REAL HYPERSURFACES OF A NONFLAT COMPLEX SPACE FORM IN TERMS OF THE RICCI TENSOR

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Abstract. We know the fact that there are no real hypersurfaces with parallel Ricci tensors in a nonflat complex space form (cf. [5]). In this paper we investigate real hypersurfaces in a nonflat complex space form using some conditions of the Ricci tensor $S$ which are weaker than $VS = 0$. We characterize Hopf hypersurfaces of a non-flat complex space form.

0 Introduction

A Kähler manifold of constant holomorphic sectional curvature $c$ is called a complex space form, which is denoted by $M_n(c)$. A complete and simply connected complex space forms are isometric to a complex projective space $CP_n$, a complex Euclidean space $C^n$ or a complex hyperbolic space $CH_n$ as $c > 0$, $c = 0$ or $c < 0$.

Let $M$ be a real hypersurface of $M_n(c)$. Then $M$ has an almost contact metric structure $(\varphi, \xi, \eta, g)$ induced from the complex structure $J$ and the Kähler metric of $M_n(c)$ (for details see §1). The structure vector $\xi$ is said to be principal if $A\xi = \alpha \xi$ is satisfied, where $A$ is the shape operator of $M$ and $\alpha = \eta(A\xi)$. A real hypersurfaces is said to be a Hopf hypersurface if the structure vector $\xi$ of $M$ is principal.

Typical examples of real hypersurfaces in $CP_n$ are homogeneous ones which are orbits under subgroups of $PU(n + 1)$. The complete classification of them was obtained by Takagi [10] as follows:

THEOREM T [10]. Let $M$ be a homogeneous real hypersurface of $CP_n$. Then $M$ is a tube of radius $r$ over one of the following Kähler submanifolds:
(A1) a hyperplane $\mathbb{C}P_{n-1}$, where $0 < r < \frac{n}{2}$,
(A2) a totally geodesic $\mathbb{C}P_{k}$ $(1 \leq k \leq n-2)$, where $0 < r < \frac{n}{2}$,
(B) a complex quadric $Q_{n-1}$, where $0 < r < \frac{n}{4}$,
(C) $\mathbb{C}P_{1} \times \mathbb{C}P_{(n-1)/2}$, where $0 < r < \frac{n}{4}$ and $n \geq 3$ is odd,
(D) a complex Grassmann $G_{2,5}C$, where $0 < r < \frac{n}{4}$ and $n = 9$,
(E) a Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \frac{n}{5}$ and $n = 15$.

Also Berndt [1] classified all Hopf real hypersurfaces in $CH_n$ with constant principal curvatures as follows:

**Theorem B [1].** Let $M$ be a real hypersurface of $CH_n$. Then $M$ has constant principal curvatures and $\xi$ is principal if and only if $M$ is locally congruent to one of the following:

(A0) a self-tube, that is, a horosphere,
(A1) a geodesic hypersphere, or a tube over a hyperplane $CH_{n-1}$,
(A2) a tube over a totally geodesic $CH_k$ $(1 \leq k \leq n-2)$,
(B) a tube over a totally real hyperbolic space $RH_n$.

Let $\nabla$ and $S$ be the Levi-Civita connection and the Ricci tensor of $M$, respectively. There are many studies about Ricci tensors of real hypersurfaces (cf. [2], [3], [4], [5], [6], [7], [8], [9]). Very important fact is that there are no real hypersurfaces with parallel Ricci tensors $S$ (that is, $\nabla_{X}S = 0$ for each vector field $X$ tangent to $M$) in $M_{n}(c)$, $c \neq 0$, $n \geq 3$ (cf. [5]). Especially, there exist no Einstein real hypersurfaces $M$ in $M_{n}(c)$, $c \neq 0$, $n \geq 3$. So, it is natural to investigate real hypersurfaces $M$ by using some conditions (on the derivatives of $S$) which are weaker than $\nabla S = 0$.

Recently, the first author, Hwang and Kim proved the following theorem:

**Theorem 0.1.** Let $M$ be a real hypersurface in a nonflat complex space form. If the Ricci tensor $S$ of $M$ satisfies $\nabla_{\xi}S = 0$, $\nabla_{\partial_{\xi}}S = 0$ and $S_{\xi} = g(S_{\xi}, \xi)\xi$, then $M$ is locally congruent to one of the homogeneous real hypersurfaces of Theorem $T$ and Theorem $B$.

In this paper we pay particular attention to the fact that for each Hopf hypersurface $M$ in $M_{n}(c)$, $c \neq 0$ the characteristic vector $\xi$ of $M$ is an eigenvector of the Ricci tensor $S$ of $M$. So it is natural to consider a problem that if the vector $\xi$
is an eigenvector of the Ricci tensor $S$ of a real hypersurface $M$ in $M_n(c)$, $c \neq 0$, is $M$ a Hopf hypersurface?

The purpose of this paper is to establish the following theorem which gives a partial answer to this problem:

**Theorem 4.1.** Let $M$ be a real hypersurface in $M_n(c)$, $c > 0$. If it satisfies $\nabla_{\phi \gamma \zeta} S = 0$ and at the same time satisfies $S \xi = \sigma \xi$ for some constant $\sigma$, then $M$ is a Hopf hypersurface.

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1 Preliminaries

Let $M$ be a real hypersurface immersed in a complex space form $(M_n(c), G)$ with almost complex structure $J$ and the Kähler metric $G$ of constant holomorphic sectional curvature $c$, and let $C$ be a unit normal vector field on $M$. The Riemannian connection $\tilde{\nabla}$ in $M_n(c)$ and $\nabla$ in $M$ are related by the following formulas for any vector fields $X$ and $Y$ on $M$:

\[
\tilde{\nabla}_Y X = \nabla_Y X + g(A Y, X) C, \tag{1.1}
\]

\[
\tilde{\nabla}_XC = -AX, \tag{1.2}
\]

where $g$ denotes the Riemannian metric on $M$ induced from that $G$ of $M_n(c)$ and $A$ is the shape operator of $M$ in $M_n(c)$. An eigenvector $X$ of the shape operator $A$ is called a principal curvature vector. Also an eigenvalue $\lambda$ of $A$ is called a principal curvature. It is known that $M$ has an almost contact metric structure induced from the almost complex structure $J$ on $M_n(c)$, that is, we define a tensor field $\phi$ of type $(1,1)$, a vector field $\xi$, an 1-form $\eta$ on $M$ by $g(\phi X, Y) = G(JX, Y)$ and $g(\xi, X) = \eta(X) = G(JX, C)$. Then we have

\[
\phi^2 X = -X + \eta(X) \xi, \quad g(\xi, \xi) = 1, \quad \phi \xi = 0. \tag{1.3}
\]

From (1.1) we see that

\[
(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y) \xi, \tag{1.4}
\]

\[
\nabla_X \xi = \phi AX. \tag{1.5}
\]

Since the ambient space is of constant holomorphic sectional curvature $c$, equations of the Gauss and Codazzi are respectively given by
\[ R(X, Y)Z = \frac{c}{4} \{ g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \]
\[ - 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY, \] (1.6)

\[ (\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4} \{ \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \} \] (1.7)

for any vector fields \(X, Y\) and \(Z\) on \(M\), where \(R\) denotes the Riemannian curvature tensor of \(M\). We shall denote the Ricci tensor of type \((1,1)\) by \(S\). Then it follows from (1.6) that

\[ SX = \frac{c}{4} \{ (2n + 1)X - 3\eta(X)\xi \} + hAX - A^2 X, \] (1.8)

where \(h = \text{trace } A\). Further, using (1.5), we obtain

\[ (\nabla_X S)Y = -\frac{3}{4} c \{ g(\phi AX, Y)\xi + \eta(Y)\phi AX \} + (Xh)AY \]
\[ + (hI - A)(\nabla_X A)Y - (\nabla_Y A)AY, \] (1.9)

where \(I\) is the identity map.

To write our formulas in convention forms, we denote \(\alpha = \eta(A\xi), \beta = \eta(A^2 \xi), \mu^2 = \beta - \alpha^2\) and \(\nabla f\) by the gradient vector field of a function \(f\) on \(M\). In the following, we use the same terminology and notation as above unless otherwise stated.

If we put \(U = \nabla\xi\), then \(U\) is orthogonal to the structure vector field \(\xi\). Then it is, using (1.3) and (1.5), seen that

\[ \phi U = -A\xi + \alpha \xi, \] (1.10)

which shows that \(g(U, U) = \beta - \alpha^2\). By the definition of \(U\), (1.3) and (1.5) it is verified that

\[ g(\nabla_X \xi, U) = g(A^2 \xi, X) - \alpha g(A\xi, X). \] (1.11)

Now, differentiating (1.10) covariantly along \(M\) and using (1.4) and (1.5), we find

\[ \eta(X)g(\nabla U + \nabla\alpha, Y) + g(\phi X, \nabla Y U) \]
\[ = g((\nabla X A)X, \xi) - g(A\phi AX, Y) + \alpha g(A\phi X, Y), \] (1.12)

which enables us to obtain

\[ (\nabla \xi A)\xi = 2AU + \nabla\alpha \] (1.13)
Real hypersurfaces of a nonflat complex space form

because of (1.7). From (1.12) we also have

\[ \nabla_\xi U = 3\phi AU + \alpha A\xi - \beta \xi + \phi \nabla \alpha, \]  

(1.14)

where we have used (1.3), (1.5) and (1.11).

If \( A\xi - g(A\xi, \xi)\xi \neq 0 \), then we can put

\[ A\xi = \alpha \xi + \mu W, \]  

(1.15)

where \( W \) is a unit vector field orthogonal to \( \xi \). Then from (1.10) it is seen that \( U = \mu \phi W \) and hence \( g(U, U) = \mu^2 \), and \( W \) is also orthogonal to \( U \). Thus, we see, making use of (1.5), that

\[ \mu g(\nabla_X W, \xi) = g(AU, X). \]  

(1.16)

2 Real Hypersurfaces Satisfying \( S\xi = g(S\xi, \xi)\xi \)

Let \( M \) be a real hypersurface of a nonflat complex space form \( M_n(c) \). If it satisfies

\[ S\xi = g(S\xi, \xi)\xi, \]  

(2.1)

then we have by (1.8)

\[ A^2\xi = hA\xi + (\beta - h\alpha)\xi, \]  

(2.2)

where we have put \( g(S\xi, \xi) = \sigma \),

\[ \beta - h\alpha = \frac{c}{2}(n - 1) - \sigma. \]  

(2.3)

In what follows we assume that \( \mu \neq 0 \) on \( M \), that is, \( \xi \) is not a principal curvature vector field and we put \( \Omega = \{ p \in M | \mu(p) \neq 0 \} \). Then \( \Omega \) is an open subset of \( M \), and from now on we discuss our arguments on \( \Omega \).

From (1.15) and (2.2), we see that

\[ AW = \mu \xi + (h - \alpha)W \]  

(2.4)

and hence

\[ A^2 W = hAW + (\beta - h\alpha)W \]  

(2.5)

because of \( \mu \neq 0 \).

Now, differentiating (2.4) covariantly along \( \Omega \), we find

\[ (\nabla_X A) W + A\nabla_X W = (X\mu)\xi + \mu \nabla_X \xi + X(h - \alpha)W + (h - \alpha)\nabla_X W. \]  

(2.6)

By taking the inner product with \( W \) in the last equation, we obtain
\[ g((\nabla X A)W, W) = -2g(AX, U) + Xh - X\alpha \]  \hfill (2.7)

since \( W \) is a unit vector field orthogonal to \( \xi \). We also have by applying \( \xi \) to (2.6)
\[ \mu g((\nabla X A)W, \xi) = (h - 2\alpha)g(AX, U) + \mu(X\mu), \]  \hfill (2.8)

where we have used (1.16), which together with the Codazzi equation (1.7) gives
\[ \mu(\nabla_W A)\xi = (h - 2\alpha)AU - \frac{c}{2}U + \mu
abla\mu, \]  \hfill (2.9)
\[ \mu(\nabla_\xi A)W = (h - 2\alpha)AU - \frac{c}{4}U + \mu\nabla\mu. \]  \hfill (2.10)

Replacing \( X \) by \( \xi \) in (2.6) and taking account of (2.10), we find
\[ (h - 2\alpha)AU - \frac{c}{4}U + \mu\nabla\mu + \mu\{\nabla_\xi W - (h - \alpha)\nabla_\xi W\}\]
\[ = \mu(\xi\mu)\xi + \mu^2 U + \mu(\xi h - \xi\alpha)W. \]  \hfill (2.11)

By the way, from \( g(U, X) = -\mu W \) we have
\[ g(AX, X)\xi - \phi\nabla_X U = (X\mu)W + \mu\nabla_X W. \]
Replacing \( X \) by \( \xi \) in this and using (1.10) and (1.14), we get
\[ \mu\nabla_\xi W = 3AU - \alpha U + \nabla\alpha - (\xi\alpha)\xi - (\xi\mu)W, \]  \hfill (2.12)
which implies
\[ W\alpha = \xi\mu. \]  \hfill (2.13)

From the last equations, it follows that
\[ 3A^2 U - 2hAU + A\nabla\alpha + \frac{1}{2}\nabla\beta - h\nabla\alpha + \left(\alpha h - \beta - \frac{c}{4}\right)U \]
\[ = 2\mu(W\alpha)\xi + \mu(\xi h)W - (h - 2\alpha)(\xi\alpha)\xi, \]  \hfill (2.14)
which enables us to obtain
\[ \xi\beta = 2\alpha(\xi\alpha) + 2\mu(W\alpha). \]  \hfill (2.15)

Differentiating (2.2) covariantly and making use of (1.5), we get
\[ (\nabla_X A)A\xi + A(\nabla_X A)\xi + A^2\phi AX - hA\phi AX \]
\[ = (Xh)A\xi + h(\nabla_X A)\xi + X(\beta - h\alpha)\xi + (\beta - h\alpha)\phi AX, \]  \hfill (2.16)
which together with (1.7) implies that
\[
\frac{c}{4} \{ u(Y)\eta(X) - u(X)\eta(Y) \} + \frac{c}{2} (h - \alpha) g(\phi Y, X) - g(A^2 \phi AX, Y) \\
+ g(A^2 \phi AY, X) + 2hg(\phi AX, AY) - (\beta - h\alpha) \{ g(\phi AY, X) - g(\phi AX, Y) \} \\
= g(A Y, (\nabla_X A)\xi) - g(AX, (\nabla_Y A)\xi) + (Yh) g(A\xi, X) - (Xh) g(A\xi, Y) \\
+ Y(\beta - ha)\eta(X) - X(\beta - ha)\eta(Y),
\]
(2.17)
where we have defined an 1-form \( u \) by \( u(X) = g(U, X) \) for any vector field \( X \). If we replace \( X \) by \( \mu W \) to both sides of (2.17) and take account of (1.13), (2.4), (2.5), (2.8) and (2.9), then we obtain
\[
(3\alpha - 2h) A^2 U + 2 \left( h^2 + \beta - 2h\alpha + \frac{c}{4} \right) AU + (h - \alpha) \left( \beta - h\alpha - \frac{c}{2} \right) U \\
= \mu A\nabla\mu + (ah - \beta)\nabla\alpha - \frac{1}{2} (h - \alpha)\nabla\beta + \mu^2 \nabla h \\
- \mu (Wh) A\xi - \mu W(\beta - ha)\xi.
\]
(2.18)
Using (1.15), the equation (2.16) can be written as
\[
A(\nabla_X A)\xi + (\alpha - h)(\nabla_X A)\xi + \mu(\nabla_X A)W \\
= (Xh) A\xi + X(\beta - ha)\xi + (\beta - h\alpha)\phi AX + hA\phi AX - A^2 \phi AX.
\]
Thus, replacing \( X \) by \( \alpha \xi + \mu W \) in this and making use of (1.5), (1.13), (1.15) and (2.7)–(2.9), we find
\[
2h A^2 U + 2 \left( ah - \beta - h^2 - \frac{c}{4} \right) AU + \left( h^2\alpha - h\beta + \frac{c}{2} h - \frac{3}{4} c\alpha \right) U \\
= g(A\xi, \nabla h) A\xi - \frac{1}{2} A\nabla\beta + \frac{1}{2} (h - 2\alpha)\nabla\beta + \beta\nabla\alpha \\
- \mu^2 \nabla h + g(A\xi, \nabla(\beta - ha))\xi.
\]
(2.19)

3 Real Hypersurfaces Satisfying \( \nabla_{\phi \nabla \xi} S = 0 \) and \( S\xi = g(S\xi, \xi)\xi \)

We continue now, our arguments under the same hypothesis \( S\xi = g(S\xi, \xi)\xi \) as in section 2. Furthermore, suppose that \( \nabla_{\phi \nabla \xi} S = 0 \), that is, \( \nabla W S = 0 \) since we now suppose that \( \mu \neq 0 \).

Then, by replacing \( X \) by \( W \), we have from (1.9)
\[- \frac{3}{4} c(h - \alpha)(u(Y)\xi + \eta(Y)U) + \mu(Wh)AY + \mu h(\nabla W A)Y \]
\[= \mu A(\nabla W A)Y - \mu(\nabla W A)AY, \quad (3.1)\]
where we have used (1.5) and (2.4). If we replace \( Y \) by \( W \) and make use of (2.7) and (2.9), then we find
\[(Wh)AW = hAU - \frac{c}{2} U - 2A^2 U + \frac{1}{2} \nabla \beta - \alpha \nabla h + A \nabla h - A \nabla \alpha \quad (3.2)\]
because of \( \mu \neq 0 \).

In the following we assume that \( \sigma \) is constant on \( M \) and then \( \beta - h \alpha = \) constant. In this case we notice here that the following fact:

**Remark 3.1.** \( h - \alpha \neq 0 \) on \( \Omega \).

In fact, if not, then we have \( h = \alpha \) and hence \( \beta - \alpha^2 = \) constant, because \( \sigma = \) constant. Thus (3.2) implies \( Wh = W \alpha = 0 \) and hence
\[2A^2 U = \alpha AU - \frac{c}{2} U. \quad (3.3)\]
Further, (2.14) and (2.18) turns out respectively to
\[2A^2 U - 2\alpha AU + \left( \alpha^2 - \beta - \frac{c}{4} \right) U = -A \nabla \alpha + (\xi \alpha)A\xi, \quad (3.4)\]
\[\alpha A^2 U + 2\left( \beta - \alpha^2 + \frac{c}{4} \right) AU = 0. \quad (3.5)\]
It is, using (3.3)–(3.5), verified that \( \alpha \neq 0 \) on this set.

Combining (3.3) with (3.5), we see that
\[\alpha AU = 2\left( \alpha^2 - \beta - \frac{c}{4} \right) U \quad (3.6)\]
and thus \( AU = vU \) because of \( \alpha \neq 0 \), where we have put
\[\alpha v = 2\left( \alpha^2 - \beta - \frac{c}{4} \right). \quad (3.7)\]
From this and (3.3), we obtain
Therefore \( v = \text{constant} \neq 0 \) because of (3.3). Hence it is, using (3.7), seen that \( \alpha = \text{constant} \) and thus

\[
3v^2 - 2\alpha v + \alpha^2 - \beta - \frac{c}{4} = 0,
\]

which together with (3.7) and (3.8), produces a contradiction. Consequently \( h - \alpha \neq 0 \) on \( \Omega \) is proved. In what follows we assume that \( h - \alpha \neq 0 \) is satisfied everywhere.

Differentiating (2.1) covariantly, we find

\[
(\nabla_X S)e; + \nabla_X e; = \sigma \nabla_X \xi
\]

because \( \sigma = \text{constant} \) is assumed, which together with hypothesis \( \nabla_w S = 0 \) yields

\[
\nabla_w \xi = \sigma \nabla_w \xi.
\]

By the way we have \( \mu \nabla_w \xi = (h - \alpha)U \) with the aid of (1.5) and (2.4), (3.9) implies \( SU = \sigma U \) because of Remark 3.1. Hence (1.8) leads to

\[
A^2 U = hAU + \left( \beta - h\alpha + \frac{3}{4} c \right) U.
\]

From (2.3) we have

\[
\nabla \beta = \alpha \nabla h + h \nabla \alpha.
\]

Thus (2.15) is reduced to

\[
2\mu(W\alpha) = (h - 2\alpha)(\xi \alpha) + \alpha(\xi h).
\]

Using (1.15), (3.10) and (3.12), the equation (2.14) turns out to be

\[
hAU + 2(\beta - h\alpha + c) U = (\xi h) A\xi - A\nabla \alpha + h \nabla \alpha - \frac{1}{2} \nabla \beta.
\]

From (2.19) and (2.10), we also find

\[
\left( 2\beta - 2h\alpha + \frac{c}{2} \right) AU + \left\{ h(h\alpha - \beta) + \frac{c}{4}(3\alpha - 8h) \right\} U + g(A\xi, \nabla h) A\xi
\]

\[
= \frac{1}{2} A\nabla \beta - \beta \nabla \alpha + \left( \alpha - \frac{1}{2} h \right) \nabla \beta + \mu^2 \nabla h.
\]
Because of (3.2) and (3.10), we see that

\[(Wh)AW = -hAU - 2(\beta - h\alpha + c)U + A\nabla h - A\nabla \alpha + \frac{1}{2}\nabla \beta - \alpha \nabla h,\]

which together with (3.10) and (3.11) gives

\[A\nabla h = (Wh)AW + (\xi h)A\xi.\]  

(3.15)

Making use of (3.13) and (3.15), we have from (3.14)

\[
(4\beta - 4h\alpha + h^2 + c)AU + \left(\frac{3}{2} c \alpha - 2ch\right)U
= a(Wh)AW - \{(\alpha - h)(\xi h) + 2\mu(Wh)\}A\xi
+ \left(2ah - 2\beta - \frac{1}{2} h^2\right)\nabla \alpha + \left(2\beta - \frac{3}{2} h\alpha\right)\nabla h.
\]

(3.16)

If we use (2.2), (2.5) and (3.10), then above equation implies

\[
\frac{3}{4} c \left\{(4\beta - 4h\alpha + h^2 + c)AU + \left(\frac{3}{2} c \alpha - 2ch\right)U\right\}
= \left(2ah - 2\beta - \frac{1}{2} h^2\right)\{A^2\nabla \alpha - hA\nabla \alpha - (\beta - h\alpha)\nabla \alpha\}
+ \left(2\beta - \frac{3}{2} h\alpha\right)\{A^2\nabla h - hA\nabla h - (\beta - h\alpha)\nabla h\},
\]

which together with (3.15) yields

\[
\frac{3}{4} c \left\{(4\beta - 4h\alpha + h^2 + c)AU + \frac{c}{2} (3\alpha - 4h) U\right\}
= \left(2ah - 2\beta - \frac{1}{2} h^2\right)\{A^2\nabla \alpha - hA\nabla \alpha - (\beta - h\alpha)\nabla \alpha\}
+ \left(2\beta - \frac{3}{2} h\alpha\right)(\beta - h\alpha)\{(Wh)W + (\xi h)\xi - \nabla h\}.  
\]

(3.17)

On the other hand, we have from (3.13)

\[
A^2\nabla \alpha - hA\nabla \alpha + (h^2 + 2\beta - 2h\alpha + 2c)AU + h\left(\beta - h\alpha + \frac{3}{4} c\right)U
= (\xi h)A^2\xi - \frac{1}{2} A\nabla \beta,
\]
where we have used (3.10), or using (3.11) and (3.14),

\[ A^2 \nabla \alpha - hA \nabla \alpha + (\beta - ha) \nabla \alpha \]

\[ = \left( 4\alpha - 4\beta - h^2 - \frac{5}{2}c \right) AU + \frac{c}{4} (5h - 3a) U \]

\[ - \frac{1}{2} h^2 \nabla \alpha + \left( \beta - \frac{1}{2} \alpha \right) \nabla h + (\xi h) A^2 \xi - g(A \xi, \nabla h) A \xi. \]  

(3.18)

If we take the inner product \( \xi \) with this and make use of (1.15) and (2.2), then we obtain

\[ \mu \lambda (Wh) = \left( 2ha - 2\beta - \frac{1}{2}h^2 \right) (\xi \alpha) + \left( 2\beta - \frac{1}{2}ha - a^2 \right) (\xi h). \]  

(3.19)

Substituting (3.18) into (3.17) and taking account of (3.16), we find

\[ \frac{3}{2} c \left\{ cAU + c \left( 3\alpha - 4h \right) U + (h - a) \left( 2ah - 2\beta - \frac{1}{2}h^2 \right) U \right\} \]

\[ = h(h - a)(\beta - ha) \{ \nabla h - (\xi h) \xi - (Wh) W \}. \]  

(3.20)

Applying \( A \) to both sides of this and using (3.10) and (3.15), we have

\[ \left\{ \frac{c}{2} (3\alpha - 2h) + (h - a) \left( 2ah - 2\beta - \frac{1}{2}h^2 \right) \right\} AU + c \left( \beta - ha + \frac{3}{4}c \right) U = 0. \]  

(3.21)

**Lemma 3.1.** Let \( M \) be a real hypersurface of \( M_n(c) \) \( (c \neq 0) \). If it satisfies \( \nabla \omega S = 0 \) and \( S \xi = \sigma \xi \) for some constant \( \sigma \), then we have

\[ AU = \lambda U \]  

(3.22)

on \( \Omega \), where \( \mu^2 \lambda = g(AU, U) \).

**Proof.** Let \( \Omega_0 \) be a set of points in \( M \) such that \( \|AU - \lambda U\| \neq 0 \) on \( \Omega \) and suppose that \( \Omega_0 \) be nonempty. If \( \beta - ha + \frac{3}{4}c \neq 0 \), then we have from (3.21)

\[ \frac{c}{2} (3\alpha - 2h) + (h - a) \left( 2ah - 2\beta - \frac{1}{2}h^2 \right) \neq 0 \]

and hence (3.22) is valid. Thus it is, using (3.21), seen that

\[ \beta - ha + \frac{3}{4}c = 0 \]  

(3.23)

and therefore \( h(h^2 - ah - c) = 0 \) on \( \Omega_0 \). So we have
on $\Omega_0$. In fact, if not, then we have $h = 0$. Thus (3.10) and (3.23) are respectively to

\[ AU = 0, \quad \beta + \frac{3}{4}c = 0. \]

Hence (3.13) becomes $2(\beta + c)U + A\nabla \alpha = 0$. But, by (3.14) we have $\nabla \alpha = \alpha U$. Combining the last two equations, we obtain $\beta + c = 0$, a contradiction. Thus (3.24) is accomplished.

Differentiating (3.24), and using (3.23), we find

\[ 2h \nabla h = \alpha \nabla h + h \nabla \alpha = \nabla \beta. \] (3.25)

From this and (3.15) we obtain

\[ A \nabla \beta = 2h \{(Wh)AW + (\xi h)A\xi\}. \] (3.26)

If we take account of (3.23)–(3.26), then (3.14) turns out to be

\[ -cAU + \frac{c}{4}(3\alpha - 5h)U = (h - \alpha)(\xi h)A\xi - \mu(Wh)A\xi + h(Wh)AW \\
+ (\mu^2 + \alpha h - c)\nabla h - \beta \nabla \alpha. \] (3.27)

On the other hand, we have from (3.13)

\[ h^2 AU + \frac{c}{2} h U = (\alpha - h)(\xi h)A\xi + (\alpha - 2h)(Wh)AW + c \nabla h \]

because of (3.24)–(3.26). Comparing with the last two equations, it follows that

\[ (h^2 - c)AU + \frac{3}{4}c(\alpha - h)U \\
= (\alpha - h)(Wh)AW - \mu(Wh)A\xi + (\beta - \alpha^2 + \alpha h)\nabla h - \beta \nabla \alpha. \]

Applying this by $hA$ and making use of (2.2), (2.5) and (3.23), we find

\[ \left\{ h^2(h^2 - c) + \frac{3}{4}ch(\alpha - h) \right\} AU \\
= h(\alpha - h)(Wh)\left\{ hAW - \frac{3}{4}cW \right\} - \mu h(Wh)\left( hA\xi - \frac{3}{4}c\xi \right) \\
+ h(\beta - \alpha^2 + \alpha h)A \nabla h - \beta hA \nabla \alpha, \]
which together with (3.15) and (3.23)–(3.25) implies that

$$\left\{ (ah + c)ah - \frac{3}{4}c^2 \right\} AU$$

$$= h(\alpha - h)(Wh) \left( hAW - \frac{3}{4}cW \right) - \mu h(Wh) \left( hA\xi - \frac{3}{4}c\xi \right)$$

$$+ \frac{3}{4}c(\alpha - h) \{(Wh)AW + (\xi h)A\xi \}.$$  

(3.28)

Furthermore, using (2.2) and (2.5), we have from (3.28)

$$\left\{ (ah + c)ah - \frac{3}{4}c^2 \right\} AU = 0$$

because $U$ is orthogonal to $\xi$ and $W$. Hence we have

$$(\alpha^3 + c\alpha)h + c\alpha^2 - \frac{3}{4}c^2 = 0$$

on $\Omega_0$. Since $c \neq 0$, it follows that

$$h = \frac{\frac{3}{4}c^2 - c\alpha^2}{\alpha(\alpha^2 + c)}.$$  

(3.29)

From this and (3.24) we have $12\alpha^4 + 52c\alpha^2 - 9c^2 = 0$ on $\Omega_0$. So we see that $\nabla\alpha = 0$ and hence $\nabla h = 0$ because of (3.29). Thus (3.27) becomes $AU = \frac{1}{4}(3\alpha - 5h)U$ on $\Omega_0$. Therefore $\Omega_0$ is void. This completes the proof.

**Lemma 3.2.** Under the same assumptions as those stated in Lemma 3.1, we have $\xi\alpha = 0$, $W\alpha = 0$, $\xi h = 0$ and $Wh = 0$ on $\Omega$.

**Proof.** As in the proof of Lemma 3.1, it is sufficient to show that the following two cases:

Case 1. $\beta - h\alpha + \frac{3}{4}c = 0$ and $h^2 - h\alpha - c = 0$,

Case 2. $\frac{1}{2}(3\alpha - 2h) + (h - \alpha)(2ah - 2\beta - \frac{1}{2}h^2) \neq 0$.

Case 1: By taking the inner product with $\xi$ in (3.14), we obtain

$$\mu(h - \alpha)(Wh) = \left( 2\alpha^2 - 3h\alpha + \frac{7}{4}c \right)(\xi h) + \left( h\alpha - \frac{3}{4}c \right)(\xi\alpha).$$  

(3.30)

From (3.19) we have
\[ \mu \alpha (W h) = \frac{1}{2} (h \alpha - 2 c)(\xi \alpha) + \frac{1}{2} (3 h \alpha - 2 \alpha^2 - 3 c)(\xi h). \]  \hfill (3.31)

Using (3.24), (3.30) and (3.31), we are led to

\[ \{(\xi h)^2 + (\xi \alpha)^2\} (25 h \alpha + 14 c - 3 \alpha^2) = 0. \]  \hfill (3.32)

So, on the set of points satisfying \(25 h \alpha + 14 c - 3 \alpha^2 \neq 0\),

\[ \xi h = \xi \alpha = 0. \]

On account of Remark 3.1 and (3.30), we deduce that \(W h = 0\).

Further, from (3.12), we get \(W \alpha = 0\) since \(\mu \neq 0\).

If \(2 h \alpha + 14 c - 3 \alpha^2 \equiv 0\), then \(\alpha \neq 0\) since \(c \neq 0\). So, we have

\[ h = \frac{3 \alpha^2 - 14 c}{25 \alpha}. \]  \hfill (3.33)

Combining this with (3.24), we see that

\[(3 \alpha^2 - 14 c)^2 - 25 \alpha^2 (3 \alpha^2 - 14 c) - 625 \alpha c^2 \equiv 0. \]

Therefore we have \(V \alpha = 0\). So we have \(V h = 0\) by (3.33).

Case 2: Putting \(\beta - h \alpha + \frac{2}{3} c = c^r\), (3.21) is reduced to

\[ \left\{ \frac{c}{2} (3 \alpha - 2 h) + (h - \alpha) \left( \frac{3}{2} c - 2 c^r - \frac{1}{2} h^2 \right) \right\} AU + cc^r U = 0. \]

From this we have

\[ AU = \lambda U, \quad \lambda = \frac{-2 c c^r}{c (3 \alpha - 2 h) + (h - \alpha) (3 c - 4 c^r - h^2)}. \]

Therefore we are led to the following equation by (3.10):

\[ (4 c^r + h^2) \{(4 c^r + h^2) x^2 - 2 h (4 c^r + h^2) x + h^2 (4 c^r + h^2) - c^2\} = 0. \]  \hfill (3.34)

If \(4 c^r + h^2 \equiv 0\), then \(h = \text{constant}\). So, using (3.19), we are led to \(\xi \alpha = 0\) since \(c \neq 0\). Furthermore, from (3.12), we have \(W \alpha = 0\).
If $4c' + h^2 \neq 0$, then from (3.34) we have

$$
(4c' + h^2)\alpha^2 - 2h(4c' + h^2)\alpha + h^2(4c' + h^2) - c^2 = 0. \tag{3.35}
$$

Differentiating both sides of (3.35), we obtain

$$
(\alpha - h)(4c' + h^2)\nabla\alpha + \{h\alpha^2 - (4c' + h^2)\alpha + 2h(2c' + h^2)\}\nabla h = 0. \tag{3.36}
$$

By taking the inner products with $\xi$ in (3.14), we obtain

$$
\mu\alpha(W\alpha) - \mu h(W\alpha) = \left(-\alpha^2 + h\alpha + 2c' - \frac{3}{2}c\right)(\xi h)
+ \left(h\alpha - 2c' + \frac{3}{2}c - h^2\right)(\xi\alpha). \tag{3.37}
$$

By our assumption (3.19) is reduced to

$$
\mu\alpha(W\alpha) = \left(\frac{3}{2}h\alpha - \alpha^2 + 2c' - \frac{3}{2}c\right)(\xi h) - \left(\frac{1}{2}h^2 + 2c' - \frac{3}{2}c\right)(\xi\alpha). \tag{3.38}
$$

Using (3.36) and (3.37), we obtain

$$
2\mu(h^2 + 2c')(\alpha - h)(W\alpha)
= \left\{-2h(h^2 + 4c')\alpha + (h^2 + 4c')(h^2 + h\alpha + 2c' - \frac{3}{2}c) - c^2\right\}(\xi h)
+ (h^2 + 4c')(h\alpha - 2c' + \frac{3}{2}c - h^2)(\xi\alpha). \tag{3.39}
$$

Making use of (3.35), we have from (3.38) and (3.39)

$$
\left[-2(h^2 + 2c')\alpha^3 + 2h(3h^2 + 7c')\alpha^2 + \left\{-4h^4 - \left(8c' + \frac{3}{2}c\right)h^2 + c^2\right\}\alpha
+ (3c - 4c')h(h^2 + 2c')\right](\xi h)
- \left\{h^2\left(2h^2 - \frac{3}{2}c + 8c'\right)\alpha
+ h(c^2 - 10c'h^2 - 2h^4 + 3ch^2 - 8c'^2 + 6cc')\right\}(\xi\alpha) = 0. \tag{3.40}
$$

From (3.36) we have

$$
(\alpha - h)(h^2 + 4c')(\xi\alpha) + \{h\alpha^2 - (4c' + 3h^2)\alpha + 2h(2c' + h^2)\}(\xi h) = 0. \tag{3.41}
$$

From (3.40) and (3.41) we obtain
\{(\xi h)^2 + (\xi \alpha)^2\}
\times \left[ (\alpha - h)(h^2 + 4c') \left\{-2(h^2 + 2c')\alpha^3 + 2h(3h^2 + 7c')\alpha^2 \\
- 4h^4\alpha - \left(8c' + \frac{3}{2}c\right)h^2\alpha + \alpha^2\alpha + (3c - 4c')h(h^2 + 2c') \right\} \\
+ (h\alpha^2 - (4c' + 3h^2)\alpha + 2h(2c' + h^2)) \left\{ h^2 \left(2h^2 - \frac{3}{2}c + 8c'\right)\alpha \\
+ h(c^2 - 10c'h^2 - 2h^4 + 3ch^2 - 8c'^2 + 6c'c) \right\} \right] = 0. \quad (3.42)

If \((\xi h)^2 + (\xi \alpha)^2 \neq 0\), then from (3.42) we have
\((-12h^2c' - 2h^4 - 16c^2)\alpha^4 \\
+ \left( -\frac{3}{2}h^3c + 72hc^2 + 58h^3c' + 10h^5 \right)\alpha^3 \\
+ \left( 2h^2c^2 + 3h^4c + \frac{9}{2}h^2c + 4c'e^2 - 88c'h^4 \\
- 6h^4 - 14h^6 - 24c'h^2 - 128c'^2h^2 + 6c'ch^2 \right)\alpha^2 \\
+ \left( -18c'ch - 8c'e^2h + 6h^5 + 62c'h^5 - 3c^2h - 2h^3c^2 \\
+ 24c'^2h + 10h^7 + 88c'h^3 - 9ch^3 - \frac{3}{2}ch^5 + 30c'c^3 \right)\alpha \\
+ 6c'ch^4 + 4c'e^2h^2 - 4h^8 - 24c'h^6 - 32c'h^4 + 2c^2h^4 + 3ch^6 = 0. \quad (3.43)

Using Sylvester’s elimination method to (3.35) and (3.43), we deduce that
\((-24c'c' - 7c^2 + 16c'^2)h^{20} + (-57c^2c + 72c'c + 384c^3 - 48c'^2 \\
+ 21c^2 + 36c^3 - 120c'e^2)h^{18} + f(h) = 0, \quad (3.44)

where \(f(h)\) is the polynomial of \(h\) of degree \(\leq 16\). (We use a computer to calculate this.)

We can check that the coefficients of \(h^{20}\) and \(h^{18}\) does not vanish simultaneously since \(c \neq 0\). (We use a computer to check this.)

By the above argument, we know that (3.44) is a non-trivial algebraic equation of \(h\). So, we arrive at \(h = \text{constant}\). From (3.41), we have \(\xi \alpha = 0\). These
are contradictions. So, we have \( \xi \alpha = \xi h = 0 \). Furthermore, using (3.12) and (3.39), we arrive at \( W\alpha = Wh = 0 \). We have thus proved the lemma. ■

4 Proof of the Theorem

We continue our discussion under the same assumption of §3. First, we prove the following two lemmas:

**Lemma 4.1.** Let \( \lambda \) be a principal curvature corresponding to \( U \). Then \( \lambda \) does not vanish identically on \( \Omega = \{ p \in M \mid \mu(p) \neq 0 \} \).

**Proof.** From Lemma 3.1 and (3.10) the following equation holds on \( \Omega \):

\[
\lambda^2 = \lambda h + \beta - h\alpha + \frac{3}{4} c.
\] (4.1)

By Lemma 3.2, (3.15) becomes

\[
A \nabla h = 0, \quad \lambda(Uh) = 0.
\] (4.2)

Because of Lemma 3.1 and Lemma 3.2, (3.13) and (3.16) are reduced respectively to

\[
\{h\lambda + 2(\beta - h\alpha + c)\} U = -A\nabla \alpha + \frac{1}{2}(h\nabla \alpha - \alpha \nabla h),
\] (4.3)

\[
\theta U = \left(2ah - 2\beta - \frac{1}{2} h^2\right) \nabla \alpha + \left(2\beta - \frac{3}{2} h\alpha\right) \nabla h,
\] (4.4)

where we define \( \theta \) by \( \theta = (4\beta - 4h\alpha + h^2 + c)\lambda + \frac{3}{2} c\alpha - 2ch \).

From (3.11) and Lemma 3.2, we have \( \xi \beta = 0 \). Therefore it is seen, using Lemma 3.2, that

\[
\xi \theta = 0.
\]

From this and Lemma 3.1, we see, making use of (4.4), that

\[
\theta du(\xi, X) = 0
\] (4.5)

for any vector fields \( X \) on \( \Omega \), where \( u \) is defined by \( u(X) = g(U, X) \), and exterior derivation \( du \) of \( u \) is given by

\[
du(\xi, X) = \frac{1}{2} \{ \xi u(X) - Xu(\xi) - u([\xi, X])]\}. 
\]
On the other hand, using (1.15) and $AU = \lambda U$, the equation (1.14) turns out to be

$$\nabla_x U = \mu(\alpha - 3\lambda)W - \mu^2 \xi + \phi \nabla \alpha,$$

which together with (1.11) and (2.2) implies that

$$du(\xi, X) = (h - 3\lambda)\mu w(X) + g(\phi \nabla \alpha, X), \tag{4.6}$$

where $w(X) = g(W, X)$.

If $\lambda = 0$, then by (3.1) we have

$$\beta = -\frac{3}{4}c. \tag{4.7}$$

Thus (4.3) and (4.4) becomes respectively

$$cU = -2A\nabla \alpha + h\nabla \alpha - a\nabla h, \tag{4.8}$$

$$(3c - 4ch)U = (3c - h^2)\nabla \alpha - (3c - h\alpha)\nabla h. \tag{4.9}$$

Because of Lemma 3.1 and (4.2), we see, using (4.9), that

$$(3c - h^2)A\nabla \alpha = 0. \tag{4.10}$$

If the set of points satisfying $A\nabla \alpha \neq 0$ is not empty, then on that set we have

$h = \text{constant}$

because of (4.10). So, from (4.9), we are led to

$$\nabla \alpha = 0.$$

This is a contradiction. So, we obtain

$$A\nabla \alpha = 0 \quad \text{on } \Omega. \tag{4.11}$$

Thus (4.7) becomes

$$cU = h\nabla \alpha - a\nabla h.$$

So, we have

$$du(\xi, X) = 0$$

because of Lemma 3.2. Therefore (4.6) means that

$$\phi \nabla \alpha = \mu(h - 3\lambda)W.$$

Since $\xi \alpha = 0$, it follows that
Real hypersurfaces of a nonflat complex space form

\[ \nabla \alpha = hU. \quad (4.12) \]

So, from (4.8), we have

\[ \alpha \nabla h = (h^2 - c)U. \quad (4.13) \]

Combining last two equations with (3.2) and (3.11), we obtain

\[ A\nabla \beta = 0, \quad A\nabla \mu = 0. \]

Thus (2.18) with \( AU = 0 \) and (4.7) implies

\[ \frac{5}{4} c(h - \alpha)U = \frac{3}{4} c \nabla \alpha - \frac{1}{2} (h - \alpha) \{ \alpha \nabla h + h \nabla \alpha \} \]

\[ + \left( h \alpha - \frac{3}{4} c - \alpha^2 \right) \nabla h. \quad (4.14) \]

Substituting (4.12) and (4.13) in the right-hand side of (4.14), we are led to

\[ (h - \alpha)^2 = c. \quad (4.15) \]

Combining this with (4.12) and (4.13), we have

\[ \alpha(h - \alpha) = 0. \]

Since \( h - \alpha \neq 0 \), we have

\[ \alpha = 0. \quad (4.16) \]

So, (4.12) implies that \( h = 0 \). These are contradictions. We have thus proved the lemma.

**Lemma 4.2.** \( \theta = 0 \) on \( \Omega \).

**Proof.** If not, then from (4.5) we have

\[ du(\xi, X) = 0. \]

By (4.6), we obtain

\[ \nabla \alpha = (h - 3\lambda)U. \quad (4.17) \]

Hence (4.3) is reduced to

\[ \alpha \nabla h = \{ h^2 - 7\lambda h + 6\lambda^2 - 4(\beta - h\alpha + c) \} U. \quad (4.18) \]

Applying \( A \) to both sides of (4.18), we have
\[ 4(\beta - h\alpha) = h^2 - 7h\lambda + 6\lambda^2 - 4c \] (4.19)

since \( A\nabla h = 0 \) and \( \lambda \neq 0 \) on \( \Omega \).

Combining (4.19) with (4.1), we are led to
\[ 2\lambda^2 - 3\lambda h + h^2 - c = 0. \] (4.20)

Differentiating both sides of (4.20), we obtain
\[ (4\lambda - 3h)\nabla \lambda + (2h - 3\lambda)\nabla h = 0. \] (4.21)

On the other hand, from (4.1) we have
\[ (2\lambda - h)\nabla \lambda = \lambda \nabla h. \] (4.22)

Combining (4.22) with (4.21), we are led to
\[ (h - \lambda)^2 \nabla \lambda = 0. \]

Furthermore, we have
\[ \nabla \lambda = 0 \]

since \( h \neq \lambda \) by (4.20) and \( c \neq 0 \). So, from (4.22) we obtain
\[ \nabla h = 0 \] (4.23)

since \( \lambda \neq 0 \) by Lemma 4.1. Thus (4.4) becomes
\[ (4\beta - 4h\alpha + h^2 + c)\lambda + \frac{3}{2} c\alpha - 2ch = (h - 3\lambda) \left( 2ch - 2\beta - \frac{1}{2} h^2 \right). \] (4.24)

Differentiating both sides of (4.24), we have
\[ \nabla \alpha = 0 \] (4.25)

since \( c \neq 0 \).

From (4.4), (4.23) and (4.25), we are led to
\[ \theta = 0. \]

This is a contradiction. We have thus proved the lemma.

Finally, we prove

**Theorem 4.1.** Let \( M \) be a real hypersurface in \( M_n(c) \), \( c > 0 \). If it satisfies \( \nabla_{\theta \nu \xi} S = 0 \) and at the same time satisfies \( S \xi = \sigma \xi \) for some constant \( \sigma \), then \( M \) is a Hopf hypersurface.
PROOF. By Lemma 4.2 and (4.1), we have
\[
\lambda(4\lambda^2 - 4h\lambda + h^2 - 2c) = \frac{c}{2}(4h - 3\alpha), \tag{4.26}
\]
\[
\left(2\alpha h - 2\beta - \frac{1}{2}h^2\right)\nabla \alpha + \left(2\beta - \frac{3}{2}h\alpha\right)\nabla h = 0. \tag{4.27}
\]
Applying $A$ to both sides of (4.27) and using (4.2), we obtain
\[
\left(2\alpha h - 2\beta - \frac{1}{2}h^2\right)A\nabla \alpha = 0.
\]
Now, suppose that $A\nabla \alpha \neq 0$, then we have
\[
2\alpha h - 2\beta - \frac{1}{2}h^2 = 0.
\]
From this and our assumption $\sigma = \text{constant}$, we have
\[
\nabla h = 0. \tag{4.28}
\]
Differentiating both sides of (4.1), we obtain
\[
(h - 2\lambda)\nabla \lambda = 0. \tag{4.29}
\]
From (4.28) and (4.29), we are led to
\[
\nabla \lambda = 0. \tag{4.30}
\]
Thus from (4.26) we see that
\[
\nabla \alpha = 0.
\]
This contradicts to $A\nabla \alpha = 0$. So, we have
\[
A\nabla \alpha = 0, \quad U\alpha = 0 \tag{4.31}
\]
since $\lambda \neq 0$.

Using (4.2) and (4.31) and applying $U$ to both sides of (4.3), we have
\[
h\lambda + 2(\beta - h\alpha + c) = 0. \tag{4.32}
\]
From (4.1) and (4.32), we obtain
\[
\lambda^2 = \frac{1}{2}h\lambda - \frac{1}{4}c. \tag{4.33}
\]
Substituting (4.33) to both sides of (4.26), we are led to
\[ \alpha = h + 2\lambda \quad (4.34) \]

since \( c \neq 0 \).

Combining (4.34) with (4.32), we have

\[ g(U, U) = \beta - \alpha^2 = -7\lambda^2 - \frac{9}{4}c < 0. \]

This is a contradiction. The theorem is now proved by all the above arguments.

References


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