On some noncoercive boundary value problems for the Laplacian

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31. On Some Noncoercive Boundary Value Problems for the Laplacian

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1. Introduction. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with boundary \( \Gamma \) of class \( C^\infty \). \( \overline{\Omega} = \Omega \cup \Gamma \) is a \( C^\infty \)-manifold with boundary. Let \( a \), \( b \) and \( c \) be real valued \( C^\infty \)-functions on \( \Gamma \), let \( \mathbf{n} \) be the unit exterior normal to \( \Gamma \) and let \( \alpha \) and \( \beta \) be real \( C^\infty \)-vector fields on \( \Gamma \).

We shall consider the following boundary value problem: For given functions \( f \) defined on \( \Omega \) and \( \phi \) defined on \( \Gamma \) find \( u \) in \( \Omega \) such that

\[
(\lambda - \Delta)u = f \quad \text{in} \quad \Omega,
\]

\[
\mathbf{n} \cdot \nabla u + (\alpha + i\beta)u + (b + ic)u = \phi \quad \text{on} \quad \Gamma.
\]

Here \( \lambda \geq 0 \) and \( \Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \cdots + \partial^2/\partial x_n^2 \). The problem (*) in the case that \( b(x) \equiv 0 \) on \( \Gamma \), i.e., the oblique derivative problem was investigated by many authors (cf. [2], [6], [7], [8]), but the problem (*) in the case that

\[ a(x) \equiv 0 \\text{on} \\Gamma \]

was treated by a few authors, e.g., Vainberg and Grušin [12] (see also [5]), whose results we shall first describe briefly. For each real \( s \), we shall denote by \( H^s(\Omega) \) (resp. \( H^s(\Gamma) \)) the Sobolev space on \( \Omega \) (resp. \( \Gamma \)) of order \( s \) and by \( \| \cdot \|_s \) (resp. \( | \cdot |_s \)) its norm.

If \( a(x) > |\beta(x)| \) on \( \Gamma \), then the problem (*) is coercive and the following results are valid for all \( s > \frac{3}{2} \) (cf. [9]):

i) For every solution \( u \in H^s(\Omega) \) of (*) with \( f \in H^{s-2}(\Omega) \) and \( \phi \in H^{s-3/2}(\Gamma) \) we have \( u \in H^t(\Omega) \) and an a priori estimate

\[
\| u \|_t \leq C \| f \|_{s-2} + |\phi|_{s-3/2} + \| u \|_s
\]

where \( t < s \) and \( C > 0 \) is a constant depending only on \( \lambda, s \) and \( t \).

ii) If \( f \in H^{s-2}(\Omega) \), \( \phi \in H^{s-3/2}(\Gamma) \) and \( (f, \phi) \) is orthogonal to some finite dimensional subspace of \( C^{\infty}(\overline{\Omega}) \oplus C^{\infty}(\Gamma) \), then there is a solution \( u \in H^s(\Omega) \) of (*).

iii) If \( \lambda > 0 \) is sufficiently large, then we can omit \( \| u \|_t \) in the right hand side of (1) and for every \( f \in H^{s-2}(\Omega) \) and every \( \phi \in H^{s-3/2}(\Gamma) \) there is a unique solution \( u \in H^s(\Omega) \) of (*).

If \( a(x) \geq |\beta(x)| \) on \( \Gamma \) and \( a(x) = |\beta(x)| \) holds at some points of \( \Gamma \), then the problem (*) is noncoercive. Vainberg and Grušin [12] treated the problem (*) in the case that \( n = 2 \), \( a(x) \equiv 1 \), \( \alpha(x) \equiv 0 \), \( |\beta(x)| \equiv 1 \) on \( \Gamma \). Under the assumption that \( b(x) + ic(x) \neq 0 \) on \( \Gamma \), they proved smoothness, an a priori estimate and existence theorems for the solutions of
(\star), which involve a loss of 1 derivative compared with the results i) and ii) (see [12], Theorem 19).

In this note we shall treat the problem (\star) in the case that \( n \) is arbitrary and that \( \alpha(x) \geq |\beta(x)| \) on \( \Gamma \). Under the assumptions expressed in terms of differential geometry such as the second fundamental form of the hypersurface \( \Gamma \subset \mathbb{R}^n \), the mean curvature of \( \Gamma \), the divergence of the vector field \( \alpha \) and so on (see (B-1), (B-2), (B-1), (B-2) and (C)), we shall give smoothness, an \textit{a priori} estimate and existence theorems for the solutions of (\star), which involve a loss of 1 derivative compared with the results i), ii) and iii) (Theorem 1 and Theorem 2). Even in the case that \( \beta(x) \equiv 0 \) on \( \Gamma \) and hence that \( \alpha(x) \geq 0 \) on \( \Gamma \), these results are new (cf. [2], [7], [8]). The details will be given somewhere else.

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2. Preliminaries. Since \( \lambda \geq 0 \), for every \( \phi \in C^{n}(\Gamma) \) we can uniquely solve the Dirichlet problem:

\[
\begin{aligned}
(\lambda - \Delta)w &= 0 \quad \text{in } \Omega, \\
w &= \phi \quad \text{on } \Gamma,
\end{aligned}
\]

hence we can define the Poisson operator \( \mathcal{P}(\lambda) \) by \( w = \mathcal{P}(\lambda)\phi \). The mapping \( T(\lambda) : \phi \mapsto \mathcal{B}\mathcal{P}(\lambda)\phi \big|_{\Gamma} \) is a first order pseudodifferential operator on \( \Gamma \) (cf. [5], [6], [12]) and the problem (\star) can be reduced to the study of \( T(\lambda) \) by the same argument as the proof of Theorem 2.2 of Taira [11] (cf. [6], [7], [12]). The \textit{principal} symbol of \( T(\lambda) \) is

\[
(\alpha(x) |\xi| - \beta(x, \xi)) + i\alpha(x, \xi)
\]

(see [5], § 3). Here \( x = (x_1, x_2, \ldots, x_n) \) are some local coordinates in \( \Gamma \) and \( \xi = (\xi_1, \xi_2, \ldots, \xi_n) \) are the corresponding dual coordinates in the cotangent space \( T^*\Gamma \) and \( |\xi| \) is the length of \( \xi \) with respect to the Riemannian metric of \( \Gamma \) induced by the natural metric of \( \mathbb{R}^n \), and \( \alpha(x, \xi) \) (resp. \( \beta(x, \xi) \)) is the principal symbol of the vector field \( \alpha(x)/i \) (resp. \( \beta(x)/i \)).

Let \( \Lambda = (1 - \Delta')^{1/2} \) where \( \Delta' \) is the Laplace-Beltrami operator corresponding to the Riemannian metric of \( \Gamma \). To apply Theorem 3.1 of Melin [10] to \( \Re (\Lambda^{s-1}T(\lambda)) \) where \( s \geq 3/2 \) (see Proposition), we have to make a digression. Let \( p_1(x, \xi) = \alpha(x)|\xi| - \beta(x, \xi) \). Then \( p_1(x, \xi) \geq 0 \) on the space of non zero cotangent vectors \( T^*\Gamma \setminus 0 \) if and only if \( \alpha(x) \geq |\beta(x)| \) on \( \Gamma \). Hence we assume that \( p_1 \geq 0 \) on \( T^*\Gamma \setminus 0 \). Let \( \Sigma = \{\rho \in T^*\Gamma \setminus 0 ; p_1(\rho) = 0\} \). For every tangent vector \( u \) of \( T^*\Gamma \) at \( \rho \in \Sigma \), let \( v \) be some vector field on \( T^*\Gamma \) equal to \( u \) at \( \rho \) and define a quadratic form \( a_\rho(u, u) \) by the equation:

\[
a_\rho(u, u) = (v^\rho p_1)_\rho.
\]

Since \( p_1 \geq 0 \) on \( T^*\Gamma \setminus 0 \), it follows that \( a_\rho(u, u) \) is independent of the choice of \( v \). Let \( \hat{\Sigma}(T^*\Gamma) \) be the complexification of the tangent space \( T_\rho(T^*\Gamma) \) of \( T^*\Gamma \) at \( \rho \in \Sigma \). We consider the symplectic form
\[ \sigma = \sum_{j=1}^{n-1} d\zeta_j \wedge dx_j \quad \text{on } T^*\Gamma \]

and the quadratic form \( a_\rho \) as bilinear forms on \( \tilde{T}_\rho(T^*\Gamma) \times \tilde{T}_\rho(T^*\Gamma) \). Since \( \sigma \) is non-degenerate, we can define for every \( \rho \in \Sigma \) a linear map \( A_\rho: \tilde{T}_\rho(T^*\Gamma) \to \tilde{T}_\rho(T^*\Gamma) \) by the equation:

\[ a_\rho(u, \tilde{v}) = a_\rho(u, \tilde{v}), \quad u, \tilde{v} \in \tilde{T}_\rho(T^*\Gamma). \]

It is easily seen that the spectrum of \( A_\rho \) is situated on the imaginary axis, symmetrically around the origin (see [10], § 2). For every \( \rho \in \Sigma \), we shall denote by \( \text{Tr} H_\rho(\rho) \) the sum of the positive elements in \( i \cdot \text{Spectrum} (A_\rho) \) where each eigenvalue is counted with its multiplicity.

The subprincipal symbol of \( \text{Re} (T(\lambda)) \) is

\[ b(x) - \frac{1}{2} \text{div} \alpha(x) + \frac{1}{2} a(x)(|\xi|^{-2} \omega_\alpha(\xi, \xi) - (n-1)M(x)) \]

(cf. [5], § 3). Here \( \text{div} \alpha \) is the divergence of the vector field \( \alpha \) and \( M(x) \) is the mean curvature at \( x \) of the hypersurface \( \Gamma \subset \mathbb{R}^n \) and \( \omega_\alpha \) is the second fundamental form at \( x \) of \( \Gamma \), and \( \zeta \in T_x \Gamma \) is the tangent vector of \( \Gamma \) at \( x \) corresponding to \( \xi \in T^*_x \Gamma \) by the duality between \( T_x \Gamma \) and \( T^*_x \Gamma \) with respect to the Riemannian metric of \( \Gamma \), where \( T_x \Gamma \) (resp. \( T^*_x \Gamma \)) is the tangent (resp. cotangent) space of \( \Gamma \) at \( x \). Further, the subprincipal symbol of \( \text{Re} (A^{s-1}T(\lambda)) \) on \( \Sigma = \{(x, \xi) \in T^*\Gamma \setminus 0; a(x) |\xi| - \beta(x, \xi) = 0 \} \) is

\[
\left( b(x) - \frac{1}{2} \text{div} \alpha(x) \right) |\xi|^{s-3} + \frac{1}{2} a(x)(|\xi|^{-2} \omega_\alpha(\xi, \xi) - (n-1)M(x)) |\xi|^{s-3} \\
+ \frac{1}{2} [\{ |\xi|^{s-1}, \alpha(x, \xi) \}] - \frac{1}{2} a(x, \xi) \text{ div } \delta_\xi(x). \]

Here

\[ \{ |\xi|^{s-2}, \alpha(x, \xi) \} = \sum_{j=1}^{n-1} \left( \frac{\partial}{\partial \xi_j} (|\xi|^{s-2}) \frac{\partial}{\partial x_j} \alpha(x, \xi) - \frac{\partial}{\partial \xi_j} \alpha(x, \xi) \frac{\partial}{\partial x_j} (|\xi|^{s-2}) \right) \]

and

\[ \delta_\xi(x) = \sum_{j=1}^{n-1} \frac{\partial}{\partial \xi_j} (|\xi|^{s-2}) \frac{\partial}{\partial x_j} \]

is a real \( C^\infty \)-vector field on \( \Gamma \) defined for \( \xi \neq 0 \) (cf. [1], Proposition 5.2.1).

3. Results. Applying Theorem 3.1 of Melin [10] to \( \text{Re} (A^{s-1}T(\lambda)) \) where \( s \geq 3/2 \) and by the same argument as the proof of Theorem 6 of Fujiwara [4], we can obtain

Proposition. Let \( s \geq 3/2, t < s-3/2 \). There exist constants \( C_3 > 0 \) and \( C_6 \) depending only on \( \lambda, s \) and \( t \) such that the estimate

\[ (3) \quad \text{Re} (A^{s-1}T(\lambda)\phi, \phi) \geq C_3 |\phi|^{s-1/2} - C_6 |\phi|^t \]

holds for all \( \phi \in C^\infty(\Gamma) \) if and only if the following assumptions (A), (B-1), and (B-2), hold:

(A) \[ a(x) \geq |\beta(x)| \quad \text{on } \Gamma. \]

(B-1), At every point \( x \in \Gamma \) where \( a(x) = 0 \), the inequality

\[ 2b(x) - \text{div} \alpha(x) + \{ |\xi|^{s-3}, \alpha(x, \xi) \} - \alpha(x, \xi) \text{ div } \delta_\xi(x) > 0 \]
holds for all $\xi \in T^*_x \Gamma$ with $|\xi| = 1$ (see (2)).

(B-2). At every point $x \in \Gamma$ where $a(x) = |\beta(x)| > 0$, the inequality

$$\text{Tr} \, H_p(x, \xi) + 2b(x) - \text{div} \, a(x) + a(x) \left( \omega_2 \left( \frac{\beta(x)}{a(x)}, \frac{\beta(x)'}{a(x)} \right) - (n-1) M(x) \right)$$

$$+ \left[ |\xi|^{s-3}, a(x, \xi) \right] - a(x, \xi) \text{ div} \, \delta(x) > 0$$

holds for $\xi \in T^*_x \Gamma$ corresponding to $\beta(x)/a(x) \in T_x \Gamma$ by the duality between $T^*_x \Gamma$ and $T_x \Gamma$ with respect to the Riemannian metric of $\Gamma$ (see (2)).

Furthermore, if $\lambda > 0$ is sufficiently large, then we can omit $|\phi|_t$ in the right hand side of (3).

Remark 1. It follows from the assumption (A) that at every point $x \in \Gamma$ where $a(x) = 0$, $\text{Tr} \, H_p(x, \xi) = 0$ for all $\xi \in T^*_x \Gamma$ with $|\xi| = 1$.

Remark 2. If the set $\Gamma_0 = \{x \in \Gamma; a(x) = |\beta(x)|\}$ is an $(n-2)$-dimensional regular submanifold of $\Gamma$ and the vector field $a$ is transversal to $\Gamma_0$, then for every $s \geq 3/2$ we can construct a $C^\omega$-function $h$ on $\Gamma$ such that $h(x) > 0$ on $\Gamma$ and that the estimate (3) hold with $|A|^{s-2} T(\lambda)$ replaced by $h|A|^{s-2} T(\lambda)$ (cf. [8], Lemma 4).

By the same argument as the proof of Theorem 2.2 of Taira [11], we can obtain from Proposition

Theorem 1. Assume that

(a) $a(x) \geq |\beta(x)|$ on $\Gamma$

and that the assumptions (B-1) and (B-2) hold for some $s > 3/2$.

Then we have:

i) for every solution $u \in H^{s-1}(\Omega)$ of (1) with $f \in H^{s-1}(\Omega)$ and $\phi \in H^{s-1/2}(\Gamma)$ we have an a priori estimate:

$$\|u\|_{s-1} \leq C(t)(|f|_{s-1} + |\phi|_{s-1/2} + \|u\|_t)$$

where $t < s - 1$ and $C(t)$ is a constant depending only on $\lambda, s$ and $t$;

iii) if $\lambda > 0$ is sufficiently large, then we can omit $\|u\|_t$ in the right hand side of (4) and for every $f \in H^{s-1}(\Omega)$ and every $\phi \in H^{s-1/2}(\Gamma)$ there is a unique solution $u \in H^{s-1}(\Omega)$ of (1).

Remark 3. Further, we can prove that if $f \in H^{s-1}(\Omega)$, $\phi \in H^{s-1/2}(\Gamma)$ and $(f, \phi)$ is orthogonal to some finite dimensional subspace of $H^s_0(\Omega) \oplus H^{s+1/2}(\Gamma)$ where $H^s_0(\Omega)$ is the dual space of $H^{s-1}(\Omega)$, then there is a solution $u \in H^{s-1}(\Omega)$ of (1).

Remark 4. If the assumptions (B-1) and (B-2) hold for all $s > 3/2$, then by the same argument as the proof of Theorem 7.4 of Egorov and Kondrat’ev [2] we can prove that every solution $u \in H^{s-1}(\Omega)$ of (1) with $f \in H^{s-1}(\Omega)$ and $\phi \in H^{s-1/2}(\Gamma)$ belongs to $H(\Omega)$.

Further, applying Theorem 1 of Fediû [3] to $T(\lambda)$, we can obtain

Theorem 2. Assume that

(a) $a(x) \geq |\beta(x)|$ on $\Gamma$

and that the following assumptions (B-1), (B-2) and (C) hold:
(B-1) At every point $x \in \Gamma$ where $a(x) = 0$, $b(x) > 0$.
(B-2) At every point $x \in \Gamma$ where $a(x) = |\beta(x)| > 0$, the inequality
\[ \text{Tr } H_p(x, \xi) + 2b(x) - \text{div } a(x) \]
\[ + a(x) \left( \frac{\beta(x)}{a(x)}, \frac{\beta(x)}{a(x)} \right) - (n-1)M(x) > 0 \]
holds for $\xi \in T_x^* \Gamma$ corresponding to $|\beta(x)/a(x)| \in T_x \Gamma$.

(C) There exists a constant $C_0 > 0$ such that the inequality
\[ |d\alpha(x, \xi)| \leq C_0(a(x) - \beta(x, \xi)) \]
holds for all $x \in \Gamma$ and all $\xi \in T_x^* \Gamma$ with $|\xi| = 1$. Here $d\alpha$ is the exterior derivative of $\alpha(x, \xi)$ and $|d\alpha|$ is the length of the cotangent vector $d\alpha$ of $T^* \Gamma$ with respect to the natural metric of $T^* \Gamma$ induced by the Riemannian metric of $\Gamma$.

Then the assumptions (B-1) and (B-2) hold for all $s$ (hence by Theorem 1 we have for all $s > \frac{3}{2}$ the results i) and iii) and we have for all $s > \frac{3}{2}$:

i)" for every solution $u \in H^s(\Omega)$ of (*) with $f \in H^{s-1}(\Omega)$ and $\phi \in H^{s-\frac{3}{2}}(\Gamma)$ where $t < s - 1$, we have $u \in H^{s-1}(\Omega)$;

ii)" if $f \in H^{s-\frac{3}{2}}(\Omega)$, $\phi \in H^{s-\frac{3}{2}}(\Gamma)$ and $(f, \phi)$ is orthogonal to some finite dimensional subspace of $C^\infty(\overline{\Omega}) \oplus C^\infty(\Gamma)$, then there is a solution $u \in H^{s-\frac{3}{2}}(\Omega)$ of (*).

Remark 5. The example of Kato [8] shows that the assumption (C) is necessary for Theorem 2 to be valid.

Remark 6. In the case that $n = 2$, the inequality (5) is reduced to the following inequality (6):
\[ \text{Tr } H_p(x, \xi) + 2b(x) - \text{div } a(x) > 0, \]

since
\[ \omega_x \left( \frac{\beta(x)}{a(x)}, \frac{\beta(x)}{a(x)} \right) - (n-1)M(x) = 0. \]

References


