J. Math. Soc. Japan
Vol. 65, No. 4 (2013) pp. 1135–1151
doi: 10.2969/jmsj/06541135

Lagrangian Floer homology of a pair of real forms in Hermitian symmetric spaces of compact type

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(Received Aug. 11, 2011)
(Revised Feb. 3, 2012)

Abstract. In this paper we calculate the Lagrangian Floer homology $HF(L_0, L_1 : \mathbb{Z}_2)$ of a pair of real forms (L_0, L_1) in a monotone Hermitian symmetric space M of compact type in the case where L_0 is not necessarily congruent to L_1. In particular, we have a generalization of the Arnold-Givental inequality in the case where M is irreducible. As its application, we prove that the totally geodesic Lagrangian sphere in the complex hyperquadric is globally volume minimizing under Hamiltonian deformations.

1. Introduction and main results.

Let (M, ω) be a symplectic manifold, i.e., M is a smooth manifold with a closed nondegenerate 2-form ω. Let L be a Lagrangian submanifold in M, i.e., $\dim_{\mathbb{R}} L = (1/2) \dim_{\mathbb{R}} M$ and ω vanishes on L. For a pair of closed Lagrangian submanifolds (L_0, L_1) in M, we can define Lagrangian Floer homology $HF(L_0, L_1 : \mathbb{Z}_2)$ with coefficient $\mathbb{Z}_2$ under some appropriate topological conditions.

In 1988, Floer [**7**] defined the homology when $\pi_2(M, L_i) = 0$, $i = 0, 1$, and proved that it is isomorphic to the singular homology group $H_*(L_0, \mathbb{Z}_2)$ of L_0 in the case where L_0 is *Hamiltonian isotopic* to L_1. As a result, he solved affirmatively the so called Arnold conjecture for Lagrangian intersections in that case (see [**3**] and [**7**]). A symplectic diffeomorphism ϕ of (M, ω) is called *Hamiltonian* if ϕ is represented by the time-1 map of the flow $\{\phi_t\}$ of a time dependent Hamiltonian vector field on M, i.e., $(d/dt)\phi_t(x) = X_{H_t}(\phi_t(x))$, $\phi_0(x) = x$, where X_{H_t} is defined by the equation $\omega(X_{H_t}, \cdot) = dH_t$ for a smooth function $H : [0, 1] \times M \to \mathbb{R}$. We denote by $\mathrm{Ham}(M, \omega)$ the set of all Hamiltonian diffeomorphisms of M.

2010 *Mathematics Subject Classification.* Primary 53D40; Secondary 53D12.
Key Words and Phrases. Lagrangian Floer homology, Hermitian symmetric space, real form, 2-number, Arnold-Givental inequality.
The first author was partly supported by the Grant-in-Aid for Young Scientists (B) (No. 22740043), MEXT.
The second author was partly supported by the Grant-in-Aid for Young Scientists (B) (No. 23740057), JSPS.
The third author was partly supported by the Grant-in-Aid for Science Research (C) (No. 21540063), JSPS.

After that, Givental [**11**] and Chang-Jiang [**5**] proved the conjecture for $L = \mathbb{R}P^n \subset M = \mathbb{C}P^n$ independently. (See also [**18**]). In the same paper, Givental posed the following conjecture which generalizes the above results by Floer and himself.

Conjecture 1 (Arnold-Givental). *Let (M, ω) be a symplectic manifold and $\tau : M \to M$ be an anti-symplectic involution of M. Assume that the fixed point set $L = \mathrm{Fix}(\tau)$ is not empty and compact. Then for any $\phi \in \mathrm{Ham}(M, \omega)$ such that the Lagrangian submanifold L and its image ϕL intersect transversally, the inequality*

$$\#(L \cap \phi L) \geq SB(L, \mathbb{Z}_2)$$

holds, where $SB(L, \mathbb{Z}_2)$ denotes the sum of $\mathbb{Z}_2$-Betti numbers of L.

Note that the assumption of Conjecture 1 admits many explicit examples. For instance, any real form L of Hermitian symmetric spaces of compact type is included.

The first substantial progress towards Conjecture 1 was made by Y.-G. Oh [**19**]. He solved the Arnold-Givental conjecture affirmatively for *monotone* real forms of Hermitian symmetric spaces of compact type (see Corollary 6 below). To prove it, he improved Floer's construction so as to apply the Lagrangian Floer homology theory to the case of monotone Lagrangian submanifolds (see [**17**] and Section 2 in this paper). After that, Frauenfelder [**10**] proved the Arnold-Givental conjecture for some class of Lagrangian submanifolds in Marsden-Weinstein quotients, which are fixed point sets of some anti-symplectic involution. Recently, Fukaya, Oh, Ohta and Ono [**8**] proved Conjecture 1 in a considerably more general setting, but the general case is still an open problem.

For a pair of Lagrangian submanifolds (L_0, L_1), where L_0 is not Hamiltonian isotopic to L_1 in a symplectic manifold (M, ω), there are relatively few examples where it is known how to calculate $HF(L_0, L_1 : \mathbb{Z}_2)$. Recently, explicit calculations of the Floer homology of Lagrangian submanifolds in toric Fano manifolds have been intensively studied (see [**1**], [**2**] and [**9**]).

In this paper, we shall focus on real forms L_0, L_1 of a Hermitian symmetric space of compact type and calculate the Lagrangian Floer homology $HF(L_0, L_1 : \mathbb{Z}_2)$ in a unified method. Let (M, J_0, ω) be a Hermitian symmetric space of compact type. A submanifold L of M is called a *real form* if there exists an anti-holomorphic involutive isometry σ of M satisfying

$$L = \{x \in M \mid \sigma(x) = x\}.$$

Note that a real form of a Hermitian symmetric space is a totally geodesic Lagrangian submanifold. We denote by $I_0(M)$ the identity component of the holomorphic isometry group of M. For two subsets A, B of M, we say that A is *congruent* to B if there exists $g \in I_0(M)$ such that $B = gA$. Let $L = \mathrm{Fix}(\sigma)$ be a real form and g a holomorphic isometry of M. Then $gL = \mathrm{Fix}(g\sigma g^{-1})$ is also a real form of M.

The following is the main result.

THEOREM 2. *Let (M, J_0, ω) be a Hermitian symmetric space of compact type which is monotone as a symplectic manifold. Let L_0, L_1 be real forms of M such that L_0 intersects L_1 transversally. Assume that the minimal Maslov numbers of L_0 and L_1 are greater than or equal to 3. Then we have*

$$HF(L_0, L_1 : \mathbb{Z}_2) \cong \bigoplus_{p \in L_0 \cap L_1} \mathbb{Z}_2[p].$$

That is, the intersection $L_0 \cap L_1$ itself becomes a basis of the Floer homology $HF(L_0, L_1 : \mathbb{Z}_2)$.

If M is irreducible, then the assumptions for M, L_0 and L_1 are satisfied automatically except for only one case (see Section 3). Moreover, using the structure of the transverse intersection $L_0 \cap L_1$ which was examined by Tanaka and Tasaki [**23**, Section 5], Theorem 2 yields

THEOREM 3. *Let M be an irreducible Hermitian symmetric space of compact type and L_0, L_1 be real forms of M which intersect transversally. Then the following results hold.*

(1) *If $M = G^{\mathbb{C}}_{2m}(\mathbb{C}^{4m})$ $(m \geq 2)$, L_0 is congruent to $G^{\mathbb{H}}_m(\mathbb{H}^{2m})$ and L_1 is congruent to $U(2m)$, then we have*

$$HF(L_0, L_1 : \mathbb{Z}_2) \cong (\mathbb{Z}_2)^{2^m},$$

where $2^m < \binom{2m}{m} = \#_2 L_0 < 2^{2m} = \#_2 L_1$. Here $\#_2 L$ denotes the 2-number of L.

(2) *Otherwise, we have*

$$HF(L_0, L_1 : \mathbb{Z}_2) \cong (\mathbb{Z}_2)^{\min\{\#_2 L_0, \#_2 L_1\}}.$$

In the case where M is non-irreducible, although we must determine the rank of $HF(L_0, L_1 : \mathbb{Z}_2)$ case by case (see Section 5), at least we can state in general the following fact by combining Theorem 2 with Lemma 3.1 in [**24**].

Corollary 4. *Let M be a monotone Hermitian symmetric space of compact type and L_0, L_1 be real forms of M whose minimal Maslov numbers are greater than or equal to* 3*. Then for any $\phi \in \mathrm{Ham}(M, \omega)$ we have*

$$L_0 \cap \phi L_1 \neq \emptyset.$$

Theorems 2 and 3 can be regarded as a solution for a problem proposed by Y.-G. Oh in [**17**, Section 6]. Here we review the definition of 2-number introduced by Chen and Nagano [**6**]. A subset S in a Riemannian symmetric space M is called an *antipodal set*, if the geodesic symmetry s_x fixes every point of S for any point x of S. The 2-*number* $\#_2 M$ of M is defined as the supremum of the cardinalities of antipodal sets in M, which is known to be finite. An antipodal set in M is said to be *great* if its cardinality attains $\#_2 M$. Takeuchi [**22**] proved that if L is a symmetric R-space, then

$$\#_2 L = SB(L, \mathbb{Z}_2)$$

holds. Note that any real form of Hermitian symmetric spaces of compact type is a symmetric R-space, which is shown in [**21**]. These facts and the invariance of $HF(L_0, L_1 : \mathbb{Z}_2)$ under Hamiltonian isotopies of M imply

Corollary 5. *Let M be an irreducible Hermitian symmetric space of compact type and (L_0, L_1) be a pair of real forms of M. Then for any $\phi \in \mathrm{Ham}(M, \omega)$ such that L_0 and ϕL_1 intersect transversally, the following inequalities hold.*

(1) *If $M = G^{\mathbb{C}}_{2m}(\mathbb{C}^{4m})$ $(m \geq 2)$, L_0 is congruent to $G^{\mathbb{H}}_m(\mathbb{H}^{2m})$ and L_1 is congruent to $U(2m)$, then we have*

$$\#(L_0 \cap \phi L_1) \geq 2^m.$$

(2) *Otherwise, we obtain*

$$\#(L_0 \cap \phi L_1) \geq \min\{SB(L_0, \mathbb{Z}_2), SB(L_1, \mathbb{Z}_2)\}. \tag{1.1}$$

As the case (1) above shows, in general, we cannot estimate $\#(L_0 \cap \phi L_1)$ by the sum of $\mathbb{Z}_2$-Betti numbers of L_0 or L_1. The estimate in the case (1) is sharp. Note that we can construct many examples which do not satisfy inequality (1.1) for the reducible case (see Section 5). We call inequality (1.1) *the generalized Arnold-Givental inequality.* Indeed, (1.1) yields

Corollary 6 (Oh [**19**] and [**17**, Theorem 1.3]). *Let (M, J_0, ω) be an irre-*

ducible Hermitian symmetric space of compact type and σ be an anti-holomorphic involutive isometry of M. Then Conjecture 1 *is true for the real form $L = \mathrm{Fix}(\sigma)$ of M.*

REMARK 7. Real forms of Hermitian symmetric spaces of compact type M are classified by Leung [**14**] and Takeuchi [**21**]. If M is irreducible, then real forms L_0 and L_1 of M, which are not congruent each other, are given in the list below. Hence we can apply Theorem 3 and Corollary 5 to the following.

M	L_0	L_1	$\#(L_0 \cap L_1)$
$G_{2q}^{\mathbb{C}}(\mathbb{C}^{2m+2q})$	$G_q^{\mathbb{H}}(\mathbb{H}^{m+q})$	$G_{2q}^{\mathbb{R}}(\mathbb{R}^{2m+2q})$	$\binom{m+q}{q}$
$G_n^{\mathbb{C}}(\mathbb{C}^{2n})$	$U(n)$	$G_n^{\mathbb{R}}(\mathbb{R}^{2n})$	2^n
$G_{2m}^{\mathbb{C}}(\mathbb{C}^{4m})$	$G_m^{\mathbb{H}}(\mathbb{H}^{2m})$	$U(2m)$	2^m
$SO(4m)/U(2m)$	$U(2m)/Sp(m)$	$SO(2m)$	2^m
$Sp(2m)/U(2m)$	$Sp(m)$	$U(2m)/O(2m)$	2^m
$Q_n(\mathbb{C})$	$S^{k,n-k}$	$S^{l,n-l}$	$2k+2$ (if $k \leq l$)
$E_6/T \cdot Spin(10)$	$F_4/Spin(9)$	$G_2^{\mathbb{H}}(\mathbb{H}^4)/\mathbb{Z}_2$	3
$E_7/T \cdot E_6$	$T \cdot (E_6/F_4)$	$(SU(8)/Sp(4))/\mathbb{Z}_2$	8

Here, $G_r^{\mathbb{K}}(\mathbb{K}^{n+r})$ denotes the Grassmann manifold of r-planes in $\mathbb{K}^{n+r}$ over the field $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. We denote the n-dimensional complex hyperquadric and a real form of it by $Q_n(\mathbb{C})$ and $S^{k,n-k} = (S^k \times S^{n-k})/\mathbb{Z}_2$, respectively (see [**24**]).

This paper is organized as follows. The calculation of $HF(L_0, L_1 : \mathbb{Z}_2)$ is based on the Floer homology theory for monotone Lagrangian submanifolds as developed in [**17**]. Section 2 contains an overview of the Lagrangian Floer theory. In Section 3, we give a criterion for a Hermitian symmetric space of compact type M to be monotone (see Proposition 10). Using it, we can also calculate some examples in the case where M is *non-irreducible.* They are exhibited in Section 5. In Section 4, we shall prove Theorem 2. There, we see that a geodesic symmetry of a Hermitian symmetric space M induces a free $\mathbb{Z}_2$-action on the space of J_0-holomorphic strips. In the last section, as an application of inequality (1.1), we prove that a real form $S^{0,n}$ of $Q_n(\mathbb{C})$ is globally volume minimizing under Hamiltonian deformations.

2. Lagrangian Floer homology.

In this section, we briefly review the Lagrangian Floer theory as developed in [**17**] (see also [**1**]). Let (M, ω) be a closed symplectic manifold, L_0 and L_1 two closed Lagrangian submanifolds which intersect transversally. We choose a time-dependent family $J = \{J_t\}_{0 \leq t \leq 1}$ of almost complex structures on M compatible with the symplectic form ω. The Floer chain complex $CF(L_0, L_1)$ is the vector space over $\mathbb{Z}_2$ generated by the finitely many elements of $L_0 \cap L_1$. A *J-holomorphic strip* is a map $u : \mathbb{R} \times [0,1] \to M$ which satisfies the equation

$$\bar{\partial}_J u := \frac{\partial u}{\partial s} + J_t(u)\frac{\partial u}{\partial t} = 0 \tag{2.2}$$

with the following boundary conditions:

$$u(\cdot, 0) \in L_0,\ \ u(\cdot, 1) \in L_1, \tag{2.3}$$

$$u(-\infty, \cdot), u(+\infty, \cdot) \in L_0 \cap L_1, \tag{2.4}$$

where $\mathbb{R} \times [0,1]$ is regarded as a subset of $\mathbb{C}$ with coordinates $s + \sqrt{-1}t$. A solution of the equation (2.2) with Lagrangian boundary condition (2.3) satisfies the asymptotic condition (2.4) if and only if the energy of u

$$E(u) = \frac{1}{2}\int_{\mathbb{R}\times[0,1]} \left(\left|\frac{\partial u}{\partial s}\right|^2 + \left|\frac{\partial u}{\partial t}\right|^2 \right)$$

is finite. The space of all J-holomorphic strips that connect $p \in L_0 \cap L_1$ to $q \in L_0 \cap L_1$ is denoted by $\tilde{\mathcal{M}}_J(L_0, L_1 : p, q)$. We set

$$\tilde{\mathcal{M}}_J(L_0, L_1) := \bigcup_{p,q \in L_0 \cap L_1} \tilde{\mathcal{M}}_J(L_0, L_1 : p, q).$$

A family of almost complex structures J is said to be *regular* if the linearization $D_u\bar{\partial}_J$ of $\bar{\partial}_J$ is surjective for all $u \in \tilde{\mathcal{M}}_J(L_0, L_1)$. For a regular J, each $\tilde{\mathcal{M}}_J(L_0, L_1 : p, q)$ is a finite-dimensional smooth manifold, with connected components of different dimensions. We denote by $\mathcal{J}^{reg}$ the set of all regular almost complex structures on M. The set $\mathcal{J}^{reg}$ is a set of the second category in the set of families of almost complex structures on M. From now on we assume $J \in \mathcal{J}^{reg}$. If $u \in \tilde{\mathcal{M}}_J(L_0, L_1 : p, q)$, then

$$\dim(T_u\tilde{\mathcal{M}}_J(L_0, L_1 : p, q)) = \mathrm{Index}(D_u\bar{\partial}_J).$$

The right hand side is the index of $D_u\bar{\partial}_J$. It is the spectral flow of $\bar{\partial}_J$ along u and is equal to the Maslov index $\mu(u)$ of u.

The moduli space $\tilde{\mathcal{M}}_J(L_0, L_1)$ has a natural action of $\mathbb{R}$ by translation in the first variable. Hence, we define

$$\mathcal{M}_J(L_0, L_1 : p, q) := \tilde{\mathcal{M}}_J(L_0, L_1 : p, q)/\mathbb{R},$$
$$\mathcal{M}_J(L_0, L_1) := \tilde{\mathcal{M}}_J(L_0, L_1)/\mathbb{R}.$$

An *isolated trajectory* is a trajectory u in $\tilde{\mathcal{M}}_J(L_0, L_1)$ such that the equivalence class $[u]$ is a 0-dimensional component of $\mathcal{M}_J(L_0, L_1)$. The boundary operator $\partial : CF(L_0, L_1) \to CF(L_0, L_1)$ is defined by

$$\partial(p) = \sum_{q \in L_0 \cap L_1} n(p, q) \cdot q,$$

where $n(p, q)$ is the mod-2 number of isolated trajectories in $\tilde{\mathcal{M}}_J(L_0, L_1 : p, q)$.

To define the Floer homology group with coefficients in $\mathbb{Z}_2$

$$HF(L_0, L_1 : \mathbb{Z}_2) := \frac{\mathrm{Ker}(\partial)}{\mathrm{Im}(\partial)},$$

we must assume some topological conditions on M, L_0 and L_1. For a closed Lagrangian submanifold L in a symplectic manifold (M, ω), two homomorphisms

$$I_{\mu,L} : \pi_2(M, L) \to \mathbb{Z}, \quad I_\omega : \pi_2(M, L) \to \mathbb{R}$$

are defined as follows. For a smooth map $w : (D^2, \partial D^2) \to (M, L)$, $I_{\mu,L}(w)$ is defined to be the Maslov number of the bundle pair $(w^*TM, (w|\partial D^2)^*TL)$ and I_ω is defined by $I_\omega(w) = \int_{D^2} w^*\omega$. Then L is said to be *monotone* if there exists a constant $\alpha > 0$ such that $I_\omega = \alpha I_{\mu,L}$. The *minimal Maslov number* Σ_L of L is defined to be the positive generator of $\mathrm{Im}(I_{\mu,L}) \subset \mathbb{Z}$. Oh proved the following

Theorem 8 ([**17**, Theorems 4.4, 5.1]). *Let (L_0, L_1) be a pair of monotone Lagrangian submanifolds which intersect transversally. Suppose that $\Sigma_{L_i} \geq 3$ for $i = 0, 1$ and $\mathrm{Im}(\pi_1(L_i)) \subset \pi_1(M)$ is a torsion subgroup for at least one of $i = 0, 1$. Then there exists a dense subset $\mathcal{J}' \subset \mathcal{J}^{reg}$ such that if $J \in \mathcal{J}'$, then we have*

(1) *∂ is well-defined,*
(2) *$\partial \circ \partial = 0$,*
(3) *$HF(L_0, L_1 : \mathbb{Z}_2)$ is independent of J and Hamiltonian isotopies.*

Let M be a Hermitian symmetric space of compact type. Since M is simply connected, the condition that $\mathrm{Im}(\pi_1(L_i)) \subset \pi_1(M)$ is automatically satisfied. Therefore, to apply Theorem 8 to real forms L_0, L_1 of M, it suffices to assume that L_0 and L_1 are monotone and $\Sigma_{L_i} \geq 3$ for $i = 0, 1$. Moreover, we can specify the case where a real form L does not satisfy the condition that $\Sigma_L \geq 3$ from arguments in Section 3. If M is irreducible, the only exceptional case is $L = \mathbb{R}P^1$ in $M = \mathbb{C}P^1$, where $\Sigma_L = 2$. Hence, a real form L of a Hermitian symmetric space M of compact type does not satisfy the condition that $\Sigma_L \geq 3$ if and only if M has $\mathbb{C}P^1$ as an irreducible factor and $\mathbb{R}P^1 \subset \mathbb{C}P^1$ is an irreducible factor of the real form L.

Remark 9. If the assumption that $\Sigma_{L_i} \geq 3$ is not satisfied, then we have to analyze the structure of disc bubbles to prove that $\partial \circ \partial = 0$. It requires the classification of holomorphic discs with Maslov index 2 (see [**1**]).

3. Monotonicity and minimal Maslov number of a real form.

In this section, let (M, J, ω) be a compact Kähler manifold with complex structure J and Kähler form ω. The first Chern class of (M, J, ω) is denoted by $c_1(M) := c_1(TM, J)$. Then two homomorphisms

$$I_c : \pi_2(M) \to \mathbb{Z}, \quad I_\omega : \pi_2(M) \to \mathbb{R}$$

are defined as follows. For a smooth map $u : S^2 \to M$ which represents an element $A \in \pi_2(M)$, $I_c(A)$ is defined to be the Chern number $c_1(A) := \langle c_1(M), [u] \rangle$ and I_ω is defined by $I_\omega(A) = \int_{S^2} u^*\omega$ as in the case of Lagrangian submanifolds. Then (M, J, ω) is said to be *monotone* if there exists a positive constant $\alpha > 0$ such that $I_\omega = \alpha I_c$. The *minimal Chern number* Γ_{c_1} of M is defined to be the positive generator of the subgroup $I_c(\pi_2(M))$ of $\mathbb{Z}$.

The Ricci form ρ of (M, J, ω) is a closed $(1,1)$-form on M and $\rho/2\pi$ represents the first Chern class $c_1(M) \in H^2(M, \mathbb{Z})$. (M, J, ω) is called *Kähler-Einstein* if there exists a constant c such that $\rho = c\omega$. It is straightforward to check that a Kähler-Einstein manifold (M, J, ω) with a positive Ricci constant c is monotone. In particular, an *irreducible* Hermitian symmetric space of compact type is monotone.

Now we shall give a useful criterion for a Hermitian symmetric space of compact type to be monotone. A Hermitian symmetric space of compact type (M, J_0, ω) can be decomposed as

$$(M, J_0, \omega) \cong (M_1, J_1, \omega_1) \times (M_2, J_2, \omega_2) \times \cdots \times (M_k, J_k, \omega_k),$$

where each (M_i, J_i, ω_i) is an irreducible one. Then the Kähler form ω and the Ricci form ρ of M are represented as $\omega = \omega_1 \oplus \omega_2 \oplus \cdots \oplus \omega_k$ and $\rho = \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_k$, respectively.

Proposition 10. *Let (M, J_0, ω) be a Hermitian symmetric space of compact* *type. Then (M,ω) is monotone as a symplectic manifold if and only if (M,J_0,ω) is a Kähler-Einstein manifold with a positive Ricci constant.*

PROOF. Since each irreducible component M_i of M is Kähler-Einstein, there exist constants $c_i>0$ such that $\rho_i=c_i\omega_i$ for $i=1,2,\dots,k$. Then we have

$$[\omega]=[\omega_1]+[\omega_2]+\cdots+[\omega_k]=\frac{2\pi}{c_1}c_1(M_1)+\frac{2\pi}{c_2}c_1(M_2)+\cdots+\frac{2\pi}{c_k}c_1(M_k).$$

If M is Kähler-Einstein, i.e., $c_1=\cdots=c_k=:c$, then $[\omega]=(2\pi/c)c_1(M)$ holds. It shows that M is monotone.

Conversely, if M is monotone, then there exists a constant $\alpha>0$ such that

$$\frac{2\pi}{c_1}c_1(M_1)+\frac{2\pi}{c_2}c_1(M_2)+\cdots+\frac{2\pi}{c_k}c_1(M_k)=\alpha(c_1(M_1)+c_1(M_2)+\cdots+c_1(M_k)).$$

It yields that $2\pi/c_i=\alpha$ for $i=1,2,\dots,k$, and hence M is a Kähler-Einstein manifold with positive Ricci constant $2\pi/\alpha$. $\square$

The following formula is necessary for us to ensure the monotonicity for a real form L and to estimate its minimal Maslov number Σ_L (see [**17**, Lemma 2.1]).

LEMMA 11 (Viterbo). *Let (M,J,ω) be a compact Kähler manifold and L a closed Lagrangian submanifold. Let $w,w':(D^2,\partial D^2)\to(M,L)$ be smooth maps of pairs satisfying $w|_{\partial D^2}=w'|_{\partial D^2}$. If we define a map u from $S^2=D^2\cup\overline{D^2}$ to M as*

$$u(z)=\begin{cases} w(z), & z\in D^2,\\ w'(z), & z\in\overline{D^2},\end{cases}$$

then we have

$$I_{\mu,L}(w)-I_{\mu,L}(w')=2c_1([u]).$$

COROLLARY 12. *Let (M,J,ω) be a monotone compact Kähler manifold. Then the fixed point set $L=\mathrm{Fix}(\sigma)$ of an involutive anti-holomorphic isometry σ is monotone.*

PROOF. For any $A\in\pi_2(M,L)$, we take a smooth map $w:(D^2,\partial D^2)\to(M,L)$ as a representative of A. Then we can define another smooth map $w'=\sigma\circ w:(D^2,\partial D^2)\to(M,L)$. By Lemma 11, we have

$$I_{\mu,L}(w) = c_1([u]). \tag{3.5}$$

Since M is monotone, there exists a constant $\alpha > 0$ such that $\int_{S^2} u^*\omega = \alpha c_1([u])$, and hence $\int_{S^2} u^*\omega = \alpha I_{\mu,L}(w)$. The left hand side of this equation is equal to $2I_\omega(A)$. Therefore, $I_\omega(A) = (\alpha/2)I_{\mu,L}(w)$. That is, L is a monotone Lagrangian submanifold with the monotone constant $\alpha/2$. □

The definitions of minimal Maslov and Chern numbers and equality (3.5) immediately imply

Corollary 13. *For a compact Kähler manifold* (M, J, ω), *the minimal Chern number* Γ_{c_1} *of* M *and the minimal Maslov number* Σ_L *of a real form* L *of* M *satisfy*

$$\Sigma_L \geq \Gamma_{c_1}.$$

Therefore, to apply Theorem 8 to a pair of real forms (L_0, L_1) of a Hermitian symmetric space M of compact type, it suffices to *assume that* M *is monotone* (it is equivalent that M is Kähler-Einstein) *and each* L_i (it is automatically monotone Lagrangian submanifold) *satisfies* $\Sigma_{L_i} \geq 3$.

The minimal Chern numbers of irreducible Hermitian symmetric spaces M of compact type are calculated as follows (see [**4**, p. 521]).

M	Γ_{c_1}
$U(m+n)/(U(m) \times U(n))$	$m+n$
$SO(2n)/U(n)$	$2n-2$
$Sp(n)/U(n)$	$n+1$
$SO(n+2)/(SO(2) \times SO(n))$	n
$E_6/T \cdot Spin(10)$	12
$E_7/T \cdot E_6$	18

Therefore, any real form L of M satisfies that $\Sigma_L \geq 3$ except for $L = \mathbb{R}P^1$ in $M = \mathbb{C}P^1 = U(2)/(U(1) \times U(1))$. This case is treated in [**18**, Section 5] independently.

4. Calculation of the Floer homology.

In this section, we consider a monotone Hermitian symmetric space (M, J_0, ω) of compact type with the standard complex structure J_0 and the standard Kähler form ω. Let L_0 and L_1 be real forms of M which intersect transversally and satisfy that $\Sigma_{L_i} \geq 3$ for $i = 0, 1$. We take $J_0 = J_t$ for all $t \in [0, 1]$. The following result ensures that J_0 can be used to calculate the Floer homology $HF(L_0, L_1 : \mathbb{Z}_2)$ [**20**, Main Theorem].

THEOREM 14 (Regularity [**20**]). *Let (M, J_0, ω) be a Kähler manifold with non-negative holomorphic bisectional curvature. Let L_0 and L_1 be closed totally geodesic Lagrangian submanifolds in M which intersect transversally. Then the complex structure J_0 is regular, i.e., the linearization $D_u\bar{\partial}_{J_0}$ of $\bar{\partial}_{J_0}$ is surjective for all $u \in \tilde{\mathcal{M}}_{J_0}(L_0, L_1)$.*

We apply the above theorem to the case where (M, J_0, ω) is a Hermitian symmetric space of compact type. By the same argument as [**19**, Proposition 4.5] for the case where $(L_0, L_1) = (L, \phi(L))$, Theorem 14 yields

PROPOSITION 15 (Compactness). *Let (M, J_0, ω) be a monotone Hermitian symmetric space of compact type. Let L_0 and L_1 be real forms of M which intersect transversally. In addition, assume that $\Sigma_{L_i} \geq 3$ for $i = 0, 1$. Then the 0-dimensional part of $\mathcal{M}_{J_0}(L_0, L_1)$ is compact and the 1-dimensional part of $\mathcal{M}_{J_0}(L_0, L_1)$ is compact up to the splitting of two isolated trajectories. Therefore, $\partial_{J_0}^2 = 0$.*

The following Theorem by Tanaka and Tasaki is essential for calculation.

THEOREM 16 (Theorem 1.1 in [**23**]). *Let M be a Hermitian symmetric space of compact type and L_0 and L_1 real forms which intersect transversally. Then the intersection $L_0 \cap L_1$ is an antipodal set of L_0 and L_1.*

The geodesic symmetry s_p at any point p of a Hermitian symmetric space is a holomorphic isometry. In Theorem 16 the intersection $L_0 \cap L_1$ is also an antipodal set in M, because L_0 and L_1 are totally geodesic, which yields the following:

LEMMA 17. *Under the assumption of Theorem 16, for any $p \in L_0 \cap L_1$, where $L_0 \cap L_1$ is not empty by Lemma 3.1 in [**24**], the geodesic symmetry s_p satisfies*

$$s_p(L_0) = L_0, \quad s_p(L_1) = L_1, \quad s_p(q) = q \quad (q \in L_0 \cap L_1).$$

PROOF. Since a real form of M is totally geodesic, we have $L_i = \mathrm{Exp}_p(T_pL_i)$ for $i = 0, 1$. Remark that $s_p^2 = \mathrm{id}_M$. Since the differential map of s_p satisfies $(ds_p)_p = -1$, we obtain

$$s_p(L_i) = \mathrm{Exp}_p((ds_p)_pT_pL_i) = \mathrm{Exp}_p(T_pL_i) = L_i.$$

By Theorem 16, the intersection $L_0 \cap L_1$ is an antipodal set of L_0 and L_1. Hence, by definition, $s_x y = y$ holds for any $x, y \in L_0 \cap L_1$. In particular, we have $s_p(q) = q$ for $q \in L_0 \cap L_1$. □

Now we shall calculate $HF(L_0, L_1 : \mathbb{Z}_2)$. By assumption, the intersection $L_0 \cap L_1$ consists of finite points. We choose any two points $p, q \in L_0 \cap L_1$. By Lemma 17, we see that p, q are fixed points of the action of s_p. Let u be a J_0-holomorphic strip in $\tilde{\mathcal{M}}_{J_0}(L_0, L_1 : p, q)$. It satisfies the boundary conditions

$$u(s,0) \in L_0, \quad u(s,1) \in L_1, \quad u(-\infty,t) = p, \quad u(+\infty,t) = q.$$

Using the holomorphic isometry s_p, let us define another holomorphic map $\bar{u}(s,t) := s_p(u(s,t))$ from $\mathbb{R} \times [0,1]$ to M. By Lemma 17, real forms L_0 and L_1 are invariant under the action of s_p. Hence, the holomorphic map $\bar{u}$ satisfies that

$$\bar{u}(s,0) = s_p(u(s,0)) \in L_0, \quad \bar{u}(s,1) = s_p(u(s,1)) \in L_1$$

and $\bar{u}(-\infty,t) = s_p(u(-\infty,t)) = s_p(p) = p$, $\bar{u}(+\infty,t) = s_p(u(+\infty,t)) = s_p(q) = q$. It says that the holomorphic map $\bar{u}$ also belongs to $\tilde{\mathcal{M}}_{J_0}(L_0, L_1 : p, q)$. Moreover, $s_p \circ \bar{u} = u$ and we see that $[\bar{u}] \neq [u] \in \mathcal{M}_{J_0}(L_0, L_1 : p, q)$ from the definition of the map s_p, and hence the moduli space $\mathcal{M}_{J_0}(L_0, L_1 : p, q)$ possesses a free $\mathbb{Z}_2$-action induced from s_p. Since the 0-dimensional part of the moduli space $\mathcal{M}_{J_0}(L_0, L_1 : p, q)$ is compact by Proposition 15, it contains an even number of elements. Therefore, we obtain

Proposition 18 (Vanishing). *Under the same assumptions as in Proposition* 15, *the number of* 0*-dimensional components of* $\mathcal{M}_{J_0}(L_0, L_1 : p, q)$ *are even and so the boundary operator* $\partial : CF(L_0, L_1) \to CF(L_0, L_1)$ *vanishes.*

Thus we complete the proof of Theorem 2.

Remark 19. If real forms L_0 and L_1 are congruent, then the above calculation provides us with an alternative proof of the known fact that

$$HF(L_0, L_0 : \mathbb{Z}_2) \cong (\mathbb{Z}_2)^{\#_2 L_0} = (\mathbb{Z}_2)^{SB(L_0, \mathbb{Z}_2)},$$

because the intersection $L_0 \cap L_1$ is a great antipodal set of L_0 (and L_1), which is proved in [**23**, Theorem 1.3].

5. Some examples for the reducible case.

Let (M, J_0, ω) be an irreducible Hermitian symmetric space of compact type and $\sigma : M \to M$ an involutive anti-holomorphic isometry. Since the product $M \times M$ of M is a Kähler-Einstein manifold with positive Ricci constant, we can apply Theorem 2 to real forms of $M \times M$. Since $(x, y) \mapsto (\sigma(y), \sigma(x))$ is an involutive anti-holomorphic isometry of $M \times M$, whose fixed point set

$$D_\sigma(M) = \{(x, \sigma(x)) \mid x \in M\}$$

is a real form of $M \times M$. On the other hand, for real forms L_0 and L_1 of M, we see that $L_0 \times L_1$ is a real form of $M \times M$. Then

$$(L_0 \times L_1) \cap D_\sigma(M) = \{(x, \sigma(x)) \mid x \in L_0 \cap \sigma^{-1}(L_1)\}.$$

The condition that two real forms $L_0 \times L_1$ and $D_\sigma(M)$ of $M \times M$ intersect transversally is equivalent to the condition that two real forms L_0 and $\sigma^{-1}(L_1)$ of M intersect transversally. In this situation, we obtain

$$\#\{(L_0 \times L_1) \cap D_\sigma(M)\} = \#\{L_0 \cap \sigma^{-1}(L_1)\}.$$

Moreover, $\sigma^{-1}(L_1)$ is congruent to L_1.

Example 20. Let M be the complex projective space $\mathbb{C}P^n$. Real forms L_0 and L_1 of M are congruent to $\mathbb{R}P^n$. Then $\#\{(L_0 \times L_1) \cap D_\sigma(M)\} = \#\{L_0 \cap \sigma^{-1}(L_1)\} = n + 1$. By Lemma 1.1 in [**6**], we have

$$\#_2(L_0 \times L_1) = \#_2(L_0)\#_2(L_1) = (n + 1)^2, \quad \#_2(D_\sigma(M)) = \#_2 M = n + 1,$$

and hence the intersection number of the two real forms is equal to smaller 2-number $n + 1$. Moreover, we can easily check that $I_{\mu, D_\sigma(M)} = 2(n + 1)$ and $I_{\mu, L_0 \times L_1} \geq 3$ for $n \geq 2$. By Theorem 2, we have

$$HF(L_0 \times L_1, D_\sigma(M) : \mathbb{Z}_2) \cong (\mathbb{Z}_2)^{n+1}$$

for $n \geq 2$. When $n = 1$, two real forms $L_0 \times L_1 = \mathbb{R}P^1 \times \mathbb{R}P^1 \cong T^2$ and $D_\sigma(M) \cong S^2$ can be regarded as real forms of 2-dimensional complex hyperquadric $Q_2(\mathbb{C}) \cong \mathbb{C}P^1 \times \mathbb{C}P^1$. Although $\Sigma_{T^2} = 2$, we can also prove that $HF(S^2, T^2 : \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ by combining the arguments in Section 4 and in [**1**]. Hence, the pair $(L_0 \times L_1, D_\sigma(M))$ satisfies the generalized Arnold-Givental inequality (1.1).

Example 21. Put $M = Q_n(\mathbb{C})$. Assume that real forms L_0, L_1 of M are congruent to $S^{k,n-k}, S^{l,n-l}$ $(0 \leq k \leq l \leq [n/2])$, respectively. Then by a result in [**24**], we have

$$\#\{(L_0 \times L_1) \cap D_\sigma(M)\} = \#\{L_0 \cap \sigma^{-1}(L_1)\} = 2(k+1).$$

If $n \geq 3$, then minimal Maslov numbers $I_{\mu, D_\sigma(M)}$ and $I_{\mu, L_0 \times L_1}$ are greater than or equal to 3. By Theorem 2, we obtain

$$HF(L_0 \times L_1, D_\sigma(M) : \mathbb{Z}_2) \cong (\mathbb{Z}_2)^{2(k+1)}.$$

Note that $\#_2(L_0 \times L_1) = 4(k+1)(l+1)$ and $\#_2(D_\tau(M)) = 2([n/2]+1)$. Hence the intersection number of the two real forms coincides with $\min\{\#_2(L_0 \times L_1), \#_2(D_\sigma(M))\}$ only in the case where $k = l = [n/2]$, otherwise the intersection number is smaller than it.

In this way, we can construct many pairs of real forms which do not satisfy the generalized Arnold-Givental inequality (1.1).

6. A volume estimate for a real form under Hamiltonian isotopies.

In general, a closed Lagrangian submanifold L in a Kähler manifold (M, J, ω) is said to be *Hamiltonian volume minimizing* if it satisfies

$$\mathrm{vol}(\phi L) \geq \mathrm{vol}(L)$$

for any Hamiltonian diffeomorphism $\phi \in \mathrm{Ham}(M, \omega)$ (see [**16**]). Non-trivial known examples of Hamiltonian volume minimizing Lagrangian submanifolds are very few. It is known that the real form $\mathbb{R}P^n$ in the complex projective space $\mathbb{C}P^n$ and real form $S^1 \times S^1$ in $S^2 \times S^2$ are Hamiltonian volume minimizing Lagrangian submanifolds (see [**16**] and [**13**]). Since $S^2 \times S^2$ is isomorphic to $Q_2(\mathbb{C})$, it is worthwhile to verify which real form of the complex hyperquadric $Q_n(\mathbb{C})$ is Hamiltonian volume minimizing. In fact, the Hamiltonian stabilities of real forms of $Q_n(\mathbb{C})$ were determined by Oh [**16**].

Here, we give a lower bound of the volume of the image $\phi(S^{k,n-k})$ of a real form $S^{k,n-k} = (S^k \times S^{n-k})/\mathbb{Z}_2$ of $Q_n(\mathbb{C})$ by any $\phi \in \mathrm{Ham}(Q_n(\mathbb{C}), \omega)$. By the generalized Arnold-Givental inequality (1.1), we have

$$\#(S^{0,n} \cap \phi S^{k,n-k}) \geq \min\{SB(S^{0,n}, \mathbb{Z}_2), SB(S^{k,n-k}, \mathbb{Z}_2)\} = 2. \tag{6.6}$$

Moreover, we use the following Crofton type formula.

Theorem 22 (Le [**15**]). *Let N be an n-dimensional real submanifold in $Q_n(\mathbb{C}) \cong \widetilde{G_n}(\mathbb{R}^{n+2})$. Then*

$$\int_{SO(n+2)} \#(gS^n \cap N) d\mu_{SO(n+2)}(g) \leq 2 \frac{\mathrm{vol}(SO(n+2))}{\mathrm{vol}(S^n)} \mathrm{vol}(N) \tag{6.7}$$

holds.

Proposition 23. *For any $\phi \in \mathrm{Ham}(Q_n(\mathbb{C}), \omega)$, we have $\mathrm{vol}(\phi S^{k,n-k}) \geq \mathrm{vol}(S^n)$.*

Proof. Put $N = \phi S^{k,n-k}$ $(k = 0, 1, \ldots, [n/2])$. Then (6.7) and (6.6) yield

$$\begin{aligned} \mathrm{vol}(\phi S^{k,n-k}) &\geq \frac{\mathrm{vol}(S^n)}{2\mathrm{vol}(SO(n+2))} \int_{SO(n+2)} \#(gS^n \cap \phi S^{k,n-k}) d\mu_{SO(n+2)}(g) \\ &\geq \frac{\mathrm{vol}(S^n)}{2\mathrm{vol}(SO(n+2))} \int_{SO(n+2)} 2 d\mu_{SO(n+2)}(g) \\ &= \mathrm{vol}(S^n). \end{aligned}$$

$\square$

Gluck, Morgan and Ziller [**12**] proved that $S^{0,n} = S^n$ in $Q_n(\mathbb{C})$ is volume minimizing in its homology class when n is even. On the other hand, since the homology $H_k(Q_n(\mathbb{C}))$ vanishes when k is odd, $S^{0,n}$ can not be homologically volume minimizing in $Q_n(\mathbb{C})$ in the case where n is odd. At least, we can conclude from Proposition 23 the following

Corollary 24. *A real form $S^{0,n}$ of the complex hyperquadric $Q_n(\mathbb{C})$ is Hamiltonian volume minimizing.*

Acknowledgements. We would like to thank Professor Yoshihiro Ohnita for pointing out that Proposition 23 produces new examples of Hamiltonian volume minimizing Lagrangian submanifolds as in Corollary 24. We would like to thank Professor Martin Guest for some helpful comments on this paper. The authors are also grateful to the referee, whose useful comments improved the manuscript.

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