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著者別名	初貝 安弘
journal or publication title	Physical review B
volume	91
number	8
page range	085112
year	2015-02
権利	(C)2015 American Physical Society
URL	http://hdl.handle.net/2241/00124354

doi: 10.1103/PhysRevB.91.085112

Survival of sharp $n = 0$ Landau levels in massive tilted Dirac fermions: Role of the generalized chiral operator

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(Received 23 October 2014; revised manuscript received 4 February 2015; published 18 February 2015)

An anomalously sharp (δ-function-like) $n = 0$ Landau level in the presence of disorder is usually considered to be a manifestation of the massless Dirac fermions in magnetic fields. This property persists even when the Dirac cone is tilted, which has been shown by Kawarabayashi *et al.* [Phys. Rev. B **83**, 153414 (2011)] to be a consequence of a "generalized chiral symmetry." Here we pose the question of whether this property will be washed out when the tilted Dirac fermion becomes massive. Surprisingly, the levels continue to be δ-function-like, although the mass term that splits $n = 0$ Landau levels may seem to degrade the anomalous sharpness. This has been shown both numerically for a tight-binding model and analytically in terms of the Aharonov-Casher argument extended to the massive tilted Dirac fermions. A key observation is that, while the generalized chiral symmetry is broken by the mass term, the $n = 0$ Landau level continues to accommodate eigenstates of the generalized chiral operator, resulting in the robustness against chiral-symmetric disorders. Mathematically, the conventional and generalized chiral operators are related to each other via a nonunitary transformation, with which the split, nonzero-energy $n = 0$ wave functions of the massive system are just gauge-transformed zero-mode wave functions of the massless system. The message is that the chiral symmetry, rather than a simpler notion of the sublattice symmetry, is essential for the robustness of the $n = 0$ Landau level.

DOI: 10.1103/PhysRevB.91.085112 PACS number(s): 71.70.Di, 73.22.Pr

I. INTRODUCTION

After the experimental discovery of graphene [1,1], fascination with the massless Dirac fermions has become one of the central interests in condensed-matter physics [2]. The physics of zero-gap semiconductors actually has a long history of studies, starting with the theoretical work by Wallace, and is now described in condensed-matter textbooks [3]. There are various spin-offs, among which is the topological insulator with the quantized spin Hall effect, where the topological property of Dirac fermions plays a fundamental role [4–7]. In the context of zero-gap semiconductors, the first topological insulator, HgTe-CdTe, was realized by making the mass negative [3,8]. Quantum phase transitions of fermions associated with gap closing and opening can be described by a Dirac fermion in terms of reversing the sign of the mass [9]. We then have realizations of diverse quantum phases, such as chiral spin states, flux phases, and nodal fermions. Another important class of the Dirac fermions is an organic material, α-(BEDT-TTF)$_2$I$_3$ [10–12], where the Dirac cone dispersion is substantially tilted. In a broader context, anisotropic superconductors with d-wave symmetry have Dirac cones in the dispersion for the Bogoliubov quasiparticle, which serve as another class of Dirac fermions in two dimensions [13,14].

While the massless Dirac cone in graphene is related to the honeycomb lattice structure, the gap closing itself can be analyzed more generally in terms of the level crossing in quantum mechanics. According to the von Neumann–Wigner theorem, a degeneracy point has generically codimension three [15–17]. This indicates that the existence of massless Dirac cones in three spatial dimensions is rather natural. Conversely, in two dimensions a Dirac cone is an accident unless some symmetry exists. The chiral symmetry [18] is often evoked for graphene as represented by the honeycomb lattice, for which the symmetry is usually regarded as nothing but the sublattice symmetry against the sign change of the wave function on one of the sublattices in a bipartite lattice structure. Hence it is a usual practice to attribute the reason why graphene realizes the massless Dirac fermions to the honeycomb structure. In two-dimensional systems with a chiral symmetry, one can also prove the fermion-doubling theorem as a two-dimensional analog of the Nielsen-Ninomiya theorem conceived for four dimensions [17,19–21], which dictates that the number of Dirac cones has to be even. It also brings a supersymmetric (SUSY) structure in the one-particle Hamiltonian [22–24]. In the case of graphene this is why we have two Dirac cones at valleys K and K'. Thus in the physics of graphene the chiral symmetry is important [18,20,21,25,26]. We can even use the chiral symmetry to discuss the topological nature of the system [18,25]. For instance, in a d-wave superconductor, the chiral symmetry translates into the time-reversal symmetry in the Bogoliubov Hamiltonian [21,27], which protects the existence of nodes in the gap.

Now, in two dimensions the Dirac cone is, in general, tilted, as in the case of the organic material, where the conventional chiral symmetry is broken [28]. One may then wonder if the existence of a Dirac cone itself suffices for the topological properties even when the chiral symmetry is apparently absent. The present authors have revealed that the notion of the chiral symmetry can actually be extended to accommodate the tilted cones [28], where the tilted cones has a symmetry against the "generalized chiral operator," and have demonstrated some of its consequences both analytically and numerically. Most importantly, if we look at the $n = 0$

Landau level (right at the Dirac point) in magnetic fields, its density of states remains δ-function-like even in the presence of disorder, while one might assume that this property would be specific to vertical Dirac cones.

In the present paper we pose the question of whether the anomalous property of the $n = 0$ Landau level will be washed out when the tilted Dirac fermion becomes massive. While this question may seem too detailed, it is actually not since from this we can clarify an important question: are the existence of zero modes and the chiral symmetry one and the same? While for a vertical cone they are obviously the same, in a massive case the $n = 0$ Landau level splits into two with nonzero energies, so that one might imagine that the two properties should differ from each other in this case. Surprisingly, we find that the levels, now split, do remain δ-function-like. This has been shown analytically in terms of the Aharonov-Casher argument, which is known to construct wave functions in the zero-mode Landau level in vertical cones and is here extended to the massive tilted Dirac fermions. A key observation is that, while the generalized chiral *symmetry* is broken by the mass term, the $n = 0$ Landau level continues to accommodate eigenstates of the generalized chiral *operator*, ensuring the robustness against chiral-symmetric disorders. Mathematically, the conventional and generalized chiral operators are found to be related to each other via a nonunitary transformation, with which we can identify the split, nonzero-energy wave functions of the massive system as *gauge-transformed zero-mode wave functions* of the massless ones. The anomalously sharp Landau level is confirmed by a numerical result for a model tight-binding system for disorders that respect the generalized-chiral symmetry, in sharp contrast to the disorders that do not.

We can visualize the point as follows. While the conventional chiral symmetry dictates that each wave function in the $n = 0$ Landau level has nonzero amplitudes only on the A sublattice (or B sublattice), the wave function for tilted Dirac fermions is not an eigenstate of the sublattice symmetry, so that it has amplitudes on both of the sublattices (or two components of the spinor). This may seem to suggest that the sharpness of the $n = 0$ Landau level is degraded for tilted Dirac fermions when we make the fermion massive by introducing a staggered potential over the A and B sublattices. If this were the case, the tilted cone should differ from the vertical cone, and the sharpness of the $n = 0$ Landau level would be affected by the staggered potential. The present result shows that this is not the case. Thus the message of the present work is that (i) the (generalized) chiral symmetry, rather than a simpler notion of the sublattice symmetry, is essential for the robustness of the $n = 0$ Landau level, which is why (ii) the chiral operator plays a crucial role even in the massive case.

Since the presence of Dirac cones is accidental in two-dimensional (2D) systems unless there is some symmetry protection, it is natural to expect an energy gap in Dirac fermion systems. Hence the massive Dirac fermion with tilting in 2D is a generic and common problem. In fact, extensive studies are now going on for massive Dirac-fermion materials such as molybdenum disulfide compounds [29], as well as several organic materials with substantially tilted Dirac cones. The insulating phase in such organic materials could be a candidate for the massive and tilted Dirac fermions [12,30,31]. In a completely different area, the massive and tilted Dirac fermions may be realized in cold atoms in optical lattices, where the Dirac cones are often tilted and the parameters are more controllable than in solid-state materials [32–35].

In this paper we start in Sec. II with a numerical result for a lattice model that has tilted Dirac cones, from which we find the anomalous sharpness of the $n = 0$ Landau levels is surprisingly unaffected by the introduction of the mass term for the case of the spatially smooth (long-range) disorder, as long as the disorder respects the chiral symmetry (as is the case with random magnetic fields introduced there). We further find numerically that, for spatially uncorrelated (short-range) disorder, the anomalous sharpness of the $n = 0$ Landau levels is unexpectedly recovered as the staggered potential is increased. This is just the *opposite* of the case of massless cones, where the sharpness of the $n = 0$ Landau level is degraded for spatially uncorrelated disorder [28]. The recovery of the sharpness of the $n = 0$ Landau level for uncorrelated disorder has also been reported for shifted Dirac cones [36]. In the present case, the energies of the two $n = 0$ Landau levels associated with the two valleys are split by the mass term (i.e., the staggered potential), although the Dirac cones themselves are not shifted in energy. The present recovery of the sharpness thus indicates that the disorder-induced mixing between the split $n = 0$ Landau levels is significantly suppressed by increasing the energy splitting introduced by the mass term for a chiral-symmetry-preserving disorder. This is in sharp contrast to a potential disorder, which we also demonstrate numerically.

We then present in Sec. III the main analytic part, which provides a solution for the puzzling numerical result. Namely, in order to understand the origin of the anomalous sharpness of the $n = 0$ Landau level for massive Dirac fermions, we develop a general effective theory, first for the massless case and for the massive case in Sec. IV, and we find a simple algebraic relationship (which turns out to be nonunitary) between the generalized chiral operator and the conventional chiral operator. This enables us to discuss quantitatively the effect of the staggered potential on the anomalous sharpness of the $n = 0$ Landau levels for tilted Dirac fermions with a disorder that respects the generalized chiral symmetry. We then show that the $n = 0$ Landau levels of the massive Dirac fermions are still the eigenstate of the generalized chiral operator, where the robustness for the random gauge field is again shown analytically using the Aharonov-Casher argument. Although an *a priori* introduction of the generalized chiral operator is given in Ref. [36], here we provide a transparent and logical derivation using four-dimensional notation. This enables us to consistently describe the massless and massive Dirac fermions with tilting. Since the massive and massless Dirac cones occur as semimetals and semiconductors in 2D, the compact four-dimensional notation given here will be useful for understanding Dirac-fermion-related physics. Section V is devoted to a summary.

II. NUMERICAL RESULT FOR MASSIVE TILTED DIRAC FERMIONS

To examine the $n = 0$ Landau level for massive and tilted Dirac fermions, let us first perform a numerical analysis based on the tight-binding lattice model [28] on a two-dimensional square lattice, with the tight-binding Hamiltonian given by

$$H_{TB} = \sum_{\boldsymbol{r}} [-t c^{\dagger}_{\boldsymbol{r}+\hat{\boldsymbol{y}}} c_{\boldsymbol{r}} + (-1)^{x+y} t c^{\dagger}_{\boldsymbol{r}+\hat{\boldsymbol{x}}} c_{\boldsymbol{r}} + \text{H.c.} + t'(c^{\dagger}_{\boldsymbol{r}+\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}} c_{\boldsymbol{r}} + c^{\dagger}_{\boldsymbol{r}+\hat{\boldsymbol{x}}-\hat{\boldsymbol{y}}} c_{\boldsymbol{r}}) + \text{H.c.}].$$

Here the lattice positions are denoted by $\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}}$, with $\hat{\boldsymbol{x}}$ ($\hat{\boldsymbol{y}}$) being the unit vector in the x (y) direction and the length in units of the lattice constant of the square lattice, t is the nearest-neighbor hopping, and t' is the next nearest-neighbor hopping. The model is similar to the π flux model [37] that has a half flux per plaquette (π flux). The factor $(-1)^{x+y}$ in the nearest-neighbor hopping is the Peierls phase for the half-flux quantum. When $t' = 0$, the dispersion of the model has two massless Dirac cones at $E = 0$ around the k points $\boldsymbol{k}_0 = (0,\pm\pi/2)$, while the Dirac cones become tilted when $t' \neq 0$ [28].

The effective low-energy Hamiltonian H around the Dirac cones can then be expressed in terms of the Pauli matrices ${}^t\boldsymbol{\sigma} = (\sigma_1,\sigma_2,\sigma_3) \equiv (\sigma_x,\sigma_y,\sigma_z)$ as

$$H = (X^0\sigma_0 + \boldsymbol{X}\cdot\boldsymbol{\sigma})\delta k_x + (Y^0\sigma_0 + \boldsymbol{Y}\cdot\boldsymbol{\sigma})\delta k_y,$$

where $\delta\boldsymbol{k} = \boldsymbol{k} - \boldsymbol{k}_0$ is the deviation of the momentum from the Dirac point $\boldsymbol{k}_0$ and σ_0 is a two-dimensional unit matrix. In the case of the above tight-binding Hamiltonian H_{TB} we have ${}^t\boldsymbol{X} = (0,2t,0)$ and ${}^t\boldsymbol{Y} = (\mp 2t,0,0)$, and $(X^0,Y^0) = (0,\pm 4t')$, which makes the Dirac cone indeed tilted.

Now we make the fermions massive by introducing a mass term. This can be readily done by introducing a staggered potential, and the massive Hamiltonian $H_{TB}(m)$ reads

$$H_{TB}(m) = H_{TB} + mc^2 \sum_{\boldsymbol{r}} (-1)^{x+y} c^{\dagger}_{\boldsymbol{r}} c_{\boldsymbol{r}}, \tag{1}$$

where A (B) sublattice site energies are elevated (lowered). The effective low-energy Hamiltonian becomes

$$H(m) = H + mc^2\sigma_z,$$

where the term $mc^2\sigma_z$ makes the Dirac fermions massive with a gap at the Dirac point. In the case of the usual vertical Dirac fermions, the mass term can be expressed in terms of the conventional chiral operator $\Gamma \propto \sigma_z$. The chiral operator is generally defined as an operator that anticommutes with the Hamiltonian H, which, for the vertical Dirac cone with $X^0 = Y^0 = 0$, is given by $\Gamma = \hat{\boldsymbol{n}}_0 \cdot \boldsymbol{\sigma}$, with $\hat{\boldsymbol{n}}_0 \equiv \boldsymbol{X}\times\boldsymbol{Y}/|\boldsymbol{X}\times\boldsymbol{Y}|$ [28,38]. For the present model, the conventional chiral operator is simply $\Gamma = \pm\sigma_z$, where the plus (minus) sign applies to the valley around $\boldsymbol{k}_0 = (0,\pi/2)$ [$(0,-\pi/2)$]. The Hamiltonian $H(m)$ is then expressed with Γ as

$$H(m) = \begin{cases} H + mc^2\Gamma & \text{for} \quad \boldsymbol{k} = (0,\pi/2), \\ H - mc^2\Gamma & \text{for} \quad \boldsymbol{k} = (0,-\pi/2). \end{cases} \tag{2}$$

We then apply an external magnetic field to carry out exact numerical diagonalization for a finite system in the presence of disorder. The magnetic field is taken into account by the Peierls phase, $t(t') \to t(t')e^{2\pi i\theta}$, where the summation of θ along a loop is given by the enclosed magnetic flux ϕ in units of the flux quantum h/e. The disorder is introduced here as a random component $\delta\phi(\boldsymbol{r})$ in the magnetic flux $\phi(\boldsymbol{r}) = \phi + \delta\phi(\boldsymbol{r})$ piercing each square plaquette, where ϕ denotes the uniform component. The random component $\delta\phi(\boldsymbol{r})$ is assumed to obey a Gaussian distribution with a variance σ and a spatial correlation,

$$\langle\delta\phi(\boldsymbol{r})\delta\phi(\boldsymbol{r}')\rangle = \sigma^2 \exp(-|\boldsymbol{r}-\boldsymbol{r}'|^2/4d^2),$$

where d is the correlation length. We have chosen randomness in the magnetic field since disorder in the gauge degrees of freedom (such as the random magnetic field) respects the generalized chiral symmetry [28].

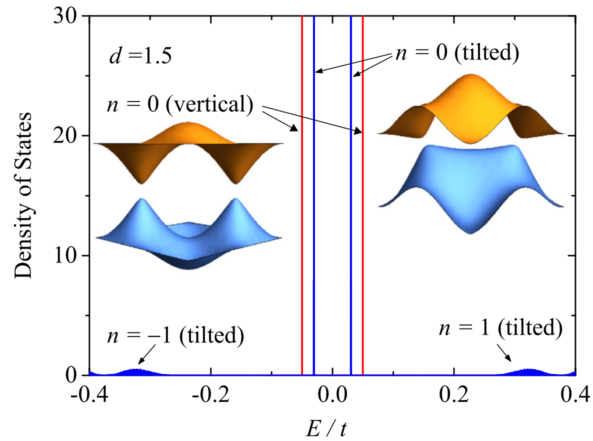

FIG. 1. (Color online) Density of states for the lattice model having tilted Dirac cones in the presence of a mass term (staggered potential) in a magnetic field [$\phi/(h/e) = 0.01$] with a spatially correlated disorder ($d = 1.5$). Landau levels for both tilted ($t'/t = 0.4$, blue) and vertical ($t'/t = 0$, red) cones are shown. The amplitude of disorder is taken here to be $\sigma/(h/e) = 0.0029$, the mass $mc^2/t = 0.05$, and the system size is 30×30 with an average over 5000 samples.

In Fig. 1, we show the density of states of the system with tilted Dirac cones in a finite magnetic field [$\phi/(h/e) = 0.01$] for the case of spatially correlated disorder ($d = 1.5$ in units of the lattice constant). For comparison, we also display the result for the case of vertical Dirac cones. We can immediately notice that the introduction of the mass term $mc^2\sigma_z$ does not affect the anomalous sharpness of the split $n = 0$ Landau levels even for the tilted cones as in the vertical cones. Since the other Landau levels (e.g., $n = \pm 1$) are broadened, the $n = 0$ levels do stand out.

A further surprise occurs when we examine the robustness of the split $n = 0$ Landau levels against the spatially uncorrelated disorder ($d/a = 0$). For the massless ($m = 0$) case, uncorrelated disorder degrades the sharp $n = 0$ Landau levels due to the intervalley scattering [26,39]. However, we can see in Fig. 2 that the anomalous sharpness is actually recovered as the mass is made heavier with the level splitting becoming wider. In the massive case, each $n = 0$ Landau level is associated with one of the two Dirac cones. (See Fig. 5 below.) The present result indicates that the mixing between the Dirac cones is effectively suppressed when the $n = 0$ Landau levels are split by the staggered potential. This reminds us of our previous work, where we introduced a model in which the two Dirac cones remain massless but are shifted in energy with a complex hopping. There the robustness is recovered

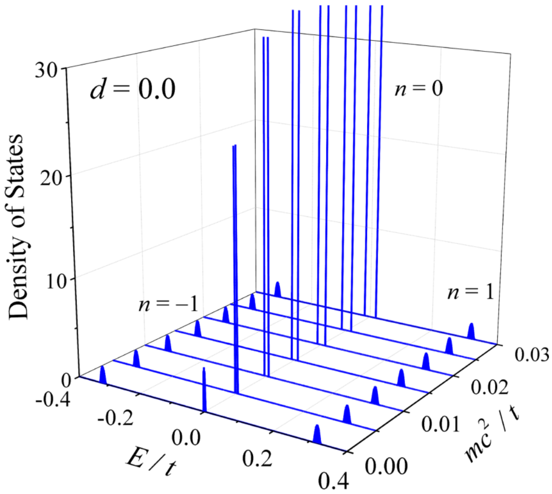

FIG. 2. (Color online) For the spatially uncorrelated disorder ($d = 0$), the density of states for the lattice model with tilted Dirac cones is plotted against the mass (staggered potential) for the same magnetic field and amplitude of disorder as in Fig. 1.

even for short-range disorder [36]. The present result indicates that a similar suppression of the mixing is at work, where the energy offset comes not from the shifted cones but from a mass gap.

We can show that the situation becomes completely different for a spatially uncorrelated potential disorder which does not respect the generalized chiral symmetry. For the present lattice model, a potential disorder can be represented by random site energies as $\sum_{\boldsymbol{r}} \varepsilon_{\boldsymbol{r}} c_{\boldsymbol{r}}^\dagger c_{\boldsymbol{r}}$ in place of the random component of the magnetic field. We then find, as clearly shown in Fig. 3, that the recovery of the anomalous sharpness of the $n = 0$ Landau level is completely absent for the case of potential disorder, even though the mixing between two valleys is suppressed by the mass term. This suggests that, although the mass term (staggered potential) naively breaks the generalized chiral symmetry of the system, whether the disorder respects the generalized chiral symmetry continues to be crucial for the anomalous sharpness of the $n = 0$ Landau levels of the massive tilted Dirac fermions.

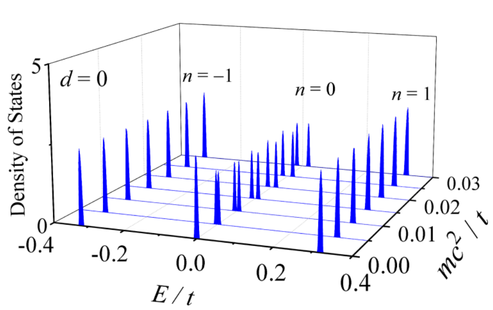

FIG. 3. (Color online) The density of states for uncorrelated potential disorder, instead of the random magnetic field, is plotted against the mass (staggered potential) for the same uniform magnetic field as in Figs. 1 and 2. The random potential is assumed to obey a Gaussian distribution with zero mean and a variance of $0.02t$.

III. TILTED MASSLESS DIRAC FERMIONS

A. A general formulation

To understand the robustness of the zero modes for massive and tilted Dirac fermions [40–42], let us first summarize a general effective theory for massless and tilted Dirac fermions from the viewpoint of the generalized chiral symmetry. For this purpose we introduce a compact four-dimensional notation to make the discussion transparent. As a generic band structure of semiconductors, let us consider a two-band Hamiltonian H_g in a 2×2 matrix form focusing on the valence and conduction bands. Since the Hamiltonian is Hermitian, it is expanded by $\sigma_0, \sigma_1, \sigma_2$, and σ_3 as

$$H_g(\boldsymbol{k}) = \sigma_0 R_0(\boldsymbol{k}) + \boldsymbol{\sigma} \cdot \boldsymbol{R}(\boldsymbol{k}),$$

where ${}^t\boldsymbol{R}(\boldsymbol{k}) = (R_1(\boldsymbol{k}), R_2(\boldsymbol{k}), R_3(\boldsymbol{k}))$ are real. The energy dispersions are given by

$$E_\pm(\boldsymbol{k}) = R_0(\boldsymbol{k}) \pm |\boldsymbol{R}(\boldsymbol{k})|,$$

where $|\boldsymbol{R}| = \sqrt{R_1^2 + R_2^2 + R_3^2}$, with the energy gap $E_g(\boldsymbol{k})$ for each momentum $\boldsymbol{k}$ being $E_g(\boldsymbol{k}) = 2|\boldsymbol{R}(\boldsymbol{k})|$. We have a semiconductor under the condition

$$E_-(\boldsymbol{k}_v) \leqslant E_+(\boldsymbol{k}_c),$$

where $\boldsymbol{k}_v$ ($\boldsymbol{k}_c$) are the wave numbers in the valence (conduction) band.

In the case of a zero-gap semiconductor, the energy gap vanishes at some momentum $\boldsymbol{k}_0$. Expanding the Hamiltonian around $\boldsymbol{k}_0$, we have an effective Hamiltonian ($H_g \approx H$),

$$H = (X^0\sigma_0 + \boldsymbol{X} \cdot \boldsymbol{\sigma})\delta k_x + (Y^0\sigma_0 + \boldsymbol{Y} \cdot \boldsymbol{\sigma})\delta k_y,$$

where $\delta\boldsymbol{k} = \boldsymbol{k} - \boldsymbol{k}_0$, $X^0 = \partial_{k_x} R_0|_{\boldsymbol{k}_0}$, $Y^0 = \partial_{k_y} R_0|_{\boldsymbol{k}_0}$, and the three-dimensional vectors ${}^t\boldsymbol{X} = (X^1, X^2, X^3)$ and ${}^t\boldsymbol{Y} = (Y^1, Y^2, Y^3)$ are defined by $\boldsymbol{X} = \partial_x \boldsymbol{R}|_{\boldsymbol{k}_0}$, $\boldsymbol{Y} = \partial_y \boldsymbol{R}|_{\boldsymbol{k}_0}$. The terms that contain X^0 or Y^0 induce tilting of the Dirac cones, while when $(X^0, Y^0) = 0$, the Dirac cones can be anisotropic but vertical.

With an effective momentum around the gapless point, $\boldsymbol{p} = \hbar\delta\boldsymbol{k} = {}^t(p_x, p_y)$, we have

$$H = \hbar^{-1}(\sigma_\mu X^\mu, \sigma_\mu Y^\mu)\boldsymbol{p}, \tag{3}$$

where a summation over repeated indices is assumed.

Now, let us introduce a four-dimensional notation to simplify the calculation. For this purpose, we introduce, on top of the "contravariant" four-dimensional vectors ${}^tX = (X^0, X^1, X^2, X^3)$ and ${}^tY = (Y^0, Y^1, Y^2, Y^3)$, the conjugated (or "covariant") vectors $\bar{X}$ and $\bar{Y}$, defined as

$$\bar{X} = (X_0, X_1, X_2, X_3) = {}^tXg = (-X^0, X^1, X^2, X^3),$$

where $g = \mathrm{diag}(-1,1,1,1)$ is a metric. Now we have a simple identity (see Appendix A),

$$\begin{aligned}(\bar{X}^\mu\sigma_\mu)(\sigma_\nu Y^\nu) &= \bar{X}Y\sigma_0 + i\boldsymbol{n}\cdot\boldsymbol{\sigma},\\ (\bar{Y}^\mu\sigma_\mu)(\sigma_\nu X^\nu) &= \bar{Y}X\sigma_0 - i\boldsymbol{n}\cdot\boldsymbol{\sigma},\\ \boldsymbol{n} &= \boldsymbol{X}\times\boldsymbol{Y} + i\boldsymbol{\eta},\end{aligned} \tag{4}$$

where $\boldsymbol{\eta} = X^0\boldsymbol{Y} - Y^0\boldsymbol{X}$. Note that, while we have $\bar{X}Y = X^\mu Y_\mu = \bar{Y}X$, $\boldsymbol{n}$ is antisymmetric against $X \leftrightarrow Y$. Its norm becomes (see Appendix B)

$$\boldsymbol{n}^2 = (\bar{X}X)(\bar{Y}Y) - (\bar{Y}X)(\bar{X}Y) \equiv (\hbar c)^4,$$

where the Fermi velocity of the Dirac fermions c is defined. When $\boldsymbol{n}^2 > 0$, the velocity c is real.

By introducing the covariant notation, the discussion becomes transparent. For the usual (vertical) Dirac cones, it is known that considering a squared Hamiltonian H^2 facilitates the analysis. In the present case of tilted cones, this has to be modified. We can instead note that it is useful to define a Hamiltonian conjugate to Eq. (3) as

$$\bar{H} = \hbar^{-1}(\bar{X}^\mu \sigma_\mu, \bar{Y}^\mu \sigma_\mu)\boldsymbol{p} = H - 2H_0,$$

where

$$H_0 = \hbar^{-1}\sigma_0(X^0, Y^0)\boldsymbol{p}.$$

Now we can consider a product, $\bar{H}H$, a "contraction" in the present four-dimensional representation. The expression can be put in the form

$$\bar{H}H = \hbar^{-2}\boldsymbol{p}^\dagger \mathcal{G}\boldsymbol{p},$$

where $\mathcal{G}$ is a 4×4 matrix composed of 2×2 Pauli matrices and is expressed, with formula (4), as

$$\begin{aligned}\mathcal{G} &= \begin{pmatrix} \bar{X}^\mu \sigma_\mu \\ \bar{Y}^\mu \sigma_\mu \end{pmatrix} (\sigma_\nu X^\nu, \sigma_\nu Y^\nu) \\ &= \begin{pmatrix} \bar{X}X\sigma_0 & \bar{X}Y\sigma_0 + i\boldsymbol{n}\cdot\boldsymbol{\sigma} \\ \bar{Y}X\sigma_0 - i\boldsymbol{n}\cdot\boldsymbol{\sigma} & \bar{Y}Y\sigma_0 \end{pmatrix}.\end{aligned}$$

The determinant of $\mathcal{G}$ vanishes since its rank is 2, which can be confirmed directly by evaluating the determinant. We then have

$$\begin{aligned}\bar{H}H &= c^2 \boldsymbol{p}^\dagger \Xi \boldsymbol{p}\, \sigma_0, \\ \Xi &= \frac{1}{(\hbar c)^2}\begin{pmatrix} \bar{X}X & \bar{X}Y \\ \bar{Y}X & \bar{Y}Y \end{pmatrix},\end{aligned} \tag{5}$$

where we have used $[p_x, p_y] = 0$ and we can note that $\det \Xi = 1$.

From the Schrödinger equation, $H\Psi = E\Psi$, and Eq. (5), we have

$$\bar{H}H\Psi = E(H - 2H_0)\Psi = (E^2 - 2EH_0)\Psi,$$

which reduces to

$$\sigma_0[c^2\boldsymbol{p}^\dagger \Xi \boldsymbol{p} + 2(E/\hbar)(X_0, Y_0)\boldsymbol{p}]\Psi = E^2\Psi. \tag{6}$$

By completing the square, we have (see details in Appendix C) a simple bilinear form,

$$c_r^2 \boldsymbol{p}_E^\dagger \Xi \boldsymbol{p}_E \sigma_0 \Psi = E^2\Psi, \tag{7}$$

where

$$\begin{aligned}c_r &= c\left[\frac{\mathrm{Re}\,\boldsymbol{n}^2}{(\mathrm{Re}\,\boldsymbol{n})^2}\right]^{1/2} \equiv \frac{c}{\cosh q}, \\ \boldsymbol{p}_E &= \boldsymbol{p} + \Delta\boldsymbol{p}_E, \\ \Delta\boldsymbol{p}_E &= E\frac{1}{c^2\hbar}\Xi^{-1}\begin{pmatrix} X^0 \\ Y^0 \end{pmatrix}.\end{aligned} \tag{8}$$

This implies the equienergy contour is an ellipse centered at $\Delta\boldsymbol{p}_E$. (See Fig. 4 and also Appendices A–F.) The role of the parameter q appearing in the renormalization factor for the velocity c will become apparent when we discuss the relationship between the generalized chiral operator and the conventional chiral operator Γ in Sec. IV.

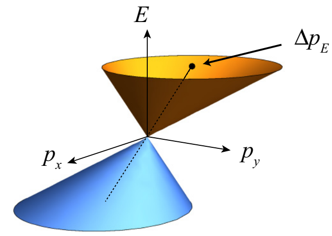

FIG. 4. (Color online) A tilted Dirac dispersion and its cross section (an ellipse) with a constant-energy plane. The center of the ellipse is given by Δp_E.

B. Landau levels and the generalized chiral operator

Having formulated the case in zero magnetic field, let us move on to the Landau states when we apply an external magnetic field for the tilted Dirac fermion. In terms of the dynamical momentum π_μ with $\mu = x, y$,

$$\pi_\mu = p_\mu - eA_\mu, \quad p_\mu = -i\hbar\partial_\mu,$$

where e is an elementary charge and A_μ is a vector potential which describes a magnetic field perpendicular to the two-dimensional system as

$$B = \partial_x A_y - \partial_y A_x.$$

The dynamical momentum satisfies

$$[\pi_x, \pi_y] = i\hbar eB = i(\hbar/\ell_B)^2,$$

where $\ell_B = \sqrt{\hbar/eB}$ is the magnetic length. We may choose $eB > 0$ without loss of generality. With a substitution $\boldsymbol{p} \to \boldsymbol{\pi} = \boldsymbol{p} - e\boldsymbol{A}$ we have the Hamiltonian

$$H = \hbar^{-1}(\sigma_\mu X^\mu, \sigma_\mu Y^\mu)\boldsymbol{\pi}$$

and its conjugate,

$$\bar{H} = \hbar^{-1}(\sigma_\mu \bar{X}^\mu, \sigma_\mu \bar{Y}^\mu)\boldsymbol{\pi} = H - 2\sigma_0(X^0, Y^0)\boldsymbol{\pi}.$$

Since π_x and π_y no longer commute in a magnetic field, we have an extra term proportional to $\boldsymbol{n}\cdot\boldsymbol{\sigma}$ for $\bar{H}H$:

$$\begin{aligned}\bar{H}H &= \hbar^{-2}\boldsymbol{\pi}^\dagger \mathcal{G}\boldsymbol{\pi} \\ &= c^2\boldsymbol{\pi}^\dagger \Xi \boldsymbol{\pi}\sigma_0 + i\hbar^{-2}\boldsymbol{n}\cdot\boldsymbol{\sigma}[\pi_x, \pi_y] \\ &= c^2\boldsymbol{\pi}^\dagger \Xi \boldsymbol{\pi}\, \sigma_0 - \ell_B^{-2}\boldsymbol{n}\cdot\boldsymbol{\sigma}.\end{aligned}$$

From the Schrödinger equation, $H\Psi = E\Psi$, and the relation above, we have

$$\left[c^2\boldsymbol{\pi}^\dagger \Xi \boldsymbol{\pi} + \frac{2E}{\hbar}(X_0, Y_0)\boldsymbol{\pi}\right]\sigma^0\Psi - \ell_B^{-2}\boldsymbol{n}\cdot\boldsymbol{\sigma}\Psi = E^2\Psi. \tag{9}$$

We can readily complete the square to arrive at

$$c_r^2[\boldsymbol{\pi}_E^\dagger \Xi \boldsymbol{\pi}_E - (\hbar/\ell_B)^2\gamma]\Psi = E^2\Psi,$$

where $\boldsymbol{\pi}_E = \boldsymbol{\pi} + \Delta\boldsymbol{p}_E$, with $\Delta\boldsymbol{p}_E$ and c_r defined in Eq. (8). An important ingredient is the generalized chiral operator γ, defined by

$$\gamma = \frac{\boldsymbol{n}\cdot\boldsymbol{\sigma}}{(\hbar c)^2}, \tag{10}$$

which has eigenvalues ± 1 because $\mathrm{Tr}\gamma = 0$ and satisfies

$$\det\gamma = -\boldsymbol{n}^2/(\hbar c)^4 = -1.$$

However, the operator is not Hermitian in general.

If we denote the right eigenstates of γ as $|\chi_\pm\rangle$ with

$$\gamma|\chi_\pm\rangle = \pm|\chi_\pm\rangle,$$

the wave function is expressed as $\Psi_\pm = |\chi_\pm\rangle\psi_\pm$. Then the Schrödinger equation is reduced to a scalar equation,

$$c_r^2[\boldsymbol{\pi}_E^\dagger\Xi\boldsymbol{\pi}_E \mp (\hbar/\ell_B)^2]\psi_\pm = E^2\psi_\pm.$$

If we note that the first term, $c_r^2\boldsymbol{\pi}_E^\dagger\Xi\boldsymbol{\pi}_E$, may be mathematically regarded, by replacing c_r^2 with $1/(2m^*)$, as a Hamiltonian for anisotropic fermions with a parabolic dispersion with an effective mass m^* in a magnetic field (Appendix E), we can introduce a single-component Landau wave function ψ_n that satisfies

$$\left[\frac{1}{2m^*}\boldsymbol{\pi}^\dagger\Xi\boldsymbol{\pi}\right]\psi_n = \hbar\omega_C\left(n+\frac{1}{2}\right)\psi_n,$$

where the effective cyclotron frequency is

$$\omega_C = \frac{eB}{m^*} = 2c_r^2 eB.$$

The squared energy then has the spectrum $\hbar\omega_C[(n+\frac{1}{2})\mp\frac{1}{2}]$; that is, the energy itself has a Dirac Landau level structure,

$$E_n = \pm c_r\sqrt{2\hbar eBn}, \quad n = 0,1,2,\ldots.$$

Note that the $n=0$ Landau level is given by the eigenstate of γ with the eigenvalue $+1$.

C. Generalized chiral symmetry

Let us discuss here the generalized chiral operator $\gamma = \boldsymbol{n}\cdot\boldsymbol{\sigma}/(\hbar c)^2$ defined in Sec. III B assuming that c is real (i.e., assuming there are Dirac cones). Since γ is antisymmetric against $X \rightleftarrows Y$, we have from Eq. (4)

$$2i(\hbar c)^2\gamma = \bar{x}y - \bar{y}x, \quad 2i(\hbar c)^2\gamma^\dagger = x\bar{y} - y\bar{x},$$

where $x = x^\dagger \equiv \sigma_\mu X^\mu$, $\bar{x} = \bar{x}^\dagger = \bar{X}^\mu\sigma_\mu$, etc. The Hamiltonian can be expressed as $H = x\pi_x + y\pi_y$, so that we obtain

$$2i(\hbar c)^2 H\gamma = (x\bar{x}y - x\bar{y}x)\pi_x + (y\bar{x}y - y\bar{y}x)\pi_y,$$
$$2i(\hbar c)^2\gamma^\dagger H = (x\bar{y}x - y\bar{x}x)\pi_x + (x\bar{y}y - y\bar{x}y)\pi_y.$$

Since $\bar{x}x = X^\mu X_\mu\sigma_0$ commutes with y and $\bar{y}y$ commutes with x, γ and H have an anticommutation relation, defined as

$$\{H,\gamma\}_R \equiv H\gamma + \gamma^\dagger H = 0,$$

which we have called the generalized chiral symmetry [28]. Note again that

$$\mathrm{Tr}\,\gamma = 0, \quad \det\gamma = -1, \quad \gamma^2 = (\gamma^\dagger)^2 = \sigma_0, \ \ \gamma^\dagger \neq \gamma.$$

The generalized chiral symmetry is essential for showing that the zero modes are generally eigenstates of the generalized chiral operator [28].

D. Robust zero modes

Now, let us focus on the zero modes (zero-energy states). There is a long history of study of the zero modes in massless Dirac fermions, notably the well-known work of Aharonov and Casher [26,28,43,44]. For $E=0$ states, the Schrödinger equation $H\Psi = 0$ reduces to

$$c^2[\boldsymbol{\pi}^\dagger\Xi\boldsymbol{\pi} - (\hbar/\ell_B)^2\gamma]\Psi = 0.$$

If we take the eigenstates $|\chi_+\rangle$ of the generalized chiral operator with the eigenvalue $+1$ with $\Psi = |\chi_+\rangle\psi_+$, ψ_+ satisfies

$$\left[(\pi_x,\pi_y)\Xi\begin{pmatrix}\pi_x\\ \pi_y\end{pmatrix} + i[\pi_x,\pi_y]\right]\psi_+ = 0$$

since $[\pi_x,\pi_y] = i(\hbar/\ell_B)^2$. The matrix Ξ, being real symmetric, can be diagonalized with an orthogonal matrix V_Ξ as

$$\Xi = {}^tV_\Xi \mathrm{diag}\,(\xi_1,\xi_2)V_\Xi, \tag{11}$$

where ${}^tV_\Xi V_\Xi = \sigma_0, \xi_1 > 0, \xi_2 > 0$, and $\xi_1\xi_2 = \det\Xi = 1$. Here we have assumed $\det V_\Xi = 1$ without loss of generality since, if $\det V_\Xi = -1$, $V_\Xi\sigma_x$ diagonalizes Ξ with ξ_1 and ξ_2 being interchanged. Then we can define a new momentum,

$$\boldsymbol{\Pi} = \begin{pmatrix}\sqrt{\xi_1} & 0\\ 0 & \sqrt{\xi_2}\end{pmatrix}V_\Xi\boldsymbol{\pi},$$

which preserves the commutator,

$$\begin{aligned}[\Pi_1,\Pi_2] &= \sqrt{\xi_1\xi_2}\sum_{i,j}V_{\Xi 1i}V_{\Xi 2j}[\pi_i,\pi_j]\\ &= (V_{\Xi 11}V_{\Xi 22} - V_{\Xi 12}V_{\Xi 21})[\pi_x,\pi_y]\\ &= \det V_\Xi[\pi_x,\pi_y]\\ &= [\pi_x,\pi_y].\end{aligned}$$

The zero-mode equation now reads

$$D^\dagger D\psi_+ = 0,$$

where

$$D = \Pi_1 + i\Pi_2.$$

Since $D^\dagger D$ is semipositive definite, we have

$$D\psi_+ = 0.$$

Noting that this is a first-order differential equation, we have an explicit solution (which is given below) like the one in the discussion by Aharanov-Casher [28,43]. This guarantees the stability of the zero modes. This argument is only possible for a real c^2 (where the Dirac operator is an elliptic one), which explicitly indicates that the index theorem for the elliptic operator is indeed relevant [28,39].

IV. MASSIVE AND TILTED DIRAC FERMIONS

A. General properties

Now we come to the massive case in question. Our motivation is to clarify the origin of the numerically observed anomalous robustness of the split $n=0$ Landau levels for the massive and tilted Dirac fermions. The generalized chiral operator γ introduced in Sec. III B can be expressed with the normalized vector $\hat{\boldsymbol{n}} = \boldsymbol{n}/\Delta$, which changes Eq. (10) to

$$\gamma = \hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma},$$

where we have introduced the norm of the vector $\boldsymbol{n}$,

$$\Delta = \sqrt{\boldsymbol{n}^2} = \sqrt{(\mathrm{Re}\,\boldsymbol{n})^2 - (\mathrm{Im}\,\boldsymbol{n})^2} = (\hbar c)^2.$$

Recall that in Eq. (4) the real and imaginary parts of $\boldsymbol{n}$ are given by $\mathrm{Re}\,\boldsymbol{n} = \boldsymbol{X} \times \boldsymbol{Y}$ and $\mathrm{Im}\,\boldsymbol{n} = \boldsymbol{\eta} = (X_0\boldsymbol{Y} - Y_0\boldsymbol{X})$, so that they are orthogonal to each other $[(\mathrm{Re}\,\boldsymbol{n}) \cdot (\mathrm{Im}\,\boldsymbol{n}) = 0]$. The conventional chiral operator Γ is expressed in a similar form in terms of a real vector $\hat{\boldsymbol{n}}_0 = \mathrm{Re}\,\boldsymbol{n}/\Delta_0$, with $\Delta_0 = |\mathrm{Re}\,\boldsymbol{n}| = |\boldsymbol{X} \times \boldsymbol{Y}|$, as

$$\Gamma = \hat{\boldsymbol{n}}_0 \cdot \boldsymbol{\sigma}.$$

We can then relate Γ to the generalized chiral operator γ as

$$\begin{aligned}\gamma &= (\hat{\boldsymbol{n}}_0 \cdot \boldsymbol{\sigma})(\hat{\boldsymbol{n}}_0 \cdot \boldsymbol{\sigma})(\hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma}) \\ &= \Gamma\,[(\hat{\boldsymbol{n}}_0 \cdot \hat{\boldsymbol{n}})\sigma_0 + i\boldsymbol{\sigma} \cdot (\hat{\boldsymbol{n}}_0 \times \hat{\boldsymbol{n}})] \\ &= \Gamma\,[(\hat{\boldsymbol{n}}_0 \cdot \mathrm{Re}\,\hat{\boldsymbol{n}})\sigma_0 - \boldsymbol{\sigma} \cdot (\hat{\boldsymbol{n}}_0 \times \mathrm{Im}\,\hat{\boldsymbol{n}})] \\ &= \Gamma\,[\Delta_0/\Delta - \boldsymbol{\sigma} \cdot (\mathrm{Re}\,\boldsymbol{n} \times \mathrm{Im}\,\boldsymbol{n})/(\Delta_0\Delta)]\,,\end{aligned}$$

where we have inserted $\Gamma^2 = 1$ in the first line, used the formula above Eq. (A1) in the second line, and used the fact that $\hat{\boldsymbol{n}}_0 \perp \mathrm{Im}\,\hat{\boldsymbol{n}}$ in the third line. Since $|\mathrm{Re}\,\boldsymbol{n} \times \mathrm{Im}\,\boldsymbol{n}| = |\mathrm{Re}\,\boldsymbol{n}||\mathrm{Im}\,\boldsymbol{n}| = \Delta_0\sqrt{\Delta_0^2 - \Delta^2}$, we end up with a compact expression,

$$\gamma = \Gamma(\cosh q - \boldsymbol{\tau} \cdot \boldsymbol{\sigma} \sinh q) = \Gamma e^{-q\boldsymbol{\tau}\cdot\boldsymbol{\sigma}},$$

where the parameter q is defined in Eq. (8) or, equivalently, $\tanh q = \sqrt{\Delta_0^2 - \Delta^2}/\Delta = |\boldsymbol{\eta}|/|\boldsymbol{X} \times \boldsymbol{Y}|$ and the unit vector $\boldsymbol{\tau}$ is given by

$$\boldsymbol{\tau} = \frac{\mathrm{Re}\,\boldsymbol{n} \times \mathrm{Im}\,\boldsymbol{n}}{|\mathrm{Re}\,\boldsymbol{n} \times \mathrm{Im}\,\boldsymbol{n}|} = \frac{(\boldsymbol{X} \times \boldsymbol{Y}) \times \boldsymbol{\eta}}{|(\boldsymbol{X} \times \boldsymbol{Y}) \times \boldsymbol{\eta}|}.$$

Note that the parameter q is real as long as $\Delta^2 \geqslant 0$, which is equivalent to the ellipticity of the Hamiltonian (3) where the index theorem is relevant.

We can also note that $\{\Gamma, \boldsymbol{\tau} \cdot \boldsymbol{\sigma}\} = 0$ since $\boldsymbol{\tau}$ is normal to $\boldsymbol{X} \times \boldsymbol{Y}$, and we have a suggestive representation,

$$\gamma = \Gamma e^{-q\boldsymbol{\tau}\cdot\boldsymbol{\sigma}} = e^{q\boldsymbol{\tau}\cdot\boldsymbol{\sigma}}\Gamma = e^{q\boldsymbol{\tau}\cdot\boldsymbol{\sigma}/2}\Gamma\, e^{-q\boldsymbol{\tau}\cdot\boldsymbol{\sigma}/2}.$$

This immediately implies that the eigenstates $|\pm\rangle$ of the conventional (Hermitian) chiral operator Γ (with $\Gamma|\pm\rangle = \pm|\pm\rangle$) can be related to the right eigenstates $|\chi_\pm\rangle$ of the generalized (non-Hermitian) chiral operator as

$$|\chi_\pm\rangle = \frac{1}{\sqrt{\cosh q}} e^{q\boldsymbol{\tau}\cdot\boldsymbol{\sigma}/2}|\pm\rangle.$$

The normalization factor $1/\sqrt{\cosh q}$ is introduced since $\langle +|\exp[q(\boldsymbol{\tau} \cdot \boldsymbol{\sigma})]|+\rangle = \langle -|\exp[q(\boldsymbol{\tau} \cdot \boldsymbol{\sigma})]|-\rangle = \cosh q$. On the other hand, we can readily verify the relation

$$\gamma^\dagger \Gamma \gamma = \Gamma, \tag{12}$$

which guarantees that

$$\langle \chi_+|\Gamma|\chi_-\rangle = \langle \chi_-|\Gamma|\chi_+\rangle = 0.$$

The diagonal matrix elements are evaluated as

$$\langle \chi_\pm|\Gamma|\chi_\pm\rangle = \pm\frac{1}{\cosh q} = \pm\frac{\Delta}{|\boldsymbol{X} \times \boldsymbol{Y}|}.$$

B. Symmetry breaking and robust zero modes

The relations obtained above are useful in considering the effects of the mass term (i.e., a staggered field $\propto \Gamma$), which breaks the generalized chiral symmetry into $\{H,\gamma\}_R \neq 0$ for the Hamiltonian H. For the vertical Dirac cones [24,45,46], the effect of the staggered potential is rather trivial since the states in the $n = 0$ Landau level are also eigenstates of the chiral operator Γ, with their energies simply shifted according to their eigenvalues of Γ. In sharp contrast, tilted Dirac cones have the states in the $n = 0$ Landau level that reside on both of the sublattices and are not the eigenstates of Γ. This is why the effects of the staggered potential become nontrivial for the tilted cone. We now employ the representation of Γ in terms of the generalized chiral bases to explore the effects of the staggered potential on the $n = 0$ Landau level. Essentially, we shall show that the states in the $n = 0$ Landau level remain eigenstates of γ even in the presence of the staggered potential.

For a typical source of a mass gap, we can again introduce a chiral-symmetry-breaking term $mc^2\Gamma$ in the Hamiltonian as

$$H(m) = H + mc^2\Gamma.$$

For the massless, tilted cones, we have shown that it is useful to consider $\bar{H}H$. Let us extend this argument to the massive case by considering $\bar{H}(m)H(m)$. Amazingly, we can simplify this into

$$\bar{H}(m)H(m) = (\bar{H} + mc^2\Gamma)(H + mc^2\Gamma) \quad = \bar{H}H + m^2c^4,$$

where cross terms between $\bar{H}(m)$ and $H(m)$ vanish because the unperturbed Hamiltonian without tilting, $H_C = H - H_0$, is chiral symmetric with $\{H_C,\Gamma\} = 0$. Now, following the case without tilting, let us assume that the $n = 0$ Landau state is $\Psi^m = |\chi_+\rangle\psi_+^m$. Then the Schrödinger equation, $H(m)\Psi^m = E\Psi^m$, implies

$$\begin{aligned}\bar{H}(m)H(m)\Psi^m &= (E^2 - 2EH_0)\Psi^m \\ &= c^2[\boldsymbol{\pi}^\dagger\Xi\boldsymbol{\pi} - (\hbar/\ell_B)^2\gamma + m^2c^2]\Psi^m,\end{aligned}$$

which leads to

$$c_r^2[\boldsymbol{\pi}_E^\dagger\Xi\boldsymbol{\pi}_E - (\hbar/\ell_B)^2 + m^2c^2]\psi_+^m = E^2\psi_+^m. \tag{13}$$

It is clear from this equation that the symmetry-breaking term $mc^2\Gamma$ indeed opens a gap $\pm mcc_r$ in the absence of a magnetic field. We can cast this into

$$c_r^2(D_E^\dagger D_E + m^2c^2)\psi_+^m = E^2\psi_+^m,$$

where

$$D_E = \Pi_{1,E} + i\Pi_{2,E}, \quad \boldsymbol{\Pi}_E = \begin{pmatrix}\sqrt{\xi_1} & 0 \\ 0 & \sqrt{\xi_2}\end{pmatrix} V_\Xi\boldsymbol{\pi}_E,$$

with ξ_1,ξ_2 given in Eq. (11). Then $D_E^\dagger D_E$ is semipositive definite, and the wave function in the $n = 0$ Landau level is specified by

$$D_E\psi_+^m = 0,$$

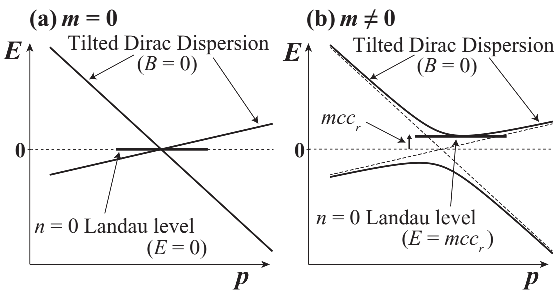

FIG. 5. The $n = 0$ Landau level (horizontal lines) for tilted Dirac fermions. (a) While its energy is zero for the massless ($m = 0$) case, (b) it is shifted to $+mcc_r$ in the presence of the mass term $mc^2\Gamma$ ($m \neq 0$) when $eB > 0$.

which has the energy

$$E = mcc_r = mc^2/\cosh q,$$

where we have used Eq. (8) and chosen the positive sign for the energy since it should tend to $+mc^2$ when the tilting becomes zero (see Fig. 5). As long as the velocity c is real, D and D_E (also $\boldsymbol{\pi}$ and $\boldsymbol{\pi}_E$) are simply related through a shift in the momentum by $\Delta \boldsymbol{p}_E$, which indicates ψ_+ and ψ_+^m are also related via a gauge transformation,

$$\psi_+^m = e^{-i\Delta \boldsymbol{p}_E \cdot \boldsymbol{r}/\hbar}\psi_+. \quad (14)$$

Since $[D_E, D_E^\dagger] = -2i[\Pi_1, \Pi_2] = 2eB\hbar$, we have Landau levels for generic Dirac fermions with mass and tilting as

$$E_n = \begin{cases} mcc_r, & n = 0, \\ \pm c_r\sqrt{2eB\hbar n + m^2c^2}, & n = 1,2,\ldots. \end{cases}$$

This is a condensed-matter realization of the anomaly [47,48]. As for a Dirac cone of a specific lattice model, the chiral operator Γ and the mass term are determined in a model-dependent way for each valley. See, for example, Eq. (2). This implies the $n = 0$ Landau level is not valley degenerate in two-dimensional semiconductors with a small gap. To grasp the role of the generalized chiral operator and relation (12) more explicitly, let us write the wave function ψ in the chiral basis as $\Psi^m = |\chi_+\rangle\psi_+^m + |\chi_-\rangle\psi_-^m$. Then the Schrödinger equation, $H(m)\Psi^m = E\Psi^m$, becomes

$$\begin{pmatrix} \langle\chi_+|mc^2\Gamma|\chi_+\rangle & \langle\chi_+|H|\chi_-\rangle \\ \langle\chi_-|H|\chi_+\rangle & \langle\chi_-|mc^2\Gamma|\chi_-\rangle \end{pmatrix} \begin{pmatrix} \psi_+^m \\ \psi_-^m \end{pmatrix} = E \begin{pmatrix} 1 & \beta \\ \beta^* & 1 \end{pmatrix} \begin{pmatrix} \psi_+^m \\ \psi_-^m \end{pmatrix},$$

where $\beta = \langle\chi_+|\chi_-\rangle$. Due to the generalized chiral symmetry, H appears only in the off-diagonal elements, while relation (12) guarantees that Γ appears only in the diagonal elements. From the explicit form of the matrix elements for Γ, the equation is simplified to

$$\begin{pmatrix} mcc_r & \boldsymbol{\alpha}\cdot\boldsymbol{\pi}_E \\ \boldsymbol{\alpha}^*\cdot\boldsymbol{\pi}_E & -mcc_r \end{pmatrix} \begin{pmatrix} \psi_+^m \\ \psi_-^m \end{pmatrix} = E \begin{pmatrix} \psi_+^m \\ \psi_-^m \end{pmatrix}, \quad (15)$$

with

$${}^t\boldsymbol{\alpha} \equiv (\alpha_X, \alpha_Y) = \hbar^{-1}(\langle\chi_+|X^\mu\sigma_\mu|\chi_-\rangle, \langle\chi_+|Y^\mu\sigma_\mu|\chi_-\rangle)$$

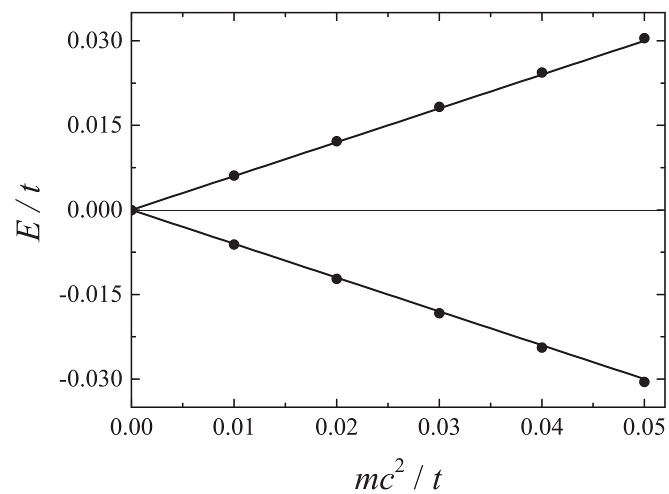

FIG. 6. The energies for the split $n = 0$ Landau levels as a function of the mass term mc^2 for the tight-binding lattice model, Eq. (1) (solid circles). Solid lines represent the energies expected from the effective theory, $\pm mcc_r = \pm mc^2\sqrt{1 - 4(t'/t)^2}$ ($= \pm 0.6mc^2$ here).

(see Appendix F). Then we find that the normalizable wave functions for $eB > 0$ at $E = \pm mcc_r$ should have [28]

$$\psi_-^m = 0, \quad \boldsymbol{\alpha}^*\cdot\boldsymbol{\pi}_E\psi_+^m = 0,$$

which indicates that the eigenstates at the bottom of the upper band ($E = mcc_r$) are indeed the eigenstate of γ with an eigenvalue of $+1$ ($\Psi^m = |\chi_+\rangle\psi_+^m$) because $\psi_-^m = 0$. Namely, the generalized chiral operator continues to commute with the Hamiltonian, within the $n = 0$ Landau subspace, even for massive Dirac fermions.

In other words, for tilted Dirac fermions, the wave functions of the $n = 0$ Landau levels for massless fermions ($m = 0$) and those for massive fermions ($m \neq 0$) are related through the gauge transformation (14). We can therefore conclude that the robustness of the $n = 0$ Landau level at $E = 0$ against disorder that respects the generalized chiral symmetry persists in the cases where its energy is shifted to $E = mcc_r$ by the mass term $mc^2\Gamma$.

In the tight-binding lattice model discussed in Sec. II, we have two valleys, for which the signs of the symmetry-breaking term $mc^2\Gamma$ are opposite. The signs of the energy shift are therefore opposite for these two Dirac cones in the lattice model, which is actually seen as the split zero modes shown in Fig. 1. We show in Fig. 6 the energies of the split $n = 0$ Landau levels obtained for the tight-binding lattice model (1) as a function of mc^2. They excellently agree with the analytical formula $\pm mcc_r$ given by the effective theory.

V. SUMMARY

We have investigated the robustness of the zero modes for massive and tilted Dirac fermions in a magnetic field. It is demonstrated numerically that the anomalous robustness of zero modes against disorder in gauge degrees of freedom is preserved for massive and tilted Dirac fermions. Notably, for the massive fermions, the robustness appears even in the case of the short-range disorder, in contrast to the case of massless Dirac fermions. We have also presented a general formulation for the generic two-dimensional massless and massive Dirac fermions in which a simple algebraic transformation between the generalized chiral operator and the conventional chiral operator has been obtained. Based on the low-energy effective theory, we have explicitly discussed the applicability of the argument by Aharonov and Casher to show the robust zero modes of the massive and tilted Dirac fermions, where the wave function of the $n = 0$ Landau level for the massive case is related to that for the massless case through a gauge transformation. The present numerical and analytical results for tilted Dirac fermions, where the chiral symmetry and the sublattice symmetry are distinguished, clearly suggest that the generalized chiral symmetry, rather than the sublattice symmetry, is indeed a key ingredient for the robust zero modes for the generic Dirac fermions in two dimensions.

ACKNOWLEDGMENTS

We thank Y. Ono and T. Morimoto for useful discussions and comments. This work was supported in part by JSPS Grants No. 26247064 and No. 25610101 (Y.H.) and Grant No. 25107005 from MEXT (Y.H., H.A.).

APPENDIX A: FOUR-DIMENSIONAL NOTATION

Let us elaborate the general formulation for a four-dimensional real vector, $X = {}^t(X_0, X_1, X_2, X_3)$ with a metric $g = \mathrm{diag}\,(-1,1,1,1)$, which defines

$$\bar{X} = (X^0, X^1, X^2, X^3) = {}^tXg = (-X_0, X_1, X_2, X_3).$$

An inner product of the two four-vectors $\bar{X}$ and Y is expressed as

$$\bar{X}Y = \bar{Y}X = X_\mu Y^\mu = -X^0Y^0 + \boldsymbol{X}\cdot\boldsymbol{Y}.$$

For example, the norm of the four-vector $\bar{X}X$ is given as

$$\bar{X}X = X^\mu X_\mu = |\boldsymbol{X}|^2 - X_0^2.$$

Noting that for three-dimensional vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ one has $(\boldsymbol{X}\cdot\boldsymbol{\sigma})(\boldsymbol{Y}\cdot\boldsymbol{\sigma}) = (\boldsymbol{X}\cdot\boldsymbol{Y})\sigma_0 + i(\boldsymbol{X}\times\boldsymbol{Y})\cdot\boldsymbol{\sigma}$, we have a simple formula,

$$\begin{aligned}(\bar{X}^\mu\sigma_\mu)(\sigma_\mu Y^\mu) &= (-X^0\sigma_0 + \boldsymbol{X}\cdot\boldsymbol{\sigma})(Y^0\sigma_0 + \boldsymbol{Y}\cdot\boldsymbol{\sigma})\\ &= \bar{X}Y\sigma_0 + i\boldsymbol{n}(X,Y)\cdot\boldsymbol{\sigma},\end{aligned} \tag{A1}$$

where

$$\boldsymbol{n}(X,Y) = \boldsymbol{X}\times\boldsymbol{Y} + i\boldsymbol{\eta}(X,Y), \tag{A2}$$

$$\boldsymbol{\eta}(X,Y) = X^0\boldsymbol{Y} - \boldsymbol{X}Y^0. \tag{A3}$$

Note that $\bar{X}Y = X\bar{Y} \equiv X^\mu Y_\mu = -X^0Y^0 + \boldsymbol{X}\cdot\boldsymbol{Y}$ is symmetric, while $\boldsymbol{n}(X,Y)$ is antisymmetric when one exchanges X and Y.

Also, noting that

$$\begin{aligned}\det A^\mu\sigma_\mu &= \det\begin{pmatrix} A^0 + A^3 & A^1 - iA^2 \\ A^1 + iA^2 & A^0 - A^3\end{pmatrix}\\ &= (A^0)^2 - |\boldsymbol{A}|^2 = -\bar{A}A,\end{aligned}$$

we have, by defining $\sigma_{\boldsymbol{A}} \equiv \boldsymbol{A}\cdot\boldsymbol{\sigma}/|\boldsymbol{A}|$,

$$A^\mu\sigma_\mu = A^0\sigma_0 + \boldsymbol{A}\cdot\boldsymbol{\sigma} = \sqrt{\bar{A}A}e^{\phi_A\sigma_A}\sigma_{\boldsymbol{A}},$$

where $e^{\phi_A\sigma_A} = \sigma_0\cosh\phi_A + \sigma_{\boldsymbol{A}}\sinh\phi_A$ with $\cosh\phi_A = |\boldsymbol{A}|/\sqrt{\bar{A}A}$ and $\sinh\phi_A = A_0/\sqrt{\bar{A}A}$.

APPENDIX B: DETERMINANT OF Ξ

Let us evaluate here the determinant in the discussion as

$$\begin{aligned}(c\hbar)^4 &= \begin{vmatrix}\bar{X}X & \bar{X}Y \\ \bar{Y}X & \bar{Y}Y\end{vmatrix}\\ &= \big(|\boldsymbol{X}|^2 - X_0^2\big)\big(|\boldsymbol{Y}|^2 - Y_0^2\big) - (\boldsymbol{X}\cdot\boldsymbol{Y} - X_0Y_0)^2\\ &= |\boldsymbol{X}\times\boldsymbol{Y}|^2 - |X^0\boldsymbol{Y} - \boldsymbol{X}Y^0|^2\\ &= \mathrm{Re}\,(\boldsymbol{n}\cdot\boldsymbol{n}) = \boldsymbol{n}^2,\end{aligned}$$

where $\mathrm{Im}\,(\boldsymbol{n}\cdot\boldsymbol{n}) = 2(\boldsymbol{X}\times\boldsymbol{Y})\cdot(X^0\boldsymbol{Y} - \boldsymbol{X}Y^0) = 0$.

It is also evaluated by the expansion of the minors as

$$\begin{aligned}\begin{vmatrix}\bar{X}X & \bar{X}Y \\ \bar{Y}X & \bar{Y}Y\end{vmatrix} &= \det\left[\begin{pmatrix} -X^0 & X_x & X_y & X_z \\ -Y^0 & Y_x & Y_y & Y_z\end{pmatrix}\begin{pmatrix} X^0 & Y^0 \\ X_x & Y_x \\ X_y & Y_y \\ X_z & Y_z\end{pmatrix}\right]\\ &= \begin{vmatrix}-X^0 & X_x \\ -Y^0 & Y_x\end{vmatrix}\begin{vmatrix}X^0 & Y^0 \\ X_x & Y_x\end{vmatrix}\\ &\quad + \begin{vmatrix}-X^0 & X_y \\ -Y^0 & Y_y\end{vmatrix}\begin{vmatrix}X^0 & Y^0 \\ X_y & Y_y\end{vmatrix}\\ &\quad + \begin{vmatrix}-X^0 & X_z \\ -Y^0 & Y_z\end{vmatrix}\begin{vmatrix}X^0 & Y^0 \\ X_z & Y_z\end{vmatrix}\\ &\quad + \det\left[\begin{pmatrix} X_x & X_y & X_z \\ Y_x & Y_y & Y_z\end{pmatrix}\begin{pmatrix} X_x & Y_x \\ X_y & Y_y \\ X_z & Y_z\end{pmatrix}\right]\\ &= -(X^0Y_x - Y^0X_x)^2 - (X^0Y_y - Y^0X_y)^2\\ &\quad -(X^0Y_z - Y^0X_z)^2 + |\boldsymbol{X}\times\boldsymbol{Y}|^2\\ &= |\boldsymbol{X}\times\boldsymbol{Y}|^2 - |X^0\boldsymbol{Y} - \boldsymbol{X}Y^0|^2\\ &= \mathrm{Re}\,\boldsymbol{n}^2 = \boldsymbol{n}^2.\end{aligned}$$

APPENDIX C: COMPLETING THE SQUARE

Here let us show details for deriving Eq. (7) by completing the square. We start with

$$\begin{aligned}&(X^0, Y^0)\Xi^{-1}\begin{pmatrix}X^0 \\ Y^0\end{pmatrix}(\hbar c)^2\\ &= (X^0, Y^0)\begin{pmatrix}\bar{Y}Y & -\bar{X}Y \\ -\bar{Y}X & \bar{X}X\end{pmatrix}\begin{pmatrix}X^0 \\ Y^0\end{pmatrix}\\ &= |X^0\boldsymbol{Y} - Y^0\boldsymbol{X}|^2 = |\boldsymbol{\eta}(X,Y)|^2 = (\mathrm{Im}\,\boldsymbol{n})^2.\end{aligned}$$

Then we have

$$\begin{aligned}&c^2\boldsymbol{p}^\dagger\Xi\boldsymbol{p} + 2(E/\hbar)(X^0, Y^0)\boldsymbol{p}\\ &= \left[c\boldsymbol{p}^\dagger + \frac{E}{c\hbar}(X^0, Y^0)\Xi^{-1}\right]\Xi\left[c\boldsymbol{p} + \frac{E}{c\hbar}\Xi^{-1}\begin{pmatrix}X^0 \\ Y^0\end{pmatrix}\right]\\ &\quad - \frac{E^2}{(c\hbar)^2}(X^0, Y^0)\Xi^{-1}\begin{pmatrix}X^0 \\ Y^0\end{pmatrix}\\ &= c^2\boldsymbol{p}_E^\dagger\Xi\boldsymbol{p}_E^\dagger - \frac{(\mathrm{Im}\,\boldsymbol{n})^2}{(c\hbar)^4}E^2,\end{aligned}$$

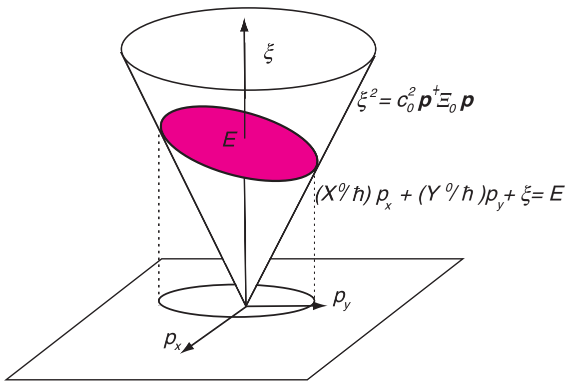

FIG. 7. (Color online) Geometrical meaning of the tilted Dirac cones in (p_x,p_y,ξ) space.

where

$$\boldsymbol{p}_E = \boldsymbol{p} + \Delta \boldsymbol{p}_E, \quad \Delta \boldsymbol{p}_E = E \frac{1}{c^2\hbar} \Xi^{-1} \begin{pmatrix} X^0 \\ Y^0 \end{pmatrix}.$$

Also note that

$$1 + \frac{(\mathrm{Im}\, \boldsymbol{n})^2}{(c\hbar)^4} = \frac{\boldsymbol{n}^2 + (\mathrm{Im}\, \boldsymbol{n})^2}{\boldsymbol{n}^2} = \frac{(\mathrm{Re}\, \boldsymbol{n})^2}{\mathrm{Re}\, (\boldsymbol{n}^2)}$$
$$= (\cosh q)^2.$$

APPENDIX D: THREE-DIMENSIONAL REPRESENTATION

In the main text, we have given a compact treatment of the tilted Dirac cone physics with a four-dimensional representation. Here let us show how a three-dimensional treatment is feasible but cumbersome. The Schrödinger equation for the two-component spinor Ψ is given as

$$H\Psi = [\hbar^{-1}\sigma_0(X^0,Y^0)\boldsymbol{p} + H_C^0]\Psi = E\Psi,$$

where ${}^t\boldsymbol{p} = (p_x,p_y)$ and $H_C^0 = \hbar^{-1}(\boldsymbol{X}\cdot\boldsymbol{\sigma},\boldsymbol{Y}\cdot\boldsymbol{\sigma})\boldsymbol{p}$. The equation is written as $H_C^0\Psi = (E-z)\Psi$, with

$$z = \hbar^{-1}(X^0,Y^0)\boldsymbol{p}.$$

Using it twice, one has

$$\left(H_C^0\right)^2\Psi = (E-z)^2\Psi.$$

Since $(H_C^0)^2 = [c_0^2\boldsymbol{p}^\dagger\Xi_0\boldsymbol{p}]\sigma_0 \propto \sigma_0$ [49], we have a scalar equation for Ψ,

$$\hbar^{-2}[(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{p}]^2 = c_0^2\boldsymbol{p}^\dagger\Xi_0\boldsymbol{p} = (E-z)^2, \tag{D1}$$

where [50]

$$\Xi_0 = \frac{1}{(\hbar c_0)^2}\begin{pmatrix} \boldsymbol{X}\cdot\boldsymbol{X} & \boldsymbol{X}\cdot\boldsymbol{Y} \\ \boldsymbol{X}\cdot\boldsymbol{Y} & \boldsymbol{Y}\cdot\boldsymbol{Y}\end{pmatrix}, \tag{D2}$$

$$c_0^2 = |\boldsymbol{X}\times\boldsymbol{Y}|/\hbar^2. \tag{D3}$$

The "light velocity" c_0 is chosen so that $\det\Xi_0 = 1$.

Geometrically (see Fig. 7), a constant-energy curve $E(p_x,p_y) = \mathrm{const}$ in (p_x,p_y,ξ) space is given by the intersection of the cone and the plane,

$$\xi^2 = c_0^2\boldsymbol{p}^\dagger\Xi_0\boldsymbol{p},$$
$$(X^0/\hbar)p_x + (Y^0/\hbar)p_y + \xi = E,$$

which can be a parabola, an ellipse, a hyperbola, or a point. Any intersection of the cone and the plane is an ellipse if the slope of the plane does not exceed that of the cone, which guarantees that the energy dispersion is given by the Dirac cone.

When the Dirac cone is not tilted, that is, $X^0 = Y^0 = 0$, the energy dispersion is given by $E = z$. Since Ξ_0 is a real symmetric matrix with $\mathrm{Tr}\,\Xi_0 > 0$, it is diagonalized by the orthogonal matrix V as

$$\Xi_0 = V^\dagger \mathrm{diag}\left(\xi_1^0,\xi_2^0\right)V,$$

where $\xi_1^0 > 0, \xi_2^0 > 0$ and $\xi_1^0\xi_2^0 = \det\Xi_0 = 1$. Now we have

$$E = \pm c_0\bar{P},$$

where $\bar{P} = \sqrt{\xi_1^0 P_1^2 + \xi_2^0 P_2^2}$, $\boldsymbol{P} = V\boldsymbol{p}$, and c_0 is the Dirac fermion velocity without tilting.

For the tilted case with finite X^0 and/or $Y^0 \neq 0$, we need to complete the square by rewriting Eq. (D1). Here let us complete the square in Eq. (D1). If we expand the right-hand side as

$$\begin{aligned} c_0^2\boldsymbol{p}^\dagger\Xi_0\boldsymbol{p} &= \hbar^{-2}\boldsymbol{p}^\dagger\begin{pmatrix}\boldsymbol{X}\\ \boldsymbol{Y}\end{pmatrix}(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{p} \\ &= \left\{E - \left[\left(\frac{X_0}{\hbar}\right)p_x + \left(\frac{Y_0}{\hbar}\right)p_y\right]\right\}^2 \\ &= \Bigg[E^2 - 2E\hbar^{-1}(X^0,Y^0)\boldsymbol{p} \\ &\quad + \hbar^{-2}\boldsymbol{p}^\dagger\begin{pmatrix}X^0\\ Y^0\end{pmatrix}(X^0,Y^0)\boldsymbol{p}\Bigg], \end{aligned}$$

we have

$$\left[c^2\boldsymbol{p}^\dagger\Xi\boldsymbol{p} + 2\left(\frac{E}{\hbar}\right)(X^0,Y^0)\boldsymbol{p}\right]\Psi = E^2\Psi,$$

where

$$\begin{aligned}\Xi &= \frac{1}{(\hbar c)^2}\begin{pmatrix} -X^0X^0 + \boldsymbol{X}\cdot\boldsymbol{X} & -X^0Y^0 + \boldsymbol{X}\cdot\boldsymbol{Y} \\ -X^0Y^0 + \boldsymbol{X}\cdot\boldsymbol{Y} & -Y^0Y^0 + \boldsymbol{Y}\cdot\boldsymbol{Y}\end{pmatrix} \\ &= \frac{1}{(\hbar c)^2}\begin{pmatrix}\bar{X}X & \bar{X}Y \\ \bar{X}Y & \bar{Y}Y\end{pmatrix}.\end{aligned}$$

The equation coincides with Eq. (6) in the four-dimensional notation in the text.

Although complicated, one can perform a similar process with a magnetic field as

$$\begin{aligned} H &= \hbar^{-1}[\sigma_\mu(X^\mu,Y^\mu)\boldsymbol{\pi}] = H_0 + H_C, \\ H_0 &= \hbar^{-1}\sigma_0(X^0,Y^0)\boldsymbol{\pi}, \\ H_C &= \hbar^{-1}(\boldsymbol{X}\cdot\boldsymbol{\sigma},\boldsymbol{Y}\cdot\boldsymbol{\sigma})\boldsymbol{\pi}, \end{aligned}$$

where $\boldsymbol{\pi} = \boldsymbol{p} - e\boldsymbol{A}$ is the dynamical momentum. The Schrödinger equation reads

$$H_C\Psi = (E - Z)\Psi,$$

with $Z = (X^0/\hbar)\pi_x + (Y^0/\hbar)\pi_y$. Using the relations

$$\begin{aligned}[H_C, Z] &= [\hbar^{-1}[(\boldsymbol{X}\cdot\boldsymbol{\sigma})\pi_x + (\boldsymbol{Y}\cdot\boldsymbol{\sigma})\pi_y], Z] \\ &= \hbar^{-2}[(Y^0\boldsymbol{X} - X^0\boldsymbol{Y})\cdot\boldsymbol{\sigma}][\pi_x,\pi_y] \\ &= -i\ell_B^{-2}(\mathrm{Im}\,\boldsymbol{n})\cdot\boldsymbol{\sigma}\end{aligned}$$

and

$$\begin{aligned}H_C^2 &= \hbar^{-2}[(\boldsymbol{X}\cdot\boldsymbol{\sigma})\pi_x + (\boldsymbol{Y}\cdot\boldsymbol{\sigma})\pi_y]^2 \\ &= \left(\boldsymbol{\pi}^\dagger c_0^2 \Xi_0 \boldsymbol{\pi}\right)\sigma_0 + i\hbar^{-2}(\boldsymbol{X}\times\boldsymbol{Y})\cdot\boldsymbol{\sigma}[\pi_x,\pi_y] \\ &= \left(\boldsymbol{\pi}^\dagger c_0^2 \Xi_0 \boldsymbol{\pi}\right)\sigma_0 - \ell_B^{-2}(\mathrm{Re}\,\boldsymbol{n})\cdot\boldsymbol{\sigma},\end{aligned}$$

we have

$$\begin{aligned}H_C^2\Psi &= H_C\,[(E-Z)\Psi] \\ &= \{(E-Z)H_C + [H_C, E-Z]\}\,\Psi \\ &= \left[(E-Z)^2 + i\ell_B^{-2}(\mathrm{Im}\,\boldsymbol{n})\cdot\boldsymbol{\sigma}\right]\Psi.\end{aligned}$$

This implies

$$\left[\left(\boldsymbol{\pi}^\dagger c_0^2 \Xi_0 \boldsymbol{\pi}\right)\sigma_0 - \ell_B^{-2}\boldsymbol{n}\cdot\boldsymbol{\sigma}\right]\Psi = (E-Z)^2\Psi.$$

Similar to the case without magnetic field, one has

$$[c^2\boldsymbol{\pi}^\dagger\Xi\boldsymbol{\pi} + 2(E/\hbar)(X_0,Y_0)\boldsymbol{\pi}]\sigma^0\Psi - \ell_B^{-2}\boldsymbol{n}\cdot\boldsymbol{\sigma}\Psi = E^2\Psi,$$

which coincides with Eq. (9) in the text.

APPENDIX E: LANDAU LEVELS FOR AN ANISOTROPIC MASS

Let us summarize the standard Landau quantization of electrons with parabolic dispersion with anisotropic masses (effective mass approximation) described by the following Hamiltonian:

$$H = \boldsymbol{\pi}^\dagger \frac{1}{2m^*}\Xi_L\boldsymbol{\pi},$$

with $\boldsymbol{\pi} = \boldsymbol{p} - e\boldsymbol{A} = \boldsymbol{\pi}^\dagger$, rot $\boldsymbol{A} = B\hat{z}$, and

$$\Xi_L = \begin{pmatrix}\xi_x & \xi_{xy}\\ \xi_{xy} & \xi_y\end{pmatrix},$$

where

$$\left(\frac{\ell_B}{\hbar}\right)^2[\pi_x,\pi_y] = i,\ \ \ell_B = \sqrt{\frac{\hbar}{eB}}.$$

Here we have assumed $eB > 0$ without loss of generality. Since the matrix Ξ_L is real symmetric, it is diagonalized by the orthogonal matrix V as

$$\begin{aligned}&\Xi_L = V^\dagger\Xi_D V,\\ &\Xi_D = \mathrm{diag}(\xi_X,\xi_Y),\quad \xi_X\xi_Y = \det\Xi_L,\quad \xi_X + \xi_Y = \mathrm{Tr}\,\Xi_L,\\ &V = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix},\quad {}^\exists\theta\in\mathbb{R}.\end{aligned}$$

Then we have

$$H = \boldsymbol{\Pi}^\dagger\Xi_D\boldsymbol{\Pi},\quad \boldsymbol{\Pi}\equiv V\boldsymbol{\pi},\quad \left(\frac{\ell_B}{\hbar}\right)^2[\Pi_X,\Pi_Y] = i.$$

Now defining a bosonic operator (with $[a,a^\dagger] = 1$),

$$a = \frac{1}{\sqrt{2}}\frac{\ell_B}{\hbar}(\Pi_X + i\Pi_Y),$$

the Hamiltonian is written as

$$H = \frac{\hbar\omega}{4}[\xi_X(a+a^\dagger)^2 - \xi_Y(a-a^\dagger)^2],$$

with $\omega = eB/m^*$.

Now we define a new bosonic operator ($[b,b^\dagger] = 1$) as

$$a = ub + v^*b^\dagger,\quad a^\dagger = u^*b^\dagger + vb,$$

with $\quad [a,a^\dagger] = [ub + v^*b^\dagger, u^*b^\dagger + vb] = |u|^2 - |v|^2 = 1.$ Here we choose

$$\begin{aligned}&\xi_X(u+v)^2 = \xi_Y(u-v)^2,\quad u+v = C\sqrt{\xi_Y},\\ &u - v = -C\sqrt{\xi_X}.\end{aligned}$$

Assuming $\xi_X,\xi_Y > 0$ and imposing $|u|^2 - |v|^2 = 1$, we have $|C|^2 = 1/\sqrt{\xi_X\xi_Y} = 1/(\det\Xi_L)^{1/2}$ and therefore arrive at

$$u = \frac{\sqrt{\xi_X} + \sqrt{\xi_Y}}{2(\det\Xi_L)^{1/4}},\quad v = \frac{-\sqrt{\xi_X} + \sqrt{\xi_Y}}{2(\det\Xi_L)^{1/4}}.$$

Finally, the Hamiltonian is written as

$$\begin{aligned}&H = \frac{1}{2}\hbar\omega(bb^\dagger + b^\dagger b)|C|^2(\xi_X\xi_Y) = \hbar\omega_\Xi\left(b^\dagger b + \frac{1}{2}\right),\\ &\omega_\Xi = \omega\sqrt{\det\Xi_L} = \frac{eB}{m^*}\sqrt{\det\Xi_L} = \frac{eB}{m^*}\sqrt{\xi_X\xi_Y}.\end{aligned}$$

APPENDIX F: DERIVATION OF EQ. (15)

The equation above Eq. (15) can be expressed, by introducing a dynamical momentum $\boldsymbol{\pi}'_E = \boldsymbol{\pi} + \Delta\boldsymbol{p}'_E$ in terms of a real vector $\Delta\boldsymbol{p}'_E$ satisfying the relation $\boldsymbol{\alpha}\cdot\Delta\boldsymbol{p}'_E = -E\beta$, as

$$\begin{pmatrix}mcc_r & \boldsymbol{\alpha}\cdot\boldsymbol{\pi}'_E\\ \boldsymbol{\alpha}^*\cdot\boldsymbol{\pi}'_E & -mcc_r\end{pmatrix}\begin{pmatrix}\psi_+^m\\ \psi_-^m\end{pmatrix} = E\begin{pmatrix}\psi_+^m\\ \psi_-^m\end{pmatrix}.$$

We can show that $\Delta\boldsymbol{p}'_E = \Delta\boldsymbol{p}_E$ by multiplying the matrix on the left-hand side of the equation once again to get

$$\left[\mathrm{Im}(\alpha_X\alpha_Y^*)\left(\boldsymbol{\pi}'^\dagger_E\Xi'\boldsymbol{\pi}'_E \mp \hbar^2/\ell_B^2\right) + (mcc_r)^2\right]\psi_\pm^m = E^2\psi_\pm^m,$$

with

$$\Xi' = \frac{1}{\mathrm{Im}(\alpha_X\alpha_Y^*)}\begin{pmatrix}|\alpha_X|^2 & \mathrm{Re}(\alpha_X\alpha_Y^*)\\ \mathrm{Re}(\alpha_X\alpha_Y^*) & |\alpha_Y|^2\end{pmatrix}.$$

Comparing this with Eq. (13), we can see that $\Xi = \Xi'$, $\Delta\boldsymbol{p}_E = \Delta\boldsymbol{p}'_E$, and $c_r^2 = \mathrm{Im}(\alpha_X\alpha_Y^*)$.

[1] K. S. Novoselov, A. K. Geim, S. V. Morozov, D. Jiang, M. I. Katsnelson, I. V. Grigorieva, S. V. Dubonos, and A. A. Firsov, Nature (London) **438**, 197 (2005).
[2] *Physics of Graphene*, edited by H. Aoki and M. S. Dresselhaus (Springer, Heidelberg and New York, 2014).
[3] W. A. Harrison, *Electronic Structure and Properties of Solids* (Dover, New York, 1989).
[4] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 146802 (2005).
[5] J. E. Moore, Nature (London) **464**, 194 (2010).
[6] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010).
[7] X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. **83**, 1057 (2011).
[8] B. A. Bernevig, T. L. Hughes, and S. C. Zhang, Science **314**, 1757 (2006).
[9] Y. Hatsugai, M. Kohmoto, and Y.-S. Wu, Phys. Rev. B **54**, 4898 (1996).
[10] S. Katayama, A. Kobayashi, and Y. Suzumura, J. Phys. Soc. Jpn. **75**, 054705 (2006).
[11] A. Kobayashi, S. Katayama, Y. Suzumura, and H. Fukuyama, J. Phys. Soc. Jpn. **76**, 034711 (2007).
[12] K. Kajita, Y. Nishio, N. Tajima, Y. Suzumura, and A. Kobayashi, J. Phys. Soc. Jpn. **83**, 072002 (2014).
[13] P. A. Lee, Phys. Rev. Lett. **71**, 1887 (1993).
[14] Y. Hatsugai and P. A. Lee, Phys. Rev. B **48**, 4204 (1993).
[15] M. V. Berry, Proc. R. Soc. London, Ser. A **392**, 45 (1984).
[16] C. Herring, Phys. Rev. **52**, 365 (1937).
[17] Y. Hatsugai, New J. Phys. **12**, 065004 (2010).
[18] Y. Hatsugai and H. Aoki, in *Physics of Graphene*, edited by H. Aoki and M. S. Dresselhaus (Springer, Heidelberg and New York, 2014), p. 213.
[19] H. B. Nielsen and M. Ninomiya, Nucl. Phys. B **185**, 20 (1981).
[20] Y. Hatsugai, T. Fukui, and H. Aoki, Eur. Phys. J. Spec. Top. **148**, 133 (2007).
[21] Y. Hatsugai, Solid State Commun. **149**, 1061 (2009).
[22] E. Witten, Nucl. Phys. B **202**, 253 (1982).
[23] M. Ezawa, Phys. E (Amsterdam, Neth.) **40**, 269 (2007).
[24] J. Kailasvuori, Europhys. Lett. **87**, 47008 (2009).
[25] Y. Hatsugai, T. Fukui, and H. Aoki, Phys. Rev. B **74**, 205414 (2006).
[26] T. Kawarabayashi, Y. Hatsugai, and H. Aoki, Phys. Rev. Lett. **103**, 156804 (2009).
[27] S. Ryu and Y. Hatsugai, Phys. Rev. Lett. **89**, 077002 (2002).
[28] T. Kawarabayashi, Y. Hatsugai, T. Morimoto, and H. Aoki, Phys. Rev. B **83**, 153414 (2011).
[29] R. Suzuki, M. Sakano, Y. J. Zhang, R. Akashi, D. Morikawa, A. Harasawa, K. Yaji, K. Kuroda, K. Miyamoto, T. Okuda, K. Ishizaka, R. Arita, and Y. Iwasa, Nat. Nanotechnol. **9**, 611 (2014).
[30] A. Kobayashi, S. Katayama, and Y. Suzumura, J. Phys. Soc. Jpn. **74**, 2897 (2005).
[31] T. Morinari and Y. Suzumura, J. Phys. Soc. Jpn. **83**, 094701 (2014).
[32] H. Watanabe, Y. Hatsugai, and H. Aoki, Phys. Rev. B **82**, 241403 (2010).
[33] L. Tarruell, D. Greif, T. Uehlinger, G. Jotzu, and T. Esslinger, Nature (London) **483**, 302 (2012).
[34] N. Goldman, A. Kubasiak, A. Bermudez, P. Gaspard, M. Lewenstein, and M. A. Martin-Delgado, Phys. Rev. Lett. **103**, 035301 (2009).
[35] F. Mei, S.-L. Zhu, X.-L. Feng, Z.-M. Zhang, and C. H. Oh, Phys. Rev. A **84**, 023622 (2011).
[36] T. Kawarabayashi, T. Honda, H. Aoki, and Y. Hatsugai, in *The Physics of Semiconductors: Proceedings of the 31st International Conference on the Physics of Semiconductors (ICPS) 2012*, AIP Conf. Proc. No. 1566 (AIP, New York, 2013), p. 283.
[37] Y. Hatsugai and M. Kohmoto, Phys. Rev. B **42**, 8282 (1990).
[38] Y. Hatsugai, J. Phys. Conf. Series **334**, 012004 (2011).
[39] T. Kawarabayashi, Y. Hatsugai, T. Morimoto, and H. Aoki, Int. J. Mod. Phys. Conf. Series **11**, 145 (2012).
[40] M. O. Goerbig, J.-N. Fuchs, G. Montambaux, and F. Piéchon, Phys. Rev. B **78**, 045415 (2008).
[41] T. Morinari, T. Himura, and T. Tohyama, J. Phys. Soc. Jpn. **78**, 023704 (2009).
[42] T. Morinari and T. Tohyama, J. Phys. Soc. Jpn. **79**, 044708 (2010).
[43] Y. Aharonov and A. Casher, Phys. Rev. A **19**, 2461 (1979).
[44] A. W. W. Ludwig, M. P. A. Fisher, R. Shankar, and G. Grinstein, Phys. Rev. B **50**, 7526 (1994).
[45] F. D. M. Haldane, Phys. Rev. Lett. **61**, 2015 (1988).
[46] J. N. Fuchs, F. Piéchon, M. O. Goerbig, and G. Montambaux, Eur. Phys. J. B **77**, 351 (2010).
[47] G. W. Semenoff, Phys. Rev. Lett. **53**, 2449 (1984).
[48] R. Jackiw, Phys. Rev. D **29**, 2375 (1984).
[49] Using a well-known formula, we have $(H_C^0)^2 = \hbar^{-2}[\boldsymbol{\sigma} \cdot (\boldsymbol{X},\boldsymbol{Y})\boldsymbol{p}]^2 = \hbar^{-2}[(\boldsymbol{\sigma} \cdot \boldsymbol{X})p_x + (\boldsymbol{\sigma} \cdot \boldsymbol{Y})p_y]^2 = \hbar^{-2}(\boldsymbol{X} \cdot \boldsymbol{X} p_x^2 + 2\boldsymbol{X} \cdot \boldsymbol{Y} p_x p_y + \boldsymbol{Y} \cdot \boldsymbol{Y} p_y^2)\sigma_0 = |(\boldsymbol{X},\boldsymbol{Y})\boldsymbol{p}|^2\sigma_0 = [c_0^2 \boldsymbol{p}^\dagger \Xi_0 \boldsymbol{p}] + \sigma_0$.
[50] Here $\det \Xi_0 = \frac{1}{(c_0\hbar)^4}[|\boldsymbol{X}|^2|\boldsymbol{Y}|^2 - (\boldsymbol{X} \cdot \boldsymbol{Y})^2] = \frac{1}{(c_0\hbar)^4}|\boldsymbol{X} \times \boldsymbol{Y}|^2 = 1$.