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On a class of hypoelliptic differential operators with double characteristics

Dedicated to Professor Mutsuhide Matsumura on his 60th birthday in 1991

By Kazuaki TAIRA

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Introduction and results.

This paper is devoted to the study of hypoellipticity for second-order degenerate elliptic differential operators $P(x, D)$ with real coefficients on $\mathbb{R}^n$ of the form:

$$P(x, D) = \frac{\partial^2}{\partial x_1^2} + \sum_{i, j=2}^{n} \frac{\partial}{\partial x_i} (a^{ij}(x) \frac{\partial}{\partial x_j}) + \sum_{i=1}^{n} b^i(x) \frac{\partial}{\partial x_i} + c(x),$$

where:

1) The $a^{ij}$ are the components of a $C^\infty$ symmetric contravariant tensor of type $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ on $\mathbb{R}^n$, and

$$\sum_{i, j=2}^{n} a^{ij}(x) \xi_i \xi_j \geq 0 \quad \text{on} \quad T^*(\mathbb{R}^n).$$

Here $T^*(\mathbb{R}^n)$ is the cotangent bundle of $\mathbb{R}^n$.

2) $b^i \in C^\infty(\mathbb{R}^n)$.

3) $c \in C^\infty(\mathbb{R}^n)$.

Let $u$ be a distribution on an open subset $\Omega$ of $\mathbb{R}^n$. The singular support of $u$, denoted by $\text{sing supp } u$, is the complement of the largest open subset of $\Omega$ on which $u$ is of class $C^\infty$. A differential operator $P(x, D)$ is said to be hypoelliptic in $\Omega$ if it satisfies the condition:

$$\text{sing supp } u = \text{sing supp } Pu \quad \text{for all} \quad u \in \mathcal{D}'(\Omega).$$

This condition is equivalent to the following:

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K. TAIRA

\[
\begin{cases}
\text{For any open subset } \Omega' \text{ of } \Omega, \text{ we have } \\
u \in \mathcal{D}'(\Omega), \ Pu \in C^\infty(\Omega') \implies u \in C^\infty(\Omega').
\end{cases}
\]

We say that \(P(x, D)\) is \textit{globally hypoelliptic} in \(\Omega\) if it satisfies the weaker condition:

\[
u \in \mathcal{D}'(\Omega), \ Pu \in C^\infty(\Omega) \implies u \in C^\infty(\Omega).
\]

To state our fundamental hypothesis for the operator \(P(x, D)\), we let

\[
\Phi = \frac{\partial}{\partial x_1} \otimes \frac{\partial}{\partial x_1} + \sum_{i,j=2}^{n} a^{ij}(x) \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j},
\]

which lies in the space \(\Gamma(\mathbb{R}^n, \mathcal{T}(\mathbb{R}^n) \otimes \mathcal{T}(\mathbb{R}^n))\) of \(C^\infty\) symmetric contravariant tensor fields of type \((\begin{array}{l}20\end{array})\) on \(\mathbb{R}^n\). Here the notation \(\otimes_s\) stands for the symmetric tensor product:

\[
\frac{\partial}{\partial x_i} \otimes_s \frac{\partial}{\partial x_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_j} \otimes \frac{\partial}{\partial x_i} \right).
\]

Denote by \(\Gamma(\mathbb{R}^n, \mathcal{T}^*(\mathbb{R}^n))\) (resp. \(\Gamma(\mathbb{R}^n, \mathcal{T}(\mathbb{R}^n))\)) the space of \(C^\infty\) covariant (resp. contravariant) vector fields on \(\mathbb{R}^n\). Then, making use of \(\Phi\), we can define a mapping

\[
\Psi : \Gamma(\mathbb{R}^n, \mathcal{T}^*(\mathbb{R}^n)) \longrightarrow \Gamma(\mathbb{R}^n, \mathcal{T}(\mathbb{R}^n))
\]

\[
\zeta \longmapsto \Phi(\zeta, \cdot).
\]

In terms of local coordinates \(x=(x_1, x_2, \cdots, x_n)\), we have for \(\zeta = \sum_{i=1}^{n} \zeta_i dx_i\)

\[
\Psi(\zeta) = \zeta_1 \frac{\partial}{\partial x_1} + \sum_{i,j=2}^{n} a^{ij}(x) \zeta_i \frac{\partial}{\partial x_j}.
\]

We let

\[
X_1 = \text{the image of } \Psi
\]

\[
= \{\Psi(\zeta) ; \zeta \in \Gamma(\mathbb{R}^n, \mathcal{T}^*(\mathbb{R}^n))\}.
\]

Further we define the \textit{drift vector field} \(X_0\) by

\[
X_0 = \sum_{i=2}^{n} b^i(x) \frac{\partial}{\partial x_i}.
\]

The fundamental hypothesis for the operator \(P(x, D)\) is the following:

(H) \text{The Lie algebra } \mathcal{L}(X) \text{ over } \mathbb{R} \text{ generated by } X=X_1 \cup X_0 \text{ has rank } n \text{ outside a closed subset } S \text{ of the hypersurface } \{x=(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n ; x_1=0\}.

By the celebrated theorem of Hörmander (\cite[Theorem 1.1]{Hr1}), we know that the operator \(P(x, D)\) is hypoelliptic outside the set \(S\). Furthermore, Oleinik and Radkevič proved (cf. \cite[Theorem 2.6.3]{OR}; \cite[Theorem 3]{A}) that:
If the set $S$ is compact in $\Omega$, then the operator $P(x, D)$ is globally hypoelliptic in $\Omega$.

The purpose of this paper is to give sufficient conditions for hypoeptilicity for the operator $P(x, D)$ under condition (H). Some previous results in this direction are due to Fedii [F], Kusuoka-Stroock [KS], Morimoto [Mo], Hoshiro [Ho] and also Morioka [Ma]. The results here extend and improve substantially those results in a unified theory.

To state hypotheses for the $a^{ij}$, we let
\[
\alpha(x, \xi') = \alpha(x_{1}, x', \xi') = \sum_{i,j=1}^{n} a^{ij}(x, x') \xi_{i} \xi_{j},
\]
where
\[
x' = (x_{2}, \ldots, x_{n}), \quad \xi' = (\xi_{2}, \ldots, \xi_{n}),
\]
and the variable $x_{1}$ is considered as a parameter.

For the $a^{ij}$, we assume that:
(A.1) There exists a constant $a_{0}>0$ such that
\[
\sum_{i,j=2}^{n} \left| \frac{\partial^{2} \alpha}{\partial x_{i} \partial \xi_{j}}(x_{1}, x', \xi') \right|^{2} \leq a_{0} \alpha(x_{1}, x', \xi') \text{ on } T^{*}(R^{n-1}).
\]
This condition is satisfied if $\alpha(x, \xi')$ is diagonal, that is, if $a^{ij}(x) = 0$ for $i \neq j$.

(A.2) The function
\[
\mu(x) = \mu(x_{1}, x') = \min_{i' \rightarrow 1} \alpha(x_{1}, x', \xi')
\]
is Lipschitz continuous in the variable $x_{1}$ and is of class $C^{\infty}$ in the variables $x'$, and satisfies the condition:
\[
\mu(x_{1}, x') > 0 \quad \text{outside the set } S.
\]
We remark that condition (A.2) implies that the operator $P(x, D)$ is elliptic outside the set $S$, so condition (H) is satisfied.

For the $b^{i}$, we assume that:
(B) There exists a constant $b_{0}>0$ such that
\[
\sum_{i=2}^{n} |b^{i}(x)| \leq b_{0} \sqrt{\mu(x)} \quad \text{on } R^{n}.
\]

Now we can state our main result (cf. [WS, Theorem 4.9]):

**Theorem 1.** Assume that conditions (A.1), (A.2) and (B) are satisfied and that
\[
(0.1) \quad \lim_{x_{1} \rightarrow 0} \frac{\tilde{\lambda}(x_{1}, x') \log \mu(x_{1}, x')}{\sqrt{\lambda(x_{1}, x')}} = 0
\]
uniformly in the variables $x' = (x_{2}, \ldots, x_{n})$ over compact subsets of $R^{n-1}$ which
intersect the set $S$, where

$$\lambda(x_1, x') = \sum_{i=1}^{n} a^{i}(x_1, x'),$$

$$\tilde{\lambda}(x_1, x') = \int_{0}^{x_1} \lambda(t, x')dt .$$

Then the operator $P(x, D)$ is hypoelliptic in $\mathbb{R}^n$, that is,

$$\text{sing supp } Pu = \text{sing supp } u \quad \text{for all } u \in \mathcal{D}'(\mathbb{R}^n).$$

**Remark.** If the function $\lambda(x_1, x')$ is monotone increasing for $x_1 > 0$ and is monotone decreasing for $x_1 < 0$, that is, if we have

$$x_1 \lambda_x(x_1, x') \geq 0 \quad \text{on } \mathbb{R}^n,$$

then the above condition (0.1) may be replaced by the following simpler one:

(0.1')

$$\lim_{x_1 \to 0} \sqrt{\lambda(x_1, x')} x_1 \log \mu(x_1, x') = 0 .$$

In fact, it suffices to note that we have

$$|\tilde{\lambda}(x_1, x')| = \left| \int_{0}^{x_1} \lambda(t, x')dt \right| \leq \lambda(x_1, x') |x_1| \quad \text{on } \mathbb{R}^n .$$

Thus Theorem 1 is a generalization of Theorem 4 of Hoshiro [Ho2].

**Example 1.** Consider the following operator $P(x, D)$ on $\mathbb{R}^3$:

$$P(x, D) = \frac{\partial^2}{\partial x_1^2} + \frac{\partial}{\partial x_2} (f(x) \frac{\partial}{\partial x_2}) + \frac{\partial}{\partial x_3} (g(x) \frac{\partial}{\partial x_3}) ,$$

where $f$ and $g$ are non-negative functions on $\mathbb{R}^3$ such that

$$f(x_1, x_2, x_3) > 0 \quad \text{for } x_1 \neq 0 ,$$

$$g(x_1, x_2, x_3) > 0 \quad \text{for } x_1 \neq 0 .$$

Then the operator $P(x, D)$ is hypoelliptic in $\mathbb{R}^3$ if the following two conditions are satisfied:

$$\lim_{\substack{x_1 \to 0}} \frac{\int_{0}^{x_1} f(t, x')dt \log g(x_1, x')}{\sqrt{f(x_1, x')}} = 0 .$$

$$\lim_{\substack{x_1 \to 0}} \frac{\int_{0}^{x_1} g(t, x')dt \log f(x_1, x')}{\sqrt{g(x_1, x')}} = 0 .$$

Here the convergence is uniform in the variables $x' = (x_2, x_3)$ over compact subsets of $\mathbb{R}^3$. 
Hypoelliptic differential operators

Our method can be applied to the study of hypoellipticity for second-order degenerate parabolic differential operators $Q(x, D)$ with real coefficients on $\mathbb{R}^n$ of the form:

$$Q(x, D) = \frac{\partial^2}{\partial x_1^2} + \sum_{i, j=2}^{n-1} \frac{\partial}{\partial x_i} \left( a^{ij}(x) \frac{\partial}{\partial x_j} \right) + b^n(x) \frac{\partial}{\partial x_n} + c(x),$$

where:

1) The $a^{ij}$ are the components of a $C^\infty$ symmetric contravariant tensor of type $(\begin{array}{l}20\end{array})$ on $\mathbb{R}^n$, and

$$\sum_{i, j=2}^{n-1} a^{ij}(x) \xi_i \xi_j \geq 0 \quad \text{on} \quad T^*(\mathbb{R}^n).$$

2) $b^n \in C^\infty(\mathbb{R}^n)$.

3) $c \in C^\infty(\mathbb{R}^n)$.

The next result is due to Oleinik and Radkevič (cf. [OR, Theorem 2.6.3]; [A, Theorem 2]):

If condition (H) is satisfied for $Q(x, D)$ and the set $S$ is compact in $\Omega$, and if $b^n(x) \neq 0$ on $S$, then the operator $Q(x, D)$ is globally hypoelliptic in $\Omega$.

Now we let

$$\alpha(x, \xi^\nu) = \alpha(x_1, x', \xi^\nu) = \sum_{i, j=2}^{n-1} a^{ij}(x_1, x') \xi_i \xi_j,$$

where

$$\xi^\nu = (\xi_2, \cdots, \xi_{n-1}).$$

For the $a^{ij}$, we assume that:

(\text{A.1'}) There exists a constant $a_0 > 0$ such that

$$\sum_{i, j=2}^{n-1} \left| \frac{\partial^2 \alpha}{\partial x_i \partial \xi_j}(x_1, x', \xi^\nu) \right|^2 \leq a_0 \alpha(x_1, x', \xi^\nu) \quad \text{on} \quad T^*(\mathbb{R}^{n-1}).$$

This condition is satisfied if $\alpha(x, \xi^\nu)$ is diagonal, that is, if $a^{ij}(x) = 0$ for $i \neq j$.

(\text{A.2'}) The function

$$\mu(x) = \mu(x_1, x') = \min_{i, j=1} \alpha(x_1, x', \xi^\nu)$$

is Lipschitz continuous in the variable $x_1$ and is of class $C^\infty$ in the variables $x'$, and satisfies the condition:

$$\mu(x_1, x') > 0 \quad \text{outside the set} \quad S.$$

For the function $b^n$, we assume that:

(\text{B'}) $b^n(x_1, x') \neq 0$ outside the set $S$, and either $b^n(x) \geq 0$ on $\mathbb{R}^n$ or $b^n(x) \leq 0$ on $\mathbb{R}^n$.

We remark that conditions (\text{A.2'}) and (\text{B'}) imply that condition (H) is satisfied.
Then we have the following:

**Theorem 2.** Assume that conditions (A.1'), (A.2') and (B') are satisfied and that

\[
(0.2a) \quad \lim_{x_{1} \to 0} \frac{\tilde{\lambda}(x_{1}, x') \log|b^{n}(x_{1}, x')|}{\sqrt{\tilde{\lambda}(x_{1}, x')}} = 0,
\]

\[
(0.2b) \quad \lim_{x_{1} \to 0} \frac{\overline{b}^{n}(x_{1}, x')^{2} \log\mu(x_{1}, x')}{b^{n}(x_{1}, x')} = 0,
\]

\[
(0.2c) \quad \lim_{x_{1} \to 0} \frac{\lambda(x_{1}, x') \log\mu(x_{1}, x')}{b^{n}(x_{1}, x')} = 0,
\]

where

\[
\lambda(x_{1}, x') = \sum_{i=2}^{n-1} a^{i}(x_{1}, x'),
\]

\[
\tilde{\lambda}(x_{1}, x') = \int_{0}^{x_{1}} \lambda(t, x') dt,
\]

\[
\overline{b}^{n}(x_{1}, x') = \int_{0}^{x_{1}} b^{n}(t, x') dt.
\]

Here the convergence is uniform in the variables \(x'=(x_{2}, \cdots, x_{n})\) over compact subsets of \(\mathbb{R}^{n-1}\) which intersect the set \(S\).

Then the operator \(Q(x, D)\) is hypoelliptic in \(\mathbb{R}^{n}\).

The next example is a generalization of Theorem 4 of Hoshiro [Ho1].

**Example 2.** Consider the following operator \(Q(x, D)\) on \(\mathbb{R}^{3}\):

\[
Q(x, D) = \frac{\partial^{2}}{\partial x_{1}^{2}} + \frac{\partial}{\partial x_{2}}(f(x) \frac{\partial}{\partial x_{2}}) + g(x) \frac{\partial}{\partial x_{3}},
\]

where \(f\) and \(g\) are non-negative functions on \(\mathbb{R}^{3}\) such that

\[
f(x_{1}, x_{2}, x_{3}) > 0 \quad \text{for} \quad x_{1} \neq 0,
\]

\[
g(x_{1}, x_{2}, x_{3}) > 0 \quad \text{for} \quad x_{1} \neq 0.
\]

Then the operator \(Q(x, D)\) is hypoelliptic in \(\mathbb{R}^{3}\) if the following two conditions are satisfied:

\[
\lim_{x_{1} \to 0} \frac{\int_{0}^{x_{1}} f(t, x') dt \log g(x_{1}, x')}{\sqrt{f(x_{1}, x')}} = 0,
\]

\[
\lim_{x_{1} \to 0} \frac{\left(\int_{0}^{x_{1}} g(t, x') dt\right)^{2} \log f(x_{1}, x')}{g(x_{1}, x')} = 0.
\]
Here the convergence is uniform in the variables \( x' = (x_2, x_3) \) over compact subsets of \( R^2 \). (We remark that condition (0.2c) is superfluous for Example 2, since one may take \( \mu = \lambda \) in inequality (3.9) in the proof of Theorem 2.)

The rest of this paper is organized as follows. In Section 1, we consider a family of modifications \( P_{\Lambda_\delta}(x, D) \) of the operator \( P(x, D) \) which is adapted to the study of hypoellipticity. The operators \( P_{\Lambda_\delta}(x, D) \) are introduced in the study of propagation of singularities for hyperbolic pseudodifferential operators with double characteristics by Kajitani-Wakabayashi [KW]. We give a general criterion for hypoellipticity for the operator \( P(x, D) \) under a weaker condition \((\text{H}')\) in terms of the operators \( P_{\Lambda_\delta}(x, D) \) (Theorem 1.1). This criterion is more useful if it is combined with the well-known Poincaré inequality (Corollary 1.2).

Sections 2 and 3 are devoted to the proof of Theorems 1 and 2, respectively, indicating applications of such a criterion to the study of hypoellipticity for the operators \( P(x, D) \) and \( Q(x, D) \). The proof follows the pattern given in Section 5 of Kajitani-Wakabayashi [KW]. That is, we calculate the symbol of the operators \( P_{\Lambda_\delta}(x, D) \) in question, and then apply a sharpened form of Gårding’s inequality due to Fefferman-Phong [FP] (Theorem 2.1 and Corollary 2.2) to the operators \( P_{\Lambda_\delta}(x, D) \). It is Lemma 2.4 that allows us to make good use of the Fefferman-Phong inequality.

This paper is inspired by the work of Wakabayashi and Suzuki [WS]. It is a genuine pleasure to acknowledge the great debt which I owe to S. Wakabayashi, with whom I had extensive and fruitful conversations while working on this paper. I am also grateful to T. Hoshino, Y. Morimoto and N. Iwasaki for some useful comments.

1. A criterion for hypoellipticity.

In this section we give a general criterion for hypoellipticity for the operator \( P(x, D) \) which is a variant of Theorem 1.2 of Kajitani-Wakabayashi [KW]. For the sake of completeness, we reproduce here its proof due to Wakabayashi (cf. [WS, Theorem 1.1]).

First we recall the definition of the symbol class \( S^{m}_{1,0}(R^n \times R^n) \) for \( m \in R \). We say that a \( C^m \) function \( p(x, \xi) \) on the cotangent bundle \( T^*(R^n) \) belongs to the class \( S^{m}_{1,0}(R^n \times R^n) \) if, for any multi-indices \( \alpha \) and \( \beta \), there exists a constant \( C_{\alpha, \beta} > 0 \) such that

\[
|\partial_{\xi}^\alpha \partial_x^\beta p(x, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|^2)^{(m-|\alpha|)/2} \quad \text{for all} \quad (x, \xi) \in T^*(R^n).
\]

Here we have identified the cotangent bundle \( T^*(R^n) \) with the space \( R^*(R^n) \times R^n \).

Let \( \lambda(\xi) \) be a real-valued symbol in the class \( S^{m}_{1,0}(R^n \times R^n) \) such that
\[ \lambda(\xi) = \begin{cases} 
\langle \xi' \rangle & \text{if } |\xi'| \geq \frac{1}{2} |\xi| \text{ and } |\xi| \geq 4, \\
\frac{1}{4} \langle \xi \rangle & \text{if } |\xi'| \leq \frac{1}{4} |\xi| \text{ and } |\xi| \geq 4, 
\end{cases} \]

and that
\[ \frac{1}{4} \langle \xi \rangle \leq \lambda(\xi) \leq \langle \xi \rangle, \quad \lambda(\xi) \geq 1, \]

where
\[ \xi = (\xi_1, \xi'), \quad \xi' = (\xi_2, \cdots, \xi_n), \]
\[ \langle \xi' \rangle = (1 + |\xi'|^2)^{1/2}, \]
\[ \langle \xi \rangle = (1 + |\xi|^2)^{1/2}. \]

Let \( x^0 = (x^0_1, x^0_2, \cdots, x^0_n) \) be a point of a subset \( T \) of the \((n-k)\) dimensional surface \( \{x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n ; x_1 = x_2 = \cdots = x_k = 0\} \). If \( 0 \leq \delta \leq 1, a \geq 0, N \geq 0 \) and \( s \in \mathbb{R} \), we let
\[ A_\delta(x'', \xi) = A_\delta(x'', \xi ; a, N, s) = (-s + a |x'' - x^{0''}|^2) \log \lambda(\xi) + N \log (1 + \delta \lambda(\xi)), \]

where
\[ x'' = (x_{k+1}, \cdots, x_n), \]
\[ x^{0''} = (x^0_{k+1}, \cdots, x^0_n). \]

We remark that
\[ e^{A_\delta(x'', \xi)} = \lambda(\xi)^{(-s + a |x'' - x^{0''}|^2)(1 + \delta \lambda(\xi))}, \]
\[ e^{-A_\delta(x'', \xi)} = \lambda(\xi)^{(s - a |x'' - x^{0''}|^2)(1 + \delta \lambda(\xi))}, \]

and that
\[ |\partial_\xi^\alpha \partial_{x'}^\beta (e^{\pm A_\delta(x'', \xi)})| \leq C_{\alpha, \beta} |\xi|^{-|\alpha|} (1 + \log |\xi|)^{|\beta|} e^{\pm A_\delta(x'', \xi)}, \]

where the constant \( C_{\alpha, \beta} \) is independent of \( \delta \).

Furthermore we introduce a family of second-order pseudodifferential operators \( P_{A_\delta}(x, D) \) defined by the formula
\[ P_{A_\delta}(x, D) = e^{-A_\delta(x'', D)} P(x, D) e^{A_\delta(x'', D)}, \]

where \( e^{A_\delta(x'', D)} \) are properly supported pseudodifferential operators with symbols \( e^{\pm A_\delta(x'', \xi)} \), respectively.

Now we can state a criterion for hypoellipticity for the operator \( P(x, D) \):

**Theorem 1.1.** Assume that:

(H') The operator \( P(x, D) \) is hypoelliptic outside a closed subset \( T \) of the
Hypoelliptic differential operators

$(n-k)$ dimensional surface \( \{ x=(x', x^m) \in \mathbb{R}^n ; x'=0 \} \), where \( x^m=(x_1, \ldots, x_k) \) and \( x^m=(x_{k+1}, \ldots, x_n) \) for \( 1 \leq k \leq n-1 \).

Furthermore, assume that, for each point \( x' \) of the set \( T \), there exist an open neighborhood \( U(x') \) of \( x' \) and numbers \( a_0 \geq 0, N_0 \geq 0 \) and \( s_0 \in \mathbb{R} \) such that:

For any \( a \geq a_0 \), any \( N \geq N_0 \), and any \( s \geq s_0 \), there exist functions \( \theta(x^m) \in C^\infty(\mathbb{R}^{n-k}) \) and \( \psi(x) \in C^\infty(\mathbb{R}^n) \) with \( \text{supp}(1-\theta) \cap \{x^m\} = \emptyset \) and \( \text{supp} \psi \cap T = \emptyset \) and constants \( 0 < \delta_0 \leq 1 \) and \( C > 0 \) such that the estimate

\[
\|v\| \leq C(\|P_{\Lambda_{\delta}}(x, D)v\| + \|(1-\theta(x^m))v\| + \|\psi(x)v\|)
\]

holds for all \( v \in C^\infty_0(U(x')) \) and all \( 0 < \delta \leq \delta_0 \). Here \( \| \cdot \| \) is the norm of the space \( L^2(\mathbb{R}^n) \).

Then the operator \( P(x, D) \) is hypoelliptic in \( \mathbb{R}^n \).

**Proof.** Let \( x' \) be an arbitrary point of the set \( T \). Assume that \( u \in \mathcal{D}'(\mathbb{R}^n) \) and the function

\[ f = P(x, D)u \]

is of class \( C^\infty \) in a neighborhood of \( x' \).

Without loss of generality, one may assume that

\[ x' = (0, 0), \]
\[ U(x') = \{ x = (x', x^m) \in \mathbb{R}^n ; |x'| < 1, |x^m| < 1 \}. \]

We take three open neighborhoods \( U_1, U_2, U_3 \) of \( x'=0, 0 \) such that

\[ U_1 = \{ x = (x', x^m) \in \mathbb{R}^n ; |x'| < \frac{3}{4}, |x^m| < \frac{3}{4} \}, \]
\[ U_2 = \{ x = (x', x^m) \in \mathbb{R}^n ; |x'| < \frac{1}{2}, |x^m| < \frac{1}{2} \}, \]
\[ U_3 = \{ x = (x', x^m) \in \mathbb{R}^n ; |x'| < \frac{1}{4}, |x^m| < \frac{1}{4} \}. \]

One may assume that for some \( s' \in \mathbb{R} \)

\[ u \in \mathcal{C}''(\mathbb{R}^n) \cap H^s(\mathbb{R}^n), \]

and that the function \( f \) is of class \( C^\infty \) near the set \( U_1 \). For each \( \sigma > s' \), we can choose numbers \( a \geq a_0 \) and \( s \geq s_0 \) such that

\[
\begin{align*}
& s - \frac{1}{16} a > \sigma, \\
& s - \frac{1}{4} a < s' - 1,
\end{align*}
\]

and also choose a number \( N \geq N_0 \) such that
Now, by the calculus of pseudodifferential operators, one can find an elliptic symbol $q_{\delta}(x''', \xi)=q_{\delta}(x''', \xi; a, N, s)$ in the class $\mathcal{S}_{1,0}^{0}(\mathbb{R}^{n}\times\mathbb{R}^{n})$ such that

$e^{A_{\delta}}(x'', D)e^{-A_{\delta}}(x'', D)q_{\delta}(x'', D)\equiv I \mod \text{an operator of order } -\infty$.

If $\chi$ and $\chi_{1}$ are functions in $C_{0}^{\infty}(U_{1})$ such that

\[
\begin{align*}
\{ & \chi(x)=1 \text{ on } U_{2}, \\
& \chi_{1}(x)=1 \text{ near } supp^{\chi},
\end{align*}
\]

we let

$\nu_{\delta}=x_{1}(x)e^{-A_{\delta}}(x^{m}, D)q_{\delta}(x^{m}, D)(\chi u)$.

Then we have

\[
\begin{align*}
\|P_{1_{\delta}}(x, D)v_{\delta}-e^{-A_{\delta}}(x^{m}, D)(\chi f)-e^{-A_{\delta}}(x^{m}, D)[P, \chi]u\| & = \|e^{-A_{\delta}}(x^{m}, D)P(x, D)(e^{A_{\delta}}(x^{m}, D)\chi_{1}(x)e^{-A_{\delta}}(x^{m}, D)-I)(\chi u)\| \\
& \leq C\|u\|_{s'}.
\end{align*}
\]

Here and in the following the letter $C$ denotes a generic positive constant independent of $\delta(0<\delta\leq\delta_{0})$, and $\|\cdot\|_{s}$ is the norm of the Sobolev space $H^{s}(\mathbb{R}^{n})$ of order $s$.

Furthermore, since the operator $e^{-A_{\delta}}(x^{m}, D)$ is of order at most $s$, it follows that

\[
\|e^{-A_{\delta}}(x^{m}, D)(\chi f)\| \leq C\|\chi f\|_{s}.
\]

We also have

\[
\|e^{-A_{\delta}}(x^{m}, D)[P, \chi]u\| \leq C\|u\|_{s'}.
\]

In fact, if $\tilde{\chi}$ is a function in $C_{0}^{\infty}(U_{1})$ such that $\tilde{\chi}(x)=1$ on $U_{2}$ and $\tilde{\chi}\chi=\chi$ and if $\eta$ is a function in $C_{0}^{\infty}(\mathbb{R}^{n})$ such that $\eta(x)=1$ near $supp(1-\overline{\chi})$ and $supp\eta\cap U_{2}=\emptyset$, then it follows that

$e^{-A_{\delta}}(x^{m}, D)[P, \chi]u = e^{-A_{\delta}}(x^{m}, D)\tilde{\chi}[P, \chi]u + \eta e^{-A_{\delta}}(x^{m}, D)(1-\tilde{\chi})[P, \chi]u + (1-\eta)e^{-A_{\delta}}(x^{m}, D)(1-\overline{\chi})[P, \chi]u$.

But we remark that the operators $\tilde{\chi}[P, \chi]$ and $(1-\eta)e^{-A_{\delta}}(x^{m}, D)(1-\tilde{\chi})$ are of order $-\infty$, and the operator $\eta e^{-A_{\delta}}(x^{m}, D)(1-\overline{\chi})$ is of order at most $s'-1$, since $s-a|\chi^{m}|^{2}<s'-1$ for $|\chi^{m}|\geq1/2$. Hence we find that

$\|e^{-A_{\delta}}(x^{m}, D)[P, \chi]u\| \leq C\|u\|_{s'}$.

Therefore, we obtain from inequalities (1.2), (1.3) and (1.4) that

\[
\|P_{1_{\delta}}(x, D)v_{\delta}\| \leq C(\|\chi f\|_{s}+\|u\|_{s'}).
\]
For the term \(\|(1-\theta(x'''))v_{\delta}\|\), without loss of generality, one may assume that
\[
\text{supp}(1-\theta) \subset \left\{ x'' \in \mathbb{R}^{n-1} ; \ |x''| \geq \frac{1}{2} \right\}.
\]
Then we have
\[
(1.6) \quad \|(1-\theta(x'''))v_{\delta}\| \leq C \|u\|_{s'-1},
\]
since the operator \((1-\theta)e^{-\Lambda_{\delta}}(x^{m}, D)\) is of order at most \(s'-1\).

On the other hand, if \(\tilde{\phi}\) is a function in \(C^{\infty}(\mathbb{R}^{n})\) such that \(\tilde{\phi}\phi=\phi\) and \(\text{supp} \tilde{\phi} \cap T = \emptyset\), then it follows from condition (H) that
\[
(1.7) \quad \|\phi(x)v_{\delta}\| \leq \|\phi(x)\chi_{1}(x)e^{-\Lambda_{\delta}}(x^{m}, D)q_{\delta}(x^{m}, D)((1-\tilde{\phi})\chi u)\|
+ \|\phi(x)\chi_{1}(x)e^{-\Lambda_{\delta}}(x^{m}, D)q_{\delta}(x^{m}, D)(\tilde{\phi}(\chi u))\|
\leq C(\|u\|_{s'} + \|\tilde{\phi}(\chi u)\|_{s}),
\]
since the operator \(e^{-\Lambda_{\delta}}(x^{m}, D)q_{\delta}(x^{m}, D)(1-\tilde{\phi})\) is of order \(-\infty\) and the function \(\tilde{\phi}(\chi u)\) is of class \(C^\infty\). Further we find that the function \(\phi v_{\delta}\) is of class \(C^\infty\).

If we take another function \(\chi_{2}\) in \(C_{0}^{\infty}(U_{1})\) such that \(\chi_{2}(x)=1\) near \(\text{supp} \chi_{1}\), then we have for all \(\delta>0\)
\[
v_{\delta} = \chi_{2}\phi v_{\delta} + (1-\phi)\chi_{2}v_{\delta} \in H^{2}(\mathbb{R}^{n}) \cap \mathcal{E}'(U_{1}),
\]
since the operator \(e^{-\Lambda_{\delta}}(x^{m}, D)\) is of order \(-N\) and \(s'-(s-N)>2\). But, if \(\{w_{j}\}\) is a sequence in \(C_{0}^{\infty}(U_{1})\) such that
\[
\tilde{w}_{j} \rightarrow v_{\delta} \text{ in } H^{2}(\mathbb{R}^{n}),
\]
then it is easy to verify the following:

(a) \(\tilde{w}_{j} = \chi_{2}\phi v_{\delta} + (1-\phi)\chi_{2}w_{j} \in C_{0}^{\infty}(U_{1}) \subset C_{0}^{\infty}(U(x^{0}))\).

(b) \(\tilde{w}_{j} \rightarrow v_{\delta} \text{ in } H^{2}(\mathbb{R}^{n})\).

(c) \(P_{A_{\delta}}(x, D)\tilde{w}_{j} \rightarrow P_{A_{\delta}}(x, D)v_{\delta} \text{ in } L^{2}(\mathbb{R}^{n})\).

(d) \((1-\theta(x^{m}))\tilde{w}_{j} \rightarrow (1-\theta(x^{m}))v_{\delta} \text{ in } L^{2}(\mathbb{R}^{n})\).

(e) \(\phi(x)\tilde{w}_{j} \rightarrow \phi(x)v_{\delta} \text{ in } L^{2}(\mathbb{R}^{n})\).

This proves that estimate (1.1) remains valid for the functions \(v_{\delta}\).

Therefore, it follows from inequalities (1.5), (1.6) and (1.7) that we have for all \(0<\delta \leq \delta_{0}\)
\[
\|v_{\delta}\| \leq C(\|\chi_{2}f\|_{s'} + \|u\|_{s'} + \|\tilde{\phi}(\chi u)\|_{s}).
\]
Hence, letting \(\delta \downarrow 0\), we find that
\[
v_{\delta} \rightarrow v_{0} \text{ weakly in } L^{2}(\mathbb{R}^{n}),
\]
where
\[ v_0 = \mathcal{L}_1(x) \lambda(D)^{(s-a|x^{m}|^{2})} q_0(x^{m}, D) (\mathcal{L} u) \in L^2(\mathbb{R}^n). \]

But we remark that

\[
\begin{cases}
\lambda(\xi) \geqq 1, \\
q_0(x^{m}, \xi) = 1 + \cdots \text{ near } x^{m} = 0, \\
\mathcal{L}_1(x) = 1 \text{ near } x^{0} = 0,
\end{cases}
\]

and also that we have for \(|x^{m}| \leqq 1/4\)

\[
s-a|x^{m}|^{2} > \sigma.
\]

Thus, taking a function \(\mathcal{L}_3 \in C_0^\infty(U_3)\) such that

\[
\mathcal{L}_3(x) = 1 \text{ near } x^{0} = 0,
\]

we find that

\[
\mathcal{L}_3 u \in H^s(\mathbb{R}^n).
\]

This proves that \(u\) is of class \(C^a\) at \(x^{0} = 0\), since \(a\) is arbitrary.

The proof of Theorem 1.1 is now complete.

If we combine Theorem 1.1 with the well-known Poincaré inequality, we obtain the following useful criterion for hypoellipticity (cf. [WS], [Mo]):

**Corollary 1.2.** Assume that condition (H) is satisfied and that, for each point \(x^{0}\) of the set \(T\), there exist an open neighborhood \(U(x^{0})\) of \(x^{0}\) and numbers \(a_0 \geqq 0\), \(N_0 \geqq 0\) and \(s_0 \in \mathbb{R}\) such that:

- For any \(a \geqq a_0\), any \(N \geqq N_0\) and any \(s \geqq s_0\), there exist constants \(0 < \delta \leqq 1\), \(C_1 > 0\) and \(C_2 > 0\) such that we have for all \(v \in C_0^\infty(U(x^{0}))\) and all \(0 < \delta \leqq \delta_0\)

\[
|\langle P_{A_b}(x, D)v, v \rangle| \geqq C_1 \|D_{x_1} v\|^2 - C_2 \|v\|^2.
\]

Here \(\langle \cdot, \cdot \rangle\) is the inner product of the space \(L^2(\mathbb{R}^n)\) and \(D_{x_1} = 1/\sqrt{-1} \partial/\partial x_1\).

Then the operator \(P(x, D)\) is hypoelliptic in \(\mathbb{R}^n\).

**Proof.** First we recall the Poincaré inequality:

**Lemma 1.3.** Let \(\Omega\) be an open subset of \(\mathbb{R}^n\) such that each line parallel to some line meets \(\Omega\) in a set of width at most \(L\). Then we have for all \(u \in H^1_b(\Omega)\)

\[
\|u\| \leqq L \left( \sum_{j=1}^n \|D_{x_j} u\|^2 \right)^{1/2}.
\]

Here \(H^1_b(\Omega)\) is the closure of \(C_0^\infty(\Omega)\) in the Sobolev space \(H^1(\Omega)\).

Now, without loss of generality, one may assume that

\[
x^0 = (0, 0),
\]

\[
U(x^0) = \{x = (x^{m}, x^{m}) \in \mathbb{R}^n ; \ |x^{m}| < 1, \ |x^m| < 1 \}.
\]
We choose a function $\psi(t)$ in $C^\infty_0(\mathbb{R})$ such that
\[
\begin{cases}
0 \leq \psi(t) \leq 1 & \text{on } \mathbb{R}, \\
\text{supp } \psi \subset \{|t| \leq 1\}, \\
\psi(t) = 1 & \text{if } |t| \leq \frac{1}{2}.
\end{cases}
\]

If $v$ is a function in $C^\infty_0(U(x^0))$, then it can be decomposed as follows:
\[v = v_1 + v_2 + v_3,
\]
where
\[v_1 = \psi\left(\frac{|x'|}{d}\right)\psi(|x^m|)v,
\]
\[v_2 = (1-\psi\left(\frac{|x'|}{d}\right))\psi(|x^m|)v,
\]
\[v_3 = (1-\psi(|x^m|))v,
\]
and $d > 0$ is a small parameter and will be chosen later on.

Then, applying Poincaré’s inequality to the function $v_1$, we have
\[
||v_1|| \leq \sqrt{2} d ||D_{x_1}v_1||.
\]
But we remark that
\[D_{x_1}v_1(x) = \psi\left(\frac{|x'|}{d}\right)\psi(|x^m|)D_{x_1}v(x) + \frac{1}{d} D_{x_1}\psi\left(\frac{|x'|}{d}\right)\frac{x_1}{|x'|}\psi(|x^m|)v(x),
\]
and
\[D_{x_1}\psi\left(\frac{|x'|}{d}\right) = 0 \quad \text{for } |x^\sigma| \leq \frac{d}{2}.
\]
Thus, if we let
\[\phi_a(x) = (1-\psi\left(\frac{3|x'|}{d}\right))\phi\left(\frac{|x^m|}{3}\right),
\]
we obtain that
\[\phi_a(x) = 1 \quad \text{on supp } \left[D_{x_1}\phi\left(\frac{|x^m|}{d}\right)\phi(|x^m|)\right],
\]
so that
\[D_{x_1}\phi\left(\frac{|x^m|}{d}\right)\frac{x_1}{|x^m|}\phi(|x^m|)v(x) = D_{x_1}\phi\left(\frac{|x^m|}{d}\right)\frac{x_1}{|x^m|}\phi(|x^m|)\phi_a(x)v(x).
\]
Hence we have
\[
||D_{x_1}v_1|| \leq ||D_{x_1}v|| + \frac{1}{d} C_{\psi}\phi_a(x)v,
\]
\[ C_d = \max \left| D_x \phi \left( \frac{|x''|}{d} \right) \right|. \]

Therefore, combining inequalities (1.9) and (1.10), we obtain that
\[ \|v_1\| \leq \sqrt{2} d \|D_x v\| + \sqrt{2} C_d \|\psi_{d}(x) v\|. \]

Similarly, we have for the function \( v_2 \)
\[ \|v_2\| \leq \|\psi_{d}(x) v\|. \]

In fact, it suffices to note that
\[ \phi_{d}(x) = 1 \quad \text{on} \quad \text{supp} \left[ (1-\phi \left( \frac{|x''|}{d} \right)) \phi(|x''|) \right]. \]

Hence we have for all \( v \in C_c^\infty(U(x^0)) \)
\[
\|v\|^2 = \|v_1 + v_2 + v_3\|^2 \\
\leq 3(\|v_1\|^2 + \|v_2\|^2 + \|v_3\|^2) \\
\leq 12d^2 \|D_x v\|^2 + C'_d(\|(1-\phi(|x''|))v\|^2 + \|\psi_{d}(x) v\|^2),
\]

with a constant \( C'_d > 0 \).

On the other hand, we have by the Schwarz inequality
\[
\langle P_{\delta}(x, D)v, v \rangle \leq 4 \|P_{\delta}(x, D)v\|^2 + \|v\|^2.
\]

Therefore, combining inequalities (1.8), (1.11) and (1.12), we have for all \( v \in C_c^\infty(U(x^0)) \) and all \( 0 < \delta \leq \delta_0 \)
\[
\|v\| \leq C'_d \|P_{\delta}(x, D)v\| + \|(1-\phi(|x''|))v\| + \|\psi_{d}(x) v\|,
\]

if we take
\[
0 < d < \frac{\sqrt{C_1}}{2 \sqrt{3} \sqrt{C_2+1}}.
\]

Thus Corollary 1.2 follows from an application of Theorem 1.

2. Proof of Theorem 1.

Our proof of Theorem 1 is based on Corollary 1.2.

1) First we give a version of the criterion in Corollary 1.2 adapted to the present context.

Let \( x^*=(x_1^*, x_2^*, \ldots, x_n^*) \) be a point of a closed subset \( S \) of the hypersurface \( \{x=(x_1, x') \in \mathbb{R}^n ; x_1=0\} \), where \( x'=(x_2, \ldots, x_n) \). Without loss of generality, one may assume that
Let $\lambda(\xi)$ be a real-valued symbol in the class $S_{1,0}^{1}(\mathbb{R}^{n} \times \mathbb{R}^{n})$ such that

$$\lambda(\xi) = \begin{cases} 
\langle \xi' \rangle & \text{if } |\xi'| \geq \frac{1}{2} |\xi| \text{ and } |\xi| \geq 4, \\
\frac{1}{4} \langle \xi \rangle & \text{if } |\xi'| \leq \frac{1}{4} |\xi| \text{ and } |\xi| \geq 4,
\end{cases}$$

and that

$$\frac{1}{4} \langle \xi \rangle \leq \lambda(\xi) \leq \langle \xi \rangle, \quad \lambda(\xi) \geq 1.$$ 

If $0 \leq \delta \leq 1$, $a \geq 0$, $N \geq 0$ and $s \in \mathbb{R}$, we let

$$A_{\delta}(x', \xi) = A_{\delta}(x', \xi; a, N, s) = (-s + a |x'|^2) \log \lambda(\xi) + N \log (1 + \delta \lambda(\xi)),$$

and

$$P_{J1_{\delta}}(x, D) = e^{-\Lambda_{\delta}(x', D)}P(x, D)e^{\Lambda_{\delta}(x', D)},$$

where $e^{\pm \Lambda_{\delta}(x', \xi)}$ are properly supported pseudodifferential operators with symbols $e^{\pm \Lambda_{\delta}(x', \xi)}$, respectively:

$$e^{\Lambda_{\delta}(x', \xi)} = \lambda(\xi)^{(-s + a |x'|^2)}(1 + \delta \lambda(\xi))^N,$$

$$e^{-\Lambda_{\delta}(x', \xi)} = \lambda(\xi)^{(s - a |x'|^2)}(1 + \delta \lambda(\xi))^{-N}.$$

By virtue of Corollary 1.2 in order to prove the hypoellipticity for the operator $P(x, D)$, it suffices to show that there exists an open neighborhood $U_{\epsilon_{0}} = \{x = (x_{1}, x') \in \mathbb{R}^{n} ; |x_{1}| < \epsilon_{0}, |x'| < 1\}$ of $x^{0} = (0, 0)$ such that we have for all $v \in C_{0}^{\infty}(U_{\epsilon_{0}})$ and all $0 < \delta \leq 1$

$$\Re(P_{J_{\delta}}(x, D)v, v) \geq C_{1}||D_{x_{1}}v||^{2} - C_{2}||v||^{2},$$

with constants $C_{1} > 0$ and $C_{2} > 0$ independent of $\delta$.

2) In the proof of inequality [2.1], we make good use of the following Fefferman-Phong inequality (cf. [FP, Theorem]; [Hr2, Corollary 18.6.11]):

**Theorem 2.1.** If $p(x', \xi')$ is a symbol in the class $S_{1,0}^{1}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ such that $p(x', \xi') \geq 0$ on $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$, then we have for all $v \in C_{0}^{\infty}(\mathbb{R}^{n-1})$

$$\Re(p(x', D')v, v) \geq -C||v||^{2}.$$

Here the constant $C$ may be chosen uniformly in the $p(x', \xi')$ in a bounded subset of $S_{1,0}^{1}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$.

**Corollary 2.2.** Let $p(x_{1}, x', \xi')$ be a symbol in the class $S_{1,0}^{1}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$ such that $p(x_{1}, x', \xi') \geq 0$ on $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$, where the variable $x_{1}$ is considered as a
If the family \( \{p(x_1, x', \xi')\}_{x_1 \in \mathbb{R}} \) forms a bounded subset of \( S^s_{-n}(\mathbb{R}^{n-1} \times \mathbb{R}^n) \), then we have for all \( u \in C_c^\infty(\mathbb{R}^n) \)
\[
\text{Re}(p(x_1, x', D')u, u) \geq -C\|u\|^2.
\] Here the constant \( C \) may be chosen uniformly in the \( p(x_1, x', \xi') \).

**PROOF.** If we apply Theorem 2.1 to the functions \( u(x_1, \cdot) \in C_c^\infty(\mathbb{R}^{n-1})(x_1 \in \mathbb{R}) \), we obtain that
\[
\text{Re}\int_{\mathbb{R}^{n-1}} p(x_1, x', D')u(x_1, x')\cdot u(x_1, x')dx' \geq -C\int_{\mathbb{R}^{n-1}} |u(x_1, x')|^2 dx'.
\] Hence inequality (2.2) follows by integrating the both sides with respect to \( x_1 \).

3) In order to calculate the symbol of the operator \( P_{\Lambda_\delta}(x, D) \), we remark that the operator \( P(x, D) \) is micro-elliptic outside a conic neighborhood of a point \((x^0, \xi^0)=(x^0, 0, \xi_{2}^{0}, \cdots , \xi_{n}^{0})\) in the bundle \( T^*(\mathbb{R}^n)\backslash 0 \) of non-zero cotangent vectors. Here a conic subset \( C \) of \( T^*(\mathbb{R}^n) \) is such a set that \((x, \xi) \in C \) implies \((x, r\xi) \in C \) for all \( r > 0 \). Hence, without loss of generality, one may assume that
\[
4 \leq |\xi| \leq 2|\xi'|, \quad \xi = (\xi_1, \xi'),
\] and that
\[
\lambda(\xi) = (1+|\xi'|^2)^{1/2} = \langle\xi', \xi'\rangle,
\]
\[
\Lambda_\delta(x', \xi) = \Lambda_\delta(x', \xi') = (-s+a|x'|^2)\log\langle\xi'\rangle + N\log(1+\delta\langle\xi'\rangle).
\]
Then, for the derivatives of the symbol \( \Lambda_\delta(x', \xi') \), we have the following:
\[
\Lambda_{\delta x_j}(x', \xi') = 2ax_j \log\langle\xi'\rangle, \quad 2 \leq j \leq n,
\]
\[
\Lambda_{\delta x_j \xi_k}(x', \xi') = 2a\delta_{jk} \log\langle\xi'\rangle, \quad 2 \leq j, k \leq n,
\]
\[
\Lambda_{\delta \xi_j}(x', \xi') = \left\{(s+a|x'|^2)+N\frac{\delta\langle\xi'\rangle}{1+\delta\langle\xi'\rangle}\right\}\frac{\xi_j}{\langle\xi'\rangle}\frac{1}{\langle\xi'\rangle}, \quad 2 \leq j \leq n,
\]
\[
\Lambda_{\delta \xi_j \xi_k}(x', \xi') = \left\{(s+a|x'|^2)+N\frac{\delta\langle\xi'\rangle}{1+\delta\langle\xi'\rangle}\right\}\left(\delta_{jk} - \frac{\xi_j}{\langle\xi'\rangle}\frac{\xi_k}{\langle\xi'\rangle}\right)\frac{1}{\langle\xi'\rangle^2}, \quad 2 \leq j, k \leq n.
\]
Here and in the following, for the derivatives of a symbol \( p(x, \xi) \), we use the shorthand
\[
p_{x_i} = p_{x_i}(x, \xi) = \frac{\partial p}{\partial x_i}(x, \xi),
\]
\[
p_{\xi_i} = p_{\xi_i}(x, \xi) = \frac{\partial p}{\partial \xi_i}(x, \xi).
\]
Hypoelliptic differential operators

But, since $|\xi'| \leq |\xi| \leq 2|\xi'|$ in a conic neighborhood of $\xi^0 = (0, \xi_2^0, \cdots, \xi_n^0)$, it follows that

\[
\begin{align*}
A_{\delta x_j}(x', \xi') &\in \cap_{\rho>0} S_{1.0}^\rho(R^n \times R^n), \quad 2 \leq j \leq n, \\
A_{\delta x_j \delta x_k}(x', \xi') &\in \cap_{\rho>0} S_{1.0}^\rho(R^n \times R^n), \quad 2 \leq j,k \leq n, \\
A_{\delta \xi_j}(x', \xi') &\in S_{1.0}^{-1}(R \times R^n), \quad 2 \leq j \leq n, \\
A_{\delta \xi_j \delta \xi_k}(x', \xi') &\in S_{1.0}^{-2}(R^n \times R^n), \quad 2 \leq j,k \leq n.
\end{align*}
\]

Therefore, we find that the symbol $P_{A_{\delta}}(x, \xi)$ of $P_{A_{\delta}}(x, D)$ is given by the following (cf. [KW, Section 5]):

\[
P_{A_{\delta}}(x, \xi) = (1+q_\delta(x', \xi)) \left[ p(x, \xi) + \sqrt{-1} \sum_{j=2}^{n} (p_{\xi_j} A_{\delta x_j} - p_{x_j} A_{\delta \xi_j}) \\
- \frac{1}{2} \sum_{j,k=2}^{n} p_{\xi_j \xi_k} A_{\delta x_j} A_{\delta x_k} + \sum_{j,k=2}^{n} p_{\xi_j \delta \xi_k} A_{\delta x_j} A_{\delta \xi_k} \\
+ \frac{1}{2} \sum_{j,k=2}^{n} p_{\xi_j \delta \xi_k} A_{\delta x_j \delta \xi_k} \\
+ \sum_{j,k=2}^{n} p_{\xi_j} A_{\delta \xi_k} A_{\delta x_j \delta \xi_k} + \sum_{j,k=2}^{n} p_{x_j} A_{\delta x_k} A_{\delta \xi_j \delta \xi_k} \right]
\quad \text{mod } S_{1,0}^0(R \times R^n).
\]

Here $p(x, \xi)$ is a symbol in the class $S_{1,0}^0(R^n \times R^n)$ given by

\[
p(x, \xi) = \xi_1^2 + \sum_{i,j=2}^{n} a^{ij}(x) \xi_i \xi_j - \sqrt{-1} \sum_{i=1}^{n} \left( b^i(x) + \sum_{j=1}^{n} \frac{\partial a^{ij}}{\partial x_j}(x) \right) \xi_i - c(x),
\]

and $q_\delta(x', \xi) = q_\delta(x', \xi; a, N, s)$ is a symbol in the class $\cap_{\rho>0} S_{1,0}^{1+\rho}(R^n \times R^n)$ given by

\[
q_\delta(x', \xi) = \sqrt{-1} \sum_{j=2}^{n} A_{\delta \xi_j} A_{\delta x_j} + \frac{1}{2} \sum_{j,k=2}^{n} (A_{\delta \xi_j \delta \xi_k} + A_{\delta \xi_j \delta \xi_k}) (A_{\delta x_j x_k} - A_{\delta x_j} A_{\delta x_k}).
\]

But, since we have for $|\xi|$ sufficiently large (uniformly in $\delta>0$)

\[
\frac{1}{2} \leq |1+q_\delta(x', \xi)| \leq 2,
\]

one can find an elliptic symbol $r_\delta(x', \xi) = r_\delta(x', \xi; a, N, s)$ in the class $S_{1,0}^0(R^n \times R^n)$ such that we have for $|\xi|$ sufficiently large (uniformly in $\delta>0$)

\[
r_\delta(x', \xi)(1+q_\delta(x', \xi)) = 1.
\]

We let

\[
(2.3) \quad \tilde{P}_{A_{\delta}}(x, D) = r_\delta(x', D)P_{A_{\delta}}(x, D),
\]

where $r_\delta(x', D)$ is a properly supported, elliptic pseudodifferential operator with symbol $r_\delta(x', \xi)$ such that we have for $|\xi|$ sufficiently large (uniformly in $\delta>0$)
\[ \frac{1}{2} \leq |r_3(x', \xi)| \leq 2. \]

Then we have by a direct calculation

\[ \bar{F}_{12}(x, \xi) \equiv \xi_1^2 + \alpha(x, \xi') + \sum_{j=2}^{n} b^j(x) A_{\delta x_{j}} - \frac{1}{2} \sum_{j,k=2}^{n} a_{\xi_{j} \xi_{k}} A_{\delta x_{j}} A_{\delta x_{k}} \]

\[ + \sum_{j,k=2}^{n} a_{\xi_{j}} A_{\delta \xi_{k}} A_{\delta x_{j}} A_{\delta x_{k}} \]

\[ + \sum_{j,k=2}^{n} a_{\xi_{j} \xi_{k}} A_{\delta x_{j}} A_{\delta \xi_{k}} A_{\delta x_{j}} A_{\delta \xi_{k}} \]

\[ + \sqrt{-1} \left[ - \sum_{k=1}^{n} b^k(x) \xi_k + \sum_{j=2}^{n} \alpha_{\xi_{k}} A_{\delta x_{j} \delta \xi_{k}} - \sum_{j=2}^{n} \alpha_{x_{j}} A_{\delta \xi_{k} \delta x_{j}} \right] \]

mod $S_{1.0}^{0}(R \times R^n)$,

where

\[ \alpha(x, \xi') = \sum_{i,j=2}^{n} a^{ij}(x) \xi_i \xi_j. \]

In order to estimate the terms $\alpha_{\xi_{j}} A_{\delta \xi_{k}} A_{\delta x_{j} \delta \xi_{k}}$ and $\alpha_{x_{j}} A_{\delta \xi_{k} \delta x_{j}}$ in formula (2.4), we need the following:

**Lemma 2.3.** Let $d(x, \xi)$ and $e(x, \xi)$ be symbols in the classes $S_{1.0}^{-1+\rho}(R^n \times R^n)$ and $S_{1.0}^{-2+\rho}(R \times R^n)$ for some $0 < \rho < 1$, respectively. Then, for every $\epsilon > 0$, one can find constants $C_\epsilon > 0$ and $C'_\epsilon > 0$ such that

\[ |\alpha_{\xi_{j}}(x, \xi') d(x, \xi)| \leq \epsilon \alpha(x, \xi') + C_\epsilon \] on $T^*(R^n)$,

\[ |\alpha_{x_{j}}(x, \xi') e(x, \xi)| \leq \epsilon \alpha(x, \xi') + C'_\epsilon \] on $T^*(R^n)$.

**Proof.** Since $\alpha(x, \xi') \geq 0$ on $T^*(R^n)$, it follows from an application of Lemma 1.7.1 of Oleinik-Radkevič [OR] that

\[ |\alpha_{\xi_{j}}(x, \xi')| \leq a^{ij}(x) \alpha(x, \xi') \] on $T^*(R^n)$,

\[ |\alpha_{x_{j}}(x, \xi')| \leq 2 \left( \sup_{x \in \mathbb{R}^n} |\alpha_{x_{j}}(x, \xi')| \right) \alpha(x, \xi') \] on $T^*(R^n)$.

Thus, using the Schwarz inequality, we obtain from inequality (2.7) that for every $\epsilon > 0$

\[ |\alpha_{\xi_{j}}(x, \xi') d(x, \xi)| \leq \epsilon \alpha(x, \xi') + \frac{1}{4\epsilon} a^{ij}(x) d(x, \xi)^2 \] on $T^*(R^n)$.

This proves estimate (2.5), since $d(x, \xi)^2$ belongs to the class $S_{1.0}^{-2+\rho}(R^n \times R^n)$ for some $0 < \rho < 1$.

Similarly, estimate (2.6) can be proved by using inequality (2.8).

Now we recall that for all $\xi = (\xi_1, \xi')$ in a conic neighborhood of $\xi' = (0, \xi'')$
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\[ |\xi'| \leq |\xi| \leq 2|\xi'|, \]

and hence that

\[
\Lambda_{\delta x_{k}}(x', \xi')\Lambda_{\delta x_{j}x_{k}}(x', \xi') \in \bigcap_{\rho>0}S_{1.0}^{-1+\rho}(\mathbb{R}^{n}\cross\mathbb{R}^{n}), \quad 2 \leq j, k \leq n,
\]

\[
\Lambda_{\delta x_{k}}(x', \xi')\Lambda_{\delta \xi_{j}x_{k}}(x', \xi') \in \bigcap_{\rho>0}S_{1.0}^{-2+\rho}(\mathbb{R}^{n}\cross\mathbb{R}^{n}), \quad 2 \leq j, k \leq n.
\]

Therefore, applying Lemma 2.3 to the terms \(\alpha_{\xi_{j}}\Lambda_{\delta \xi_{k}}\Lambda_{\delta x_{j}x_{k}}\) and \(\alpha_{x_{j}}\Lambda_{\delta x_{k}}\Lambda_{\delta \xi_{j}x_{k}}\), we have for every \(\varepsilon>0\)

\[
\alpha_{\xi_{j}}(x, \xi')\Lambda_{\delta \xi_{k}}(x', \xi')\Lambda_{\delta x_{j}x_{k}}(x', \xi') \geq -\varepsilon\alpha(x, \xi') \mod S_{1,0}^{0}(\mathbb{R}^{n}\cross\mathbb{R}^{n}),
\]

\[
\alpha_{x_{j}}(x, \xi')\Lambda_{\delta x_{k}}(x', \xi')\Lambda_{\delta \xi_{j}x_{k}}(x', \xi') \geq -\varepsilon\alpha(x, \xi') \mod S_{1,0}^{0}(\mathbb{R}^{n}\cross\mathbb{R}^{n}).
\]

On the other hand, by virtue of conditions (B) and (A.1), we can estimate the terms \(b^{l}\Lambda_{\delta x_{j}}\) and \(\alpha_{\xi_{j}}\Lambda_{\delta x_{j}}\Lambda_{\delta \xi_{j}x_{k}}\) in formula (2.4) as follows:

\[
b^{l}(x)\Lambda_{\delta x_{j}}(x', \xi') \geq -\varepsilon\alpha(x, \xi') \mod S_{1,0}^{0}(\mathbb{R}^{n}\cross\mathbb{R}^{n}).
\]

\[
\alpha_{\xi_{j}}(x, \xi')\Lambda_{\delta x_{j}}(x', \xi')\Lambda_{\delta \xi_{j}x_{k}}(x', \xi') \geq -\varepsilon\alpha(x, \xi') \mod S_{1,0}^{0}(\mathbb{R}^{n}\cross\mathbb{R}^{n}).
\]

Summing up, we obtain from formula (2.4) that in a conic neighborhood of \((x^{0}, \xi^{0})\)

\[
\text{Re} \tilde{F}_{\Lambda_{\delta}}(x, \xi) \geq \xi_{1}^{2} + \frac{1}{2}\alpha(x, \xi') - C\left(\sum_{j,k=2}^{n}|a^{jk}(x)|\right)(\log \langle \xi^{f}\rangle)^{2} \mod S_{1,0}^{0}(\mathbb{R}^{n}\cross\mathbb{R}^{n}),
\]

where \(C>0\) is a constant independent of \(\delta\). But we remark that

\[
|a^{fk}(x)| \leq \sqrt{a^{jj}(x)a^{kk}(x)} \leq \frac{1}{2}(a^{jj}(x)+a^{kk}(x)).
\]

Hence we have in a conic neighborhood of \((x^{0}, \xi^{0})\)

\[
(2.9) \quad \text{Re} \tilde{F}_{\Lambda_{\delta}}(x, \xi) \geq \xi_{1}^{2} + \frac{1}{2}\alpha(x, \xi') - 2C\left(\sum_{j=2}^{n}a^{jj}(x)\right)(\log \langle \xi^{f}\rangle)^{2}
\]

\[
= \frac{1}{2}\xi_{1}^{2} + \frac{1}{2}\alpha(x, \xi') - 2C\lambda(x)(\log \langle \xi^{f}\rangle)^{2} \mod S_{1,0}^{0}(\mathbb{R}^{n}\cross\mathbb{R}^{n}),
\]

where

\[
\lambda(x) = \sum_{j=2}^{n}a^{jj}(x).
\]

4) The next lemma allows us to replace the symbol \((1/2)\xi_{1}^{2}\) in the bracket in formula (2.9) by a symbol of a pseudodifferential operator on \(\mathbb{R}^{n-1}\):

**Lemma 2.4.** Let \(F(x)\) be a non-negative \(C^{\infty}\) function on \(\mathbb{R}^{n}\) and \(l\) a positive integer. If \(a(x, x', D')\) is a properly supported, pseudodifferential operator on \(\mathbb{R}^{n-1}\) with symbol
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\[ a(x_1, x', \xi') = F(x_1, x') (\log \langle \xi' \rangle)^{\iota}, \]

where the variable \( x_1 \) is considered as a parameter, we define a formally self-adjoint operator \( \mathcal{A}(x_1, x', D') \) by the formula

\[ \mathcal{A}(x_1, x', D') = \frac{1}{2} \left[ a(x_1, x', D') + a(x_1, x', D')^* \right]. \]

Then we have for all \( u \in C_0^\infty(\mathbb{R}^n) \)

\[ (D_{x_1}^2 u, u) \geq ((\mathcal{A}_{x_1}(x, D') - \mathcal{A}(x, D')^2) u, u). \]

Here \( \mathcal{A}_{x_1}(x, D') = \partial \mathcal{A}(x, D') / \partial x_1 \).

\textbf{Proof.} Since \( \mathcal{A}^* = \mathcal{A} \), it follows that

\[ (D_{x_1}^2 u, u) = ((D_{x_1} + \sqrt{-1}\mathcal{A}(x, D'))(D_{x_1} - \sqrt{-1}\mathcal{A}(x, D')) u, u) \]

\[ + ((\mathcal{A}_{x_1}(x, D') - \mathcal{A}(x, D')^2) u, u) \]

\[ = \| (D_{x_1} - \sqrt{-1}\mathcal{A}(x, D')) u \|^2 + ((\mathcal{A}_{x_1}(x, D') - \mathcal{A}(x, D')^2) u, u) \]

\[ \geq ((\mathcal{A}_{x_1}(x, D') - \mathcal{A}(x, D')^2) u, u). \]

This proves the lemma.

\textbf{Lemma 2.4} tells us that the differential operator \( D_{x_1}^2 \) can be estimated from below by the pseudodifferential operator \( \mathcal{A}_{x_1}(x, D') - \mathcal{A}(x, D')^2 \) on \( \mathbb{R}^{n-1} \) in the sense of the inner product of \( L^2(\mathbb{R}^n) \). In terms of symbols, one may estimate the symbol \( \xi_1^2 \) as follows:

\[ \xi_1^2 \geq F_{x_1}(x_1, x') (\log \langle \xi' \rangle)^{\iota} - F(x_1, x')^2 (\log \langle \xi' \rangle)^{2\iota} \mod S^{0}_{1,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}). \]

This trick is due to Wakabayashi.

5) Now, applying \textbf{Lemma 2.4} with

\[ a(x_1, x', \xi') = 2(2C+1) \left( \int_{0}^{x_1} \lambda(t, x') \, dt \right) (\log \langle \xi' \rangle)^{\iota}, \]

we find that the symbol \( (1/2)\xi_1^2 \) may be replaced by the following:

\[ (2C+1)\tilde{\lambda}(x_1, x') (\log \langle \xi' \rangle)^{\iota} - 2(2C+1)^2 \tilde{\lambda}(x_1, x')^2 (\log \langle \xi' \rangle)^{2\iota}, \]

where

\[ \tilde{\lambda}(x_1, x') = \int_{0}^{x_1} \lambda(t, x') \, dt. \]

In view of formula [2.9], this proves that in a conic neighborhood of \( (x^0, \xi^0) \)
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\[ \text{Re} \tilde{P}_{\Lambda_{\delta}}(x, \xi) \geq \frac{1}{2} \xi_{1}^{2} + \left[ \frac{1}{2} \xi_{1}^{2} + \frac{1}{2} \alpha(x_{1}, x', \xi') - 2C \lambda(x_{1}, x')(\log \langle \xi' \rangle)^{2} \right] \]

\[ \geq \frac{1}{2} \xi_{1}^{2} + \pi(x_{1}, x', \xi') \mod S_{1.0}^{0}(R^{n-1} \times R^{n-1}), \]

where \( \pi(x_{1}, x', \xi') \) is a symbol in the class \( S_{1.0}^{0}(R^{n-1} \times R^{n-1}) \) given by the following formula:

\[ \pi(x_{1}, x', \xi') = \frac{1}{2} \alpha(x_{1}, x', \xi') + \lambda(x_{1}, x')(\log \langle \xi' \rangle)^{2} \]
\[ -2(2C + 1)^{2} \tilde{\lambda}(x_{1}, x')^{2}(\log \langle \xi' \rangle)^{4}. \]

Thus we are reduced to the positivity of the symbol \( \pi(x_{1}, x', \xi') \).

(a) First, if we have

\[ \lambda(x_{1}, x')(\log \langle \xi' \rangle)^{2} - 2(2C + 1)^{2} \tilde{\lambda}(x_{1}, x')^{2}(\log \langle \xi' \rangle)^{4} \geq 0, \]

then it follows that

\[ \pi(x_{1}, x', \xi') \geq 0. \]

(b) Next we assume that

\[ \lambda(x_{1}, x')(\log \langle \xi' \rangle)^{2} - 2(2C + 1)^{2} \tilde{\lambda}(x_{1}, x')^{2}(\log \langle \xi' \rangle)^{4} \leq 0, \]

that is,

(2.10)

\[ \log \langle \xi' \rangle \geq \frac{\sqrt{\lambda(x_{1}, x')}}{\sqrt{2(2C + 1)|\tilde{\lambda}(x_{1}, x')|}}. \]

Then we shall show that condition [0.1] implies that in a conic neighborhood of \((x^{0}, \xi^{0})\)

(2.11)

\[ \frac{1}{2} \alpha(x_{1}, x', \xi') \geq 2(2C + 1)^{2} \tilde{\lambda}(x_{1}, x')^{2}(\log \langle \xi' \rangle)^{4}, \]

which proves that

\[ \pi(x_{1}, x', \xi') \geq 0. \]

By condition (A.2), it follows that

\[ \alpha(x_{1}, x', \xi') \geq \mu(x_{1}, x')|\xi'|^{2} \text{ on } T^{*}(R^{n-1}). \]

Thus it suffices to show that

(2.12) \[ \mu(x_{1}, x')|\xi'|^{2} \geq 4(2C + 1)^{2} \tilde{\lambda}(x_{1}, x')^{2}(\log \langle \xi' \rangle)^{4}. \]

If we take the logarithm of the both sides, we obtain that

\[ \log \mu(x_{1}, x') + 2 \log |\xi'| \geq \log [4(2C + 1)^{2}] + 2 \log |\tilde{\lambda}(x_{1}, x')| + 4 \log(\log \langle \xi' \rangle). \]
This condition is satisfied if we have for $|\xi'|$ sufficiently large
\[(2.12') \quad \log \mu(x, x') + \log \langle \xi' \rangle \geq 2 \log |\tilde{\lambda}(x, x')|.
\]
Therefore, combining inequalities \[(2.10)\] and \[(2.12')\], we obtain that condition \[(2.11)\] is satisfied if we have for $|x_1|$ sufficiently small
\[
\log \mu(x, x') + \frac{\sqrt{\lambda(x_1, x')}}{\sqrt{2(2C+1)|\tilde{\lambda}(x_1, x')|}} \geq 0,
\]
since $\log |\tilde{\lambda}(x_1, x')| < 0$ for $|x_1|$ sufficiently small.

Summing up, we have proved that if the condition
\[(0.1) \quad \lim_{x_1 \to 0} \frac{\tilde{\lambda}(x_1, x') \log \mu(x_1, x')}{\sqrt{\lambda(x_1, x')}} = 0
\]
is satisfied, then we have
\[
\text{Re} \tilde{P}_{\Lambda_{\delta}}(x, \xi) \geq \frac{1}{2} \xi_{1}^{2} + \pi(x_1, x', \xi'),
\]
and further the symbol $\pi(x_1, x', \xi')$ is non-negative and forms a bounded subset of the class $S_{1.0}^{2}(R^{n-1} \times R^{n-1})$ for $|x_1| \leq \epsilon_0$ if $\epsilon_0 > 0$ is sufficiently small.

6) Therefore, applying Corollary 2.2 to the operator $\pi(x_1, x', D')$, we obtain that if $\epsilon_0 > 0$ is sufficiently small, then we have for all $v \in C_{0}^{\infty}(U_{\epsilon_0})$ and all $0 < \delta \leq 1$
\[
\text{Re}(\tilde{P}_{\Lambda_{\delta}}(x, D)v, v) \geq \frac{1}{2} \|D_{x_1}v\|^{2} - \tilde{C} \|v\|^{2},
\]
with a constant $\tilde{C} > 0$ independent of $\delta$. Hence, in view of formula \[(2.3)\], this proves inequality \[(2.1)\].

The proof of Theorem 1 is now complete.

3. Proof of Theorem 2.

The proof of Theorem 2 is essentially the same as that of Theorem 1.

1) Let $x^{0}=(x_{1}^{0}, x_{2}^{0}, \ldots, x_{n}^{0})$ be a point of a closed subset $S$ of the hypersurface $\{x=(x_{1}, x') \in R^{n} ; x_{1}=0\}$, where $x'=(x_{2}, \ldots, x_{n})$. Without loss of generality, one may assume that
\[
x^{0} = (0, 0).
\]
If $0 \leq \delta \leq 1$, $a \geq 0$, $N \geq 0$ and $s \in R$, we let
\[
A_{\delta}(x', \xi) = A_{\delta}(x', \xi; a, N, s) = (-s + a |x'|^{2}) \log \lambda(\xi) + N \log (1+\delta \lambda(\xi)),
\]
and
\[
Q_{A_{\delta}}(x, D) = e^{-A_{\delta}(x', D)}Q(x, D)e^{A_{\delta}(x', D)},
\]
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where \( e^{\pm \Lambda_{\delta}}(x', D) \) are properly supported pseudodifferential operators with symbols \( e^{\pm \Lambda_{\delta}(x', \xi)} \), respectively:

\[
\begin{align*}
  e^{\Lambda_{\delta}(x', \xi)} &= \lambda(\xi)^{-s-a|x'|^2} (1 + \delta \lambda(\xi))^{-N}, \\
  e^{-\Lambda_{\delta}(x', \xi)} &= \lambda(\xi)^{(s-a|x'|^2)} (1 + \delta \lambda(\xi))^{-N}.
\end{align*}
\]

By virtue of Corollary 1.2, it suffices to show that there exists an open neighborhood \( U_{\epsilon_0} = \{ x = (x_1, x') \in \mathbb{R}^n ; |x_1| < \epsilon_0, |x'| < 1 \} \) of \( x^0 = (0, 0) \) such that we have for all \( v \in C_c(U_{\epsilon_0}) \) and all \( 0 < \delta \leq 1 \)

\[
| (Q_{\Lambda_{\delta}}(x, D)v, v) | \geq C_1 \||D_{x_1}v||^2 - C_2 ||v||^2,
\]

with constants \( C_1 > 0 \) and \( C_2 > 0 \) independent of \( \delta \).

2) Since the operator \( Q(x, D) \) is micro-elliptic outside a conic neighborhood of a point \( (x^0, \xi^0) = (0, 0, \xi_2^0, \cdots, \xi_n^0) \) in the bundle \( T^*(\mathbb{R}^n) \setminus 0 \) of non-zero cotangent vectors, one may assume that

\[
4 \leq |\xi| \leq 2 |\xi'|, \quad \xi = (\xi_i, \xi'),
\]

and that

\[
\begin{align*}
  \Lambda(\xi) &= (1 + |\xi'|^2)^{1/2} = \langle \xi' \rangle, \\
  A_\delta(x', \xi') &= A_\delta(x' - N \log(1 + \delta \langle \xi' \rangle)).
\end{align*}
\]

Then, arguing as in the proof of Theorem 1 (cf. formula (2.4)), one can find an elliptic symbol \( s_\delta(x', \xi) = s_\delta(x', \xi ; a, N, s) \) in the class \( \mathcal{S}^{0}_{1.0}(\mathbb{R}^n \times \mathbb{R}^n) \) such that we have for \( |\xi| \) sufficiently large (uniformly in \( \delta > 0 \))

\[
s_\delta(x', \xi)Q_{\Lambda_{\delta}}(x, \xi) \equiv \xi_i + \alpha(x, \xi^n) + b^n(x)A_{\delta x_n} - \frac{1}{2} \sum_{j,k=2}^{n-1} \alpha_{\xi_j \xi_k} A_{\delta x_j} A_{\delta x_k}
\]

\[
+ \sum_{j=2}^{n-1} \frac{1}{2} \sum_{k=2}^{n-1} \alpha_{\delta x_j \delta x_k} A_{\delta x_j} A_{\delta x_k} + \sum_{j=2}^{n-1} \alpha_{\delta x_j} A_{\delta x_j} A_{\delta x_k} + \sum_{j=2}^{n-1} \alpha_{\delta x_j} A_{\delta x_j} A_{\delta x_k}
\]

\[
\equiv \sum_{j=2}^{n-1} \alpha_{\xi_j} A_{\delta x_j} - \sum_{j=2}^{n-1} \alpha_{\delta x_j} A_{\delta x_j}
\]

\[
\text{mod } \mathcal{S}^{0}_{1.0}(\mathbb{R}^n \times \mathbb{R}^n),
\]

where

\[
\alpha(x, \xi^n) = \sum_{j=2}^{n-1} a_{\xi_j} \xi_j, \quad \xi^n = (\xi_\xi, \cdots, \xi_{n-1}).
\]

We let

\[
\tilde{Q}_{\Lambda_{\delta}}(x, D) = s_\delta(x', D)Q_{\Lambda_{\delta}}(x, D),
\]

where \( s_\delta(x', D) \) is a properly supported, elliptic pseudodifferential operator with
symbol $s_\delta(x', \xi)$ such that we have for $|\xi|$ sufficiently large (uniformly in $\delta > 0$)

$$\frac{1}{2} \leq |s_\delta(x', \xi)| \leq 2.$$ 

Now we remark that

$$|\widetilde{Q}_{\Lambda}(x, D)v, v| = \left[ (\Re(\widetilde{Q}_{\Lambda}(x, D)v, v))^2 + (\Im(\widetilde{Q}_{\Lambda}(x, D)v, v))^2 \right]^{1/2} \geq \frac{\sqrt{2}}{2} (\Re(\widetilde{Q}_{\Lambda}(x, D)v, v) + |\Im(\widetilde{Q}_{\Lambda}(x, D)v, v)|).$$

First we estimate the term $|\Im(\widetilde{Q}_{\Lambda}(x, D)v, v)|$. To do so, arguing as in the proof of Lemma 2.3, we have for every $\epsilon > 0$

$$|\alpha_{x}(x, \xi^\prime) A_{\delta x}(x', \xi^\prime)| \leq \frac{1}{\epsilon} \alpha(x, \xi^\prime) + \frac{1}{\epsilon} a^2 x_n^2 a^{jj}(x)(\log |\xi'|)^2 \text{mod } S_{1,0}^{0}(\mathbb{R}^n \times \mathbb{R}^n),$$

$$|\alpha_{x}(x, \xi^\prime) A_{\delta x}(x', \xi^\prime)| \leq \frac{1}{\epsilon} \alpha(x, \xi^\prime) \text{mod } S_{1,0}^{0}(\mathbb{R}^n \times \mathbb{R}^n).$$

Here we recall that for all $\xi = (\xi_1, \xi^\prime)$ in a conic neighborhood of $\xi^0 = (0, \xi^{0\prime})$

$$|\xi^\prime| \leq |\xi| \leq 2|\xi^\prime|.$$ 

Furthermore, condition (B') implies that the function $b^n$ does not change sign. Hence, for every $\epsilon > 0$, one can find a constant $C_\epsilon > 0$ such that

$$|\Im(\widetilde{Q}_{\Lambda}(x, D)v, v)| \geq \Re(b^n(x)|D_\nu v, v| - \epsilon \Re(\alpha(x, D^n)v, v)$$

$$- C_\epsilon \sum_{j=1}^{n-1} \Re(a^{jj}(x)(\log |D'|)^2v, v),$$

where $\langle D_{x_n} \rangle$ and $\log|D'|$ are pseudodifferential operators with symbols $\langle \xi_n \rangle = (1 + \xi_n^2)^{1/2}$ and $\log((1 + |\xi'|^2)^{1/2})$, respectively.

Next we estimate the term $\Re(\widetilde{Q}_{\Lambda}(x, D)v, v)$. Similarly, applying Lemma 2.3 to the terms $\alpha_{x, j} A_{\delta x_j} A_{\delta x_j} A_{\delta x_j}$ and $\alpha_{x, j} A_{\delta x_j} A_{\delta x_j} A_{\delta x_j}$, we have for every $\epsilon > 0$

$$\alpha_{x, j} A_{\delta x_j}(x', \xi^\prime)|A_{\delta x_j}(x', \xi^\prime)| \geq - \epsilon \alpha(x, \xi^\prime) \text{mod } S_{1,0}^{0}(\mathbb{R}^n \times \mathbb{R}^n),$$

$$\alpha_{x, j} A_{\delta x_j}(x', \xi^\prime)|A_{\delta x_j}(x', \xi^\prime)| \geq - \epsilon \alpha(x, \xi^\prime) \text{mod } S_{1,0}^{0}(\mathbb{R}^n \times \mathbb{R}^n).$$

Moreover, by virtue of condition (A.1'), we can estimate the terms $\alpha_{x, j} A_{\delta x_j} A_{\delta x_j}$ as follows:

$$\alpha_{x, j} A_{\delta x_j}(x', \xi^\prime)|A_{\delta x_j}(x', \xi^\prime)| \geq - \epsilon \alpha(x, \xi^\prime) \text{mod } S_{1,0}^{0}(\mathbb{R}^n \times \mathbb{R}^n).$$

We also have

$$b^n(x)|A_{\delta x_j}(x', \xi^\prime)| \geq - 2a |x_n| |b^n(x)| \log |\xi'| \text{mod } S_{1,0}^{0}(\mathbb{R}^n \times \mathbb{R}^n).$$

Hence, arguing as in the proof of formula (2.9), we obtain that for some constants $C_1 > 0$ and $C_\delta > 0$ independent of $\delta$. 

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\[ \text{Re}(\tilde{Q}_{\Lambda_{\delta}}(x, D)v, v) \geq (D_{x}^{2}v, v) + \frac{3}{4} \text{Re}(\alpha(x, Dv, v)
\nonumber
- C_{1} \text{Re}(|b_{n}(x)| \log \langle D' \rangle v, v)
\nonumber
- C_{2} \sum_{j=2}^{n-1} \text{Re}(a^{jj}(x)(\log \langle D^{f} \rangle)^{2}v, v). \]

Therefore, we can find a second-order pseudodifferential operator \( \tilde{R}_{\Lambda_{\delta}}(x, D) \) with symbol \( \tilde{R}_{\Lambda_{\delta}}(x, \xi) \) such that

\[ |(\tilde{Q}_{\Lambda_{\delta}}(x, D)v, v)| \geq \frac{\sqrt{2}}{2} \text{Re}(\tilde{R}_{\Lambda_{\delta}}(x, D)v, v), \]

and that in a conic neighborhood of \( (x^{0}, \xi^{0}) \)

\[ \tilde{R}_{\Lambda_{\delta}}(x, \xi) \geq \xi_{1}^{2} + \frac{1}{2} \alpha(x, \xi'') + |b^{n}(x)| \langle \xi_{n} \rangle
\nonumber
- A|b^{n}(x)| \log \langle \xi' \rangle - B\lambda(x)(\log \langle \xi' \rangle)^{2}
\nonumber
\mod S_{1,0}^{0}(R^{n} \times R^{n}). \]

Here \( A > 0 \) and \( B > 0 \) are constants independent of \( \delta \), and

\[ \lambda(x) = \sum_{j=2}^{n-1} a^{jj}(x). \]

Thus we are reduced to the study of the symbol \( \tilde{R}_{\Lambda_{\delta}}(x, \xi) \).

3-i) Assume that

\[ \xi^{a} = (0, \xi_{2}^{a}, \ldots, \xi_{n-1}^{a}, \xi_{n}^{a}) \text{ with } \xi_{n}^{a} \neq 0. \]

Then we remark that, for all \( \xi \) in a conic neighborhood of \( \xi^{a} \), there exists a constant \( c_{1} > 0 \) such that

\[ c_{1} \xi' / \leq |\xi_{n}| \leq |\xi'|, \quad \xi' = (\xi_{2}, \ldots, \xi_{n-1}, \xi_{n}). \]

Hence, by formula \([3.4]\), we have for \( |\xi_{n}| \) sufficiently large (uniformly in \( \delta > 0 \))

\[ \tilde{R}_{\Lambda_{\delta}}(x, \xi) \geq \xi_{1}^{2} + \frac{1}{2} \alpha(x, \xi^{n}) + |b^{n}(x)| \langle \xi_{n} \rangle
\nonumber
- A|b^{n}(x)| \log \langle \xi_{n} \rangle - B\lambda(x)(\log \langle \xi_{n} \rangle)^{2}
\nonumber
\geq \frac{1}{2} \xi_{1}^{2} + \left[ \frac{1}{2} \xi_{1}^{2} + \frac{1}{2} |b^{n}(x_{1}, x')| |\xi_{n}| - B\lambda(x_{1}, x')(\log \langle \xi_{n} \rangle)^{2} \right]
\nonumber
\mod S_{1,0}^{0}(R^{n-1} \times R^{n-1}). \]

Therefore, arguing as in step 5) of the proof of [Theorem 1], we find that if the condition

\[ \lim_{x_{1} \to 0} \frac{\tilde{\lambda}(x_{1}, x') \log |b^{n}(x_{1}, x')|}{\sqrt{\lambda(x_{1}, x')}} = 0 \]
is satisfied, then we have in a conic neighborhood of \((x^0, \xi^0)\)
\[
R_{\Lambda}(x, \xi) \geq \frac{1}{2} \xi_1^2 + \rho_1(x, x', \xi') \mod S_{1.0}^{\bar{\Lambda}}(R^{n-1} \times R^{n-1}),
\]
and the symbol \(\rho_1(x, x', \xi')\) is non-negative and forms a bounded subset of the class \(\mathcal{S}_{1.0}^{1}(R^{n-1} \times R^{n-1})\) for \(|x_1| \leq \varepsilon_0\) if \(\varepsilon_0 > 0\) is sufficiently small.

3-ii) Assume that
\[
\xi^0 = (0, \xi^{0p}, \xi_{n}^{0}) \quad \text{with} \quad \xi^{0p} = (\xi_2^{0}, \cdots, \xi_{n-1}^{0}) \neq 0.
\]
Then we remark that, for all \(\xi\) in a conic neighborhood of \(\xi^0\), there exists a constant \(c_2 > 0\) such that
\[
c_2 |\xi'| \leq |\xi^0| \leq |\xi'|, \quad \xi' = (\xi^0, \xi_1).
\]
Hence, by formula \(3.4\), we have in a conic neighborhood of \((x^0, \xi^0)\)
\[
R_{\Lambda}(x, \xi) \geq \xi_1^2 + \frac{1}{2} \alpha(x, x', \xi^0) + |b^n(x)| \langle \xi_1 \rangle - A|b^n(x)| \log \langle \xi^0 \rangle

- B \lambda(x) (\log \langle \xi^0 \rangle)^2

\geq \frac{1}{2} \xi_1^2 + \left[ \frac{1}{2} \xi_1^2 + \frac{1}{2} \alpha(x, x', \xi^0) - A|b^n(x)| \log \langle \xi^0 \rangle - B \lambda(x) (\log \langle \xi^0 \rangle)^2 \right]

\mod S_{1.0}^{1}(R^{n-1} \times R^{n-1}).
\]
Now, applying Lemma 2.4 with
\[
a(x, x', \xi^0) = 2(A+1) \left( \int_0^{x_1} |b^n(t, x')| dt \right) \log \langle \xi^0 \rangle,
\]
we find that the symbol \((1/2)\xi_1^2\) in the bracket in formula \(3.5\) may be replaced by the following:
\[
(A+1)|b^n(x_1, x')| \log \langle \xi^0 \rangle - 2(A+1)^2 b^n(x_1, x') \log \langle \xi^0 \rangle^2,
\]
where
\[
b^n(x_1, x') = \int_0^{x_1} |b^n(t, x')| dt.
\]
This proves that in a conic neighborhood of \((x^0, \xi^0)\)
\[
R_{\Lambda}(x, \xi) \geq \frac{1}{2} \xi_1^2 + \left[ \frac{1}{2} \xi_1^2 + \frac{1}{2} \alpha(x_1, x', \xi^0) - A|b^n(x_1, x')| \log \langle \xi^0 \rangle

- B \lambda(x_1, x') (\log \langle \xi^0 \rangle)^2 \right]

\geq \frac{1}{2} \xi_1^2 + \rho_2(x_1, x', \xi^0) \mod S_{1.0}^{1}(R^{n-1} \times R^{n-1}),
\]
where \(\rho_2(x_1, x', \xi^0)\) is a symbol in the class \(\mathcal{S}_{1.0}^{1}(R^{n-1} \times R^{n-1})\) given by the follow-
ing formula:

\[ \rho_2(x_1, x', \xi^{''}) = \frac{1}{2} \alpha(x_1, x', \xi^{''}) + |b^n(x_1, x')| \log \langle \xi^{''} \rangle - C(\bar{b}^{n}(x_1, x')^{2} + \lambda(x_1, x'))(\log \langle \xi^{''} \rangle)^{2}, \]

with \[ C = \max(2(A+1)^{2}, B). \]

Thus we are reduced to the positivity of the symbol \( \rho_2(x_1, x', \xi^{''}) \).

(a) First, if we have

\[ |b^n(x_1, x')| \log \langle \xi' \rangle - C(\bar{b}^{n}(x_1, x')^{2} + \lambda(x_1, x'))(\log \langle \xi' \rangle)^{2} \geq 0, \]

then it follows that

\[ \rho_2(x_1, x', \xi^{''}) \geq 0. \]

(b) Next we assume that

\[ b^n(x_1, x')| \log \langle \xi' \rangle - C(\bar{b}^{n}(x_1, x')^{2} + \lambda(x_1, x'))(\log \langle \xi' \rangle)^{2} \leq 0, \]

that is,

(3.6) \[ \log \langle \xi' \rangle \geq \frac{|b^n(x_1, x')|}{C(\bar{b}^{n}(x_1, x')^{2} + \lambda(x_1, x'))}. \]

Then we shall show that conditions (0.2b) and (0.2c) imply that in a conic neighborhood of \((x^{0}, \xi^{0})\)

(3.7) \[ \frac{1}{2} \alpha(x_1, x', \xi^{''}) \geq C(\bar{b}^{n}(x_1, x')^{2} + \lambda(x_1, x'))(\log \langle \xi' \rangle)^{2}, \]

which proves that

(3.8) \[ \rho_2(x_1, x', \xi^{''}) \geq 0. \]

By condition (A.2'), it follows that

\[ \alpha(x_1, x', \xi^{''}) \geq \mu(x_1, x') |\xi''|^2 \text{ on } T^{*}(R^{n-1}). \]

Thus it suffices to show that

(3.9) \[ \mu(x_1, x') |\xi''|^2 \geq 2C(\bar{b}^{n}(x_1, x')^{2} + \lambda(x_1, x'))(\log \langle \xi'' \rangle)^{2}. \]

If we take the logarithm of the both sides, we obtain that

\[ \log \mu(x_1, x') + 2 \log |\xi''| \geq \log 2C + \log(\bar{b}^{n}(x_1, x')^{2} + \lambda(x_1, x')) + 2 \log (\log \langle \xi'' \rangle). \]

This condition is satisfied if we have for \( |\xi''| \) sufficiently large

(3.9') \[ \log \mu(x_1, x') + \log \langle \xi'' \rangle \geq \log(\bar{b}^{n}(x_1, x')^{2} + \lambda(x_1, x')). \]
Thus, combining inequalities (3.6) and (3.9'), we obtain that condition (3.7) is satisfied if we have for $|x_1|$ sufficiently small

$$\log \mu(x_1, x') + \frac{|b^n(x_1, x')|}{C(b^n(x_1, x')^2 + \lambda(x_1, x'))} \geq 0,$$

since $\log(b^n(x_1, x')^2 + \lambda(x_1, x')) < 0$ for $|x_1|$ sufficiently small.

Therefore, we find that the conditions

\[(0.2b) \quad \lim_{x_1 \to 0} \frac{b^n(x_1, x') \log \mu(x_1, x')}{b^n(x_1, x')} = 0,\]
\[(0.2c) \quad \lim_{x_1 \to 0} \frac{\lambda(x_1, x') \log \mu(x_1, x')}{b^n(x_1, x')} = 0\]

imply the desired condition (3.7) and hence condition (3.8).

Summing up, we have proved that if conditions (0.2a), (0.2b) and (0.2c) are satisfied, then we have

$$\hat{R}_{\delta}(x, \xi) \geq \frac{1}{2} |\xi_1|^2 + \rho(x_1, x', \xi') \mod S^{2}_{1,0}(R^{n-1} \times R^{n-1}),$$

and further the symbol $\rho(x_1, x', \xi')$ is non-negative and forms a bounded subset of the class $S^{2}_{1,0}(R^{n-1} \times R^{n-1})$ for $|x_1| \leq \varepsilon_0$ if $\varepsilon_0 > 0$ is sufficiently small.

4) Therefore, applying Corollary 2.2 to the operator $\rho(x_1, x', D')$, we obtain that if $\varepsilon_0 > 0$ is sufficiently small, then we have for all $v \in C_{0}^\infty(U_{\varepsilon_0})$ and all $0 < \delta \leq 1$

$$\text{Re}(\hat{R}_{\delta}(x, D)v, v) \geq \frac{1}{2} \|Dv\|^2 - \tilde{C} \|v\|^2,$$

with a constant $\tilde{C} > 0$ independent of $\delta$. In view of inequality (3.3') and formula (3.2), this proves inequality (3.1).

The proof of Theorem 2 is now complete.

References


Hypoelliptic differential operators


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