset{23}{0}, \dots, \underset{32}{0}, -d_1d_2d_3, \underset{34}{0}, \dots, \underset{42}{0}, d_1d_2d_3, \underset{44}{0}, \dots, \underset{53}{0} \end{array} \right) \quad (76) \end{aligned}$$

(75) and (76) imply that

$$\begin{aligned} & \left(\left(\gamma_{4312} \overset{\cdot}{\underset{24}{-}} \gamma_{4132} \right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{3142} \overset{\cdot}{\underset{24}{-}} \gamma_{1342} \right) \right) (d_1, d_2) \\ &= \mathbf{m} \left(\begin{array}{c} \underset{1}{0}, d_1, \underset{3}{0}, \dots, \underset{21}{0}, -d_2, \underset{23}{0}, -d_2, d_2, \underset{26}{0}, \dots, \underset{32}{0}, d_1d_2, \underset{34}{0}, \dots, \underset{42}{0}, \\ -d_1d_2, \underset{44}{0}, \dots, \underset{48}{0}, -d_1d_2, \underset{50}{0}, \underset{51}{0}, d_1d_2, \underset{53}{0} \end{array} \right) \quad (77) \end{aligned}$$

(74) and (77) imply that

$$\begin{aligned} & \left(\left(\left(\gamma_{2431} \overset{\cdot}{\underset{24}{-}} \gamma_{2413} \right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{2314} \overset{\cdot}{\underset{24}{-}} \gamma_{2134} \right) \right) \overset{\cdot}{\underset{1}{-}} \right) (d) \\ &= \mathbf{m} \left(\begin{array}{c} \underset{1}{0}, \dots, \underset{32}{0}, -d, \underset{34}{0}, \underset{35}{0}, d, \underset{37}{0}, -d, \underset{39}{0}, -d, d, \underset{42}{0}, d, \underset{44}{0}, \dots, \underset{48}{0}, d, \underset{50}{0}, \underset{51}{0}, -d, \underset{53}{0} \end{array} \right) \quad (78) \end{aligned}$$

7. Since

$$\begin{aligned} & \gamma_{3124}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, d_1d_3, \underset{7}{0}, d_2d_3, \underset{9}{0}, \dots, \underset{13}{0}, d_1d_2d_3, \underset{15}{0}, \dots, \underset{21}{0}, d_1d_3d_4, \underset{23}{0}, \dots, \underset{26}{0}, \\ d_2d_3d_4, \underset{28}{0}, \dots, \underset{41}{0}, d_1d_2d_3d_4, \underset{43}{0}, \dots, \underset{53}{0} \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{3142}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, d_1 d_3, \underset{7}{0}, d_2 d_3, d_2 d_4, \underset{10}{0}, \dots, \underset{13}{0}, d_1 d_2 d_3, 0, d_1 d_2 d_4, \underset{17}{0}, \dots, \underset{21}{0}, \\ d_1 d_3 d_4, \underset{23}{0}, \dots, \underset{27}{0}, d_2 d_3 d_4, \underset{29}{0}, \dots, \underset{42}{0}, d_1 d_2 d_3 d_4, \underset{44}{0}, \dots, \underset{53}{0} \end{array} \right) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{3124} \overset{\cdot}{\underset{31}{-}} \gamma_{3142} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \left(\begin{array}{c} d_2, \underset{2}{0}, d_1, \underset{4}{0}, \dots, \underset{8}{0}, -d_3, \underset{10}{0}, \dots, \underset{15}{0}, -d_2 d_3, \underset{17}{0}, \dots, \underset{26}{0}, d_1 d_3, -d_1 d_3, \\ \underset{29}{0}, \dots, \underset{41}{0}, d_1 d_2 d_3, -d_1 d_2 d_3, \underset{44}{0}, \dots, \underset{53}{0} \end{array} \right) \quad (79) \end{aligned}$$

Since

$$\begin{aligned} & \gamma_{3241}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1 d_2, d_1 d_3, d_1 d_4, d_2 d_3, \underset{9}{0}, \dots, \underset{14}{0}, d_1 d_2 d_3, \underset{16}{0}, \underset{17}{0}, d_1 d_2 d_4, \underset{19}{0}, \dots, \underset{22}{0}, \\ d_1 d_3 d_4, \underset{24}{0}, \dots, \underset{26}{0}, d_2 d_3 d_4, \underset{28}{0}, \dots, \underset{44}{0}, d_1 d_2 d_3 d_4, \underset{46}{0}, \dots, \underset{53}{0} \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{3421}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1 d_2, d_1 d_3, d_1 d_4, d_2 d_3, d_2 d_4, \underset{10}{0}, \dots, \underset{14}{0}, d_1 d_2 d_3, \underset{16}{0}, \dots, \underset{19}{0}, d_1 d_2 d_4, \\ \underset{21}{0}, 0, d_1 d_3 d_4, \underset{24}{0}, \dots, \underset{27}{0}, d_2 d_3 d_4, \underset{29}{0}, \dots, \underset{46}{0}, d_1 d_2 d_3 d_4, \underset{48}{0}, \dots, \underset{53}{0} \end{array} \right) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{3241} \overset{\cdot}{\underset{31}{-}} \gamma_{3421} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \left(\begin{array}{c} d_2, \underset{2}{0}, d_1, \underset{4}{0}, \dots, \underset{8}{0}, -d_3, \underset{10}{0}, \dots, \underset{17}{0}, d_2 d_3, \underset{19}{0}, -d_2 d_3, \underset{21}{0}, \dots, \underset{26}{0}, \\ d_1 d_3, -d_1 d_3, \underset{29}{0}, \dots, \underset{44}{0}, d_1 d_2 d_3, \underset{46}{0}, -d_1 d_2 d_3, \underset{48}{0}, \dots, \underset{53}{0} \end{array} \right) \quad (80) \end{aligned}$$

(79) and (80) imply that

$$\begin{aligned} & \left(\left(\gamma_{3124} \overset{\cdot}{\underset{31}{-}} \gamma_{3142} \right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{3241} \overset{\cdot}{\underset{31}{-}} \gamma_{3421} \right) \right) (d_1, d_2) \\ &= \mathbf{m} \left(\begin{array}{c} \underset{1}{0}, \underset{2}{0}, d_1, \underset{4}{0}, \dots, \underset{15}{0}, -d_2, \underset{17}{0}, -d_2, \underset{19}{0}, d_2, \underset{21}{0}, \dots, \underset{41}{0}, \\ d_1 d_2, -d_1 d_2, \underset{44}{0}, -d_1 d_2, \underset{46}{0}, d_1 d_2, \underset{48}{0}, \dots, \underset{53}{0} \end{array} \right) \quad (81) \end{aligned}$$

Since

$$\begin{aligned} & \gamma_{1243}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, \dots, \underset{9}{0}, d_3 d_4, \underset{10}{0}, \dots, \underset{20}{0}, d_1 d_3 d_4, \underset{22}{0}, \dots, \underset{25}{0}, d_2 d_3 d_4, \underset{27}{0}, \dots, \underset{30}{0}, \\ d_1 d_2 d_3 d_4, \underset{32}{0}, \dots, \underset{53}{0} \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{1423}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, \dots, \underset{8}{0}, d_2d_4, d_3d_4, \underset{11}{0}, \dots, \underset{15}{0}, d_1d_2d_4, \underset{17}{0}, \dots, \underset{20}{0}, d_1d_3d_4, \\ \underset{22}{0}, \dots, \underset{28}{0}, d_2d_3d_4, \underset{30}{0}, \dots, \underset{33}{0}, d_1d_2d_3d_4, \underset{35}{0}, \dots, \underset{53}{0} \end{array} \right) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{1243} \overset{\cdot}{-} \gamma_{1423} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \left(\begin{array}{c} d_2, \underset{2}{0}, d_1, \underset{4}{0}, \dots, \underset{8}{0}, -d_3, \underset{10}{0}, \dots, \underset{15}{0}, -d_2d_3, \underset{17}{0}, \dots, \underset{25}{0}, d_1d_3, \\ \underset{27}{0}, \underset{28}{0}, -d_1d_3, \underset{30}{0}, d_1d_2d_3, \underset{32}{0}, \underset{33}{0}, -d_1d_2d_3, \underset{35}{0}, \dots, \underset{53}{0} \end{array} \right) \quad (82) \end{aligned}$$

Since

$$\begin{aligned} & \gamma_{2413}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1d_2, \underset{6}{0}, d_1d_4, \underset{8}{0}, \underset{9}{0}, d_3d_4, \underset{11}{0}, d_1d_2d_3, \underset{13}{0}, \dots, \underset{17}{0}, d_1d_2d_4, \underset{19}{0}, \dots, \underset{23}{0}, \\ d_1d_3d_4, \underset{27}{0}, d_2d_3d_4, \underset{27}{0}, \dots, \underset{39}{0}, d_1d_2d_3d_4, \underset{41}{0}, \dots, \underset{53}{0} \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{4213}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1d_2, \underset{6}{0}, d_1d_4, \underset{8}{0}, d_2d_4, d_3d_4, \underset{11}{0}, d_1d_2d_3, \underset{13}{0}, \dots, \underset{19}{0}, d_1d_2d_4, \\ \underset{21}{0}, \dots, \underset{23}{0}, d_1d_3d_4, \underset{25}{0}, \dots, \underset{28}{0}, d_2d_3d_4, \underset{30}{0}, \dots, \underset{49}{0}, d_1d_2d_3d_4, \underset{51}{0}, \dots, \underset{53}{0} \end{array} \right) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{2413} \overset{\cdot}{-} \gamma_{4213} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \left(\begin{array}{c} d_2, \underset{2}{0}, d_1, \underset{4}{0}, \dots, \underset{8}{0}, -d_3, \underset{10}{0}, \dots, \underset{17}{0}, d_2d_3, \underset{19}{0}, -d_2d_3, \underset{21}{0}, \dots, \underset{25}{0}, d_1d_3, \\ \underset{27}{0}, \underset{28}{0}, -d_1d_3, \underset{30}{0}, \dots, \underset{39}{0}, d_1d_2d_3, \underset{41}{0}, \dots, \underset{49}{0}, -d_1d_2d_3, \underset{51}{0}, \dots, \underset{53}{0} \end{array} \right) \quad (83) \end{aligned}$$

(82) and (83) imply that

$$\begin{aligned} & \left(\left(\gamma_{1243} \overset{\cdot}{-} \gamma_{1423} \right) \overset{\cdot}{-} \left(\gamma_{2413} \overset{\cdot}{-} \gamma_{4213} \right) \right) (d_1, d_2) \\ &= \mathbf{m} \left(\begin{array}{c} \underset{1}{0}, \underset{2}{0}, d_1, \underset{4}{0}, \dots, \underset{15}{0}, -d_2, \underset{17}{0}, -d_2, \underset{19}{0}, d_2, \underset{21}{0}, \dots, \underset{30}{0}, d_1d_2, \underset{32}{0}, \underset{33}{0}, -d_1d_2, \\ \underset{35}{0}, \dots, \underset{39}{0}, -d_1d_2, \underset{41}{0}, \dots, \underset{49}{0}, d_1d_2, \underset{51}{0}, \dots, \underset{53}{0} \end{array} \right) \quad (84) \end{aligned}$$

(81) and (84) imply that

$$\begin{aligned}
& \left(\left(\left(\left(\gamma_{3124} \begin{smallmatrix} \cdot \\ 31 \end{smallmatrix} \gamma_{3142} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{3241} \begin{smallmatrix} \cdot \\ 31 \end{smallmatrix} \gamma_{3421} \right) \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \right) (d) \\
& \left(\left(\left(\left(\gamma_{1243} \begin{smallmatrix} \cdot \\ 31 \end{smallmatrix} \gamma_{1423} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{2413} \begin{smallmatrix} \cdot \\ 31 \end{smallmatrix} \gamma_{4213} \right) \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \right) (d) \\
& = \mathbf{m} \left(\begin{array}{cccccccccccccccccccc} 0, & \dots, & 0, & -d, & 0, & 0, & d, & 0, & \dots, & 0, & d, & 0, & d, & -d, & 0, & -d, & 0, & d, & 0, & 0, & -d, & 0, & \dots, & 0 \end{array} \right) \\
& \hspace{20em} (85)
\end{aligned}$$

8. Since

$$\begin{aligned}
& \gamma_{3241} (d_1, d_2, d_3, d_4) \\
& = \mathbf{m} \left(\begin{array}{cccccccccccccccccccc} d_1, & d_2, & d_3, & d_4, & d_1 d_2, & d_1 d_3, & d_1 d_4, & d_2 d_3, & 0, & \dots, & 0, & d_1 d_2 d_3, & 0, & 0, & d_1 d_2 d_4, & 0, & \dots, & 0, & \dots, & 0, & \dots, & 0, & \dots, & 0 \end{array} \right) \\
& \hspace{10em} d_1 d_3 d_4, 0, \dots, 0, d_2 d_3 d_4, 0, \dots, 0, d_1 d_2 d_3 d_4, 0, \dots, 0
\end{aligned}$$

and

$$\begin{aligned}
& \gamma_{3214} (d_1, d_2, d_3, d_4) \\
& = \mathbf{m} \left(\begin{array}{cccccccccccccccccccc} d_1, & d_2, & d_3, & d_4, & d_1 d_2, & d_1 d_3, & 0, & d_2 d_3, & 0, & \dots, & 0, & d_1 d_2 d_3, & 0, & d_1 d_2 d_4, & 0, & \dots, & 0, & d_1 d_3 d_4, & \dots, & 0, & \dots, & 0, & \dots, & 0 \end{array} \right) \\
& \hspace{10em} 0, \dots, 0, d_2 d_3 d_4, 0, \dots, 0, d_1 d_2 d_3 d_4, 0, \dots, 0
\end{aligned}$$

we have

$$\begin{aligned}
& \left(\gamma_{3241} \begin{smallmatrix} \cdot \\ 32 \end{smallmatrix} \gamma_{3214} \right) (d_1, d_2, d_3) \\
& = \mathbf{m} \left(\begin{array}{cccccccccccccccc} 0, & d_2, & d_1, & 0, & \dots, & 0, & d_3, & 0, & \dots, & 0, & -d_2 d_3, & d_2 d_3, & 0, & \dots, & 0, & -d_1 d_3, & d_1 d_3, & \dots, & 0, & \dots, & 0, & \dots, & 0 \end{array} \right) \\
& \hspace{10em} 0, \dots, 0, -d_1 d_2 d_3, d_1 d_2 d_3, 0, \dots, 0
\end{aligned} \tag{86}$$

Since

$$\begin{aligned}
& \gamma_{3412} (d_1, d_2, d_3, d_4) \\
& = \mathbf{m} \left(\begin{array}{cccccccccccccccccccc} d_1, & d_2, & d_3, & d_4, & 0, & d_1 d_3, & d_1 d_4, & d_2 d_3, & d_2 d_4, & 0, & \dots, & 0, & d_1 d_2 d_3, & 0, & \dots, & 0, & d_1 d_2 d_4, & \dots, & 0, & \dots, & 0, & \dots, & 0, & \dots, & 0 \end{array} \right) \\
& \hspace{10em} 0, \dots, 0, d_1 d_3 d_4, 0, \dots, 0, d_2 d_3 d_4, 0, \dots, 0, d_1 d_2 d_3 d_4, 0, \dots, 0
\end{aligned}$$

and

$$\begin{aligned}
& \gamma_{3142} (d_1, d_2, d_3, d_4) \\
& = \mathbf{m} \left(\begin{array}{cccccccccccccccccccc} d_1, & d_2, & d_3, & d_4, & 0, & d_1 d_3, & 0, & d_2 d_3, & d_2 d_4, & 0, & \dots, & 0, & d_1 d_2 d_3, & 0, & d_1 d_2 d_4, & 0, & \dots, & 0, & d_1 d_3 d_4, & \dots, & 0, & \dots, & 0, & \dots, & 0 \end{array} \right) \\
& \hspace{10em} 0, \dots, 0, d_2 d_3 d_4, 0, \dots, 0, d_1 d_2 d_3 d_4, 0, \dots, 0
\end{aligned}$$

we have

$$\begin{aligned}
& \left(\gamma_{3412} \begin{smallmatrix} \cdot \\ 32 \end{smallmatrix} \gamma_{3142} \right) (d_1, d_2, d_3) \\
& = \mathbf{m} \left(\begin{array}{cccccccccccccccc} 0, & d_2, & d_1, & 0, & \dots, & 0, & d_3, & 0, & \dots, & 0, & -d_2 d_3, & 0, & 0, & d_2 d_3, & 0, & 0, & -d_1 d_3, & d_1 d_3, & \dots, & 0, & \dots, & 0, & \dots, & 0 \end{array} \right) \\
& \hspace{10em} -d_1 d_3, d_1 d_3, 0, \dots, 0, -d_1 d_2 d_3, 0, 0, d_1 d_2 d_3, 0, \dots, 0
\end{aligned} \tag{87}$$

(86) and (87) imply that

$$\begin{aligned} & \left(\left(\gamma_{3241} \begin{smallmatrix} \cdot \\ 32 \end{smallmatrix} \gamma_{3214} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{3412} \begin{smallmatrix} \cdot \\ 32 \end{smallmatrix} \gamma_{3142} \right) \right) (d_1, d_2) \\ &= \mathbf{m} \left(\begin{array}{c} 0, 0, d_1, 0, \dots, 0, d_2, -d_2, d_2, -d_2, 0, \dots, 0, d_1 d_2, -d_1 d_2, d_1 d_2, -d_1 d_2, 0, \dots, 0 \\ 1, 2, 4, 15, 20, 42, 47, 53 \end{array} \right) \end{aligned} \quad (88)$$

Since

$$\begin{aligned} & \gamma_{2413} (d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1 d_2, 0, d_1 d_4, 0, 0, d_3 d_4, 0, d_1 d_2 d_3, 0, \dots, 0, d_1 d_2 d_4, 0, \dots, 0, \\ d_1 d_3 d_4, 0, d_2 d_3 d_4, 0, \dots, 0, d_1 d_2 d_3 d_4, 0, \dots, 0 \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{2143} (d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1 d_2, 0, \dots, 0, d_3 d_4, 0, d_1 d_2 d_3, 0, \dots, 0, d_1 d_2 d_4, 0, 0, 0, d_1 d_3 d_4, \\ 0, \dots, 0, d_2 d_3 d_4, 0, \dots, 0, d_1 d_2 d_3 d_4, 0, \dots, 0 \end{array} \right) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{2413} \begin{smallmatrix} \cdot \\ 32 \end{smallmatrix} \gamma_{2143} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \left(\begin{array}{c} 0, d_2, d_1, 0, \dots, 0, d_3, 0, \dots, 0, -d_2 d_3, d_2 d_3, 0, 0, -d_1 d_3, \\ 0, 0, d_1 d_3, 0, \dots, 0, -d_1 d_2 d_3, 0, 0, d_1 d_2 d_3, 0, \dots, 0 \end{array} \right) \end{aligned} \quad (89)$$

Since

$$\begin{aligned} & \gamma_{4123} (d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, 0, 0, d_1 d_4, 0, d_2 d_4, d_3 d_4, 0, \dots, 0, d_1 d_2 d_4, 0, \dots, 0, d_1 d_3 d_4, \\ 0, \dots, 0, d_2 d_3 d_4, 0, \dots, 0, d_1 d_2 d_3 d_4, 0, \dots, 0 \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{1423} (d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, 0, \dots, 0, d_2 d_4, d_3 d_4, 0, \dots, 0, d_1 d_2 d_4, 0, \dots, 0, d_1 d_3 d_4, \\ 0, \dots, 0, d_2 d_3 d_4, 0, \dots, 0, d_1 d_2 d_3 d_4, 0, \dots, 0 \end{array} \right) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{4123} \begin{smallmatrix} \cdot \\ 32 \end{smallmatrix} \gamma_{1423} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \left(\begin{array}{c} 0, d_2, d_1, 0, \dots, 0, d_3, 0, \dots, 0, -d_2 d_3, 0, 0, d_2 d_3, 0, -d_1 d_3, \\ 0, 0, d_1 d_3, 0, \dots, 0, -d_1 d_2 d_3, 0, \dots, 0, d_1 d_2 d_3, 0, \dots, 0 \end{array} \right) \end{aligned} \quad (90)$$

and

$$\begin{aligned} & \gamma_{3214}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{cccccccccccccccc} d_1, d_2, d_3, d_4, d_1 d_2, d_1 d_3, 0, d_2 d_3, 0, \dots, 0, d_1 d_2 d_3, 0, d_1 d_2 d_4, 0, \dots, 0, d_1 d_3 d_4, \\ 0, \dots, 0, d_2 d_3 d_4, 0, \dots, 0, d_1 d_2 d_3 d_4, 0, \dots, 0 \end{array} \right) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{3124} \dot{-}_{34} \gamma_{3214} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \left(\begin{array}{cccccccccccc} 0, 0, d_1, d_2, -d_3, 0, \dots, 0, d_1 d_3, -d_1 d_3, 0, -d_2 d_3, 0, \dots, 0, \\ d_1 d_2 d_3, 0, -d_1 d_2 d_3, 0, \dots, 0 \end{array} \right) \quad (94) \end{aligned}$$

(93) and (94) imply that

$$\begin{aligned} & \left(\left(\gamma_{3412} \dot{-}_{34} \gamma_{3421} \right) \dot{-}_1 \left(\gamma_{3124} \dot{-}_{34} \gamma_{3214} \right) \right) (d_1, d_2) \\ &= \mathbf{m} \left(\begin{array}{cccccccccccc} 0, 0, d_1, 0, \dots, 0, d_2, 0, d_2, -d_2, 0, \dots, 0, -d_1 d_2, 0, d_1 d_2, \\ 0, d_1 d_2, -d_1 d_2, 0, \dots, 0 \end{array} \right) \quad (95) \end{aligned}$$

Since

$$\begin{aligned} & \gamma_{4123}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{cccccccccccccccc} d_1, d_2, d_3, d_4, 0, 0, d_1 d_4, 0, d_2 d_4, d_3 d_4, 0, \dots, 0, d_1 d_2 d_4, 0, \dots, 0, d_1 d_3 d_4, \\ 0, \dots, 0, d_2 d_3 d_4, 0, \dots, 0, d_1 d_2 d_3 d_4, 0, \dots, 0 \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{4213}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{cccccccccccccccc} d_1, d_2, d_3, d_4, d_1 d_2, 0, d_1 d_4, 0, d_2 d_4, d_3 d_4, 0, d_1 d_2 d_3, 0, \dots, 0, d_1 d_2 d_4, \\ 0, \dots, 0, d_1 d_3 d_4, 0, \dots, 0, d_2 d_3 d_4, 0, \dots, 0, d_1 d_2 d_3 d_4, 0, \dots, 0 \end{array} \right) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{4123} \dot{-}_{34} \gamma_{4213} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \left(\begin{array}{cccccccccccc} 0, 0, d_1, d_2, -d_3, 0, \dots, 0, -d_1 d_3, 0, \dots, 0, d_2 d_3, -d_2 d_3, \\ 0, \dots, 0, d_1 d_2 d_3, 0, -d_1 d_2 d_3, 0, \dots, 0 \end{array} \right) \quad (96) \end{aligned}$$

Since

$$\begin{aligned} & \gamma_{1243}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{cccccccccccccccc} d_1, d_2, d_3, d_4, 0, \dots, 0, d_3 d_4, 0, \dots, 0, d_1 d_3 d_4, 0, \dots, 0, d_2 d_3 d_4, 0, \dots, 0, \\ d_1 d_2 d_3 d_4, 0, \dots, 0 \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{2143}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{cccccccccccccccccccc} d_1, d_2, d_3, d_4, d_1 d_2, 0, \dots, 0, d_3 d_4, 0, d_1 d_2 d_3, 0, \dots, 0, d_1 d_2 d_4, 0, \dots, 0, d_1 d_3 d_4, \\ 0, \dots, 0, d_2 d_3 d_4, 0, \dots, 0, d_1 d_2 d_3 d_4, 0, \dots, 0 \end{array} \right) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{1243} \dot{-}_{34} \gamma_{2143} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \left(\begin{array}{cccccccccccccccc} 0, 0, d_1, d_2, -d_3, 0, \dots, 0, -d_1 d_3, 0, \dots, 0, -d_2 d_3, \\ 0, \dots, 0, d_1 d_2 d_3, 0, \dots, 0, -d_1 d_2 d_3, 0, \dots, 0 \end{array} \right) \quad (97) \end{aligned}$$

(96) and (97) imply that

$$\begin{aligned} & \left(\left(\gamma_{4123} \dot{-}_{34} \gamma_{4213} \right) \dot{-}_1 \left(\gamma_{1243} \dot{-}_{34} \gamma_{2143} \right) \right) (d_1, d_2) \\ &= \mathbf{m} \left(\begin{array}{cccccccccccccccc} 0, 0, d_1, 0, \dots, 0, d_2, 0, d_2, -d_2, 0, \dots, 0, -d_1 d_2, 0, \dots, 0, \\ d_1 d_2, 0, \dots, 0, d_1 d_2, 0, -d_1 d_2, 0, \dots, 0 \end{array} \right) \quad (98) \end{aligned}$$

(95) and (98) imply that

$$\begin{aligned} & \left(\left(\left(\gamma_{3412} \dot{-}_{34} \gamma_{3421} \right) \dot{-}_1 \left(\gamma_{3124} \dot{-}_{34} \gamma_{3214} \right) \right) \dot{-} \right. \\ & \left. \left(\left(\gamma_{4123} \dot{-}_{34} \gamma_{4213} \right) \dot{-}_1 \left(\gamma_{1243} \dot{-}_{34} \gamma_{2143} \right) \right) \right) (d) \\ &= \mathbf{m} \left(\begin{array}{cccccccccccccccc} 0, \dots, 0, d, 0, \dots, 0, -d, 0, \dots, 0, -d, 0, d, 0, d, -d, -d, 0, d, 0, \dots, 0 \end{array} \right) \quad (99) \end{aligned}$$

10. Since

$$\begin{aligned} & \gamma_{4132}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{cccccccccccccccc} d_1, d_2, d_3, d_4, 0, 0, d_1 d_4, d_2 d_3, d_2 d_4, d_3 d_4, d_1 d_2 d_3, 0, \dots, 0, d_1 d_2 d_4, 0, \dots, 0, \\ d_1 d_3 d_4, 0, \dots, 0, d_2 d_3 d_4, 0, \dots, 0, d_1 d_2 d_3 d_4, 0, \dots, 0 \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{4123}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{cccccccccccccccc} d_1, d_2, d_3, d_4, 0, 0, d_1 d_4, 0, d_2 d_4, d_3 d_4, 0, \dots, 0, d_1 d_2 d_4, 0, \dots, 0, d_1 d_3 d_4, \\ 0, \dots, 0, d_2 d_3 d_4, 0, \dots, 0, d_1 d_2 d_3 d_4, 0, \dots, 0 \end{array} \right) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{4132} \overset{\cdot}{\underset{41}{-}} \gamma_{4123} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \left(\begin{array}{c} d_2, 0, 0, d_1, 0, \dots, 0, d_3, 0, \dots, 0, d_2 d_3, 0, \dots, 0, -d_1 d_3, d_1 d_3, \\ 0, \dots, 0, -d_1 d_2 d_3, d_1 d_2 d_3, 0, \dots, 0 \end{array} \right) \end{aligned} \quad (100)$$

Since

$$\begin{aligned} & \gamma_{4321} (d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1 d_2, d_1 d_3, d_1 d_4, d_2 d_3, d_2 d_4, d_3 d_4, 0, \dots, 0, d_1 d_2 d_3, 0, \dots, 0, \\ d_1 d_2 d_4, 0, \dots, 0, d_1 d_3 d_4, 0, \dots, 0, d_2 d_3 d_4, 0, \dots, 0, d_1 d_2 d_3 d_4 \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{4231} (d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1 d_2, d_1 d_3, d_1 d_4, 0, d_2 d_4, d_3 d_4, 0, 0, d_1 d_2 d_3, 0, \dots, 0, \\ d_1 d_2 d_4, 0, \dots, 0, d_1 d_3 d_4, 0, \dots, 0, d_2 d_3 d_4, 0, \dots, 0, d_1 d_2 d_3 d_4, 0, 0 \end{array} \right) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{4321} \overset{\cdot}{\underset{41}{-}} \gamma_{4231} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \left(\begin{array}{c} d_2, 0, 0, d_1, 0, \dots, 0, d_3, 0, \dots, 0, -d_2 d_3, 0, d_2 d_3, 0, \dots, 0, \\ -d_1 d_3, d_1 d_3, 0, \dots, 0, -d_1 d_2 d_3, 0, d_1 d_2 d_3 \end{array} \right) \end{aligned} \quad (101)$$

(100) and (??) imply that

$$\begin{aligned} & \left(\left(\gamma_{4132} \overset{\cdot}{\underset{41}{-}} \gamma_{4123} \right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{4321} \overset{\cdot}{\underset{41}{-}} \gamma_{4231} \right) \right) (d_1, d_2) \\ &= \mathbf{m} \left(\begin{array}{c} 0, \dots, 0, d_1, 0, \dots, 0, d_2, 0, d_2, 0, -d_2, 0, \dots, 0, -d_1 d_2, d_1 d_2, 0, d_1 d_2, 0, -d_1 d_2 \end{array} \right) \end{aligned} \quad (102)$$

Since

$$\begin{aligned} & \gamma_{1324} (d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, 0, 0, 0, d_2 d_3, 0, 0, d_1 d_2 d_3, 0, \dots, 0, d_2 d_3 d_4, 0, \dots, 0, \\ d_1 d_2 d_3 d_4, 0, \dots, 0 \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{1234} (d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(d_1, d_2, d_3, d_4, 0, \dots, 0 \right) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{1324} \dot{-}_{41} \gamma_{1234} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \left(d_2, \underset{2}{0}, \underset{3}{0}, d_1, \underset{5}{0}, \dots, \underset{7}{0}, d_3, \underset{9}{0}, \underset{10}{0}, d_2 d_3, \underset{12}{0}, \dots, \underset{26}{0}, d_1 d_3, \underset{28}{0}, \dots, \underset{31}{0}, d_1 d_2 d_3, \underset{33}{0}, \dots, \underset{53}{0} \right) \end{aligned} \quad (103)$$

Since

$$\begin{aligned} & \gamma_{3214} (d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(d_1, d_2, d_3, d_4, d_1 d_2, d_1 d_3, \underset{7}{0}, d_2 d_3, \underset{9}{0}, \dots, \underset{14}{0}, d_1 d_2 d_3, \underset{14}{0}, d_1 d_2 d_4, \underset{18}{0}, \dots, \underset{21}{0}, d_1 d_3 d_4, \right. \\ & \quad \left. \underset{23}{0}, \dots, \underset{26}{0}, d_2 d_3 d_4, \underset{28}{0}, \dots, \underset{43}{0}, d_1 d_2 d_3 d_4, \underset{45}{0}, \dots, \underset{53}{0} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{2314} (d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(d_1, d_2, d_3, d_4, d_1 d_2, d_1 d_3, \underset{7}{0}, \dots, \underset{12}{0}, d_1 d_2 d_3, \underset{12}{0}, \underset{12}{0}, d_1 d_2 d_4, \underset{18}{0}, \dots, \underset{21}{0}, d_1 d_3 d_4, \right. \\ & \quad \left. \underset{23}{0}, \dots, \underset{37}{0}, d_1 d_2 d_3 d_4, \underset{39}{0}, \dots, \underset{53}{0} \right) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{3214} \dot{-}_{41} \gamma_{2314} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \left(d_2, \underset{2}{0}, \underset{3}{0}, d_1, \underset{5}{0}, \dots, \underset{7}{0}, d_3, \underset{9}{0}, \dots, \underset{12}{0}, -d_2 d_3, \underset{14}{0}, d_2 d_3, \underset{16}{0}, \dots, \underset{26}{0}, d_1 d_3, \right. \\ & \quad \left. \underset{28}{0}, \dots, \underset{37}{0}, -d_1 d_2 d_3, \underset{39}{0}, \dots, \underset{43}{0}, d_1 d_2 d_3, \underset{45}{0}, \dots, \underset{53}{0} \right) \end{aligned} \quad (104)$$

(103) and (104) imply that

$$\begin{aligned} & \left(\left(\gamma_{1324} \dot{-}_{41} \gamma_{1234} \right) \dot{-}_1 \left(\gamma_{3214} \dot{-}_{41} \gamma_{2314} \right) \right) (d_1, d_2) \\ &= \mathbf{m} \left(\underset{1}{0}, \dots, \underset{3}{0}, d_1, \underset{5}{0}, \dots, \underset{10}{0}, d_2, \underset{12}{0}, d_2, \underset{14}{0}, -d_2, \underset{16}{0}, \dots, \underset{31}{0}, d_1 d_2, \right. \\ & \quad \left. \underset{33}{0}, \dots, \underset{37}{0}, d_1 d_2, \underset{39}{0}, \dots, \underset{43}{0}, -d_1 d_2, \underset{45}{0}, \dots, \underset{53}{0} \right) \end{aligned} \quad (105)$$

(102) and (105) imply that

$$\begin{aligned} & \left(\left(\left(\gamma_{4132} \dot{-}_{41} \gamma_{4123} \right) \dot{-}_1 \left(\gamma_{4321} \dot{-}_{41} \gamma_{4231} \right) \right) \dot{-}_1 \right. \\ & \quad \left. \left(\left(\gamma_{1324} \dot{-}_{41} \gamma_{1234} \right) \dot{-}_1 \left(\gamma_{3214} \dot{-}_{41} \gamma_{2314} \right) \right) \right) (d) \\ &= \mathbf{m} \left(\underset{1}{0}, \dots, \underset{31}{0}, -d, \underset{33}{0}, \dots, \underset{37}{0}, -d, \underset{39}{0}, \dots, \underset{43}{0}, d, \underset{45}{0}, \dots, \underset{47}{0}, -d, d, \underset{50}{0}, d, \underset{52}{0}, \underset{53}{-d} \right) \end{aligned} \quad (106)$$

11. Since

$$\begin{aligned} & \gamma_{4213}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \begin{pmatrix} d_1, d_2, d_3, d_4, d_1 d_2, 0, d_1 d_4, 0, d_2 d_4, d_3 d_4, 0, d_1 d_2 d_3, 0, \dots, 0, d_1 d_2 d_4, \\ 0, \dots, 0, d_1 d_3 d_4, 0, \dots, 0, d_2 d_3 d_4, 0, \dots, 0, d_1 d_2 d_3 d_4, 0, \dots, 0 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} & \gamma_{4231}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \begin{pmatrix} d_1, d_2, d_3, d_4, d_1 d_2, d_1 d_3, d_1 d_4, 0, d_2 d_4, d_3 d_4, 0, 0, d_1 d_2 d_3, 0, \dots, 0, \\ d_1 d_2 d_4, 0, \dots, 0, d_1 d_3 d_4, 0, \dots, 0, d_2 d_3 d_4, 0, \dots, 0, d_1 d_2 d_3 d_4, 0, 0 \end{pmatrix} \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{4213} \frac{\cdot}{42} \gamma_{4231} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \begin{pmatrix} 0, d_2, 0, d_1, 0, -d_3, 0, \dots, 0, d_2 d_3, -d_2 d_3, 0, \dots, 0, \\ d_1 d_3, -d_1 d_3, 0, \dots, 0, d_1 d_2 d_3, -d_1 d_2 d_3, 0, 0 \end{pmatrix} \quad (107) \end{aligned}$$

Since

$$\begin{aligned} & \gamma_{4132}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \begin{pmatrix} d_1, d_2, d_3, d_4, 0, 0, d_1 d_4, d_2 d_3, d_2 d_4, d_3 d_4, d_1 d_2 d_3, 0, \dots, 0, d_1 d_2 d_4, 0, \dots, 0, \\ d_1 d_3 d_4, 0, \dots, 0, d_2 d_3 d_4, 0, \dots, 0, d_1 d_2 d_3 d_4, 0, \dots, 0 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} & \gamma_{4312}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \begin{pmatrix} d_1, d_2, d_3, d_4, 0, d_1 d_3, d_1 d_4, d_2 d_3, d_2 d_4, d_3 d_4, 0, \dots, 0, d_1 d_2 d_3, 0, \dots, 0, \\ d_1 d_2 d_4, 0, \dots, 0, d_1 d_3 d_4, 0, \dots, 0, d_2 d_3 d_4, 0, \dots, 0, d_1 d_2 d_3 d_4, 0 \end{pmatrix} \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{4132} \frac{\cdot}{42} \gamma_{4312} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \begin{pmatrix} 0, d_2, 0, d_1, 0, -d_3, 0, \dots, 0, d_2 d_3, 0, 0, -d_2 d_3, 0, \dots, 0, d_1 d_3, -d_1 d_3, \\ 0, \dots, 0, d_1 d_2 d_3, 0, 0, -d_1 d_2 d_3, 0 \end{pmatrix} \quad (108) \end{aligned}$$

(107) and (108) imply that

$$\begin{aligned} & \left(\left(\gamma_{4213} \frac{\cdot}{42} \gamma_{4231} \right) \frac{\cdot}{1} \left(\gamma_{4132} \frac{\cdot}{42} \gamma_{4312} \right) \right) (d_1, d_2) \\ &= \mathbf{m} \begin{pmatrix} 0, \dots, 0, d_1, 0, \dots, 0, -d_2, d_2, -d_2, d_2, 0, \dots, 0, -d_1 d_2, d_1 d_2, -d_1 d_2, d_1 d_2, 0 \end{pmatrix} \quad (109) \end{aligned}$$

Since

$$\begin{aligned} & \gamma_{2134}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(d_1, d_2, d_3, d_4, d_1 d_2, \underset{6}{0}, \dots, \underset{11}{0}, d_1 d_2 d_3, \underset{13}{0}, \dots, \underset{16}{0}, d_1 d_2 d_4, \underset{18}{0}, \dots, \underset{35}{0}, d_1 d_2 d_3 d_4, \underset{37}{0}, \dots, \underset{53}{0} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{2314}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1 d_2, d_1 d_3, \underset{7}{0}, \dots, \underset{12}{0}, d_1 d_2 d_3, 0, 0, 0, d_1 d_2 d_4, \underset{18}{0}, \dots, \underset{21}{0}, d_1 d_3 d_4, \\ \underset{23}{0}, \dots, \underset{37}{0}, d_1 d_2 d_3 d_4, \underset{39}{0}, \dots, \underset{53}{0} \end{array} \right) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{2134} \overset{\cdot}{-} \gamma_{2314} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \left(\begin{array}{c} \underset{1}{0}, d_2, \underset{3}{0}, d_1, \underset{5}{0}, -d_3, \underset{7}{0}, \dots, \underset{11}{0}, d_2 d_3, -d_2 d_3, \underset{14}{0}, \dots, \underset{21}{0}, -d_1 d_3, \\ \underset{23}{0}, \dots, \underset{35}{0}, d_1 d_2 d_3, 0, -d_1 d_2 d_3, \underset{39}{0}, \dots, \underset{53}{0} \end{array} \right) \quad (110) \end{aligned}$$

Since

$$\begin{aligned} & \gamma_{1324}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, \underset{6}{0}, \underset{7}{0}, d_2 d_3, \underset{9}{0}, \underset{10}{0}, d_1 d_2 d_3, \underset{12}{0}, \dots, \underset{26}{0}, d_2 d_3 d_4, 0, 0, \underset{30}{0}, \\ 0, d_1 d_2 d_3 d_4, \underset{33}{0}, \dots, \underset{53}{0} \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} & \gamma_{3124}(d_1, d_2, d_3, d_4) \\ &= \mathbf{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, d_1 d_3, \underset{7}{0}, d_2 d_3, \underset{9}{0}, \dots, \underset{13}{0}, d_1 d_2 d_3, \underset{15}{0}, \dots, \underset{21}{0}, d_1 d_3 d_4, \underset{23}{0}, \dots, \underset{26}{0}, \\ d_2 d_3 d_4, \underset{28}{0}, \dots, \underset{41}{0}, d_1 d_2 d_3 d_4, \underset{43}{0}, \dots, \underset{53}{0} \end{array} \right) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{1324} \overset{\cdot}{-} \gamma_{3124} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \left(\begin{array}{c} \underset{1}{0}, d_2, \underset{3}{0}, d_1, \underset{5}{0}, -d_3, \underset{7}{0}, \dots, \underset{10}{0}, d_2 d_3, 0, 0, -d_2 d_3, \underset{15}{0}, \dots, \underset{21}{0}, \\ -d_1 d_3, \underset{23}{0}, \dots, \underset{31}{0}, d_1 d_2 d_3, \underset{33}{0}, \dots, \underset{41}{0}, -d_1 d_2 d_3, \underset{43}{0}, \dots, \underset{53}{0} \end{array} \right) \quad (111) \end{aligned}$$

(110) and (111) imply that

$$\begin{aligned} & \left(\left(\gamma_{2134} \overset{\cdot}{-} \gamma_{2314} \right) \overset{\cdot}{-} \left(\gamma_{1324} \overset{\cdot}{-} \gamma_{3124} \right) \right) (d_1, d_2) \\ &= \mathbf{m} \left(\begin{array}{c} \underset{1}{0}, \dots, \underset{3}{0}, d_1, \underset{5}{0}, \dots, \underset{10}{0}, -d_2, d_2, -d_2, d_2, \underset{15}{0}, \dots, \underset{31}{0}, -d_1 d_2, \underset{33}{0}, \dots, \underset{35}{0}, \\ d_1 d_2, 0, -d_1 d_2, \underset{39}{0}, \dots, \underset{41}{0}, d_1 d_2, \underset{43}{0}, \dots, \underset{53}{0} \end{array} \right) \quad (112) \end{aligned}$$

we have

$$\begin{aligned} & \left(\gamma_{2134} \overset{\cdot}{\underset{43}{-}} \gamma_{1234} \right) (d_1, d_2, d_3) \\ &= \mathbf{m} \left(\underset{1}{0}, \underset{2}{0}, \underset{3}{d_2}, \underset{4}{d_1}, \underset{5}{d_3}, \underset{6}{0}, \dots, \underset{11}{0}, \underset{12}{d_2 d_3}, \underset{13}{0}, \dots, \underset{16}{0}, \underset{17}{d_1 d_3}, \underset{18}{0}, \dots, \underset{35}{0}, \underset{36}{d_1 d_2 d_3}, \underset{37}{0}, \dots, \underset{53}{0} \right) \end{aligned} \quad (118)$$

(117) and (118) imply that

$$\begin{aligned} & \left(\left(\gamma_{3214} \overset{\cdot}{\underset{43}{-}} \gamma_{3124} \right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{2134} \overset{\cdot}{\underset{43}{-}} \gamma_{1234} \right) \right) (d_1, d_2) \\ &= \mathbf{m} \left(\underset{1}{0}, \dots, \underset{3}{0}, \underset{4}{d_1}, \underset{5}{0}, \dots, \underset{11}{0}, \underset{12}{-d_2}, \underset{13}{0}, \underset{14}{-d_2}, \underset{15}{d_2}, \underset{16}{0}, \dots, \underset{35}{0}, \underset{36}{-d_1 d_2}, \right. \\ & \quad \left. \underset{37}{0}, \dots, \underset{41}{0}, \underset{42}{-d_1 d_2}, \underset{43}{0}, \underset{44}{d_1 d_2}, \underset{45}{0}, \dots, \underset{53}{0} \right) \end{aligned} \quad (119)$$

(116) and (119) imply that

$$\begin{aligned} & \left(\left(\left(\gamma_{4321} \overset{\cdot}{\underset{43}{-}} \gamma_{4312} \right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{4213} \overset{\cdot}{\underset{43}{-}} \gamma_{4123} \right) \right) \overset{\cdot}{\underset{1}{-}} \right. \\ & \quad \left. \left(\left(\gamma_{3214} \overset{\cdot}{\underset{43}{-}} \gamma_{3124} \right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{2134} \overset{\cdot}{\underset{43}{-}} \gamma_{1234} \right) \right) \right) (d) \\ &= \mathbf{m} \left(\underset{1}{0}, \dots, \underset{35}{0}, \underset{36}{d}, \underset{37}{0}, \dots, \underset{41}{0}, \underset{42}{d}, \underset{43}{0}, \underset{44}{-d}, \underset{45}{0}, \dots, \underset{47}{0}, \underset{48}{d}, \underset{49}{0}, \underset{50}{-d}, \underset{51}{0}, \underset{52}{-d}, \underset{53}{d} \right) \end{aligned} \quad (120)$$

13. By (??), (??), (??), (??), (71), (78), (85), (92), (99), (106), (113) and

(120), we have

the left-hand side of the equation (36)

$$\begin{aligned}
&= \left(\lambda d.m \left(\underset{1}{0}, \dots, \underset{30}{0}, -d, \underset{32}{0}, -d, \underset{34}{0}, d, \underset{36}{0}, \dots, \underset{38}{0}, -d, \underset{42}{0}, d, \underset{46}{0}, \dots, \underset{48}{0}, d, \underset{52}{0}, \dots, \underset{53}{0}, -d \right) \right) + \\
&\left(\lambda d.m \left(\underset{1}{0}, \dots, \underset{30}{0}, d, -d, d, -d, \underset{35}{0}, \dots, \underset{40}{0}, -d, \underset{42}{0}, \dots, \underset{44}{0}, d, \underset{46}{0}, -d, \underset{48}{0}, \dots, \underset{50}{0}, d, \underset{52}{0}, \underset{53}{0} \right) \right) + \\
&\left(\lambda d.m \left(\underset{1}{0}, \dots, \underset{31}{0}, d, \underset{33}{0}, d, -d, \underset{36}{0}, \dots, \underset{38}{0}, d, \underset{40}{0}, \dots, \underset{44}{0}, -d, \underset{46}{0}, \dots, \underset{50}{0}, -d, \underset{52}{0}, d \right) \right) + \\
&\left(\lambda d.m \left(\underset{1}{0}, \dots, \underset{32}{0}, d, \underset{34}{0}, -d, -d, d, \underset{38}{0}, d, \underset{40}{0}, -d, \underset{42}{0}, \dots, \underset{45}{0}, -d, \underset{47}{0}, \dots, \underset{51}{0}, d, \underset{53}{0} \right) \right) + \\
&\left(\lambda d.m \left(\underset{1}{0}, \dots, \underset{34}{0}, d, \underset{36}{0}, -d, d, -d, d, \underset{41}{0}, \underset{42}{0}, -d, \underset{44}{0}, \underset{45}{0}, d, \underset{47}{0}, \underset{48}{0}, -d, \underset{50}{0}, \dots, \underset{53}{0} \right) \right) + \\
&\left(\lambda d.m \left(\underset{1}{0}, \dots, \underset{32}{0}, -d, \underset{34}{0}, \underset{35}{0}, d, \underset{37}{0}, -d, \underset{39}{0}, -d, d, \underset{42}{0}, d, \underset{44}{0}, \dots, \underset{48}{0}, d, \underset{50}{0}, \underset{51}{0}, -d, \underset{53}{0} \right) \right) + \\
&\left(\lambda d.m \left(\underset{1}{0}, \dots, \underset{30}{0}, -d, \underset{32}{0}, \underset{33}{0}, d, \underset{35}{0}, \dots, \underset{39}{0}, d, \underset{41}{0}, d, -d, \underset{44}{0}, -d, \underset{46}{0}, d, \underset{48}{0}, \underset{49}{0}, -d, \underset{51}{0}, \dots, \underset{53}{0} \right) \right) + \\
&\left(\lambda d.m \left(\underset{1}{0}, \dots, \underset{30}{0}, -d, \underset{32}{0}, \underset{33}{0}, d, \underset{35}{0}, \dots, \underset{39}{0}, d, \underset{41}{0}, d, -d, \underset{44}{0}, -d, \underset{46}{0}, d, \underset{48}{0}, \underset{49}{0}, -d, \underset{51}{0}, \dots, \underset{53}{0} \right) \right) + \\
&\left(\lambda d.m \left(\underset{1}{0}, \dots, \underset{33}{0}, -d, \underset{35}{0}, \underset{36}{0}, d, \underset{38}{0}, \underset{39}{0}, -d, \underset{41}{0}, \underset{42}{0}, d, -d, d, -d, \underset{47}{0}, d, \underset{49}{0}, \dots, \underset{53}{0} \right) \right) + \\
&\left(\lambda d.m \left(\underset{1}{0}, \dots, \underset{30}{0}, d, \underset{32}{0}, \dots, \underset{36}{0}, -d, \underset{38}{0}, \dots, \underset{41}{0}, -d, \underset{43}{0}, d, \underset{45}{0}, d, -d, -d, \underset{49}{0}, d, \underset{51}{0}, \dots, \underset{53}{0} \right) \right) + \\
&\left(\lambda d.m \left(\underset{1}{0}, \dots, \underset{31}{0}, -d, \underset{33}{0}, \dots, \underset{37}{0}, -d, \underset{39}{0}, \dots, \underset{43}{0}, d, \underset{45}{0}, \dots, \underset{47}{0}, -d, d, \underset{50}{0}, d, \underset{53}{0}, -d \right) \right) + \\
&\left(\lambda d.m \left(\underset{1}{0}, \dots, \underset{31}{0}, d, \underset{33}{0}, \dots, \underset{35}{0}, -d, \underset{39}{0}, d, \underset{41}{0}, \dots, \underset{43}{0}, -d, \dots, \underset{48}{0}, -d, d, -d, d, \underset{53}{0} \right) \right) + \\
&\left(\lambda d.m \left(\underset{1}{0}, \dots, \underset{35}{0}, d, \underset{37}{0}, \dots, \underset{41}{0}, d, \underset{43}{0}, -d, \underset{45}{0}, \dots, \underset{47}{0}, d, \underset{49}{0}, -d, \underset{51}{0}, -d, d \right) \right) \\
&= \lambda d.m \left(\underset{1}{0}, \dots, \underset{53}{0} \begin{pmatrix} \underset{30}{0}, \dots, \underset{31}{0}, -d + d - d + d, -d + d - d + d, -d + d + d - d, \\ -d + d + d - d, d - d - d + d, -d + d - d + d, \\ \underset{34}{d - d + d - d}, d - d - d + d, -d + d + d - d, \\ \underset{37}{d - d + d - d}, d - d - d + d, d - d - d + d, \\ \underset{40}{-d + d - d + d}, -d + d + d - d, d - d - d + d, \\ \underset{43}{-d + d - d + d}, d - d + d - d, d - d - d + d, \\ \underset{46}{-d + d + d - d}, -d + d + d - d, d - d - d + d, \\ \underset{49}{-d + d + d - d}, -d + d + d - d, d - d + d - d, \\ \underset{52}{d - d + d - d}, -d + d - d + d \end{pmatrix} \right) \\
&= \lambda d.m \left(\underset{1}{0}, \dots, \underset{53}{0} \right) \tag{121}
\end{aligned}$$

■

Remark 21 For our convenience we display the positions 31–53 in (43), (50),

(57), (64), (71), (78), (85), (92), (99), (106), (113) and (120) as a table:

	1/12	2/13	3/14	4/21	5/23	6/24	7/31	8/32	9/34	10/41	11/42	12/43
31/1243	$-d$	d					$-d$		d			
32/1324		$-d$	d							$-d$	d	
33/1342	$-d$	d		d		$-d$						
34/1423		$-d$	d				d	$-d$				
35/1432	d		$-d$	$-d$	d							
36/2134				$-d$		d					$-d$	d
37/2143				d	$-d$			d	$-d$			
38/2314					d	$-d$				$-d$	d	
39/2341	$-d$		d	d	$-d$							
40/2413					d	$-d$	d	$-d$				
41/2431	d	$-d$		$-d$		d						
42/3124							d		$-d$		$-d$	d
43/3142					$-d$	d	$-d$	d				
44/3214								$-d$	d	d		$-d$
45/3241		d	$-d$				$-d$	d				
46/3412				$-d$	d			$-d$	d			
47/3421	d	$-d$					d		$-d$			
48/4123								d	$-d$	$-d$		d
49/4132					$-d$	d				d	$-d$	
50/4213							$-d$		d		d	$-d$
51/4231		d	$-d$							d	$-d$	
52/4312				d		$-d$					d	$-d$
53/4321	$-d$		d							$-d$		d

(122)

Corollary 22 *Let M be a microlinear space with*

$$X_1, X_2, X_3, X_4 \in \mathfrak{X}(M)$$

Then we have

$$\begin{aligned}
& [X_1, [X_2, [X_3, X_4]]] + [X_1, [X_3, [X_4, X_2]]] + [X_1, [X_4, [X_2, X_3]]] + \\
& [X_2, [X_1, [X_4, X_3]]] + [X_2, [X_3, [X_1, X_4]]] + [X_2, [X_4, [X_3, X_1]]] + \\
& [X_3, [X_1, [X_2, X_4]]] + [X_3, [X_2, [X_4, X_1]]] + [X_3, [X_4, [X_1, X_2]]] + \\
& [X_4, [X_1, [X_3, X_2]]] + [X_4, [X_2, [X_1, X_3]]] + [X_4, [X_3, [X_2, X_1]]] \\
& = 0
\end{aligned}$$

Proof. Let

$$\begin{aligned}
\gamma_{1234} &= X_4 * X_3 * X_2 * X_1, \gamma_{1243} = (X_3 * X_4 * X_2 * X_1)^{\sigma_{1243}}, \\
\gamma_{1324} &= (X_4 * X_2 * X_3 * X_1)^{\sigma_{1324}}, \gamma_{1342} = (X_2 * X_4 * X_3 * X_1)^{\sigma_{1342}}, \\
\gamma_{1423} &= (X_3 * X_2 * X_4 * X_1)^{\sigma_{1423}}, \gamma_{1432} = (X_2 * X_3 * X_4 * X_1)^{\sigma_{1432}}, \\
\gamma_{2134} &= (X_4 * X_3 * X_1 * X_2)^{\sigma_{2134}}, \gamma_{2143} = (X_3 * X_4 * X_1 * X_2)^{\sigma_{2143}}, \\
\gamma_{2314} &= (X_4 * X_1 * X_3 * X_2)^{\sigma_{2314}}, \gamma_{2341} = (X_1 * X_4 * X_3 * X_2)^{\sigma_{2341}}, \\
\gamma_{2413} &= (X_3 * X_1 * X_4 * X_2)^{\sigma_{2413}}, \gamma_{2431} = (X_1 * X_3 * X_4 * X_2)^{\sigma_{2431}}, \\
\gamma_{3124} &= (X_4 * X_2 * X_1 * X_3)^{\sigma_{3124}}, \gamma_{3142} = (X_2 * X_4 * X_1 * X_3)^{\sigma_{3142}}, \\
\gamma_{3214} &= (X_4 * X_1 * X_2 * X_3)^{\sigma_{3214}}, \gamma_{3241} = (X_1 * X_4 * X_2 * X_3)^{\sigma_{3241}}, \\
\gamma_{3412} &= (X_2 * X_1 * X_4 * X_3)^{\sigma_{3412}}, \gamma_{3421} = (X_1 * X_2 * X_4 * X_3)^{\sigma_{3421}}, \\
\gamma_{4123} &= (X_3 * X_2 * X_1 * X_4)^{\sigma_{4123}}, \gamma_{4132} = (X_2 * X_3 * X_1 * X_4)^{\sigma_{4132}}, \\
\gamma_{4213} &= (X_3 * X_1 * X_2 * X_4)^{\sigma_{4213}}, \gamma_{4231} = (X_1 * X_3 * X_2 * X_4)^{\sigma_{4231}}, \\
\gamma_{4312} &= (X_2 * X_1 * X_3 * X_4)^{\sigma_{4312}}, \gamma_{4321} = (X_1 * X_2 * X_3 * X_4)^{\sigma_{4321}}
\end{aligned}$$

with

$$\begin{aligned}
\sigma_{1243} &= \begin{pmatrix} 1234 \\ 1243 \end{pmatrix}, \sigma_{1324} = \begin{pmatrix} 1234 \\ 1324 \end{pmatrix}, \sigma_{1342} = \begin{pmatrix} 1234 \\ 1423 \end{pmatrix}, \sigma_{1423} = \begin{pmatrix} 1234 \\ 1342 \end{pmatrix}, \sigma_{1432} = \begin{pmatrix} 1234 \\ 1432 \end{pmatrix}, \\
\sigma_{2134} &= \begin{pmatrix} 1234 \\ 2134 \end{pmatrix}, \sigma_{2143} = \begin{pmatrix} 1234 \\ 2143 \end{pmatrix}, \sigma_{2314} = \begin{pmatrix} 1234 \\ 3124 \end{pmatrix}, \sigma_{2341} = \begin{pmatrix} 1234 \\ 4123 \end{pmatrix}, \sigma_{2413} = \begin{pmatrix} 1234 \\ 3142 \end{pmatrix}, \\
\sigma_{2431} &= \begin{pmatrix} 1234 \\ 4132 \end{pmatrix}, \sigma_{3124} = \begin{pmatrix} 1234 \\ 2314 \end{pmatrix}, \sigma_{3142} = \begin{pmatrix} 1234 \\ 2413 \end{pmatrix}, \sigma_{3214} = \begin{pmatrix} 1234 \\ 3214 \end{pmatrix}, \sigma_{3241} = \begin{pmatrix} 1234 \\ 4213 \end{pmatrix}, \\
\sigma_{3412} &= \begin{pmatrix} 1234 \\ 3412 \end{pmatrix}, \sigma_{3421} = \begin{pmatrix} 1234 \\ 4312 \end{pmatrix}, \sigma_{4123} = \begin{pmatrix} 1234 \\ 2341 \end{pmatrix}, \sigma_{4132} = \begin{pmatrix} 1234 \\ 2431 \end{pmatrix}, \sigma_{4213} = \begin{pmatrix} 1234 \\ 3241 \end{pmatrix}, \\
\sigma_{4231} &= \begin{pmatrix} 1234 \\ 4231 \end{pmatrix}, \sigma_{4312} = \begin{pmatrix} 1234 \\ 3421 \end{pmatrix}, \sigma_{4321} = \begin{pmatrix} 1234 \\ 4321 \end{pmatrix}
\end{aligned}$$

Then it is easy to see that

$$\begin{aligned}
&[X_1, [X_2, [X_3, X_4]]] \\
&= \left(\left(\left(\gamma_{1234} \frac{\dot{}}{12} \gamma_{1243} \right) \frac{\dot{}}{1} \left(\gamma_{1342} \frac{\dot{}}{12} \gamma_{1432} \right) \right) \dot{} \left(\left(\left(\gamma_{2341} \frac{\dot{}}{12} \gamma_{2431} \right) \frac{\dot{}}{1} \left(\gamma_{3421} \frac{\dot{}}{12} \gamma_{4321} \right) \right) \right)
\end{aligned}$$

$$\begin{aligned}
&[X_1, [X_3, [X_4, X_2]]] \\
&= \left(\left(\left(\left(\gamma_{1342} \frac{\dot{}}{13} \gamma_{1324} \right) \frac{\dot{}}{1} \left(\gamma_{1423} \frac{\dot{}}{13} \gamma_{1243} \right) \right) \right) \dot{} \left(\left(\left(\left(\gamma_{3421} \frac{\dot{}}{13} \gamma_{3241} \right) \frac{\dot{}}{1} \left(\gamma_{4231} \frac{\dot{}}{13} \gamma_{2431} \right) \right) \right)
\end{aligned}$$

$$\begin{aligned}
&[X_1, [X_4, [X_2, X_3]]] \\
&= \left(\left(\left(\left(\gamma_{1423} \frac{\dot{}}{14} \gamma_{1432} \right) \frac{\dot{}}{1} \left(\gamma_{1234} \frac{\dot{}}{14} \gamma_{1324} \right) \right) \right) \dot{} \left(\left(\left(\left(\gamma_{4231} \frac{\dot{}}{14} \gamma_{4321} \right) \frac{\dot{}}{1} \left(\gamma_{2341} \frac{\dot{}}{14} \gamma_{3241} \right) \right) \right)
\end{aligned}$$

$$[X_2, [X_1, [X_4, X_3]]] \\ = \left(\left(\gamma_{2143} \begin{smallmatrix} \cdot \\ 21 \end{smallmatrix} \gamma_{2134} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{2431} \begin{smallmatrix} \cdot \\ 21 \end{smallmatrix} \gamma_{2341} \right) \right) \dot{-} \left(\left(\gamma_{1432} \begin{smallmatrix} \cdot \\ 21 \end{smallmatrix} \gamma_{1342} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{4312} \begin{smallmatrix} \cdot \\ 21 \end{smallmatrix} \gamma_{3412} \right) \right)$$

$$[X_2, [X_3, [X_1, X_4]]] \\ = \left(\left(\gamma_{2314} \begin{smallmatrix} \cdot \\ 23 \end{smallmatrix} \gamma_{2341} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{2143} \begin{smallmatrix} \cdot \\ 23 \end{smallmatrix} \gamma_{2413} \right) \right) \dot{-} \left(\left(\gamma_{3142} \begin{smallmatrix} \cdot \\ 23 \end{smallmatrix} \gamma_{3412} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{1432} \begin{smallmatrix} \cdot \\ 23 \end{smallmatrix} \gamma_{4132} \right) \right)$$

$$[X_2, [X_4, [X_3, X_1]]] \\ = \left(\left(\gamma_{2431} \begin{smallmatrix} \cdot \\ 24 \end{smallmatrix} \gamma_{2413} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{2314} \begin{smallmatrix} \cdot \\ 24 \end{smallmatrix} \gamma_{2134} \right) \right) \dot{-} \left(\left(\gamma_{4312} \begin{smallmatrix} \cdot \\ 24 \end{smallmatrix} \gamma_{4132} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{3142} \begin{smallmatrix} \cdot \\ 24 \end{smallmatrix} \gamma_{1342} \right) \right)$$

$$[X_3, [X_1, [X_2, X_4]]] \\ = \left(\left(\gamma_{3124} \begin{smallmatrix} \cdot \\ 31 \end{smallmatrix} \gamma_{3142} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{3241} \begin{smallmatrix} \cdot \\ 31 \end{smallmatrix} \gamma_{3421} \right) \right) \dot{-} \left(\left(\gamma_{1243} \begin{smallmatrix} \cdot \\ 31 \end{smallmatrix} \gamma_{1423} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{2413} \begin{smallmatrix} \cdot \\ 31 \end{smallmatrix} \gamma_{4213} \right) \right)$$

$$[X_3, [X_2, [X_4, X_1]]] \\ = \left(\left(\gamma_{3241} \begin{smallmatrix} \cdot \\ 32 \end{smallmatrix} \gamma_{3214} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{3412} \begin{smallmatrix} \cdot \\ 32 \end{smallmatrix} \gamma_{3142} \right) \right) \dot{-} \left(\left(\gamma_{2413} \begin{smallmatrix} \cdot \\ 32 \end{smallmatrix} \gamma_{2143} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{4123} \begin{smallmatrix} \cdot \\ 32 \end{smallmatrix} \gamma_{1423} \right) \right)$$

$$[X_3, [X_4, [X_1, X_2]]] \\ = \left(\left(\gamma_{3412} \begin{smallmatrix} \cdot \\ 34 \end{smallmatrix} \gamma_{3421} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{3124} \begin{smallmatrix} \cdot \\ 34 \end{smallmatrix} \gamma_{3214} \right) \right) \dot{-} \left(\left(\gamma_{4123} \begin{smallmatrix} \cdot \\ 34 \end{smallmatrix} \gamma_{4213} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{1243} \begin{smallmatrix} \cdot \\ 34 \end{smallmatrix} \gamma_{2143} \right) \right)$$

$$[X_4, [X_1, [X_3, X_2]]] \\ = \left(\left(\gamma_{4132} \begin{smallmatrix} \cdot \\ 41 \end{smallmatrix} \gamma_{4123} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{4321} \begin{smallmatrix} \cdot \\ 41 \end{smallmatrix} \gamma_{4231} \right) \right) \dot{-} \left(\left(\gamma_{1324} \begin{smallmatrix} \cdot \\ 41 \end{smallmatrix} \gamma_{1234} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{3214} \begin{smallmatrix} \cdot \\ 41 \end{smallmatrix} \gamma_{2314} \right) \right)$$

$$[X_4, [X_2, [X_1, X_3]]] \\ = \left(\left(\gamma_{4213} \begin{smallmatrix} \cdot \\ 42 \end{smallmatrix} \gamma_{4231} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{4132} \begin{smallmatrix} \cdot \\ 42 \end{smallmatrix} \gamma_{4312} \right) \right) \dot{-} \left(\left(\gamma_{2134} \begin{smallmatrix} \cdot \\ 42 \end{smallmatrix} \gamma_{2314} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{1324} \begin{smallmatrix} \cdot \\ 42 \end{smallmatrix} \gamma_{3124} \right) \right)$$

$$[X_4, [X_3, [X_2, X_1]]] \\ = \left(\left(\gamma_{4321} \begin{smallmatrix} \cdot \\ 43 \end{smallmatrix} \gamma_{4312} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{4213} \begin{smallmatrix} \cdot \\ 43 \end{smallmatrix} \gamma_{4123} \right) \right) \dot{-} \left(\left(\gamma_{3214} \begin{smallmatrix} \cdot \\ 43 \end{smallmatrix} \gamma_{3124} \right) \begin{smallmatrix} \cdot \\ 1 \end{smallmatrix} \left(\gamma_{2134} \begin{smallmatrix} \cdot \\ 43 \end{smallmatrix} \gamma_{1234} \right) \right)$$

■

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