Continuum-wise injective maps

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CONTINUUM-WISE INJECTIVE MAPS

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Abstract. We prove that for each $n \geq 1$ the set of all surjective continuum-wise injective maps from an n-dimensional continuum onto an LC^{n-1}-continuum with the disjoint $(n-1,n)$-cells property is a dense G_δ-subset of the space of all surjective maps. This generalizes a result of Espinoza and the second author [5].

1. Introduction

In this paper, all spaces are assumed to be metrizable and maps are continuous. A compact metric space is called a *compactum* and *continuum* means a connected compactum. Also, a locally connected continuum is called a *Peano continuum*. We denote the closed interval $[0,1]$ by I. An *arc* is a space which is homeomorphic to I. If X is a compactum, then 2^X denotes the space of all nonempty closed subsets of X endowed with the Hausdorff metirc and $C(X)$ is the closed subset of 2^X that consists of the subcontinua of X. If X and Y be compacta, then $C(X,Y)$ denotes the set of all continuous maps from X to Y endowed with the sup metric. Also, we denote the set of all surjective maps from X onto Y by $S(X,Y)$.

A surjective continuous map $f : I \to X$ is called an *arcwise increasing map* if for any two closed subintervals A and B of I such that $A \subsetneq B$, $f(A) \subsetneq f(B)$. The notion of arcwise increasing map was introduced in [7], by the second author, as a generalization of Eulerian path for Peano continua (see [5, Definition 3.1]). Some results related to arcwise increasing maps are obtained in [5].

A map $f : X \to Y$ between compacta is called a *continuum-wise injective map* if for each $A, B \in C(X)$ with $A \neq B$ and A is not a one point set, $f(A) \neq f(B)$. Also, a map $g : X \to Y$ between compacta is called a *hereditarily irreducible map* (see [10, p.204]) if for each $A, B \in C(X)$ with $A \subsetneq B$, $f(A) \subsetneq f(B)$. It is easy to see that a map $f : I \to X$ is an arcwise increasing map if and only if f is a surjective continuum-wise injective map. Note that every arcwise increasing map and every continuum-wise injective map are hereditarily irreducible maps.

The main aim of this paper is to prove Theorem 1.1. This result generalizes a result of Espinoza and the second author [5].

Theorem 1.1. *Let $n \geq 1$. Let X be a nondegenerate continuum with* $\dim X \leq n$ *and let Y be an LC^{n-1}-continuum with the disjoint $(n-1,n)$-cells property. Then, the set of all surjective continuum-wise injective maps from X onto Y is a dense G_δ-subset of $S(X,Y)$.*

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2. Preliminaries

In this section we give some notations and terminologies.

Let X be a space and let A be an subarc of X. Then A is called a *free arc of* X if we get an open set in X when deleting end points of A.

Let $n \geq 1$. Then, a space B is called an *n-cell* if B is homeomorphic to I^n. Also, a 0-*cell* means a one point set.

Let X be a space and $n \geq 0$. We say that X is *locally connected in dimension n* (abbreviated LC^n) if for every $x \in X$ and every neighborhood U of x in X there exists a neighborhood V of x in X such that for every $m \leq n$ and every continuous map f from an m-dimensional sphere to V, f is null-homotopic in U. Note that a continuum is a Peano continuum if and only if X is an LC^0-continuum.

Let X be a space and $m, n \geq 0$. Then, X is said to have the *disjoint (m, n)-cells property* if for every $\varepsilon > 0$, every m-cell B^m, every n-cell B^n and every two maps $f : B^m \to X$ and $g : B^n \to X$ there exist maps $f_0 : B^m \to X$ and $g_0 : B^n \to X$ such that $\rho(f, f_0) < \varepsilon$, $\rho(g, g_0) < \varepsilon$ and $f_0(B^m) \cap g_0(B^n) = \emptyset$. Let $m, m', n, n' \geq 0$, $m' \leq m$ and $n' \leq n$. Then it is easy to see that the disjoint (m, n)-cells property implies the disjoint (m', n')-cells property. Also, note that a space X has the disjoint $(0, 1)$-cells property if and only if X contains no free arcs.

Let $m \geq 0$, $n \geq 1$ and $m \leq n$. Note that I^{n+m+1} is an LC^{n-1}-continuum. In addition, by the argument of general position, we see that I^{n+m+1} has the disjoint (m, n)-cells property. Also, the compactum M_n^{n+m+1} (see [4, p.96]) is an LC^{n-1}-continuum with the disjoint (m, n)-cells property (see [1]). In particular, M_n^{2n+1} is called the *n-dimensional Menger compactum*. Furthermore, if D_i is a dendrite with the dense set of end points for each $i \leq m+1$, then the product space $\prod_{i=1}^{m+1} D_i$ is an LC^{n-1}-continuum with the disjoint (m, n)-cells property (see the proof of [2, Theorem 2.1]).

Let $f : X \to Y$ be a map and $A \subset X$. Then $f|A$ denotes the restriction of f to A. If A is a subset of a space X, then $\mathrm{Cl}_X A$ denotes the closure of A in X and $\mathrm{Int}_X A$ denotes the interior of A in X. Also, we denote the boundary of A in X by $\mathrm{Bd}_X A$.

If A is a subset of a metric space (X, d) and $\delta > 0$, then $\mathrm{diam} A$ denotes the diameter of A and $U_d(A, \delta)$ denotes the set $\{z \in X \mid \text{there exists } a \in A \text{ such that } d(a, z) < \delta\}$. If $A = \{x\}$, then we denote $U_d(A, \delta)$ by $U_d(x, \delta)$. Also, if $\mathcal{B}$ is a family of subsets of X, then define $\mathrm{mesh}\mathcal{B} = \sup\{\mathrm{diam} B | B \in \mathcal{B}\}$.

If X and Y are compacta and A and B are closed subsets of X, then $C(X, Y, A, B)$ denotes the set of all maps f from X to Y such that $f(A) \cap f(B) = \emptyset$. Let $N \subset X$ and $r > 0$. Then we denote the set $\{f \in C(X, Y) | f^{-1}(f(x)) = \{x\} \text{ for each } x \in N\}$ by $A_N(X, Y)$. If $N = \{a\}$, then we denote the set $A_N(X, Y)$ by $A_a(X, Y)$. In addition, if $r > 0$, then we denote the set $\{f \in C(X, Y) | \mathrm{diam}\, f^{-1}(f(x)) < r \text{ for each } x \in N\}$ by $A_{N,r}(X, Y)$.

If $\mathcal{K}$ is a simplicial complex, then $|\mathcal{K}|$ denotes the polyhedron of $\mathcal{K}$. For each $n \geq 0$, define $\mathcal{K}^{(n)} = \{\sigma \in \mathcal{K} | \sigma \text{ is at most } n\text{-dimensional}\}$. The elements of $\mathcal{K}^{(0)}$ is called the *vertices* of $\mathcal{K}$.

Let $\mathcal{A}$ is a finite family of subsets of X. By the *order* of $\mathcal{A}$ we mean the largest integer n such that $\mathcal{A}$ contains $n + 1$ sets with non-empty intersection. The order of $\mathcal{A}$ is denoted by $\mathrm{ord}\mathcal{A}$.

Finally, if A is a subset of X and $\mathcal{U}$ is a cover of X, then we denote the set $\bigcup\{U \in \mathcal{U} | U \cap A \neq \emptyset\}$ by $\mathrm{st}(A, \mathcal{U})$.

3. Main Theorem

In this section we prove Theorem 1.1. First, we prove Lemma 3.1. We mention that [3, Proposition 4.1.7] is more precise than Lemma 3.1. But in [3], there is no proof about the proposition. Hence, for the completeness we give the proof of Lemma 3.1.

Lemma 3.1. *Let $n \geq 1$. Let X be a compactum with $dim X \leq n$ and let Y be an LC^{n-1} compactum. Then for every $\varepsilon > 0$, there exists $\delta > 0$ satisfying the following:*

($\star$) *If f is a map from X to Y, A is a closed subset of X and $g : A \to Y$ satisfies $\rho(f|A, g) < \delta$, then there exists a continuous extension $\tilde{g} : X \to Y$ of g such that $\rho(f, \tilde{g}) < \varepsilon$.*

Proof. Let $\varepsilon > 0$. By [9, Lemma 1.1.6], we may think of Y as a subspace of a Banach space (Z, d). Let $\mathcal{S} = \{\{y\}|y \in Y\} \cup \{Y\}$. Since Y is compact, $\mathcal{S}$ is a uniformly equi-LC^{n-1} family of subsets of Z (as for the definition of *uniformly equi-LC^n*, see [8]). Hence, by [8, THEOREM 4.1] there exists $\delta > 0$ such that if $\varphi : X \to \mathcal{S}$ is lower semi-continuous and if $f : X \to Z$ satisfies $f(x) \in U_d(\varphi(x), \delta)$ for each $x \in X$, then there exists a continuous selection ℓ for φ such that $\ell(x) \in U_d(f(x), \varepsilon)$ (as for the definitions of *lower semi-continuous* and *continuous selection*, see [9]). We may assume that $\delta < \varepsilon$.

Let $f : X \to Y$, let A be a closed subset X and let $g : A \to Y$ be a map such that $\rho(f|A, g) < \delta$. Define $\varphi : X \to \mathcal{S}$ by

$$\varphi(x) = \begin{cases} \{g(x)\} & (x \in A) \\ Y & (x \notin A). \end{cases}$$

Note that $\varphi : X \to \mathcal{S}$ is lower semi-continuous and f satisfies $f(x) \in U_d(\varphi(x), \delta)$ for each $x \in X$. Hence, there there exists a continuous selection $\tilde{g}$ for φ such that $\tilde{g}(x) \in U_d(f(x), \varepsilon)$. Then $\tilde{g} : X \to Y$ is a continuous extension of g such that $\rho(f, \tilde{g}) < \varepsilon$. □

Lemma 3.2. *Let $m \geq 0$, $n \geq 1$ and $m \leq n$. Let X be a compactum with $dim X \leq n$ and let Y be an LC^{n-1}-compactum with the disjoint (m, n)-cells property. Let A, B be closed subsets of X such that $A \cap B = \emptyset$. If A is an i-cell for some $i \leq m$ and B is a j-cell for some $j \leq n$, then $C(X, Y, A, B)$ is a dense open subset of $C(X, Y)$.*

Proof. It is easy to see that $C(X, Y, A, B)$ is an open subset of $C(X, Y)$. Hence, we only show that $C(X, Y, A, B)$ is a dense subset of $C(X, Y)$. Let $\varepsilon > 0$ and $f \in C(X, Y)$. By lemma 3.1, there exists $\delta > 0$ such that if $g : A \cup B \to Y$ satisfies $\rho(f|(A \cup B), g) < \delta$, then there exists a continuous extension $\tilde{g} : X \to Y$ of g such that $\rho(f, \tilde{g}) < \varepsilon$. Note that A is an i-cell for some $i \leq m$ and B is a j-cell for some $j \leq n$. Since Y has the disjoint (m, n)-cells property, there exists $h : A \cup B \to Y$ such that $\rho(f|(A \cup B), h) < \delta$ and $h(A) \cap h(B) = \emptyset$. Then there exists a continuous extension $\tilde{h} : X \to Y$ of h such that $\rho(f, \tilde{h}) < \varepsilon$. Clearly, $\tilde{h} \in C(X, Y, A, B)$. Hence we see that $C(X, Y, A, B)$ is a dense subset of $C(X, Y)$. □

By Lemma 3.2 and Baire Category Theorem, we can get the next result.

Lemma 3.3. *Let $m \geq 0$, $n \geq 1$ and $m \leq n$. Let $\mathcal{K}$ be a simplicial complex with $dim|\mathcal{K}| \leq n$ and let Y be an LC^{n-1}-compactum with the disjoint (m, n)-cells* *property. Let $\mathcal{A}$, $\mathcal{B}$ be subcomplexes of $\mathcal{K}$ such that $|\mathcal{A}| \cap |\mathcal{B}| = \emptyset$ and dim$|\mathcal{A}| \leq m$. Then $C(|\mathcal{K}|, Y, |\mathcal{A}|, |\mathcal{B}|)$ is a dense open subset of $C(|\mathcal{K}|, Y)$.*

Lemma 3.4. *Let $m \geq 0$, $n \geq 1$ and $m \leq n$. Let X be a compactum with dim$X \leq n$ and let Y be an LC^{n-1}-compactum with the disjoint (m,n)-cells property. Let A, B be closed subsets of X such that $A \cap B = \emptyset$ and dim$A \leq m$. Then $C(X, Y, A, B)$ is a dense open subset of $C(X, Y)$.*

Proof. We only prove that $C(X, Y, A, B)$ is a dense subset of $C(X, Y)$. Let $\varepsilon > 0$ and $f \in C(X, Y)$. Since Y is LC^{n-1}, there exists $\delta > 0$ such that for every simplicial complex $\mathcal{K}$ with dim$|\mathcal{K}| \leq n$ and every subcomplex $\mathcal{S}$ of $\mathcal{K}$ with $\mathcal{K}^{(0)} \subset \mathcal{S}$, every map $h : |\mathcal{S}| \to Y$ with diam$h(\sigma \cap |\mathcal{S}|) < \delta$ for each $\sigma \in \mathcal{K}$ has a continous extension $\tilde{h} : |\mathcal{K}| \to Y$ such that diam$\tilde{h}(\sigma) < \varepsilon$ for each $\sigma \in \mathcal{K}$ (see [9, Proposition 4.2.29]). Since dim$X \leq n$, there exists an open cover $\mathcal{U}$ of X such that ord$\mathcal{U} \leq n$ and for each $U \in \mathcal{U}$, diam$f(\mathrm{st}(U, \mathcal{U})) < \min\{\delta, \varepsilon\}$.

Let $\mathcal{A} = \{U \in \mathcal{U} | U \cap A \neq \emptyset\}$ and $\mathcal{B} = \{U \in \mathcal{U} | U \cap B \neq \emptyset\}$. Since dim$A \leq m$ and $A \cap B = \emptyset$, we may assume that ord$\mathcal{A} \leq m$ and $\mathcal{A} \cap \mathcal{B} = \emptyset$. Let $N(\mathcal{U})$ be the nerve of $\mathcal{U}$ and let $k : X \to |N(\mathcal{U})|$ be the k-function of $\mathcal{U}$ (see [9, p132-134]). For each $U \in \mathcal{U}$, choose $p_U \in f(U)$. Define $\ell : |N(\mathcal{U})^{(0)}| \to Y$ by $\ell(v(U)) = p_U$ for each $U \in \mathcal{U}$ ($v(U)$ denotes the vertex of $N(\mathcal{U})$ associated with U). Then there exists a continuous extension $\tilde{\ell} : |N(\mathcal{U})| \to Y$ of ℓ such that diam$\tilde{\ell}(\sigma) < \varepsilon$ for each $\sigma \in N(\mathcal{U})$. Note that $\rho(f, k \circ \tilde{\ell}) < 2\varepsilon$. Also, note that $|N(\mathcal{A})| \cap |N(\mathcal{B})| = \emptyset$ (we consider $N(\mathcal{A})$ and $N(\mathcal{B})$ as subcomplexes of $N(\mathcal{U})$). By Lemma 3.3, there exists $m : |N(\mathcal{U})| \to Y$ such that $\rho(\tilde{\ell}, m) < \varepsilon$ and $m(|N(\mathcal{A})|) \cap m(|N(\mathcal{B})|) = \emptyset$. Then it is easy to see that $\rho(f, k \circ m) < 3\varepsilon$ and $k \circ m \in C(X, Y, A, B)$. Hence, we see that $C(X, Y, A, B)$ is a dense subset of $C(X, Y)$. □

Theorem 3.5. *Let $m \geq 0$, $n \geq 1$ and $m \leq n$. Let X be a compactum with dim$X \leq n$ and let Y be an LC^{n-1}-compactum with the disjoint (m,n)-cells property. If T is a closed subset of X such that dim$T \leq m$, then $A_T(X, Y)$ is a dense G_δ-subset of $C(X, Y)$.*

Proof. Let $k \in \mathbb{N}$ and $g \in C(X, Y)$. Let $\mathcal{C}$ be a countable closed cover of X such that mesh$\{\mathrm{st}(C, \mathcal{C}) | C \in \mathcal{C}\} < 1/k$ and if $C \in \mathcal{C}$ satisfies $C \cap T \neq \emptyset$, then $C \subset T$. Let $(D_1, D'_1), (D_2, D'_2), (D_3, D'_3), ...$ be a sequence of all pairs of members of $\mathcal{C}$ such that for every $i \in \mathbb{N}$, $D_i \subset T$ and $D_i \cap D'_i = \emptyset$. By Lemma 3.4, $C(X, Y, D_i, D'_i)$ is an open dense subset of $C(X, Y)$ for each $i \in \mathbb{N}$. Hence, by Baire Category Theorem $\bigcap_{i \in \mathbb{N}} C(X, Y, D_i, D'_i)$ is a dense G_δ-subset of $C(X, Y)$. Note that $\bigcap_{i \in \mathbb{N}} C(X, Y, D_i, D'_i) \subset A_{T,1/k}(X, Y)$. Hence, $A_{T,1/k}(X, Y)$ is a dense subset of $C(X, Y)$.

By [12, Lemma 2.3], $A_{T,1/k}(X, Y)$ is an open subset of $C(X, Y)$. Hence, by Baire Category Theorem $A_T(X, Y) = \bigcap_{k \in \mathbb{N}} A_{T,1/k}(X, Y)$ is a dense G_δ-subset of $C(X, Y)$. □

Corollary 3.6. *Let $m \geq 0$, $n \geq 1$ and $m \leq n$. Let X be a compactum with dim$X \leq n$ and let Y be an LC^{n-1} compactum with the disjoint (m,n)-cells property. If $\mathcal{F} = \{F_i\}_{i \in \mathbb{N}}$ is a family of closed subsets of X such that dim$F_i \leq m$ for each $i \in \mathbb{N}$, then $A_{\bigcup \mathcal{F}}(X, Y)$ is a dense G_δ-subset of $C(X, Y)$.*

Theorem 3.7. *Let $m \geq 0$, $n \geq 1$ and $m \leq n$. Let X be a continuum with dim$X \leq n$ and Y be an LC^{n-1} continuum with the disjoint (m,n)-cells property. If T is a* *nowhere dense closed subset of X such that $dimT \leq m$, then $S(X,Y) \cap A_T(X,Y)$ is a dense G_δ-subset of $S(X,Y)$.*

Proof. Let $\varepsilon > 0$, $k \in \mathbb{N}$ and $f \in S(X,Y)$. Then for $\varepsilon > 0$, by Lemma 3.1, there exists $\delta > 0$ satisfying $(\star)$.

Take a finite family of closed subsets $\mathcal{C} = \{C\}_{i=1}^{\ell}$ of Y such that mesh$\mathcal{C} < \delta$, $Y = \bigcup_{i=1}^{\ell} C_i$ and $Y \neq \bigcup_{i \in \{1,2,\dots,\ell\} \setminus \{j\}} C_i$ for each $j \leq \ell$. Since f is surjective, by Theorem 3.5 there exists $g_1 \in A_T(X,Y)$ such that $\rho(f, g_1) < \min\{\delta, \varepsilon\}$ and $g_1(X) \cap \mathrm{Int}_Y C_i \neq \emptyset$ for each $i \leq \ell$. Since $g_1 \in A_T(X,Y)$ and T is nowhere dense in X, for each $i \leq \ell$ there exists a point $c_i \in (g_1(X) \setminus g_1(T)) \cap \mathrm{Int}_Y C_i$. Let W be an open neighborhood of $g_1(T)$ in Y such that $\mathrm{Cl}_Y W \cap \{c_1, c_2, \dots, c_\ell\} = \emptyset$. For each $i \leq \ell$, take $d_i \in g_1^{-1}(c_i)$. Note that $g_1 \in A_{T,1/k}(X,Y)$. Hence, by [12, Lemma 2.3] there exists $\delta_1 > 0$ such that if $g' \in C(X,Y)$ satisfies $\rho(g_1, g') < \delta_1$, then $g' \in A_{T,1/k}(X,Y)$. Then for $\delta_1 > 0$, by Lemma 3.1, there exists $\delta_2 > 0$ satisfying $(\star)$.

Since T is a nowhere dense closed subset of X, there exists a closed subset H of X and $h_1 : T \cup H \to Y$ such that H is sufficiently near to T with respect to the Hausdorff metric on 2^X, $(T \cup \{d_1, d_2, \dots, d_\ell\}) \cap H = \emptyset$, $h_1|T = g_1|T$, $h_1(H)$ is a closed neighborhood of $g_1(T)$, $h_1(H) \subset W$ and $\rho(g_1|(T \cup H), h_1) < \min\{\delta, \delta_2\}$. Let $M = T \cup H \cup \{d_1, d_2, \dots, d_\ell\}$. Define $h_2 : M \to Y$ by $h_2(x) = h_1(x)$ if $x \in T \cup H$, and $h_2(x) = c_i$ if $x = d_i$ for some $i \leq \ell$.

Note that $\rho(g_1|M, h_2) < \min\{\delta, \delta_2\}$. Hence, there exists a continuous extension $g_2 : X \to Y$ of h_2 such that $\rho(g_1, g_2) < \min\{\varepsilon, \delta_1\}$. Note that $d_i \in g_2^{-1}(\mathrm{Int}_Y C_i) \setminus g_2^{-1}(\mathrm{Cl}_Y W)$ for each $i \leq \ell$. In particular, $g_2^{-1}(\mathrm{Int}_Y C_i) \setminus g_2^{-1}(\mathrm{Cl}_Y W) \neq \emptyset$ for each $i \leq \ell$. Hence, for each $i \leq \ell$ there exists a Cantor set $E_i \subset g_2^{-1}(\mathrm{Int}_Y C_i) \setminus g_2^{-1}(\mathrm{Cl}_Y W)$. We may assume that $E_i \cap E_j = \emptyset$ whenever $i \neq j$. For each $i \leq \ell$, take a continuous surjection $k_i : E_i \to \mathrm{Cl}_Y(C_i \setminus g_2(H))$. Let $D = (\bigcup_{i=1}^{\ell} E_i) \cup g_2^{-1}(\mathrm{Cl}_Y W)$. Define $h_3 : D \to Y$ by

$$h_3(x) = \begin{cases} g_2(x) & (x \in g_2^{-1}(\mathrm{Cl}_Y W) \\ k_i(x) & (x \in E_i \text{ for some } i \leq \ell). \end{cases}$$

Note that $\rho(g_2|D, h_3) < \delta$. Hence, there exists a continuous extension $g_3 : X \to Y$ of h_3 such that $\rho(g_2, g_3) < \varepsilon$.

Note that for each $x \in T$, $\mathrm{diam}(g_3^{-1}(g_3(x)) \cap g_2^{-1}(\mathrm{Cl}_Y W)) < 1/k$. Hence, by [12, Lemma 2.3] there exists $\delta_3 > 0$ such that if $g' \in C(X,Y)$ satisfies $\rho(g_3, g') < \delta_3$, then $\mathrm{diam}(g'^{-1}(g'(x)) \cap g_2^{-1}(\mathrm{Cl}_Y W)) < 1/k$ for each $x \in T$. Let $r = d(g_3(T), Y \setminus \mathrm{Int}_Y g_3(H))$. Let U be an open neighborhood of $\bigcup_{i=1}^{\ell} E_i$ in X such that if $x \in U$, then $g_3(x) \notin B(g_3(T), 2r/3)$.

For $\min\{r/3, \delta_3, \varepsilon\}$, by lemma 3.1, there exists $\delta_4 > 0$ satisfying $(\star)$. Let $J = (X \setminus (U \cup g_2^{-1}(W)) \cup T$. By Lemma 3.4 there exists $h_4 : J \to Y$ such that $h_4((X \setminus (U \cup g_2^{-1}(W))) \cap h_4(T) = \emptyset$ and $\rho(g_3|J, h_4) < \delta_4$. Let $F = J \cup H \cup \bigcup_{i=1}^{\ell} E_i$. Define $h_4' : F \to Y$ by

$$h_4'(x) = \begin{cases} g_3(x) & (x \in H \cup \bigcup_{i=1}^{\ell} E_i) \\ h_4(x) & (x \in J). \end{cases}$$

Then $\rho(g_3|F, h_4') < \delta_4$. Hence, there exists a continuous extension $g_4 : X \to Y$ of h_4' such that $\rho(g_3, g_4) < \min\{r/3, \delta_3, \varepsilon\}$.

Then it is easy to see that g_4 is surjective and $\rho(f, g_4) \le \rho(f, g_1) + \rho(g_1, g_2) + \rho(g_2, g_3) + \rho(g_3, g_4) < \varepsilon + \varepsilon + \varepsilon + \varepsilon = 4\varepsilon$. Also, we can see that $\rho(g_3, g_4) < \delta_3$ and $g_4^{-1}(g_4(x)) = g_4^{-1}(g_4(x)) \cap g_2^{-1}(\mathrm{Cl}_Y W)$ for each $x \in T$. Hence, $\mathrm{diam} g_4^{-1}(g_4(x)) < 1/k$ for each $x \in T$. Consequently, $g_4 \in S(X,Y) \cap A_{T,1/k}(X,Y)$. Therefore, $S(X,Y) \cap A_{T,1/k}(X,Y)$ is a dense subset of $S(X,Y)$.

By [12, Lemma 2.3] it is easy to see that $S(X,Y) \cap A_{T,1/k}(X,Y)$ is an open subset of $S(X,Y)$. Hence by the Baire Category Theorem $S(X,Y) \cap A_T(X,Y) = \bigcap_{k\in\mathbb{N}}(S(X,Y) \cap A_{T,1/k}(X,Y))$ is a dense G_δ-subset of $S(X,Y)$. □

Corollary 3.8. *Let $m \ge 0$, $n \ge 1$ and $m \le n$. Let X be a continuum with $\dim X \le n$ and let Y be an LC^{n-1} continuum with the disjoint (m,n)-cells property. If $\mathcal{F} = \{F_i\}_{i\in\mathbb{N}}$ is a family of nowhere dense closed subsets of X such that $dim F_i \le m$ for each $i \in \mathbb{N}$, then $S(X,Y) \cap A_{\bigcup\mathcal{F}}(X,Y)$ is a dense G_δ-subset of $S(X,Y)$.*

Before Theorem 3.9, we give a notation. If X and Y are compacta, then we denote the set of all continuum-wise injective maps from X to Y by $CI(X,Y)$.

Theorem 3.9. *Let X, Y be compacta. Then $CI(X,Y)$ is a G_δ-subset of $C(X,Y)$.*

Proof. Let d be an admissible metric on X and let H_d be the Hausdorff metric on 2^X induced by d. For each $n \in \mathbb{N}$, Let I_n be the set of all maps $f \in C(X,Y)$ satisfying the next condition:

($\sharp$) If $K, L \in C(X)$ satisfy $\mathrm{diam} K \ge 1/n$ and $H_d(K,L) \ge 1/n$, then $f(K) \ne f(L)$.

We claim that

(A) I_n is an open subset of $C(X,Y)$, and

(B) $CI(X,Y) = \bigcap_{n\in\mathbb{N}} I_n$.

First, we prove (A). We prove that $C(X,Y) \setminus I_n$ is a closed subset of $C(X,Y)$. Note that $C(X,Y) \setminus I_n$ is the set of all maps $f \in C(X,Y)$ satisfying the next condition:

($\sharp\sharp$) There exist $K, L \in C(X)$ such that $\mathrm{diam} K \ge 1/n$, $H_d(K,L) \ge 1/n$ and $f(K) = f(L)$

Let $f \in \mathrm{Cl}_{C(X,Y)}(C(X,Y) \setminus I_n)$. Then there exists a sequence of maps $\{f_i\}_{i\in\mathbb{N}} \subset C(X,Y) \setminus I_n$ such that $\lim f_i = f$. For each $i \in \mathbb{N}$ there exist $K_i, L_i \in C(X)$ such that $\mathrm{diam} K_i \ge 1/n$, $H_d(K_i, L_i) \ge 1/n$ and $f_i(K_i) = f_i(L_i)$. We may assume that $\{K_i\}_{i\in\mathbb{N}}$ converges to $K_0 \in C(X)$ and $\{L_i\}_{i\in\mathbb{N}}$ converge to $L_0 \in C(X)$ respectively. Then it is easy to see that $\mathrm{diam} K_0 \ge 1/n$, $H_d(K_0, L_0) \ge 1/n$ and $f(K_0) = f(L_0)$. Hence, $f \in C(X,Y) \setminus I_n$. Therefore $C(X,Y) \setminus I_n$ is a closed subset of $C(X,Y)$. This completes the proof of (A).

Next we prove (B). It is easy to see that $CI(X,Y) \subset \bigcap_{n\in\mathbb{N}} I_n$. So we only prove that $\bigcap_{n\in\mathbb{N}} I_n \subset CI(X,Y)$. Let $f \in \bigcap_{n\in\mathbb{N}} I_n$ and let $K, L \subset X$ be subcontinua of X such that K is not a one point set and $K \ne L$. Then, there exists $n_0 \in \mathbb{N}$ such that $\mathrm{diam} K \ge 1/n_0$ and $H_d(K,L) \ge 1/n_0$. Since $f \in I_{n_0}$, $f(K) \ne f(L)$. Hence, $f \in CI(X,Y)$ and we see that $\bigcap_{n\in\mathbb{N}} I_n \subset CI(X,Y)$. This completes the proof. □

If X and Y are compacta, then we denote the set of all hereditarily irreducible maps from X to Y by $HI(X,Y)$.

The proof of Theorem 3.10 is similar to the proof of Theorem 3.9. For the completeness, we give the proof.

Theorem 3.10. *Let X, Y be compacta. Then $HI(X,Y)$ is a G_δ-subset of $C(X,Y)$.*

Proof. Let d be an admissible metric on X and let H_d be the Hausdorff metric on 2^X induced by d. For each $n \in \mathbb{N}$, Let H_n be the set of all maps $f \in C(X,Y)$ satisfying the next condition:

($\sharp$) If $K, L \in C(X)$ satisfy $K \subset L$ and $H_d(K,L) \geq 1/n$, then $f(K) \subsetneq f(L)$.

We claim that

(A) H_n is an open subset of $C(X,Y)$, and

(B) $HI(X,Y) = \bigcap_{n\in\mathbb{N}} H_n$.

First, we prove (A). We prove that $C(X,Y) \setminus H_n$ is a closed subset of $C(X,Y)$. Note that $C(X,Y) \setminus H_n$ is the set of all maps $f \in C(X,Y)$ satisfying the next condition:

($\sharp\sharp$) There exist $K, L \in C(X)$ such that $K \subset L$, diam$H_d(K,L) \geq 1/n$ and $f(K) = f(L)$.

Let $f \in \mathrm{Cl}_{C(X,Y)}(C(X,Y) \setminus H_n)$. Then there exists a sequence of maps $\{f_i\}_{i\in\mathbb{N}} \subset C(X,Y) \setminus H_n$ such that $\lim f_i = f$. For each $i \in \mathbb{N}$ there exist $K_i, L_i \in C(X)$ such that $K_i \subset L_i$, $H_d(K_i, L_i) \geq 1/n$ and $f_i(K_i) = f_i(L_i)$. We may assume that $\{K_i\}_{i\in\mathbb{N}}$ converges to $K \in C(X)$ and $\{L_i\}_{i\in\mathbb{N}}$ converge to $L \in C(X)$ respectively. Then it is easy to see that $K \subset L$, $H_d(K,L) \geq 1/n$ and $f(K) = f(L)$. Hence, $f \in C(X,Y) \setminus H_n$. This completes the proof of (A).

Next we prove (B). It is easy to see that $HI(X,Y) \subset \bigcap_{n\in\mathbb{N}} H_n$. So we only prove that $\bigcap_{n\in\mathbb{N}} H_n \subset HI(X,Y)$. Let $f \in \bigcap_{n\in\mathbb{N}} H_n$ and let $K, L \subset X$ be subcontinua of X such that $K \subsetneq L$. Then, there exists $n_0 \in \mathbb{N}$ such that $H_d(K,L) \geq 1/n_0$. Since $f \in H_{n_0}$, $f(K) \subsetneq f(L)$. Hence, $f \in HI(X,Y)$ and we see that $\bigcap_{n\in\mathbb{N}} H_n \subset HI(X,Y)$. This completes the proof. □

Now we prove Theorem 1.1.

Proof of Theorem 1.1. By Theorem 3.9 it is sufficient to show that $CI(X,Y) \cap S(X,Y)$ is dense in $S(X,Y)$. Since dim$X \leq n$, there exists a countable base $\mathcal{U} = \{U_i\}_{i\in\mathbb{N}}$ of X such that dimBd$_X U_i \leq n-1$ for each $i \in \mathbb{N}$. Let $T = \bigcup_{i\in\mathbb{N}} \mathrm{Bd}_X U_i$. Note that Bd$_X U_i$ is a nowhere dense closed subset of X for each $i \in \mathbb{N}$. Hence, by Corollary 3.8 $A_T(X,Y) \cap S(X,Y)$ is a dense G_δ-subset of $S(X,Y)$. Note that $A_T(X,Y) \cap S(X,Y) \subset CI(X,Y) \cap S(X,Y)$. Hence we see that $CI(X,Y) \cap S(X,Y)$ is dense in $S(X,Y)$. This completes the proof. □

By using Corollary 3.6 and Theorem 3.9 we can get the next result. The proof of the next result is similar to the proof of Theorem 1.1. Hence, we omit the proof.

Theorem 3.11. *Let $n \geq 1$. Let X be a compactum with dim$X \leq n$ and let Y be an LC^{n-1} compactum with the disjoint $(n-1,n)$-cells property. Then, $CI(X,Y)$ is a dense G_δ-subset of $C(X,Y)$.*

Clearly, every continuum-wise injective map is a hereditarily irreducible map. Hence, by Theorem 1.1 and 3.10 we get the next result.

Theorem 3.12. *Let $n \geq 1$. Let X be a continuum with dim$X \leq n$ and let Y be an LC^{n-1} continuum with the disjoint $(n-1,n)$-cells property. Then, $HI(X,Y) \cap S(X,Y)$ is a dense G_δ-subset of $S(X,Y)$.*

Also, by Theorem 3.10 and 3.11 we get the next result.

Theorem 3.13. *Let $n \geq 1$. Let X be a compactum with $dim X \leq n$ and let Y be an LC^{n-1} compactum with the disjoint $(n-1, n)$-cells property. Then, $HI(X, Y)$ is a dense G_δ-subset of $C(X, Y)$.*

Next example shows that there exist a 2-dimensional continuum X and an LC^1 continuum Y with the disjoint $(1,1)$-cells property such that $HI(X, Y)$ and $HI(X, Y) \cap S(X, Y)$ are not dense in $C(X, Y)$ and $S(X, Y)$ respectively.

Example 3.14. *Let $A = [0, 1/3] \times I$, $B = [2/3, 1] \times I$ be subspaces of I^2. Let $f' : A \cup B \to I^3$ be the map defined by $f'(x, y) = (1/2, 3x, y)$ if $(x, y) \in A$ and $f'(x, y) = (3(x - 2/3), 1/2, y)$ if $(x, y) \in B$. If $f : I^2 \to I^3$ is a continuous extension of f' and $g : I^2 \to I^3$ is sufficiently near to f, then g is not a hereditarily irreducible map.*

Proof. Let $g \in C(I^2, I^3)$ be a map sufficiently near to f. Then we can see that there exists $0 < t < 1/2$ such that $g(A) \cap (I \times [t, 1-t] \times [t, 1-t])$ is a partition in $I \times [t, 1-t] \times [t, 1-t]$ between $\{0\} \times [t, 1-t] \times [t, 1-t]$ and $\{1\} \times [t, 1-t] \times [t, 1-t]$. Then, it is easy to see that there exists $0 < s < 1/2$ such that $g([2/3, 1] \times [1/2-s, 1/2+s]) \subset I \times [t, 1-t] \times [t, 1-t]$. We may assume that $g(A) \cap g([2/3, 1] \times [1/2 - s, 1/2 + s])$ is a partition in $g([2/3, 1] \times [1/2 - s, 1/2 + s])$ between $g(\{2/3\} \times [1/2 - s, 1/2 + s])$ and $g(\{1\} \times [1/2 - s, 1/2 + s])$. Then, $g^{-1}(g(A) \cap g([2/3, 1] \times [1/2 - s, 1/2 + s]))$ is a partition in $[2/3, 1] \times [1/2 - s, 1/2 + s]$ between $\{2/3\} \times [1/2 - s, 1/2 + s]$ and $\{1\} \times [1/2 - s, 1/2 + s]$. Hence, by [4, Lemma 1.8.15], there exists a nondegenerate continuum $L \subset g^{-1}(g(A) \cap g([2/3, 1] \times [1/2 - s, 1/2 + s]))$. Then, $g(L) \subset g(A)$. Let J be an arc in I^2 such that both $J \cap A$ and $J \cap L$ are one point sets. Then we can see that $A \cup J \subsetneq A \cup J \cup L$ and $g(A \cup J) = g(A \cup J \cup L)$. Hence g is not a hereditarily irreducible map. □

4. Final remarks

In this section we give some results which are related to the previous section. First we prove next result.

Proposition 4.1. *Let $m \geq 0$, $n \geq 1$ and $m \leq n$. If X is an n-dimensional continuum and Y is an LC^{n-1}-continuum with the disjoint (m, n)-cells property, then there exists a surjective map $f : X \to Y$ such that for each subcontinua $A, B \subset X$ with $dim(A \setminus B) \geq n - m$, $f(A) \neq f(B)$.*

Proof. Let $F_n = X$ and let $\mathcal{B}_n$ be a countable base for F_n such that for each $B \in \mathcal{B}_n$, $\dim \mathrm{Bd}_{F_n} B \leq n - 1$. Also, let $F_{n-1} = \bigcup_{B \in \mathcal{B}_n} \mathrm{Bd}_{F_n} B$ and $\mathcal{B}_{n-1}$ be a countable base for F_{n-1} such that for each $B \in \mathcal{B}_{n-1}$, $\dim \mathrm{Bd}_{F_{n-1}} B \leq n - 2$.

By induction, we obtain $\{F_n, F_{n-1}, ..., F_m\}$ and $\{\mathcal{B}_n, \mathcal{B}_{n-1}, ..., \mathcal{B}_m\}$ such that for each $k \in \mathbb{N}$ with $m \leq k \leq n-1$, $F_k = \bigcup_{B \in \mathcal{B}_{k+1}} \mathrm{Bd}_{F_{k+1}} B$ and $\mathcal{B}_k$ is a countable base for F_k such that for each $B \in \mathcal{B}_k$, $\dim \mathrm{Bd}_{F_k} B \leq k - 1$. Note that $\dim(F_k \setminus F_{k-1}) \leq 0$ for each $k \in \mathbb{N}$ with $m + 1 \leq k \leq n$. Since $X \setminus F_m = \bigcup_{k=m+1}^{n} (F_k \setminus F_{k-1})$, by [4, Theorem 1.5.10], $\dim(X \setminus F_m) \leq n - m - 1$.

By Corollary 3.8, there exists $f \in S(X, Y) \cap A_{F_m}(X, Y)$. Let $A, B \subset X$ be subcontinua such that $\dim(A \setminus B) \geq n - m$. Then, there exists an $(n - m)$-dimensional subcontinuum $E \subset A \setminus B$. If $E \subset X \setminus F_m$, then $\dim E \leq n - m - 1$. This is a contradiction. Therefore, $E \cap F_m \neq \emptyset$. Since $f \in A_{F_m}(X, Y)$, we can easily see that $f(A) \neq f(B)$. □

Theorem 4.2. *Let $m \geq 0$, $n \geq 1$ and $m < n$. If Y is an LC^{n-1}-compactum with the disjoint (m,n)-cells property, then dim$Y \geq m+1$.*

Proof. By Proposition 4.1 there exists $f : I^n \to Y$ such that for each subcontinua $A, B \subset X$ with $\dim(A \setminus B) \geq n-m$, $f(A) \neq f(B)$. Then, we can easily see that $\dim f^{-1}(y) \leq n-m-1$ for each $y \in Y$. By [4, Theorem 1.12.4], we see that $n = \dim I^n \leq \dim Y + \sup_{y \in Y} \dim f^{-1}(y) \leq \dim Y + n - m - 1$. Hence, $\dim Y \geq m+1$. □

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