微積分演習 第6回

$$f(x+d_1+\cdots+d_n) = f(x) + f'(x)\lambda_n^1(d_1,\cdots,d_n) + \cdots + f^{(n)}(x)\lambda_n^n(d_1,\cdots,d_n)$$

の証明.

n に関する帰納法

$n=1$ のとき. $f(x+d_1) = f(x) + f'(x)d$ 成立.

$n=k$ のとき, $f(x+d_1+\cdots+d_k) = f(x) + f'(x)\lambda_k^1(d_1,\cdots,d_k) + \cdots + f^{(k)}(x)\lambda_k^k(d_1,\cdots,d_k)$

と仮定.

$n=k+1$ のとき, $f(x+d_1+\cdots+d_{k+1})$

$$= f(x+d_{k+1}) + f'(x+d_{k+1})\lambda_k^1(d_1,\cdots,d_k) + f''(x+d_{k+1})\lambda_k^2(d_1,\cdots,d_k) + \cdots + f^{(k)}(x+d_{k+1})\lambda_k^k(d_1,\cdots,d_k)$$

$$= f(x) + f'(x)d_{k+1} + \left(f'(x) + f''(x)d_{k+1}\right)\lambda_k^1(d_1,\cdots,d_k) + \left(f''(x) + f'''(x)d_{k+1}\right)\lambda_k^2(d_1,\cdots,d_k) + \cdots\cdots + \left(f^{(k)}(x) + f^{(k+1)}(x)d_{k+1}\right)\lambda_k^k(d_1,\cdots,d_k)$$

$$= f(x) + f'(x)d_{k+1} + f'(x)\lambda_k^1(d_1,\cdots,d_k) + f''(x)d_{k+1}\lambda_k^1(d_1,\cdots,d_k) + f''(x)\lambda_k^2(d_1,\cdots,d_k) + f'''(x)d_{k+1}\lambda_k^2(d_1,\cdots,d_k) + f'''(x)\lambda_k^3(d_1,\cdots,d_k) + \cdots\cdots$$

$f^{(i)}(x)$ の係数を見て.

$$d_{k+1}\lambda_k^{i-1}(d_1,\cdots,d_k) + \lambda_k^i(d_1,\cdots,d_k) = \lambda_{k+1}^i(d_1,\cdots,d_{k+1})$$

したがって $n=k+1$ で成立.

帰納法により示された.

$x = x(t)$ 時刻の関数

$x' = ax$ 微分方程式

$$\xrightarrow{d_1} \xrightarrow{d_1+d_2} \xrightarrow{d_1+d_2+d_3}$$

$x(0) = C$ (定数)

$$\begin{aligned} x(d_1) &= x(0) + x'(0)d_1 \\ &= C + aCd_1 \\ &= C(1+ad_1) \end{aligned}$$

$$\begin{aligned} x(d_1+d_2) &= x(d_1) + x'(d_1)d_2 \\ &= x(d_1) + ax(d_1)d_2 \\ &= x(d_1)(1+ad_2) \\ &= C(1+ad_1)(1+ad_2) \end{aligned}$$

$$\begin{aligned} x(d_1+d_2+d_3) &= x(d_1+d_2) + x'(d_1+d_2)d_3 \\ &= x(d_1+d_2) + ax(d_1+d_2)d_3 \\ &= x(d_1+d_2)(1+ad_3) \\ &= C(1+ad_1)(1+ad_2)(1+ad_3) \end{aligned}$$

$$\begin{aligned} x(d_1+\cdots+d_n) &= C(1+ad_1)\cdots(1+ad_n) \\ &= C\{1 + a\underbrace{(d_1+\cdots+d_n)}_{\substack{\lambda_n^1(d_1,\cdots,d_n) \\ \| \\ d_1+\cdots+d_n}} + a^2\underbrace{(d_1d_2+\cdots\cdots)}_{\substack{\lambda_n^2(d_1,\cdots,d_n) \\ \| \\ \frac{(d_1+\cdots+d_n)^2}{2}}} + \cdots\cdots\} \\ &= C\{1 + a(d_1+\cdots+d_n) \\ &\quad + a^2\frac{(d_1+\cdots+d_n)^2}{2} + \cdots + a^n\frac{(d_1+\cdots+d_n)^n}{n!}\} \end{aligned}$$

$C \cdot e^{at}$

$x = \sin t$ とする.
$x' = \cos t$
$x'' = -\sin t$ なので $x'' = -x$ をみたしている.

$\boxed{x'' = -x}$ 解く.

$x(0) = C_1$
$x'(0) = C_2$ とする.

$$x(0) = C_1$$
$$x(d_1) = x(0) + x'(0) d_1 = C_1 + C_2 d_1$$

$$x(d_1+d_2) = x(d_1) + x'(d_1) d_2 = (C_1 + C_2 d_1) + \{C_2 + (-C_1) d_1\} d_2 = C_1(1 - d_1 d_2) + C_2(d_1 + d_2)$$

$$x(d_1+d_2+d_3) = x(d_1+d_2) + x'(d_1+d_2) d_3 = C_1(1-d_1 d_2) + C_2(d_1+d_2) + (\underbrace{x'(d_1)}_{C_2 - C_1 d_1} + \underbrace{x''(d_1) d_2}_{-x(d_1)d_2 = -(C_1 + C_2 d_1) d_2}) d_3$$

$$= C_1(\underbrace{1 - d_1 d_2 - d_1 d_3 - d_2 d_3}_{\fallingdotseq\, 1 - \frac{(d_1+d_2+d_3)^2}{2}}) + C_2(d_1+d_2+d_3 \underbrace{- d_1 d_2 d_3}_{\frac{(d_1+d_2+d_3)^3}{3!}})$$

$$\cos t = 1 - \frac{t^2}{2} + \cdots$$
$$\sin t = t - \frac{t^3}{3!} + \cdots$$

宿題
$x(d_1 + \cdots + d_4)$ を計算せよ.