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The intersection of two real forms in Hermitian symmetric spaces of compact type II

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Abstract. We minutely describe the intersection of two real forms in a non-irreducible Hermitian symmetric space M of compact type. In the case where M is irreducible we have already done it in our previous paper. In this paper we reduce the description of the intersection of two real forms to that in some special cases. This reduction is based on the information of the group of all isometries obtained by Takeuchi. We can describe the intersection in the special cases and in all cases. In particular we obtain the intersection number of two real forms in a Hermitian symmetric space of compact type.

1. Introduction.

The present paper is a sequel to our previous papers [**6**] and [**7**], in which we proved that the intersection of two real forms in a Hermitian symmetric space of compact type is an antipodal set and we determined the intersection numbers of two real forms in the irreducible Hermitian symmetric spaces of compact type. A submanifold L in a Hermitian symmetric space M is called a *real form* in M, if L is the set of fixed points of an involutive anti-holomorphic isometry of M. A subset S in a Riemannian symmetric space M is called an *antipodal set*, if $s_x y = y$ for any x, y in S, where s_x is the geodesic symmetry at x. The 2-*number* $\#_2 M$ of M is defined as the supremum of the cardinalities of antipodal sets of M. We call an antipodal set in M *great* if its cardinality attains $\#_2 M$.

In the present paper we show that any real form in a Hermitian symmetric space M of compact type is a product of real forms in some irreducible factors of M and some diagonal real forms, whose definition is given in Definition 2.4. Moreover, we can reduce the intersection of two real forms in M to that of two real forms in some irreducible factors and that of two diagonal real forms. We have already investigated the intersection of two real forms in each irreducible Hermitian symmetric space of compact type in [**6**]. We minutely investigate the intersection of two real forms in a non-irreducible Hermitian symmetric space of compact type in the present paper. For this purpose we reduce the intersection of two real forms to those in four special cases in Theorem 2.7. According to this theorem it is sufficient to investigate the intersection of two diagonal real forms in the product of two copies of an irreducible Hermitian symmetric space of compact type.

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We explain logical relations among [**6**], [**7**] and this paper. We proved that the intersection of two real forms in a Hermitian symmetric space of compact type is an antipodal set, which was stated in Theorem 1.1 of [**6**], but its proof was not complete. In [**7**] we correct the proof of Theorem 1.1 in [**6**] using Theorem 2.7 in this paper. We prove Theorem 2.7 in this paper, whose proof is independent of [**6**].

The organization of this paper is as follows. In Section 2, we consider classifications of real forms in a Hermitian symmetric space M of compact type with respect to the group $A(M)$ of all holomorphic isometries of M and its identity component $A_0(M)$. Leung [**1**] and Takeuchi [**5**] gave the classification of real forms in an irreducible Hermitian symmetric space of compact type with respect to $A(M)$. In order to compare two classifications of real forms with respect to $A(M)$ and $A_0(M)$, we use the result of Takeuchi [**4**] on $A(M)/A_0(M)$. Moreover we consider the classification of real forms in a non-irreducible Hermitian symmetric space M of compact type with respect to $A_0(M)$ in Theorem 2.6 and determine all possible pairs of two real forms in Theorem 2.7.

Theorem 2.7 implies that the intersection of two real forms in a Hermitian symmetric space of compact type is reduced to that of two real forms in some irreducible factors and that of two diagonal real forms. In Section 3 we describe the intersection of two diagonal real forms in Theorem 3.1.

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2. Real forms.

In this section we describe real forms in Hermitian symmetric spaces of compact type. Especially we minutely investigate real forms in a Hermitian symmetric space of compact type which is not irreducible.

Leung [**1**] classified real forms in irreducible Hermitian symmetric spaces of compact type. Although he stated that a real form in a non-irreducible Hermitian symmetric space of compact type is a product of real forms in irreducible factors, it is not true since we have such real forms as in Lemma 2.3.

Let $I(M)$ denote the group of all isometries of Hermitian symmetric space M of compact type and let $A(M)$ denote the group of all holomorphic isometries of M. We denote their identity components by $I_0(M)$ and $A_0(M)$ respectively. Then we have $I_0(M) = A_0(M)$. Leung [**1**] and Takeuchi [**5**] gave the classification of real forms in irreducible Hermitian symmetric spaces of compact type with respect to $A(M)$. If we consider the classification with respect to $A_0(M)$, we generally obtain more detailed classification. But we later show that the classification with respect to $A(M)$ coincides with the classification with respect to $A_0(M)$ (Proposition 2.2).

We recall the results about $I(M)/I_0(M)$ and $A(M)/A_0(M)$ obtained by Murakami [**3**] and Takeuchi [**4**].

We denote by $G_i(\mathbb{K}^n)$ the Grassmann manifold consisting of $\mathbb{K}$-subspaces of $\mathbb{K}$-dimension i in $\mathbb{K}^n$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and by $Q_j(\mathbb{C})$ the complex hyperquadric in the $(j+1)$-dimensional complex projective space.

LEMMA 2.1 ([**3**], [**4**]). *Let M be an irreducible Hermitian symmetric space of compact type. Then $I(M)/I_0(M)$ and $A(M)/A_0(M)$ are as follows.*

(A) *If* $M = Q_{2m}(\mathbb{C})$ $(m \geq 2)$ *or* $M = G_m(\mathbb{C}^{2m})$ $(m \geq 2)$, *then*

$$I(M)/I_0(M) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \quad \textit{and} \quad A(M)/A_0(M) \cong \mathbb{Z}_2.$$

(B) *Otherwise,*

$$I(M)/I_0(M) \cong \mathbb{Z}_2 \quad \textit{and} \quad A(M) = A_0(M).$$

Using this lemma we obtain the following proposition.

PROPOSITION 2.2. *The classification of real forms in an irreducible Hermitian symmetric space* M *of compact type with respect to* $A(M)$ *coincides with the classification with respect to* $A_0(M)$.

PROOF. In the case where M belongs to the class (B) in Lemma 2.1 we have nothing to prove. So we consider the class (A).

In the case of $M = Q_n(\mathbb{C})$ for general n, $Q_n(\mathbb{C})$ is holomorphically isometric to the oriented real Grassmann manifold $\tilde{G}_2(\mathbb{R}^{n+2})$ consisting of oriented linear subspaces of dimension 2 in $\mathbb{R}^{n+2}$. We regard $\tilde{G}_2(\mathbb{R}^{n+2})$ as a submanifold in $\bigwedge^2 \mathbb{R}^{n+2}$ in a natural way. We take an orthonormal basis $u_1, u_2, e_1, \ldots, e_n$ of $\mathbb{R}^{n+2}$. For $0 \leq k \leq n$ we define a submanifold $S^{k,n-k}$ of $\tilde{G}_2(\mathbb{R}^{n+2})$ by

$$S^{k,n-k} = S^k(\mathbb{R}u_1 + \mathbb{R}e_1 + \cdots + \mathbb{R}e_k) \wedge S^{n-k}(\mathbb{R}u_2 + \mathbb{R}e_{k+1} + \cdots + \mathbb{R}e_n),$$

where $S^m(V)$ is the unit hypersphere of dimension m in a real Euclidean space V of dimension $m+1$. By [**1**] and [**5**] any real form in $Q_n(\mathbb{C})$ is transformed by $A(Q_n(\mathbb{C}))$ to one of $S^{k,n-k}$ $(0 \leq k \leq [n/2])$.

In the case of $M = Q_{2m}(\mathbb{C})$ $(m \geq 2)$, $A(M)/A_0(M) \cong \mathbb{Z}_2$ by Lemma 2.1, so $A(M)$ has two connected components:

$$A(M) = A_0(M) \cup A_1(M).$$

We can see that the result of Takeuchi [**4**, p. 113] implies a $(2m+2) \times (2m+2)$ matrix

$$\phi = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{bmatrix}$$

is an element of $A_1(M)$, which preserves each real form $S^{k,2m-k}$ $(0 \leq k \leq m)$. Hence the classification of real forms with respect to $A(M)$ coincides with the classification of real forms with respect to $A_0(M)$.

In the case of $M = G_i(\mathbb{C}^n)$ for general i and n, by [**1**] and [**5**] any real form in $G_i(\mathbb{C}^n)$ is transformed by $A(G_i(\mathbb{C}^n))$ to $G_i(\mathbb{R}^n)$, $G_{i/2}(\mathbb{H}^{n/2})$ if i and n are even, or $U(n/2)$ if n is even and $i = n/2$.

In the case of $M = G_m(\mathbb{C}^{2m})$ $(m \geq 2)$, $A(M)/A_0(M) \cong \mathbb{Z}_2$ by Lemma 2.1, so $A(M)$ has two connected components:

$$A(M) = A_0(M) \cup A_1(M).$$

The canonical decomposition of the Lie algebra $\mathfrak{su}(2m)$ of $A(M)$ is as follows:

$$\begin{aligned}\mathfrak{su}(2m) &= \mathfrak{s}(\mathfrak{u}(m) \times \mathfrak{u}(m)) + \mathfrak{m}, \\ \mathfrak{m} &= \left\{ \begin{bmatrix} & Z \\ -{}^t\bar{Z} & \end{bmatrix} \,\middle|\, Z \text{ is an } m \times m \text{ complex matrix} \right\},\end{aligned}$$

which is identified with the tangent space T_oM. We can see that by the result of Takeuchi [**4**, p. 107], Proposition 4.1 and its corollary in Chapter VII in Loos [**2**] there exists $\phi \in A_1(M)$ which satisfies

$$\phi(o) = o, \quad d\phi_o \begin{bmatrix} & Z \\ -{}^t\bar{Z} & \end{bmatrix} = \begin{bmatrix} & {}^tZ \\ -\bar{Z} & \end{bmatrix}.$$

It preserves each tangent space at o of real forms $G_m(\mathbb{R}^{2m}), U(m)$, and $G_{m/2}(\mathbb{H}^m)$ with even m. Hence the classification of real forms with respect to $A(M)$ coincides with the classification of real forms with respect to $A_0(M)$. □

Lemma 2.3. *Let M_1 and M_2 be Hermitian symmetric spaces of compact type and let $\tau : M_1 \to M_2$ be an anti-holomorphic isometric map. Then the correspondence $M_1 \times M_2 \ni (x,y) \mapsto (\tau^{-1}(y), \tau(x)) \in M_1 \times M_2$ gives an involutive anti-holomorphic isometry of $M_1 \times M_2$ and the real form obtained from the map is*

$$D_\tau(M_1) = \{(x, \tau(x)) \mid x \in M_1\}.$$

For holomorphic isometries g_1 of M_1 and g_2 of M_2, we have $(g_1, g_2)D_\tau(M_1) = D_{g_2\tau g_1^{-1}}(M_1)$.

Proof. Since τ is an anti-holomorphic isometric map, the map $(x,y) \mapsto (\tau^{-1}(y), \tau(x))$ is an involutive anti-holomorphic isometry of $M_1 \times M_2$, which determines the real form $D_\tau(M_1)$. The definition of $D_\tau(M_1)$ implies the last part of the lemma. □

Definition 2.4. We call such a real form $D_\tau(M_1)$ as in Lemma 2.3 a *diagonal real form* determined by $\tau : M_1 \to M_2$.

Proposition 2.5. *Let M be an irreducible Hermitian symmetric space of compact type. Then any element of $I(M) - A(M)$ is an anti-holomorphic isometry. The connected components of $I(M) - A(M)$ corresponds to the $A_0(M \times M)$-congruent classes of diagonal real forms in $M \times M$ bijectively under the correspondence $I(M) - A(M) \ni \tau \mapsto D_\tau(M)$.*

Proof. Since each irreducible Hermitian symmetric space M of compact type has at least one real form, M has an anti-holomorphic isometry τ_0. We have $I(M) =$

$A(M) \cup \tau_0 A(M)$ because $I(M)/A(M) \cong \mathbb{Z}_2$ by Lemma 2.1. Hence each element of $I(M) - A(M) = \tau_0 A(M)$ is an anti-holomorphic isometry.

Let τ_1 and τ_2 be anti-holomorphic isometries of M. If they belong to the same connected component, there exists $g \in A_0(M)$ such that $\tau_2 = \tau_1 g$. Since $A_0(M \times M) = A_0(M) \times A_0(M)$ and $D_{\tau_2}(M) = D_{\tau_1 g}(M) = (g^{-1}, 1) D_{\tau_1}(M)$, $D_{\tau_1}(M)$ and $D_{\tau_2}(M)$ are $A_0(M \times M)$-congruent.

Conversely, if $D_{\tau_1}(M)$ and $D_{\tau_2}(M)$ are $A_0(M \times M)$-congruent, there exists $(g_1, g_2) \in A_0(M) \times A_0(M)$ such that $D_{\tau_2}(M) = (g_1, g_2) D_{\tau_1}(M) = D_{g_2 \tau_1 g_1^{-1}}(M)$ and so $\tau_2 = g_2 \tau_1 g_1^{-1}$. Hence τ_1 and τ_2 belong to the same connected component. Therefore the correspondence of the connected component containing $\tau \in I(M) - A(M)$ to the $A_0(M \times M)$-congruent class of $D_\tau(M)$ is a bijection. ☐

THEOREM 2.6. *A real form in a Hermitian symmetric space M of compact type is a product of real forms in irreducible factors of M and diagonal real forms determined from irreducible factors of M.*

PROOF. Let M be a Hermitian symmetric space of compact type and let L be a real form in M. M is decomposed as

$$M = M_1 \times \cdots \times M_r,$$

where M_i's are irreducible Hermitian symmetric spaces of compact type. $I_0(M_i)$ is a compact simple Lie group and we have

$$I_0(M) = I_0(M_1) \times \cdots \times I_0(M_r),$$

which is the decomposition of $I_0(M)$ as a product of compact simple Lie groups. We denote the Lie algebras of $I_0(M), I_0(M_1), \ldots, I_0(M_r)$ by $\mathfrak{g}, \mathfrak{g}_1, \ldots, \mathfrak{g}_r$ respectively. Then we have

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r,$$

which is the decomposition of $\mathfrak{g}$ as a direct sum of compact simple ideals.

Let $\tau : M \to M$ be an involutive anti-holomorphic isometry of M which determines L. If we take $o \in L$, τ induces a linear transformation $d\tau_o : T_o M \to T_o M$ of $T_o M$ which is the differential of τ at o since $\tau(o) = o$. We define an involutive automorphism I_τ of $I_0(M)$ by

$$I_\tau : I_0(M) \to I_0(M) \; ; \; g \mapsto \tau g \tau^{-1}.$$

The differential $dI_\tau : \mathfrak{g} \to \mathfrak{g}$ is an involutive automorphism. And the image $dI_\tau(\mathfrak{g}_i)$ of each simple ideal $\mathfrak{g}_i$ is a simple ideal of $\mathfrak{g}$. Hence $dI_\tau(\mathfrak{g}_i) = \mathfrak{g}_j$ for some j. That is, either dI_τ preserves a simple factor or dI_τ exchanges two simple factors. Putting

$$o = (o_1, \ldots, o_r) \qquad (o_i \in M_i)$$

and

$$\tilde{M}_i = \{o_1\} \times \cdots \times \{o_{i-1}\} \times M_i \times \{o_{i+1}\} \times \cdots \times \{o_r\},$$

we describe an arbitrary point of $\tilde{M}_i$ as

$$(e, \ldots, e, g_i, e, \ldots, e)\ o \qquad (g_i \in I_0(M_i))$$

where e denotes the identity element. If $I_\tau(I_0(M_i)) = I_0(M_j)$, we have

$$\tau\left((e, \ldots, e, g_i, e, \ldots, e)o\right) = I_\tau\left((e, \ldots, e, g_i, e, \ldots, e)\right) o \in \tilde{M}_j$$

hence $\tau(\tilde{M}_i) = \tilde{M}_j$. If $i = j$, τ preserves $\tilde{M}_i$ and if $i \neq j$, τ maps $\tilde{M}_i$ to $\tilde{M}_j$ and $\tilde{M}_j$ to $\tilde{M}_i$. If $i = j$, the i-th factor $L \cap \tilde{M}_i$ of L coincides with $F(\tau|_{\tilde{M}_i}, \tilde{M}_i)$. Let

$$\begin{aligned}(M_i \times M_j)^\sim = \{o_1\} \times \cdots \times \{o_{i-1}\} \times M_i \times \{o_{i+1}\} \times \cdots \times \{o_{j-1}\} \\ \times M_j \times \{o_{j+1}\} \times \cdots \times \{o_r\}.\end{aligned}$$

If $i \neq j$, the (i,j)-th factor $L \cap (M_i \times M_j)^\sim$ of L is the fixed point set of an involutive anti-holomorphic isometry

$$(x_i, x_j) \mapsto (\tau(x_j), \tau(x_i))$$

of $(M_i \times M_j)^\sim \cong M_i \times M_j$ and it is identified with $D_{\tau|_{\tilde{M}_i}}(\tilde{M}_i)$. Hence we conclude that L is a product of some real forms of irreducible factors of M and some diagonal real forms determined from irreducible factors of M. □

Theorem 2.7. *Let M be a Hermitian symmetric space of compact type and*

$$M = M_1 \times \cdots \times M_m$$

be a decomposition of M into irreducible factors. Then two real forms L_1 and L_2 in M are decomposed as

$$L_1 = L_{1,1} \times \cdots \times L_{1,n}, \quad L_2 = L_{2,1} \times \cdots \times L_{2,n}$$

and for each a $(1 \leq a \leq n)$ the pair of $L_{1,a}$ and $L_{2,a}$ are one of the following.

(1) *Two real forms in M_i for some i $(1 \leq i \leq m)$.*
(2) *After renumbering irreducible factors of M if necessary,*

$$N_1 \times D_{\tau_2}(M_2) \times D_{\tau_4}(M_4) \times \cdots \times D_{\tau_{2s}}(M_{2s})$$

and

$$D_{\tau_1}(M_1) \times D_{\tau_3}(M_3) \times \cdots \times D_{\tau_{2s-1}}(M_{2s-1}) \times N_{2s+1},$$

where $\tau_i : M_i \to M_{i+1}$ $(1 \leq i \leq 2s)$ *is an anti-holomorphic isometric map which determines* $D_{\tau_i}(M_i)$*, and* $N_1 \subset M_1$ *and* $N_{2s+1} \subset M_{2s+1}$ *are real forms. The intersection of these two real forms is*

$$\big\{(x, \tau_1(x), \tau_2\tau_1(x), \ldots, \tau_{2s}\cdots\tau_1(x)) \mid x \in N_1 \cap (\tau_{2s}\cdots\tau_1)^{-1}(N_{2s+1})\big\}.$$

Here $(\tau_{2s}\cdots\tau_1)^{-1}(N_{2s+1})$ *is a real form in* M_1 *and the intersection of the two real forms mentioned above is homothetic to the intersection of two real forms* N_1 *and* $(\tau_{2s}\cdots\tau_1)^{-1}(N_{2s+1})$ *in* M_1*.*

(3) *After renumbering irreducible factors of* M *if necessary,*

$$N_1 \times D_{\tau_2}(M_2) \times D_{\tau_4}(M_4) \times \cdots \times D_{\tau_{2s-2}}(M_{2s-2}) \times N_{2s}$$

and

$$D_{\tau_1}(M_1) \times D_{\tau_3}(M_3) \times \cdots \times D_{\tau_{2s-3}}(M_{2s-3}) \times D_{\tau_{2s-1}}(M_{2s-1}),$$

where $\tau_i : M_i \to M_{i+1}$ $(1 \leq i \leq 2s-1)$ *is an anti-holomorphic isometric map which determines* $D_{\tau_i}(M_i)$*, and* $N_1 \subset M_1$ *and* $N_{2s} \subset M_{2s}$ *are real forms. The intersection of these two real forms is*

$$\big\{(x, \tau_1(x), \tau_2\tau_1(x), \ldots, \tau_{2s-1}\cdots\tau_1(x)) \mid x \in N_1 \cap (\tau_{2s-1}\cdots\tau_1)^{-1}(N_{2s})\big\}.$$

Here $(\tau_{2s-1}\cdots\tau_1)^{-1}(N_{2s})$ *is a real form in* M_1 *and the intersection of the two real forms mentioned above is homothetic to the intersection of two real forms* N_1 *and* $(\tau_{2s-1}\cdots\tau_1)^{-1}(N_{2s})$ *in* M_1*.*

(4) *After renumbering irreducible factors of* M *if necessary,*

$$D_{\tau_1}(M_1) \times D_{\tau_3}(M_3) \times \cdots \times D_{\tau_{2s-1}}(M_{2s-1})$$

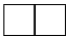
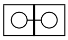
and

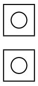
$$D_{\tau_2}(M_2) \times D_{\tau_4}(M_4) \times \cdots \times D_{\tau_{2s}}(M_{2s}),$$

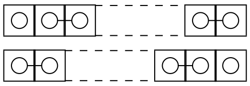
where $\tau_i : M_i \to M_{i+1}$ $(1 \leq i \leq 2s-1)$ *and* $\tau_{2s} : M_{2s} \to M_1$ *are anti-holomorphic isometric maps which determine* $D_{\tau_i}(M_i)$ $(1 \leq i \leq 2s)$*. The intersection of these two real forms is*

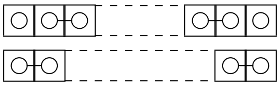
$$\big\{(x, \tau_1(x), \tau_2\tau_1(x), \ldots, \tau_{2s-1}\cdots\tau_1(x)) \mid (x, \tau_{2s}^{-1}(x)) \in D_{\tau_{2s-1}\cdots\tau_1}(M_1) \cap D_{\tau_{2s}^{-1}}(M_1)\big\}.$$

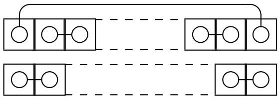
Here $D_{\tau_{2s-1}\cdots\tau_1}(M_1)$ *and* $D_{\tau_{2s}^{-1}}(M_1)$ *are diagonal real forms in* $M_1 \times M_{2s}$ *and the intersection of the two real forms mentioned above is homothetic to the intersection of these two diagonal real forms.*

We denote an irreducible Hermitian symmetric space of compact type by $\square$ and a real form in it by . We denote a product of two irreducible Hermitian symmetric spaces of compact type by  and we denote a product of real forms in each irreducible factor by  and a diagonal real form by . We express a real form in a product of more than two irreducible Hermitian symmetric spaces of compact type similarly. Then the result in Theorem 2.7 is expressed as follows.

(1) 

(2) 

(3) 

(4) 

Proof. By Theorem 2.6 L_i is a product of real forms in irreducible factors of M and diagonal real forms determined from irreducible factors of M. There are two possibilities one of which is the case when M_1-component of L_1, that is $L_1 \cap \tilde{M}_1$, is a real form in M_1 and the other is the case where M_1-component of L_1 is a part of a diagonal real form.

We consider the case where the M_1-component of L_1 is a real form of M_1. If the M_1-component of L_2 is also a real form of M_1, it is the case of (1). If the M_1-component of L_2 is a part of diagonal real form, after renumbering irreducible factors of M, a diagonal real form $D_{\tau_1}(M_1)$ with anti-holomorphic isometric map $\tau_1 : M_1 \to M_2$ is $M_1 \times M_2$-component of L_1. If the M_2-component of L_1 is a real form of M_2, it is the case of (3) where $s = 1$. If the M_2-component of L_1 is a part of diagonal real form, after renumbering irreducible factors of M, a diagonal real form $D_{\tau_2}(M_2)$ determined by anti-holomorphic isometric isomorphism $\tau_2 : M_2 \to M_3$ is $M_2 \times M_3$-component of L_1. Iterating these procedures, we obtain the case of (2) or (3).

We consider the case where the M_1-component of L_1 is a part of diagonal real form. After renumbering irreducible factors of M, a diagonal real form $D_{\tau_1}(M_1)$ determined by anti-holomorphic isometric isomorphism $\tau_1 : M_1 \to M_2$ is $M_1 \times M_2$-component of L_1. If the M_1-component of L_2 is a real form in M_1 and M_2-component of L_2 is also a real form in M_2, it is the case of (3) where $s = 1$. If the M_1-component of L_2 is a real form in M_1 and M_2-component of L_2 is a part of diagonal real form, it is the case of (2) or (3). If the M_1-component of L_2 is a part of diagonal real form, there are two possibilities. One is that the other part of the diagonal real form is contained in M_2. The other is that the other part of the diagonal real form is contained in another irreducible factor of M. The former is the case of (4) where $s = 1$ and the latter is the case of (2), (3) or (4).

In the case of (2), we obtain that the intersection of the two real forms is

$$\{(x, \tau_1(x), \tau_2\tau_1(x), \ldots, \tau_{2s} \cdots \tau_1(x)) \mid x \in N_1 \cap (\tau_{2s} \cdots \tau_1)^{-1}(N_{2s+1})\},$$

where $(\tau_{2s} \cdots \tau_1)^{-1}(N_{2s+1})$ is a real form in M_1 and the above intersection of two real forms is homothetic to the intersection of two real forms in an irreducible factor of M.

In the case of (3), we obtain that the intersection of the two real forms is

$$\{(x, \tau_1(x), \tau_2\tau_1(x), \ldots, \tau_{2s-1}\cdots\tau_1(x)) \mid x \in N_1 \cap (\tau_{2s-1}\cdots\tau_1)^{-1}(N_{2s})\},$$

where $(\tau_{2s-1}\cdots\tau_1)^{-1}(N_{2s})$ is a real form in M_1 and the above intersection of two real forms is homothetic to the intersection of two real forms in an irreducible factor of M.

In the case of (4), we obtain that the intersection of the two real forms is

$$\{(x, \tau_1(x), \tau_2\tau_1(x), \ldots, \tau_{2s-1}\cdots\tau_1(x)) \mid (x, \tau_{2s}^{-1}(x)) \in D_{\tau_{2s-1}\cdots\tau_1}(M_1) \cap D_{\tau_{2s}^{-1}}(M_1)\}.$$

$D_{\tau_{2s-1}\cdots\tau_1}(M_1)$ and $D_{\tau_{2s}^{-1}}(M_1)$ are diagonal real forms in $M_1 \times M_{2s}$ and the intersection is homothetic to these diagonal real forms in $M_1 \times M_{2s}$. □

3. The intersection of two diagonal real forms.

According to Theorem 2.7 we can reduce the intersection of two real forms in a non-irreducible Hermitian symmetric space of compact type to

(1) the intersection of two real forms in an irreducible Hermitian symmetric space of compact type,
(2) the intersection of two diagonal real forms in the product of two copies of an irreducible Hermitian symmetric space of compact type.

Since we already investigated (1) in our previous paper [**6**], it is sufficient to investigate (2).

THEOREM 3.1. *Let M_1, M_2 be irreducible Hermitian symmetric spaces of compact type which are holomorphically isometric. We take two anti-holomorphic isometric maps $\tau_1 : M_1 \to M_2$ and $\tau_2 : M_2 \to M_1$. We assume that the intersection of $D_{\tau_1}(M_1)$ and $D_{\tau_2^{-1}}(M_1)$ is discrete. Then we have the following.*

(1) *If $M_1 = Q_{2m}(\mathbb{C})$ $(m \geq 2)$ and $\tau_2\tau_1$ does not belong to $A_0(M_1)$,*

$$\#(D_{\tau_1}(M_1) \cap D_{\tau_2^{-1}}(M_1)) = 2m < 2m + 2 = \#_2 M_1.$$

(2) *If $M_1 = G_m(\mathbb{C}^{2m})$ $(m \geq 2)$ and $\tau_2\tau_1$ does not belong to $A_0(M_1)$,*

$$\#(D_{\tau_1}(M_1) \cap D_{\tau_2^{-1}}(M_1)) = 2^m < \binom{2m}{m} = \#_2 M_1.$$

(3) *Otherwise, $D_{\tau_1}(M_1) \cap D_{\tau_2^{-1}}(M_1)$ is a great antipodal set of $D_{\tau_1}(M_1)$ and $D_{\tau_2^{-1}}(M_1)$, thus*

$$\#(D_{\tau_1}(M_1) \cap D_{\tau_2^{-1}}(M_1)) = \#_2 M_1.$$

PROOF. If $\tau_2\tau_1$ belongs to $A_0(M_1)$, $D_{\tau_1}(M_1)$ and $D_{\tau_2^{-1}}(M_1)$ are congruent by Lemma 2.3. Their intersection $D_{\tau_1}(M_1) \cap D_{\tau_2^{-1}}(M_1)$ is a great antipodal set in $D_{\tau_1}(M_1)$ and $D_{\tau_2^{-1}}(M_1)$ by Theorem 1.3 in [**6**] and

$$\#(D_{\tau_1}(M_1) \cap D_{\tau_2^{-1}}(M_1)) = \#_2 M_1.$$

If $\tau_2\tau_1$ does not belong to $A_0(M_1)$, then $M_1 = Q_{2m}(\mathbb{C})$ $(m \geq 2)$, $G_m(\mathbb{C}^{2m})$ $(m \geq 2)$ by Lemma 2.1.

We assume that $M_1 = Q_{2m}(\mathbb{C})$. We prove

$$(*) \qquad \#(D_{\tau_1}(M_1) \cap D_{\tau_2^{-1}}(M_1)) = 2m$$

for $m \geq 1$ by induction on m.

In the case of $m = 1$, we have $Q_2(\mathbb{C}) = \mathbb{C}P^1 \times \mathbb{C}P^1$. We denote by $z = (z_0, z_1)$ the homogeneous coordinate of $\mathbb{C}P^1$ and define

$$\tau : \mathbb{C}P^1 \to \mathbb{C}P^1;\ [z] \mapsto [\bar{z}]$$

which is an anti-holomorphic isometry of $\mathbb{C}P^1$.

$$I(\mathbb{C}P^1) = A_0(\mathbb{C}P^1) \cup \tau A_0(\mathbb{C}P^1)$$

is the decomposition of $I(\mathbb{C}P^1)$ into the union of connected components.

We define

$$\alpha : \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^1 \times \mathbb{C}P^1; (x, y) \mapsto (y, x),$$

which is a holomorphic isometry of $\mathbb{C}P^1 \times \mathbb{C}P^1$. We write $A_0 = A_0(\mathbb{C}P^1)$ for simplicity. We obtain

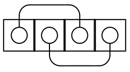
$$A(\mathbb{C}P^1 \times \mathbb{C}P^1) = (A_0 \times A_0) \cup \alpha(A_0 \times A_0),$$

where $A_0(\mathbb{C}P^1 \times \mathbb{C}P^1) = A_0 \times A_0$, and the set of all anti-holomorphic isometries of $\mathbb{C}P^1 \times \mathbb{C}P^1$ is

$$(\tau A_0 \times \tau A_0) \cup \alpha(\tau A_0 \times \tau A_0).$$

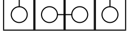
The assumption that $\tau_2\tau_1 \notin A_0(\mathbb{C}P^1 \times \mathbb{C}P^1)$ implies $\tau_2\tau_1 \in \alpha(A_0 \times A_0)$, thus τ_1 and τ_2^{-1} belong to different connected components $\tau A_0 \times \tau A_0$ and $\alpha(\tau A_0 \times \tau A_0)$. So we may suppose that $\tau_1 \in \tau A_0 \times \tau A_0$ and $\tau_2^{-1} \in \alpha(\tau A_0 \times \tau A_0)$. $D_{\tau_1}(\mathbb{C}P^1 \times \mathbb{C}P^1)$ is congruent with $D_{\tau\times\tau}(\mathbb{C}P^1 \times \mathbb{C}P^1)$ and $D_{\tau_2^{-1}}(\mathbb{C}P^1 \times \mathbb{C}P^1)$ is congruent with $D_{\alpha(\tau\times\tau)}(\mathbb{C}P^1 \times \mathbb{C}P^1)$. Their diagrams are

$D_{\tau\times\tau}(\mathbb{C}P^1 \times \mathbb{C}P^1)$:



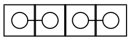
,

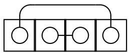
$D_{\alpha(\tau\times\tau)}(\mathbb{C}P^1 \times \mathbb{C}P^1)$:



.

We exchange the second and the third irreducible factors and obtain

$D_{\tau\times\tau}(\mathbb{C}P^1 \times \mathbb{C}P^1)$:  ,

$D_{\alpha(\tau\times\tau)}(\mathbb{C}P^1 \times \mathbb{C}P^1)$:  ,

which is the case (4) in Theorem 2.7. Since $A(\mathbb{C}P^1) = A_0(\mathbb{C}P^1)$, we have

$$\#(D_{\tau_1}(\mathbb{C}P^1 \times \mathbb{C}P^1) \cap D_{\tau_2^{-1}}(\mathbb{C}P^1 \times \mathbb{C}P^1)) = \#_2\mathbb{C}P^1 = 2.$$

Therefore we obtain $(*)$ in the case of $m = 1$.

Two real forms treated above are essentially same as those in Example 4.7 in [**6**].

Now we move to the case of $m \geq 2$. We may suppose $o \in D_{\tau_1}(M_1) \cap D_{\tau_2^{-1}}(M_1)$. This implies $\tau_1(o) = \tau_2(o) = o$. The polars of $Q_{2m}(\mathbb{C})$ are

$$M_0^+ = \{o\}, \quad M_1^+ = \{\bar{o}\}, \quad M_2^+ = Q_{2m-2}(\mathbb{C}),$$

where $\bar{o}$ denotes the pole of o and if $o = v_1 \wedge v_2$, then $\bar{o} = -v_1 \wedge v_2$. We note that $\tau_1(\bar{o}) = \bar{o}$ and $\tau_2(\bar{o}) = \bar{o}$. τ_1 and τ_2 preserve $M_2^+ = Q_{2m-2}(\mathbb{C})$. The polars of $Q_{2m}(\mathbb{C}) \times Q_{2m}(\mathbb{C})$ are given by

$$M_i^+ \times M_j^+ \quad (0 \leq i,\ j \leq 2).$$

The intersection of $D_{\tau_1}(Q_{2m}(\mathbb{C}))$ and each polar of $Q_{2m}(\mathbb{C}) \times Q_{2m}(\mathbb{C})$ is as follows.

$$\begin{aligned}
&D_{\tau_1}(Q_{2m}(\mathbb{C})) \cap \{(o,o)\} = \{(o,o)\},\\
&D_{\tau_1}(Q_{2m}(\mathbb{C})) \cap \{(\bar{o},\bar{o})\} = \{(\bar{o},\bar{o})\},\\
&D_{\tau_1}(Q_{2m}(\mathbb{C})) \cap Q_{2m-2}(\mathbb{C}) \times Q_{2m-2}(\mathbb{C}) = D_{\tau_1|_{Q_{2m-2}(\mathbb{C})}}(Q_{2m-2}(\mathbb{C})),
\end{aligned}$$

and the intersection is the empty set for the others. We obtain the intersection of $D_{\tau_2^{-1}}(Q_{2m}(\mathbb{C}))$ and each polar of $Q_{2m}(\mathbb{C}) \times Q_{2m}(\mathbb{C})$ similarly. If we put

$$\phi = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{bmatrix},$$

then the action of ϕ on $M_1 = Q_{2m}(\mathbb{C})$ is an element of $A(M_1) - A_0(M_1)$. Thus $\phi\tau_2\tau_1$ belongs to $A_0(M_1)$. If we restrict it to $Q_{2m-2}(\mathbb{C})$, then it belongs to $A_0(Q_{2m-2}(\mathbb{C}))$. Hence $\tau_2\tau_1|_{Q_{2m-2}(\mathbb{C})}$ does not belong to $A_0(Q_{2m-2}(\mathbb{C}))$. So by the assumption of induction, we have

$$\#\big(D_{\tau_1|_{Q_{2m-2}(\mathbb{C})}}(Q_{2m-2}(\mathbb{C})) \cap D_{\tau_2^{-1}|_{Q_{2m-2}(\mathbb{C})}}(Q_{2m-2}(\mathbb{C}))\big) = 2m-2.$$

Thus by Lemma 4.3 in [**6**] we have

$$\#\big(D_{\tau_1}(Q_{2m}(\mathbb{C})) \cap D_{{\tau_2}^{-1}}(Q_{2m}(\mathbb{C}))\big) = 1 + 1 + (2m-2) = 2m$$

and we complete the proof of $(*)$ by induction.

In order to calculate the intersection number of diagonal real forms in a product of two copies of the complex Grassmann manifold $G_m(\mathbb{C}^{2m})$, we investigate the action of the element ϕ of $A(G_m(\mathbb{C}^{2m})) - A_0(G_m(\mathbb{C}^{2m}))$ which is determined by

$$\phi(o) = o, \quad d\phi_o \begin{bmatrix} & Z \\ -{}^t\bar{Z} & \end{bmatrix} = \begin{bmatrix} & {}^tZ \\ -\bar{Z} & \end{bmatrix}$$

on $G_m(\mathbb{C}^{2m})$. The holomorphic isometry ϕ induces an isomorphism:

$$A_0(G_m(\mathbb{C}^{2m})) \to A_0(G_m(\mathbb{C}^{2m})); \ g \mapsto \phi g \phi^{-1}. \tag{$*$}$$

We also denote by ϕ the isomorphism of $\mathfrak{su}(2m)$ induced by the above isomorphism $(*)$. Hence the action of ϕ on $\mathfrak{su}(2m)$ is given by

$$\phi\left(\begin{bmatrix} S_1 & X \\ -{}^t\bar{X} & S_2 \end{bmatrix}\right) = \begin{bmatrix} \bar{S}_2 & {}^tX \\ -\bar{X} & \bar{S}_1 \end{bmatrix}.$$

We also denote by the same symbol ϕ the automorphism of $SU(2m)$ induced by ϕ. We denote

$$\operatorname{diag}\{x_1, \dots, x_m\} = \begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_m \end{bmatrix}.$$

Since $\big(SU(2m), S(U(m) \times U(m))\big)$ is a compact symmetric pair and

$$\left\{ \begin{bmatrix} & X \\ -X & \end{bmatrix} \,\middle|\, X = \operatorname{diag}\{x_1, \dots, x_m\}, x_i \in \mathbb{R} \right\}$$

generates a maximal torus of compact symmetric space $G_m(\mathbb{C}^{2m}) \cong SU(2m)/S(U(m) \times U(m))$, any element of $SU(2m)$ is represented as

$$\begin{bmatrix} g_1 & \\ & g_2 \end{bmatrix} \left(\exp \begin{bmatrix} & X \\ -X & \end{bmatrix} \right) \begin{bmatrix} h_1 & \\ & h_2 \end{bmatrix}$$

for some $\left[\begin{smallmatrix} g_1 & \\ & g_2 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} h_1 & \\ & h_2 \end{smallmatrix}\right] \in S(U(m) \times U(m))$ and $X = \operatorname{diag}\{x_1, \dots, x_m\}$, $x_i \in \mathbb{R}$. If we set C and S as

$$C = \operatorname{diag}\{\cos x_1, \dots, \cos x_m\}, \quad S = \operatorname{diag}\{\sin x_1, \dots, \sin x_m\},$$

then

$$\exp\begin{bmatrix} & X \\ -X & \end{bmatrix} = \begin{bmatrix} C & S \\ -S & C \end{bmatrix}.$$

Hence

$$\phi\left(\begin{bmatrix} g_1 & \\ & g_2 \end{bmatrix}\begin{bmatrix} C & S \\ -S & C \end{bmatrix}\begin{bmatrix} h_1 & \\ & h_2 \end{bmatrix}\right) = \begin{bmatrix} \bar{g}_2 & \\ & \bar{g}_1 \end{bmatrix}\begin{bmatrix} C & S \\ -S & C \end{bmatrix}\begin{bmatrix} \bar{h}_2 & \\ & \bar{h}_1 \end{bmatrix}.$$

Any point in $G_m(\mathbb{C}^{2m})$ is obtained from the origin $o = \mathbb{C}^m = \langle e_1, \dots, e_m\rangle_{\mathbb{C}}$ of $G_m(\mathbb{C}^{2m})$ by the action of $SU(2m)$. Thus the action of ϕ on $G_m(\mathbb{C}^{2m})$ is given by

$$\phi\left(\begin{bmatrix} g_1 & \\ & g_2 \end{bmatrix}\begin{bmatrix} C & S \\ -S & C \end{bmatrix} o\right) = \begin{bmatrix} \bar{g}_2 & \\ & \bar{g}_1 \end{bmatrix}\begin{bmatrix} C & S \\ -S & C \end{bmatrix} o.$$

Now we describe the polars of $G_m(\mathbb{C}^{2m})$ with respect to o and investigate the action of ϕ on each polar. The polars of $G_m(\mathbb{C}^{2m})$ are

$$\begin{aligned} M_0^+ &= \{o\}, \\ M_j^+ &= G_{m-j}(\langle e_1, \dots, e_m\rangle_{\mathbb{C}}) \times G_j(\langle e_{m+1}, \dots, e_{2m}\rangle_{\mathbb{C}}) \quad (1 \leq j \leq m-1), \\ M_m^+ &= \{\langle e_{m+1}, \dots, e_{2m}\rangle_{\mathbb{C}}\}. \end{aligned}$$

We express these polars as the orbits of $S(U(m) \times U(m))$. If we put $x_1 = \cdots = x_{m-j} = 0$ and $x_{m-j+1} = \cdots = x_m = -\pi/2$, then

$$\begin{bmatrix} C & S \\ -S & C \end{bmatrix} e_i = e_i \qquad (1 \leq i \leq m-j),$$

$$\begin{bmatrix} C & S \\ -S & C \end{bmatrix} e_i = e_{m+i} \quad (m-j+1 \leq i \leq m).$$

So we have

$$\begin{bmatrix} C & S \\ -S & C \end{bmatrix} o = \langle e_1, \dots, e_{m-j}, e_{m+m-j+1}, \dots, e_{2m}\rangle_{\mathbb{C}}$$

and

$$S(U(m) \times U(m))\begin{bmatrix} C & S \\ -S & C \end{bmatrix} o = G_{m-j}(\mathbb{C}^m) \times G_j(\mathbb{C}^m) = M_j^+.$$

The image of

$$(g_1\langle e_1, \dots, e_{m-j}\rangle_{\mathbb{C}}, g_2\langle e_{m+m-j+1}, \dots, e_{2m}\rangle_{\mathbb{C}}) \in M_j^+$$

under ϕ is

$$\begin{aligned}
&\phi(g_1\langle e_1,\dots,e_{m-j}\rangle_{\mathbb{C}}, g_2\langle e_{m+m-j+1},\dots,e_{2m}\rangle_{\mathbb{C}}) \\
&\quad = (\bar{g}_2\langle e_1,\dots,e_{m-j}\rangle_{\mathbb{C}}, \bar{g}_1\langle e_{m+m-j+1},\dots,e_{2m}\rangle_{\mathbb{C}}) \in M_j^+
\end{aligned}$$

and each polar M_j^+ is preserved by the action of ϕ because ϕ fixes o.

From the above we know the action more precisely. If we put

$$\psi = \begin{bmatrix} & 1_m \\ 1_m & \end{bmatrix},$$

then the action of ψ on $G_m(\mathbb{C}^{2m})$ is a holomorphic isometry. And we have

$$\begin{aligned}
\bar{g}_2\langle e_1,\dots,e_{m-j}\rangle_{\mathbb{C}} &= \psi\bar{g}_2\langle e_{m+m-j+1},\dots,e_{2m}\rangle_{\mathbb{C}}^{\perp}, \\
\bar{g}_1\langle e_{m+m-j+1},\dots,e_{2m}\rangle_{\mathbb{C}} &= \psi\bar{g}_1\langle e_1,\dots,e_{m-j}\rangle_{\mathbb{C}}^{\perp},
\end{aligned}$$

where $\perp$ in the right hand side denote the orthogonal complement in $\langle e_{m+1},\dots,e_{2m}\rangle_{\mathbb{C}}$ and in $\langle e_1,\dots,e_m\rangle_{\mathbb{C}}$ respectively. Thus we have

$$\phi(V_1,V_2) = \left(\psi\bar{V}_2^{\perp}, \psi\bar{V}_1^{\perp}\right) \quad ((V_1,V_2)\in G_{m-j}(\mathbb{C}^m)\times G_j(\mathbb{C}^m)).$$

Hence ϕ exchanges the irreducible factors of $M_j^+ = G_{m-j}(\mathbb{C}^m)\times G_j(\mathbb{C}^m)$.

Now we come to the position to prove that

$$\#(D_{\tau_1}(M_1)\cap D_{\tau_2^{-1}}(M_1)) = 2^m$$

for $M_1 = G_m(\mathbb{C}^{2m})$ $(m\geq 2)$. We may assume that $(o,o)\in D_{\tau_1}(M_1)\cap D_{\tau_2^{-1}}(M_1)$. The polars of $M_1\times M_2$ with respect to (o,o) are given by $M_j^+\times M_k^+$ $(0\leq j,k\leq m)$.

The intersection of $D_{\tau_1}(M_1)$ and each polar of $M_1\times M_2$ is given by the following.

$$D_{\tau_1}(M_1)\cap\left(M_j^+\times M_j^+\right) = \begin{cases} M_0^+\times M_0^+ & (j=0), \\ D_{\tau_1|_{M_j^+}}\left(M_j^+\right) & (1\leq j\leq m-1), \\ M_m^+\times M_m^+ & (j=m), \end{cases}$$

where M_0^+ and M_m^+ consist of a single point and the intersection is the empty set for the others.

Similarly, the intersection of $D_{\tau_2^{-1}}(M_1)$ and each polar of $M_1\times M_2$ is given as follows.

$$D_{\tau_2^{-1}}(M_1)\cap\left(M_j^+\times M_j^+\right) = \begin{cases} M_0^+\times M_0^+ & (j=0), \\ D_{\tau_2^{-1}|_{M_j^+}}\left(M_j^+\right) & (1\leq j\leq m-1), \\ M_m^+\times M_m^+ & (j=m), \end{cases}$$

and the intersection is the empty set for the others.

By the assumption that $\tau_2\tau_1 \notin A_0(M_1)$ and Lemma 2.1 (4), τ_1 and τ_2 belong to different connected components, thus τ_2 and $\phi\tau_1$ belong to the same connected component.

Since $(o,o) \in D_{\tau_1}(M_1) \cap D_{\tau_2^{-1}}(M_1)$, we have $\tau_1(o) = \tau_2(o) = o$. So τ_i preserves each polar M_j^+ for $i = 1,2$. By a similar argument in the proof of Theorem 2.6 we can see that τ_i preserves or exchanges two irreducible factors of $M_j^+ = G_{m-j}(\mathbb{C}^m) \times G_j(\mathbb{C}^m)$. If τ_1 preserves two irreducible factors, then $\phi\tau_1$ exchanges two irreducible factors. We see that τ_2 also exchanges two irreducible factors since τ_2 belongs to the same connected component as $\phi\tau_1$. Similarly, if τ_1 exchanges two irreducible factors, then τ_2 preserves two irreducible factors. So this case reduces to the case where τ_1 preserves two irreducible factors. Thus we can write

$$\begin{aligned} \tau_1(x_1,x_2) &= (\tau_{11}(x_1), \tau_{12}(x_2)), \\ \tau_2(x_1,x_2) &= (\tau_{22}(x_2), \tau_{21}(x_1)), \end{aligned} \qquad ((x_1,x_2) \in G_{m-j}(\mathbb{C}^m) \times G_j(\mathbb{C}^m)),$$

where

$$\begin{aligned} \tau_{11} &: G_{m-j}(\mathbb{C}^m) \to G_{m-j}(\mathbb{C}^m), \\ \tau_{12} &: G_j(\mathbb{C}^m) \to G_j(\mathbb{C}^m), \\ \tau_{21} &: G_{m-j}(\mathbb{C}^m) \to G_j(\mathbb{C}^m), \\ \tau_{22} &: G_j(\mathbb{C}^m) \to G_{m-j}(\mathbb{C}^m) \end{aligned}$$

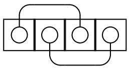
are all anti-holomorphic isometric maps. Using these we obtain

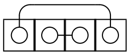
$$\begin{aligned} D_{\tau_1|_{M_j^+}}(M_j^+) &= \{(x, \tau_1(x)) \mid x \in M_j^+\} \\ &= \{(x_1, x_2, \tau_{11}(x_1), \tau_{12}(x_2)) \mid x_1 \in G_{m-j}(\mathbb{C}^m), x_2 \in G_j(\mathbb{C}^m)\} \end{aligned}$$

and

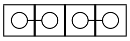
$$\begin{aligned} D_{\tau_2|_{M_j^+}}(M_j^+) &= \{(x, \tau_2(x)) \mid x \in M_j^+\} \\ &= \{(x_1, x_2, \tau_{22}(x_2), \tau_{21}(x_1)) \mid x_1 \in G_{m-j}(\mathbb{C}^m), x_2 \in G_j(\mathbb{C}^m)\}. \end{aligned}$$

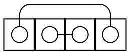
Their diagrams are

$D_{\tau_1|_{M_j^+}}(M_j^+)$:  ,

$D_{\tau_2|_{M_j^+}}(M_j^+)$:  .

We exchange the second and the third irreducible factors and obtain

$D_{\tau_1|_{M_j^+}}(M_j^+)$:  ,

$D_{\tau_2|_{M_j^+}}(M_j^+)$:  ,

which is the case (4) in Theorem 2.7.

Hence the pair of two diagonal real forms in $M_j^+ \times M_j^+$ given in the above belongs to the case (4) in Theorem 2.7. In this case

$$\#\big(D_{\tau_1|_{M_j^+}}(M_j^+) \cap D_{\tau_2|_{M_j^+}}(M_j^+)\big) = \#\big(D_{\tau_{12}\tau_{22}^{-1}\tau_{11}}(G_{m-j}(\mathbb{C}^m)) \cap D_{\tau_{21}}(G_{m-j}(\mathbb{C}^m))\big).$$

If $\tau_{21}^{-1}\tau_{12}\tau_{22}^{-1}\tau_{11}$ belongs to $I_0(G_{m-j}(\mathbb{C}^m))$, then $D_{\tau_{12}\tau_{22}^{-1}\tau_{11}}(G_{m-j}(\mathbb{C}^m))$ and $D_{\tau_{21}}(G_{m-j}(\mathbb{C}^m))$ are congruent in $M_j^+ = G_{m-j}(\mathbb{C}^m) \times G_j(\mathbb{C}^m)$ by Lemma 2.3, hence

$$\#\big(D_{\tau_{12}\tau_{22}^{-1}\tau_{11}}(G_{m-j}(\mathbb{C}^m)) \cap D_{\tau_{21}}(G_{m-j}(\mathbb{C}^m))\big) = \#_2 G_{m-j}(\mathbb{C}^m) = \binom{m}{j}$$

by Theorem 1.3 in [**6**]. For this purpose we will prove $\tau_{21}^{-1}\tau_{12}\tau_{22}^{-1}\tau_{11}$ belongs to $I_0(G_{m-j}(\mathbb{C}^m))$.

Since τ_2 and $\phi\tau_1$ belong to the same connected component of $I(M_1)$ and $\tau_2(o) = \phi\tau_1(o) = o$, there is an element $k \in I_0(M_1)$ satisfying $\tau_2 = \phi\tau_1 k$ and $k(o) = o$. We can express the action of ϕ on $M_j^+ = G_{m-j}(\mathbb{C}^m) \times G_j(\mathbb{C}^m)$ as

$$\phi(x_1, x_2) = (\phi_2(x_2), \phi_1(x_1)) \quad (x_1 \in G_{m-j}(\mathbb{C}^m), x_2 \in G_j(\mathbb{C}^m)),$$

where $\phi_1 : G_{m-j}(\mathbb{C}^m) \to G_j(\mathbb{C}^m)$ and $\phi_2 : G_j(\mathbb{C}^m) \to G_{m-j}(\mathbb{C}^m)$ are holomorphic isometric maps and we have $\phi_1\phi_2 = \mathrm{id}$ and $\phi_2\phi_1 = \mathrm{id}$, by the description of ϕ obtained above.

Since

$$\begin{aligned}(\tau_{22}(x_2), \tau_{21}(x_1)) &= \phi\tau_1 k(x_1, x_2)\\ &= (\phi_2\tau_{12}k_2(x_2), \phi_1\tau_{11}k_1(x_1))\end{aligned}$$

where $k(x_1, x_2) = (k_1(x_1), k_2(x_2))$ for $(x_1, x_2) \in G_{m-j}(\mathbb{C}^{2m}) \times G_j(\mathbb{C}^m)$, we have

$$\tau_{21} = \phi_1\tau_{11}k_1, \quad \tau_{22} = \phi_2\tau_{12}k_2.$$

Hence

$$\begin{aligned}\tau_{21}^{-1}\tau_{12}\tau_{22}^{-1}\tau_{11} &= (\phi_1\tau_{11}k_1)^{-1}\tau_{12}(\phi_2\tau_{12}k_2)^{-1}\tau_{11}\\ &= k_1^{-1}\tau_{11}^{-1}\phi_1^{-1}\tau_{12}k_2^{-1}\tau_{12}^{-1}\phi_2^{-1}\tau_{11}.\end{aligned}$$

Because $\tau_{12}k_2^{-1}\tau_{12}^{-1} \in I_0(G_j(\mathbb{C}^m))$, we have

$$\phi_1^{-1}\tau_{12}k_2^{-1}\tau_{12}^{-1}\phi_2^{-1} = \phi_2\tau_{12}k_2^{-1}\tau_{12}^{-1}\phi_2^{-1} \in I_0(G_j(\mathbb{C}^m))$$

and $\tau_{11}^{-1}\phi_1^{-1}\tau_{12}k_2^{-1}\tau_{12}^{-1}\phi_2^{-1}\tau_{11} \in I_0(G_{m-j}(\mathbb{C}^m))$ hence $\tau_{21}^{-1}\tau_{12}\tau_{22}^{-1}\tau_{11} \in I_0(G_{m-j}(\mathbb{C}^m))$. So we have

$$\#\big(D_{\tau_1|_{M_j^+}}(M_j^+) \cap D_{\tau_2^{-1}|_{M_j^+}}(M_j^+)\big) = \#_2 G_{m-j}(\mathbb{C}^m) = \binom{m}{j}.$$

Thus by Lemma 4.3 in [**6**] we obtain

$$\begin{aligned}\#(D_{\tau_1}(G_m(\mathbb{C}^{2m})) \cap D_{\tau_2^{-1}}(G_m(\mathbb{C}^{2m})) &= \sum_{j=0}^{m} \#\big(D_{\tau_1|_{M_j^+}}(M_j^+) \cap D_{\tau_2^{-1}|_{M_j^+}}(M_j^+)\big) \\ &= \sum_{j=0}^{m} \binom{m}{j} = 2^m.\end{aligned}$$

□

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