SUBSPACES OF THE SORGENFREY LINE AND THEIR PRODUCTS

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Abstract. In this article we study the products of subspaces of the Sorgenfrey line $\mathcal{S}$. Using an idea by D. K. Burke and J. T. Moore [2] we prove in particular the following:

Let $X_i$, $i = 1, \ldots, n$, $n \geq 1$, be subspaces of $\mathcal{S}$, where each $X_i$ is uncountable. Then $X_1 \times \cdots \times X_n \times \mathbb{Q}$ can be embedded in $\mathcal{S}^{n+1}$ but can not be embedded in $\mathcal{S}^n$, where $\mathbb{Q}$ is the space of rational numbers with the natural topology.

This statement strengthens [2, Theorem 2.1].

1 Introduction

All spaces considered here are assumed to be completely regular. Recall (see for example [4]) that the Sorgenfrey line $\mathcal{S}$ is the real line $\mathbb{R}$ with the topology whose base is the family $\{(a, b) : a, b \in \mathbb{R} \text{ with } a < b\}$. It is well known that $\mathcal{S}$ is a first-countable, hereditarily Lindelöf, hereditarily separable, Baire space such that the product $\mathcal{S}^2$ is not normal. The space $\mathcal{S}$ has different nice properties (see for example [1], [2], [3], [8]). In particular, D. K. Burke and J. T. Moore proved the following [2, Theorem 2.1].

If $X_0, \ldots, X_n$, $n \geq 1$, are uncountable subspaces of $\mathcal{S}$ then the product $X_0 \times \cdots \times X_n$ can not be embedded in $\mathcal{S}^n$.

This result shows that

(a) for any uncountable subspace $X$ of $\mathcal{S}$, $X^n$ is homeomorphic to $X^m$ iff $n = m$ where $n, m$ are positive integers;

(b) for a subspace $X$ of $\mathcal{S}$ if the subspace $X^n$ of $\mathcal{S}^n$ can be embedded in $\mathcal{S}^{n-1}$ then $X$ is countable.

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Using an idea of their proof we shall prove the following.

Define $S^{-1} = \emptyset$ and $S^0 = \emptyset$, where $\emptyset$ is the space of rational numbers with the natural topology. Put also $q(m, n, p) = m + 1$ if $n, m > 0$, and $q(m, n, p) = m$ otherwise, where $m, n, p$ are integers $\geq 0$.

**Theorem 1.1.** Let $\mathcal{F}$ be a finite family of non-empty subsets of $\mathcal{S}$ which are either uncountable, or homeomorphic to $\emptyset$ or discrete. Let also $m$ be the number of uncountable elements of $\mathcal{F}$, $n$ the number of elements of $\mathcal{F}$ homeomorphic to $\emptyset$, $p$ the number of discrete elements of $\mathcal{F}$ and $1 \leq n + m + p$. Then the product $\prod \mathcal{F}$ of all elements of $\mathcal{F}$ can be embedded in $\mathcal{S}^q$ and can not be embedded in $\mathcal{S}^{q-1}$, where $q = q(m, n, p)$.

Observe that Theorem 1.1 strengthens the mentioned above [2, Theorem 2.1] because any uncountable subspace of $\mathcal{S}$ contains a copy of $\emptyset$ as we will see in Lemma 2.2.

Note also that any subspace of $\mathcal{S}$ is either uncountable, or countable with at least one limit point, or discrete (and of course countable).

The next result is not complete as we wanted.

**Theorem 1.2.** Let $\mathcal{F}$ be any finite family of non-empty subsets of $\mathcal{S}$. Let $m$ be the number of uncountable elements of $\mathcal{F}$, $n$ the number of elements of $\mathcal{F}$ with at least one limit point, $p$ the number of discrete elements of $\mathcal{F}$ and $1 \leq n + m + p$. If $m \leq 2$ then the product $\prod \mathcal{F}$ of all elements of $\mathcal{F}$ can be embedded in $\mathcal{S}^q$ and can not be embedded in $\mathcal{S}^{q-1}$, where $q = q(m, n, p)$.

In particular,

**Theorem 1.3.** (i) Let $X_1$ and $X_2$ be subspaces of $\mathcal{S}$. Then $X_1 \times X_2$ can be embedded in $\mathcal{S}$ iff $X_1, X_2$ are both countable or one of them is discrete.

(ii) Let $X_i, i = 1, 2, 3$, be subspaces of $\mathcal{S}$. Then $X_1 \times X_2 \times X_3$ can be embedded in $\mathcal{S}$ iff all $X_i, i = 1, 2, 3$, are countable or two of them are discrete. $X_1 \times X_2 \times X_3$ can be embedded in $\mathcal{S}^2$ iff at least two of $X_i, i = 1, 2, 3$, are countable, or one of them is discrete.

**Problem 1.1.** Can one remove the condition $m \leq 2$ in Theorem 1.2?

A positive answer on this question would also evidently strengthen Theorem 1.1.
REMARK 1.1. There is an analog of Theorem 1.2 for the space \( \mathbb{R} \) of real numbers with the natural topology. Really, define \( \mathbb{R}^{-1} = \{ \emptyset \} \) and \( \mathbb{R}^0 = \mathbb{P} \), where \( \mathbb{P} \) is the space of irrational numbers with the natural topology. Note that any subspace of \( \mathbb{R} \) is either one-dimensional (and so contains an interval), or zero-dimensional with at least one limit point, or discrete. Using in particular Brouwer theorem about the invariance of internal points and the theorem about the universality of \( \mathbb{P} \) for zero-dimensional spaces with countable bases one can prove the following:

Let \( \mathcal{F} \) be any finite family of non-empty subsets of \( \mathbb{R} \). Let \( m \) be the number of one-dimensional elements of \( \mathcal{F} \), \( n \) the number of zero-dimensional elements of \( \mathcal{F} \) with at least one limit point, \( p \) the number of discrete elements of \( \mathcal{F} \) and \( 1 \leq n + m + p \). Then the product \( \prod \mathcal{F} \) of all elements of \( \mathcal{F} \) can be embedded in \( \mathbb{R}^{q} \) and can not be embedded in \( \mathbb{R}^{q-1} \), where \( q = q(m, n, p) \).

2 Preliminaries

A subset \( A \subset \mathbb{R} \) with the topology induced from \( \mathcal{P} \) will be denoted by \( A_{\mathcal{P}} \). The notation \( X \approx Y \) means that the spaces \( X \) and \( Y \) are homeomorphic. Our terminology follows [4].

We will continue with some properties of subspaces of \( \mathcal{P} \).

Countable subspaces properties:

(1) Every countable subspace of \( \mathcal{P}^k \), \( k \geq 1 \), has a countable base (readily);

(2) Every countable space with a countable base can be embedded in \( \mathcal{P} \) (see for example [6, Theorem 2, page 296]);

(3) Every countable space with a countable base and which has no isolated points is homeomorphic to \( \mathcal{P} \) (see for example [7, Theorem 1.9.6]);

LEMMA 2.1. (i) \( \mathcal{P} \approx \mathcal{P}_{\mathcal{P}} \);

(ii) For every open non-empty subspace \( U \) of \( \mathcal{P} \), we have \( U \approx \mathcal{P} \);

(iii) If \( \mathcal{P} = Q_1 \cup \cdots \cup Q_n \), \( n \geq 1 \), then there is an index \( m \) and a subspace \( P \) of \( Q_m \) such that \( P \approx \mathcal{P} \);

(iv) If \( X_1, \ldots, X_n \), \( n \geq 1 \), are countable subspaces of \( \mathcal{P} \) then \( X_1 \times \cdots \times X_n \) can be embedded in \( \mathcal{P} \) (and hence in \( \mathcal{P}_{\mathcal{P}} \) and in \( \mathcal{P} \)).

PROOF. Observe that the points (i) and (ii) are simple corollaries of the properties (1) and (3). The point (iv) is a corollary of the properties (1) and (2). In order to prove the point (iii) it is enough to show that if \( \mathcal{P} = A \cup B \) then either \( A \) contains a subspace \( C \approx \mathcal{P} \) or there is an open interval \( (a, b) \subset \mathbb{R} \) such that \( (a, b) \cap \mathcal{P} \subset B \). Really, on the first step consider the system \( \nu_1 \) of open intervals...
(n, n + 1), n ∈ ℤ. Either there is an element E of v₁ disjoint from A and we have done by the point (ii) or we can choose from each interval of the system v₁ a point from A. Denote the chosen set by A₁. On the second step consider the system v₂ of open intervals \((a, a + \frac{1}{2₁}), (a + \frac{1}{2₁}, b)\), \((a, b) \in v₁\). Either there is an element E of v₂ disjoint from A and we have done by the point (ii) or we can choose from each interval of the system v₂ a point from A. Denote the chosen set by A₂. Continue by this way we either will find an open interval disjoint from A or construct a countable sequence \(A₁, A₂, \ldots\) of subsets of A. Observe that the system vₙ₊₁ consists of the open intervals \((a, a + \frac{1}{2ₙ}), (a + \frac{1}{2ₙ}, b)\), \((a, b) \in vₙ\). Denote \(C = \bigcup_{i=₁}^{∞} Aᵢ\). Observe that the set \(C \subset A\) is countable and without isolated points. So \(C \approx \mathbb{Z}\) by the property (3). The lemma is proved.

Uncountable subspaces properties:
(4) Every uncountable subspace A of \(\mathcal{S}^k\), \(k \geq 1\), has the weight \(wA > \aleph₀\) (readily);
(5) For every uncountable subspace A of \(\mathcal{S}\) there is a subspace \(B \subset A\) such that each open non-empty subspace of B is uncountable (see for example \([8, \text{Lemma 6.1}]\));
(6) Every uncountable subspace A of \(\mathcal{S}\) contains an infinite, closed in \(\mathcal{S}\), discrete subspace. So A is not compact \([5, \text{Corollary 1}]\).

**Lemma 2.2.** Every uncountable subspace A of \(\mathcal{S}\) contains a subspace homeomorphic to \(\mathbb{Z}\).

**Proof.** By property (5) there is a subspace B of A such that each open non-empty subspace of B is uncountable. We will construct a subspace of B which is homeomorphic to \(\mathbb{Z}\). Consider the open cover v₁ of \(\mathcal{S}\) consisting of half-open intervals \([n, n + 1), n \in ℤ\). From each element E of v₁ such that \(E \cap B \neq \emptyset\) choose a point from B. Denote the chosen set by B₁. For every \(i \geq 1\) consider the open cover vᵢ₊₁ of \(\mathcal{S}\) consisting of half-open intervals \((a, a + \frac{1}{2ᵢ}), \left[\frac{1}{2ᵢ}, b\right), (a, b) \in vᵢ\). From each element E of vᵢ₊₁ such that \(E \cap B \neq \emptyset\) choose a point from \(Bᵢ \setminus (B₁ \cup \cdots Bᵢ)\). Denote the chosen set by \(Bᵢ₊₁\). Construct the sequence of countable disjoint subsets \(B₁, B₂, \ldots\) of B. Denote \(C = \bigcup_{i=₁}^{∞} Bᵢ\). Observe that C is countable and has no isolated points. So the subspace C of A is homeomorphic to \(\mathbb{Z}\) by the properties (1) and (3). The lemma is proved.

**Remark 2.1.** Observe that every subset of \(\mathcal{S}\) is either uncountable (and hence containing according to Lemma 2.2 a lot of limit points), or countable with at least one limit point, or discrete.
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It is convenient to follow some notations and facts from [2]. An element \( x \in \mathcal{S}^n \) is viewed as a finite sequence \( x = (x_i)_{i \leq n} \). For \( 0 \leq k \leq n \), \( x \in \mathcal{S}^n \) and \( V \subset \mathcal{S}^n \) let

\[
\delta_k^n(V, x) = \{ y \in V : |\{ i \leq n : x_i \neq y_i \}| = k \}.
\]

This will be used when \( V \) is a basic open nbd of \( x \) of the form \( B_n[x, \epsilon) = \prod_{i \leq n} [x_i, x_i + \epsilon) \) for \( \epsilon > 0 \). Observe that for such \( V \), \( \{ \delta_k^n(V, x) : 0 \leq k \leq n \} \) is a partition of \( V \) such that \( \bigcup_{i=k}^n \delta_i^n(V, x) \) is open in \( \mathcal{S}^n \) for any \( k \leq n \). In addition, for \( 1 \leq k \leq n \), \( \delta_k^n(V, x) \) is the topological sum of finitely many subspaces of \( \mathcal{S}^k \) and so it can be embedded in \( \mathcal{S}^k \) (observe also that \( \delta_0^n(V, x) = \{ x \} \)).

3 Products of Subspaces of \( \mathcal{S} \)

We continue with a statement whose proof follows the base step of induction from [2, Theorem 2.1].

**Theorem 3.1.** Let \( B \) be an uncountable subspace of \( \mathcal{S} \) and for each \( b \in B \) let \( A(b) \) be a subspace of \( \mathcal{S} \) with a limit point \( p(b) \). Then the subspace \( C = \bigcup_{b \in B} (A(b) \times \{ b \}) \) of \( \mathcal{S}^2 \) can not be embedded in \( \mathcal{S} \).

**Proof.** Assume that there is an embedding \( f : C \to \mathcal{S} \) of \( C \) into \( \mathcal{S} \). Then the mapping \( g = f \times \text{id} : C \times \mathcal{S} \to \mathcal{S}^2 \) is also an embedding. Define

\[
E = \bigcup_{b \in B} A(b) \times \{(b, -b)\} \subset C \times \mathcal{S} \subset \mathcal{S}^3.
\]

Observe that \( E \) is the topological sum of subspaces \( E(b) = A(b) \times \{(b, -b)\} \approx A(b), b \in B, \) of \( \mathcal{S}^3 \), each of which embeds in \( \mathcal{S}^2 \) by \( g \). Let \( F = g(E) \subset \mathcal{S}^2 \). Observe that \( F \) is the topological sum of \( F(b) = g(E(b)) \approx A(b), b \in B. \) For each \( b \in B, \) put \( x(b) = g(\{ p(b) \} \times \{(b, -b)\}) \in F(b) \) (observe that this point is a limit point for \( F(b) \)) and choose \( \epsilon(b) > 0 \) such that \( V(b) = B_2[x(b), \epsilon(b)] \) is disjoint from \( F(b^*) \) for all \( b^* \neq b, b^* \in B. \)

Recall that the space \( \mathcal{S} \times \mathcal{R} \) is hereditarily Lindelöf. For \( j = 1, 2, \) let \( \sigma_j \) denote the topology on the product \( Z_1 \times Z_2, \) where \( Z_j = \mathcal{S} \) and \( Z_i = \mathcal{R} \) for \( i \neq j. \) These two spaces are of course homeomorphic and hereditarily Lindelöf. Observe that for every \( j = 1, 2, \) the hereditarily Lindelöf topology \( \sigma_j \) tells us that \( (\text{int}_{\sigma_j} V(b)) \cap F(b) = \emptyset \) for all but at most countably many \( b \in B. \) So, we can find \( b \in B \) such that \( F(b) \) is disjoint from the union \( (\text{int}_{\sigma_1} V(b)) \cup (\text{int}_{\sigma_2} V(b)). \) Observe also that

\[
V(b) \setminus ((\text{int}_{\sigma_1} V(b)) \cup (\text{int}_{\sigma_2} V(b))) = \delta_0^2(V(b), x(b)).
\]
But $x(b) \in V(b) \cap F(b) \subseteq \delta^{2}_0(V(b), x(b)) = x(b)$. So the point $x(b) = V(b) \cap F(b)$ is an open subset of $F(b)$. This is a contradiction because $x(b)$ is a limit point of $F(b)$. The theorem is proved.

**COROLLARY 3.1.** Let $B$ be an uncountable subspace of $\mathcal{P}$ and $A$ a subspace of $\mathcal{P}$ with a limit point $p$. Then the subspace $C = A \times B$ of $\mathcal{P}^2$ can not be embedded in $\mathcal{P}$. Moreover, there is an uncountable subset $E$ of $B$ such that for each point $q \in \{p\} \times E$, every open nbd of $q$ in $A \times E$ can not be embedded in $\mathcal{P}$.

**PROOF.** Observe that any open nbd of $p$ in $A$ has $p$ as a limit point. Apply now the property (5).

**COROLLARY 3.2.** Let $B$ be an uncountable subspace of $\mathcal{P}$ and $A$ a subspace of $\mathcal{P}$ homeomorphic to $\mathcal{P}$. Then the subspace $C = A \times B$ of $\mathcal{P}^2$ can not be embedded in $\mathcal{P}$. Moreover, if every open non-empty subspace of $B$ is uncountable then no open non-empty subspace of $A \times B$ can be embedded in $\mathcal{P}$. In general, there is a subspace $E$ of $B$ such that no open non-empty subspace of $A \times E$ can be embedded in $\mathcal{P}$.

**PROOF.** Lemma 2.1 (ii) together with the property (5) and Corollary 3.1 prove the statement.

**PROPOSITION 3.1.** Let $A$ be a discrete subspace of $\mathcal{P}$ and $B$ a subspace of $\mathcal{P}$. Then $A \times B$ can be embedded in $\mathcal{P}$.

**PROOF.** Observe first that $A$ is countable. Recall that for any $n \in \mathbb{Z}$, $[n, n+1)_{\mathcal{P}} \cong \mathcal{P}$. Note now that $\mathcal{P}$ is the topological sum of $[n, n+1)_{\mathcal{P}}$, $n \in \mathbb{Z}$, which is homeomorphic to $\mathcal{P} \times \mathbb{Z}$. From this fact the statement follows.

**PROOF OF THEOREM 1.3 (i).** By Remark 2.1 there is a decomposition of the class of all subspaces of $\mathcal{P}$ in the three disjoint subclasses. According to that there are six different types of products. Now Lemma 2.1, Corollary 3.1 and Proposition 3.1 prove the statement.

Let $p_i : \mathcal{P}^2 \to \mathcal{P}$, $i = 1, 2$, be the projections of $\mathcal{P}^2$ onto $i$-th factor or the restrictions of these projections on certain subsets of $\mathcal{P}^2$. We continue with a couple of examples following Proposition 3.1.

**EXAMPLE 3.1.** Let $A = (\{0\} \cup \{\frac{1}{i} : i = 1, 2, \ldots\}) \times \mathcal{P} \subset \mathcal{P}^2$. Recall that $A$ can not be embedded in $\mathcal{P}$ by Corollary 3.1. But $A$ is the union $A_1 \cup A_2$ of two
subspaces such that each $A_i$ can be embedded in $\mathcal{S}$. In fact, put $A_1 = \{0\} \times \mathcal{S}$ (a closed subspace of $A$) and $A_2 = \{\frac{1}{i} : i = 1, 2, \ldots\} \times S$ (an open subspace of $A$). (Observe that $\mathcal{S} \times \mathcal{S}$ cannot be written as a finite union of subspaces which can be embedded in $\mathcal{S}$ as we will see in Lemma 4.1.)

**Example 3.2.** Fix an embedding of $\mathcal{Q} = \{q_1, q_2, \ldots\}$ into $\mathcal{S}$. Define

$$A = \bigcup_{n=1}^{\infty} ([n, n+1) \times \{q_n\}) \subset \mathcal{S}^2.$$ 

Observe that $A$ is the topological sum of the subspaces $[n, n+1) \times \{q_n\}$, $n = 1, 2, \ldots$ where each term $[n, n+1) \times \{q_n\}$ is homeomorphic to $\mathcal{S}$. So $A \approx \mathcal{S}$. But $p_1(A) = \mathcal{S}$ and $p_2(A) = \mathcal{Q}$. Moreover, for every point $q \in \mathcal{Q}$ we have $p_1^{-1}(q) \approx \mathcal{S}$. This example shows that the uncountability of $B$ in Theorem 3.1 is extremely essential. Compare also this example with Corollary 3.2.

We have more example concerning Theorem 3.1.

**Example 3.3.** Let $A$ be any uncountable subspace of $\mathcal{S}$. Then the subspace $B = \{(a, -a) : a \in A\}$ of $\mathcal{S}^2$, being non-Lindelof, cannot be embedded in $\mathcal{S}$. Observe that $p_1(B) = A$ and $p_2(B) = -A = \{-a : a \in A\}$. Moreover, $|p_1^{-1}(a)| = |p_2^{-1}(-a)| = 1$ for any $a \in A$. A generalization of this example: Let $E$ be a subspace of $\mathcal{S}^2$ which contains the graph of a strictly decreasing function from $F \subset \mathcal{S}$ to $\mathcal{S}$, where $F$ is an uncountable subset of $\mathcal{S}$. Then $E$ cannot be embedded in $\mathcal{S}$.

Theorem 1.3 (i) arises the following

**Problem 3.1.** Determine what subsets of $\mathcal{S}^2$ can be embedded in $\mathcal{S}$.

The proof of the following statement follows also the idea of the proof from [2, Theorem 2.1].

**Theorem 3.2.** Let $B$ be an uncountable subspace of $\mathcal{S}$ and for each $b \in B$ let $A(b)$ be a subspace of $\mathcal{S}^n$, $n \geq 2$, such that no open non-empty subspace of $A(b)$ can be embedded in $\mathcal{S}^{n-1}$. Then the subspace $C = \bigcup_{b \in B} (A(b) \times \{b\})$ of $\mathcal{S}^{n+1}$ cannot be embedded in $\mathcal{S}^n$.

**Proof.** Assume that there is an embedding $f : C \to \mathcal{S}^n$ of $C$ into $\mathcal{S}^n$. Then the mapping $g = f \times id : C \times \mathcal{S} \to \mathcal{S}^{n+1}$ is also an embedding. Define
E = \bigcup_{b \in B} A(b) \times \{(b, -b)\} \subseteq C \times P \subseteq P^{n+2}.

Observe that E is the topological sum of subspaces E(b) = A(b) \times \{(b, -b)\} 
\approx A(b), b \in B, of P^{n+2}, where each E(b) can be embedded in P^{n+1} by g. Let 
F = g(E) \subseteq P^{n+1}. Observe that F is the topological sum of F(b) = g(E(b)) \approx 
A(b), b \in B. For each b \in B, pick a point x(b) \in F(b) and choose e(b) > 0 such that 
V(b) = B_{n+1}[x(b), e(b)] is disjoint from F(b*) for all b* \neq b, b* \in B.

Recall that for any n \in \mathcal{N} the space P \times \mathbb{R}^n is hereditarily Lindelöf. For 
j = 1, \ldots, n + 1, let \sigma_j denote the topology on the product \prod_{i=1}^{n+1} Z_i where 
Z_j = P and Z_i = \mathbb{R} for i \neq j. These (n + 1) spaces are of course pairwise homeomorphic 
and hereditarily Lindelöf. Observe that for every j = 1, \ldots, n + 1, the hereditarily Lindelöf topology \sigma_j tells us that (int_\sigma V(b)) \cap F(b) = \emptyset for all but at most 
countably many b \in B. So, we can find b \in B such that F(b) is disjoint from the 
union \bigcup_{i=1}^{n+1} (int_\sigma V(b)). Observe also that 

\[ V(b) \setminus \bigcup_{i=1}^{n+1} (int_\sigma V(b)) \subseteq \bigcup_{i=0}^{n-1} \delta_i^{n+1}(V(b), x(b)). \]

So 

\[ (*) \quad x(b) \in V(b) \cap F(b) \subseteq \bigcup_{i=0}^{n-1} \delta_i^{n+1}(V(b), x(b)). \]

Now, for this b, pick the largest k < n such that F(b) \cap \delta_k^{n+1}(V(b), x(b)) \neq \emptyset.

Since 

\[ F(b) \cap \bigcup_{i=0}^{n+1} \delta_i^{n+1}(V(b), x(b)) = F(b) \cap \delta_k^{n+1}(V(b), x(b)) \]

is open in F(b) we see that 

\[ W = g^{-1}[F(b) \cap \delta_k^{n+1}(V(b), x(b))] \approx F(b) \cap \delta_k^{n+1}(V(b), x(b)) \]

is open in g^{-1}[F(b)] = E(b). Recall that W can not be embedded in P^{n-1} by 
assumption. In the same time the space F(b) \cap \delta_k^{n+1}(V(b), x(b)), which is homeomorphic 
to W, can be embedded in P^{n-1} by the construction (recall that k < n). This is a contradiction. The theorem is proved.

**Corollary 3.3.** Let X_i, i = 1, \ldots, n, n \geq 2, be subspaces of P such that 
X_1 \approx 2 and for every X_i, i \geq 2, each open non-empty subspace of X_i is uncountable. Then 
X_1 \times \cdots \times X_n can not be embedded in P^{n-1}.

**Proof.** Apply an obvious induction. The basis of the induction is Corollary 3.2.

**Corollary 3.4.** Let X_i, i = 1, \ldots, n, n \geq 2, be subspaces of P such that one 
of them is homeomorphic to 2 and the others are uncountable. Then X_1 \times \cdots \times X_n 
can not be embedded in P^{n-1}.
PROOF. Apply the property (5) and Corollary 3.3.

COROLLARY 3.5 ([2, Theorem 2.1]). Let $X_i$, $i = 1, \ldots, n$, $n \geq 2$, be uncountable subspaces of $\mathcal{S}$. Then $X_1 \times \cdots \times X_n$ can not be embedded in $\mathcal{S}^{n-1}$.

PROOF. Apply Corollary 3.4 and Lemma 2.2.

PROOF OF THEOREM 1.1. Lemma 2.1, Proposition 3.1 and Corollary 3.4 prove the statement.

Theorems 1.1 arises

PROBLEM 3.2. Determine what subsets of $\mathcal{S}^n$ can be embedded in $\mathcal{S}^k$ for $1 \leq k < n$.

Some examples of subsets of $\mathcal{S}^n$ concerning Problem 3.2:

EXAMPLE 3.4. Recall that $\mathcal{S} \approx ((0,1))_{\mathcal{S}} \approx ([0,1])_{\mathcal{S}} \approx X = \{0\} \cup \bigcup_{i=1}^{\infty} (a_i, b_i)_{\mathcal{S}}$, where $0 < b_{i+1} < a_i < b_i$ for every $i$ and $a_i \to 0$. Using this fact it is easy to establish that

(i) The subspace

$$A = ([0,1) \times \{0\} \times \{0\}) \cup ([0,1) \times \{0\} \times \{0\}) \cup ([0,1) \times \{0\} \times \{0\})$$

of $\mathcal{S}^3$ is homeomorphic to $\mathcal{S}$. Really, $A = A_1 \cup A_2 \cup A_3$, where

$$A_k = ([0,1)]_{\mathcal{S}} = \left(\{0\} \cup \bigcup_{i=1}^{\infty} \left[ \frac{1}{i+1}, \frac{1}{i} \right) \right)_{\mathcal{S}}, \quad k = 1, 2, 3.$$

For each $k = 1, 2, 3$ define a mapping $f_k : A_k \to X$ as follows. Put $f_k(0) = 0$, and for each $i \geq 1$ let $f_k^i([1/(i+1), 1/i))_{\mathcal{S}}$ be any homemorphism between $\left(\frac{1}{i+1}, \frac{1}{i}\right)_{\mathcal{S}}$ and $(a_{3(i-1)+k}, b_{3(i-1)+k})_{\mathcal{S}}$. Put $f(x) = f_k(x)$ for any point $x \in A_k$. The mapping $f$ is a homeomorphism between $A$ and $X \approx \mathcal{S}$. Observe also that

$$A = \bigcup_{i=0}^{1} \delta^3_i(V, (0,0,0)),$$

where $V = B_3((0,0,0), 1)$.

(ii) The subspace

$$B = ([0,1) \times \{0,1\} \times \{0\}) \cup ([0,1) \times \{0\} \times \{0\})$$

of $\mathcal{S}^3$ can be embedded in $\mathcal{S}^2$ but can not (readily) be embedded in $\mathcal{S}$.  

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Now we are ready to prove two statements necessary for Theorems 1.2 and 1.3 (ii).

**Theorem 3.3.** Let $B$ be an uncountable subspace of $\mathcal{S}$ and for each $b \in B$ let $A(b)$ be a subspace of $\mathcal{S}^2$ with a point $p(b)$ such that no open nbhd of $p(b)$ in $A(b)$ can be embedded in $\mathcal{S}$. Then the subspace $C = \bigcup_{b \in B} (A(b) \times \{b\})$ of $\mathcal{S}^3$ can not be embedded in $\mathcal{S}^2$.

**Proof.** Follow the proof of Theorem 3.2 but the points $x(b)$ let us pick up as in the proof of Theorem 3.1. Use then the inclusion $(\ast)$ from the proof of Theorem 3.2 and the equality $(\ast\ast)$ from Example 3.4 (i).

**Corollary 3.6.** Let $B_1$, $B_2$ be uncountable subspaces of $\mathcal{S}$ and $A$ a subspace of $\mathcal{S}$ with a limit point $p$. Then the subspace $C = A \times B_1 \times B_2$ of $\mathcal{S}^3$ can not be embedded in $\mathcal{S}^2$. Moreover, there are uncountable subsets $E_1$, $E_2$ of $B_1$, $B_2$ respectively such that for each point $q \in \{p\} \times E_1 \times E_2$, no open nbhd of $q$ in $A \times E_1 \times E_2$ can be embedded in $\mathcal{S}^2$.

**Proof.** Observe that any open nbhd of $p$ in $A$ has $p$ as a limit point. Apply now the property (5) and Corollary 3.1.

**Proof of Theorem 1.3 (ii).** Let us again use the decomposition from Remark 2.1 of the class of all subspaces of $\mathcal{S}$ in the three disjoint subclasses. According to that there are ten different types of products. Lemma 2.1, Corollary 3.6 and Proposition 3.1 prove the statement.

**Proof of Theorem 1.2.** Lemma 2.1, Proposition 3.1, Corollary 3.1 and Corollary 3.6 prove the statement.

A positive answer to the next question would give a positive answer to Problem 1.1.

**Question 3.1.** Let $n \geq 4$, $x \in \mathcal{S}^n$ and $V = B_n(x, \varepsilon)$. Can the set $\bigcup_{i=0}^{n-2} \mathcal{S}^n_i(V, x)$ be embedded in $\mathcal{S}^{n-2}$?

Recall that for $n = 2, 3$ this is right.

Now in order to get a complete picture it is time to make some obvious comments concerning infinite products of subspaces of the Sorgenfrey line.

Denote by $\emptyset$ the discrete two points space.
Proposition 3.2. Let $X$ be an uncountable space with $wX = \aleph_0$. Then $X$ cannot be embedded in $\mathcal{G}^n$ for any $n \in \mathcal{N}$. In particular, the Cantor space $\mathcal{G} = \mathcal{B}^{\aleph_0}$ and any its uncountable subspace cannot be embedded in $\mathcal{G}^n$ for any $n \in \mathcal{N}$.

Proof. Recall from (4) that any uncountable subspace $A$ of $\mathcal{G}^n$, $n \geq 1$, has $wA > \aleph_0$.

Observe that from Proposition 3.2 we have also that the Cantor space cannot be embedded in any countable union of subspaces of $\mathcal{G}^k$ for each $k \geq 1$.

Proposition 3.3. Let $\tau$, $\nu$ be two infinite cardinals and $\tau < \nu$. Then $\mathcal{B}^\nu$ cannot be embedded in $\mathcal{G}^\tau$.

Proof. Really, assume that there is an embedding $f : \mathcal{B}^\nu \to \mathcal{G}^\tau$. Then $f(\mathcal{B}^\nu) \approx \mathcal{B}^\nu$ is compact and $w(f(\mathcal{B}^\nu)) = w(\mathcal{B}^\nu) = \nu$ ([E, p. 84]). By the property (6) there are countable subspaces $Y_\alpha$, $\alpha \in \tau$, of $\mathcal{G}$ such that $f(\mathcal{B}^\nu) \subseteq \prod_{\alpha \in \tau} Y_\alpha$. Recall that by Lemma 2.1 each $Y_\alpha$, $\alpha \in \tau$, has a countable base. Hence, $w(\prod_{\alpha \in \tau} Y_\alpha) \leq \tau < \nu$ (see for example [4, Theorem 2.3.23]). This is a contradiction.

Proposition 3.4. Let $\tau$ be an infinite cardinal $\geq c$. Then $\mathcal{G}^\tau$ can be embedded in $\mathcal{B}^\tau$.

Proof. Observe that $w(\mathcal{G}) = c$. So $\mathcal{G}$ can be embedded in $\mathcal{B}^c$ and hence $\mathcal{G}^\tau$ can be embedded in $(\mathcal{B}^c)^\tau \approx \mathcal{B}^\tau$.

4 Unions of Subspaces of $\mathcal{G}^k$ and Their Products

Recall that two arrows space, shortly $TAS$, (see for example [4, Exercise 3.10.C]) defined by Alexandroff and Urysohn, is the union $X = C_0 \cup C_1 \subseteq \mathcal{R}^2$, where $C_0 = \{(x, 0) : 0 < x \leq 1\}$ and $C_1 = \{(x, 1) : 0 \leq x < 1\}$, and the topology on $X$ generated by the base consisting of sets of the form

$$\left\{(x, i) \in X : x_0 - \frac{1}{k} < x < x_0 \text{ and } i = 0, 1\right\} \cup \{(x_0, 0)\},$$

where $0 < x_0 \leq 1$ and $k = 1, 2, \ldots$, and of sets of the form

$$\left\{(x, i) \in X : x_0 < x < x_0 + \frac{1}{k} \text{ and } i = 0, 1\right\} \cup \{(x_0, 1)\},$$

where $0 \leq x_0 < 1$ and $k = 1, 2, \ldots$
It is easy to see that the TAS is compact and \( |TAS| = c \). So by the property (6) the TAS can not be embedded in \( \mathcal{S}^k \) for any \( k \geq 1 \). Observe that the TAS is the union of two copies of Sorgenfrey line. This motivates the following.

Define two sequences of classes of topological spaces as follows.

\[
\mathcal{M}_k^{\text{fin}} = \{ \text{unions of finitely many subspaces of } \mathcal{S}^k \} \quad \text{and} \\
\mathcal{M}_k = \{ \text{unions of countably many subspaces of } \mathcal{S}^k \}, \quad \text{where } k \geq 1.
\]

Put also \( \mathcal{M}_\infty = \{ \text{unions of countably many subspaces of } \mathcal{S}, \mathcal{S}^2, \mathcal{S}^3, \ldots \} \).

We start with obvious remarks about these classes.

**Proposition 4.1.**

(a) \( TAS \in \mathcal{M}_1^{\text{fin}} \);

(b) Any space \( X \) from \( \mathcal{M}_1^{\text{fin}} \) (or \( \mathcal{M}_1 \)) is hereditarily Lindelöf and hereditarily separable;

(c) \( \mathcal{M}_k^{\text{fin}} \subset \mathcal{M}_k \subset \mathcal{M}_\infty \) for any \( k \geq 1 \);

(d) If \( X \in \mathcal{M}_k^{\text{fin}}(\mathcal{M}_k) \) and \( Y \in \mathcal{M}_m^{\text{fin}}(\mathcal{M}_m) \) then \( X \times Y \in \mathcal{M}_{k+m}^{\text{fin}}(\mathcal{M}_{k+m}) \).

The following lemma is one more corollary of Theorem 3.1.

**Lemma 4.1.** Let \( B \) be an uncountable subspace of \( \mathcal{S} \) and for each \( b \in B \) let \( A(b) \) be a subspace of \( \mathcal{S} \). Let also \( C = \bigcup_{b \in B}(A(b) \times \{b\}) \).

(a) If for every \( b \in B \) we have \( A(b) \sim 2 \) and \( C = \bigcup_{i=1}^{n} Y_i \) for some \( n \geq 1 \) then there is \( k \leq n \) such that \( Y_k \) can not be embedded in \( \mathcal{S} \);

(b) If for every \( b \in B \) we have \( A(b) \) is uncountable and \( C = \bigcup_{i=1}^{n} Y_i \) then there is \( k \geq 1 \) such that \( Y_k \) can not be embedded in \( \mathcal{S} \).

**Proof.** (a) For each \( b \in B \) by Lemma 2.1 (iii) there are \( i(b) \leq n \) and subspace \( E(b) \) of \( A(b) \) such that \( E(b) \times \{b\} \subset Y_{i(b)} \) and \( E(b) \sim 2 \). Since \( B \) is uncountable then there are \( k \leq n \) and an uncountable subspace \( B_1 \) of \( B \) such that for each \( b \in B_1 \) we have \( i(b) = k \). By Theorem 3.1, \( \bigcup_{b \in B_1}(E(b) \times \{b\}) \subset Y_{i(b)} \) can not be embedded in \( \mathcal{S} \).

(b) This point is proved in the same manner as (a).

By Lemma 4.1 we have readily

**Theorem 4.1.**

(a) Let \( X \in \mathcal{M}_1^{\text{fin}} \) and \( X \) be uncountable. Then \( X \times 2 \notin \mathcal{M}_1^{\text{fin}} \) but \( X \times 2 \in \mathcal{M}_1 \).

(b) Let \( X, Y \in \mathcal{M}_1 \) and \( X, Y \) be uncountable. Then \( X \times Y \notin \mathcal{M}_1 \) but \( X \times Y \in \mathcal{M}_2 \).
What could be done else? Well, I think that it could be interesting to look what theorems from the previous section are valid for the $TAS$.

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5 References


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