ON NON-COMMUTATIVE EXTENSIONS OF $G_a$ BY $G_m$
OVER AN $F_p$-ALGEBRA

By

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Abstract. We will give an explicit description of non-commutative extensions of the additive group scheme (resp. the additive formal group scheme) by the multiplicative group scheme (resp. the multiplicative formal group scheme) over an $F_p$-algebra.

Introduction

It is an interesting problem to determine the extensions of $G$ by $H$, where $G$ and $H$ are elementary group schemes over a ring $A$. For example, when $G = G_{a,A}$ and $H = G_{m,A}$, it is well known that $\text{Ext}^1_A(G_{a,A}, G_{m,A}) = 0$ if $A$ is a field (cf. [1]) and $\text{Ext}^1_A(\hat{G}_{a,A}, \hat{G}_{m,A}) = 0$ if $A$ is a perfect field.

Sekiguchi and Suwa [3] gave an explicit description on the commutative extensions of $\hat{G}_{a,A}$ by $\hat{G}_{m,A}$ or of $G_{a,A}$ by $G_{m,A}$ when $A$ is a ring of characteristic $p > 0$. More precisely, they have constructed isomorphisms

$$\text{Coker}[F : W(A) \rightarrow W(A)] \cong H^2_0(\hat{G}_{a,A}, \hat{G}_{m,A})$$

and

$$\text{Coker}[F : \hat{W}(A) \rightarrow \hat{W}(A)] \cong H^2_0(G_{a,A}, G_{m,A}),$$

using the Artin-Hasse exponential series. Here $H^2_0(\hat{G}_{a,A}, \hat{G}_{m,A})$ stands for the second symmetric Hochschild cohomology group of $\hat{G}_{a,A}$ with coefficients in $\hat{G}_{m,A}$, which describes commutative extensions of $\hat{G}_{a,A}$ by $\hat{G}_{m,A}$. However, there may exist a non-trivial extension of $G_{a,A}$ by $G_{m,A}$ if $A$ has a nilpotent element. [3] gave also an example of non-commutative extensions of $G_{a,A}$ by $G_{m,A}$ (cf. [3, Remark 3.10]).

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In this article, we determine the non-commutative extensions of $G_{a,A}$ by $G_{m,A}$ and of $G_{a,A}$ by $G_{m,A}$ when $A$ is of characteristic $p > 0$. More precisely, we can state the main theorem as follows:

**Theorem.** Let $p$ be a prime number and $A$ an $F_p$-algebra. Then the correspondence $(a_i)_{i \geq 1} \mapsto \prod_{i \geq 1} E_p(a_i; XY^{p^i})$ induces bijective homomorphisms

$$(\text{Ker}[F : W(A) \to W(A)])^N \cong H^2(G_{a,A}, G_{m,A})/H_0^2(G_{a,A}, G_{m,A})$$

and

$$(\text{Ker}[F : \hat{W}(A) \to \hat{W}(A)])^{(N)} \cong H^2(G_{a,A}, G_{m,A})/H_0^2(G_{a,A}, G_{m,A}).$$

Here $H^2(G_{a,A}, G_{m,A})$ stands for the second Hochschild cohomology group of $G_{a,A}$ with coefficients in $G_{m,A}$, which describes central extensions of $G_{a,A}$ by $G_{m,A}$. See Sect. 2 for further details concerning notations.

After a short review on Witt vectors and the Artin-Hasse exponential series, we state and prove the main theorem.

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**Notation**

Throughout the article, $p$ denotes a prime number.

- $G_{a,Z}$: the additive group scheme over $\mathbb{Z}$
- $G_{m,Z}$: the multiplicative group scheme over $\mathbb{Z}$
- $W_Z$: the group scheme of Witt vectors over $\mathbb{Z}$
- $\hat{G}_{a,Z}$: the additive formal group scheme over $\mathbb{Z}$
- $\hat{G}_{m,Z}$: the multiplicative formal group scheme over $\mathbb{Z}$
- $\hat{W}_Z$: the formal group scheme of Witt vectors over $\mathbb{Z}$
- $H^2_0(G, H)$ denotes the Hochschild cohomology group consisting of symmetric 2-cocycles of $G$ with coefficients in $H$ for group schemes or formal group schemes $G$ and $H$.

For a commutative ring $B$, $B^\times$ denotes the multiplicative group $G_{m,Z}(B)$. 
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For a commutative group $M$, $M^N$ (resp. $M^{(N)}$) stands for $\prod_{i \in N} M_i$ (resp. $\bigoplus_{i \in N} M_i$) where $M_i = M$.

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1. Recall: Witt Vectors and the Artin-Hasse Exponential Series

We start with reviewing necessary facts on Witt vectors. For details, see [1, Chap. V] or [2, Chap. III].

1.1. For each $r \geq 0$, we denote by $\Phi_r(T) = \Phi_r(T_0, T_1, \ldots, T_r)$ the so-called Witt polynomial

$$\Phi_r(T) = T_0^{p^r} + pT_1^{p^{r-1}} + \cdots + p^r T_r$$

in $\mathbb{Z}[T] = \mathbb{Z}[T_0, T_1, \ldots]$. We define polynomials

$$S_r(X, Y) = S_r(X_0, \ldots, X_r, Y_0, \ldots, Y_r)$$

and

$$P_r(X, Y) = P_r(X_0, \ldots, X_r, Y_0, \ldots, Y_r)$$

in $\mathbb{Z}[X, Y] = \mathbb{Z}[X_0, X_1, \ldots, Y_0, Y_1, \ldots]$ inductively by

$$\Phi_r(S_0(X, Y), S_1(X, Y), \ldots, S_r(X, Y)) = \Phi_r(X) + \Phi_r(Y)$$

and

$$\Phi_r(P_0(X, Y), P_1(X, Y), \ldots, P_r(X, Y)) = \Phi_r(X)\Phi_r(Y).$$

Then as is well-known, the ring structure of the scheme of Witt vectors

$$W_\mathbb{Z} = \text{Spec } \mathbb{Z}[T_0, T_1, T_2, \ldots]$$

is given by the addition

$$T_0 \mapsto S_0(X, Y), \quad T_1 \mapsto S_1(X, Y), \quad T_2 \mapsto S_2(X, Y), \ldots$$

and the multiplication

$$T_0 \mapsto P_0(X, Y), \quad T_1 \mapsto P_1(X, Y), \quad T_2 \mapsto P_2(X, Y), \ldots$$
We denote by \( \hat{W}_Z \) the formal completion of \( W_Z \) along the zero section. \( \hat{W}_Z \) is considered as a subfunctor of \( W_Z \). Indeed, if \( A \) is a ring,

\[
\hat{W}(A) = \left\{ (a_0, a_1, a_2, \ldots) \in W(A); \quad a_i \text{ is nilpotent for all } i \text{ and } a_i = 0 \text{ for all but a finite number of } i \right\}.
\]

1.2. Let \( A \) be an \( F_p \)-algebra. The Verschiebung homomorphism \( V : W(A) \to W(A) \) is defined by

\[
(a_0, a_1, a_2, \ldots) \mapsto (0, a_0, a_1, a_2, \ldots),
\]

and the Frobenius homomorphism \( F : W(A) \to W(A) \) is defined by

\[
(a_0, a_1, a_2, \ldots) \mapsto (a_0^p, a_1^p, a_2^p, \ldots).
\]

Then it is verified without difficulty that \( F \) is a ring homomorphism. It is obvious that \( \hat{W}(A) \) is stable under \( F \).

1.3. Let \( A \) be an \( F_p \)-algebra. Then we can verify without difficulty that:

1. \( FV = VF = p; \)
2. \( V(F(a)b) = aV(b) \) for \( a, b \in W(A) \).

Let \( A \) be a ring and \( a \in A \). We denote the Witt vector \( (a, 0, 0, \ldots) \) by \([a]\). \([a]\) is called the Teichmüller lifting of \( a \). It is readily seen:

1. \( [a][b] = [ab]; \)
2. \( F[a] = [a^p]; \)
3. \( (a_0, a_1, a_2, \ldots) = \sum_{k=0}^{\infty} V^k[a_k]. \)

1.4. Let \( Z_{(p)} \) denotes the localization of \( Z \) at the prime ideal \( (p) \). Recall now the definition of the Artin-Hasse exponential series

\[
E_p(T) = \exp \left( \sum_{r \geq 0} \frac{T^{p^r}}{p^r} \right) \in Z_{(p)}[[T]].
\]

For \( U = (U_r)_{r \geq 0} \), we put

\[
E_p(U; T) = \prod_{r \geq 0} E_p(U, T^{p^r}) = \exp \left( \sum_{r \geq 0} \frac{\Phi_r(U) T^{p^r}}{p^r} \right) \in Z_{(p)}[U][[T]].
\]

It is readily seen that

\[
E_p(S(U, V); T) = E_p(U; T)E_p(V; T).
\]
1.5. Let $A$ be an $F_p$-algebra and $a = (a_r)_{r \geq 0} \in W(A)$. Then the correspondence $a \mapsto E_p(a; T)$ gives rise to isomorphisms
$$\text{Ker}[F : W(A) \to W(A)] \cong \text{Hom}_{A-gr}(\hat{G}_a, \hat{G}_m)$$
and
$$\text{Ker}[F : \hat{W}(A) \to \hat{W}(A)] \cong \text{Hom}_{A-gr}(G_a, G_m).$$
(cf. [1, Chap. II])

It should be remarked that if $a = (a_r)_{r \geq 0} \in \text{Ker}[F : W(A) \to W(A)]$, then
$$E_p(a, T) = \prod_{r \geq 0} E_p(a_r T^{p^r}) = \prod_{r \geq 0} \left( \sum_{i=0}^{p-1} \frac{(a_r T^{p^r})^i}{i!} \right).$$

2. Statement of the Theorem

First we recall Hochschild cohomology groups. For details, see [1, Chap. II.5 and Chap. III.6].

2.1. Let $A$ be a ring. We define the multiplicative groups $Z^2(G_a, G_m)$, $Z^0(G_a, G_m)$ and $B^2(G_a, G_m)$ by

$$Z^2(G_a, G_m) = \{F(X, Y) \in A[X, Y]; F(X, Y)F(X + Y, Z) = F(X, Y + Z)F(Y, Z)\},$$

$$Z^0(G_a, G_m) = \left\{ F(X, Y)F(X + Y, Z) = F(X, Y + Z)F(Y, Z), \begin{array}{l} F(X, Y) = F(Y, X) \end{array} \right\}$$

$$B^2(G_a, G_m) = \left\{ \begin{array}{l} F(X)F(Y) \in A[T], \begin{array}{l} F(T) \in A[T] \end{array} \end{array} \right\}.$$
\[ Z^2(G_{a,A}, G_{a,A}) = \{ F(X, Y) \in A[X, Y]; \]
\[ F(X, Y) + F(X + Y, Z) = F(X, Y + Z) + F(Y, Z), \]
\[ Z_0^2(G_{a,A}, G_{a,A}) = \left\{ \begin{array}{l}
F(X, Y) \in A[X, Y]; \\
F(X, Y) + F(X + Y, Z) = F(X, Y + Z) + F(Y, Z), \\
F(X, Y) = F(Y, X) 
\end{array} \right\}, \]
\[ B^2(G_{a,A}, G_{a,A}) = \{ F(X) + F(Y) - F(X + Y); F(T) \in A[T] \}. \]

Then we have
\[ B^2(G_{a,A}, G_{a,A}) \subset Z_0^2(G_{a,A}, G_{a,A}) \subset Z^2(G_{a,A}, G_{a,A}). \]

We put
\[ H^2(G_{a,A}, G_{a,A}) = Z^2(G_{a,A}, G_{a,A}) / B^2(G_{a,A}, G_{a,A}), \]
\[ H_0^2(G_{a,A}, G_{a,A}) = Z_0^2(G_{a,A}, G_{a,A}) / B^2(G_{a,A}, G_{a,A}). \]

It is well known that:

1) \( H^2(G_{a,A}, G_{m,A}) \) (resp. \( H_0^2(G_{a,A}, G_{m,A}) \)) is isomorphic to the group of classes of central (resp. commutative) extensions of \( G_{a,A} \) by \( G_{m,A} \), which split as extensions of \( A \)-schemes.

2) \( H^2(G_{a,A}, G_{a,A}) \) (resp. \( H_0^2(G_{a,A}, G_{a,A}) \)) is isomorphic to the group of classes of central (resp. commutative) extensions of \( G_{a,A} \times A \) by \( G_{a,A} \).

2.2. Let \( A \) be a ring. We define the multiplicative formal groups \( Z^2(G_{a,A}, \hat{G}_{m,A}), \)
\( Z_0^2(G_{a,A}, \hat{G}_{m,A}) \) and \( B^2(G_{a,A}, \hat{G}_{m,A}) \) by
\[ Z^2(G_{a,A}, \hat{G}_{m,A}) = \left\{ F(X, Y) \in A[[X, Y]]^X; \right\}
\[ \begin{array}{l}
F(X, Y) \equiv 1 \mod \deg 1, \\
F(X, Y)F(X + Y, Z) = F(X, Y + Z)F(Y, Z) 
\end{array} \right\}, \]
\[ Z_0^2(G_{a,A}, \hat{G}_{m,A}) = \left\{ F(X, Y) \in A[[X, Y]]^X; \right\}
\[ \begin{array}{l}
F(X, Y) \equiv 1 \mod \deg 1, \\
F(X, Y)F(X + Y, Z) = F(X, Y + Z)F(Y, Z), \\
F(X, Y) = F(Y, X) 
\end{array} \right\}, \]
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$$B^2(\hat{G}_{a,A}, \hat{G}_{m,A}) = \left\{ \frac{F(X)F(Y)}{F(X + Y)} ; F(T) \in A[[T]]^*, F(T) \equiv 1 \mod \deg 1 \right\}.$$  

Then we have

$$B^2(\hat{G}_{a,A}, \hat{G}_{m,A}) \subset Z^2(\hat{G}_{a,A}, \hat{G}_{m,A}) \subset Z^2(\hat{G}_{a,A}, \hat{G}_{m,A}).$$

We put

$$H^2(\hat{G}_{a,A}, \hat{G}_{m,A}) = Z^2(\hat{G}_{a,A}, \hat{G}_{m,A})/B^2(\hat{G}_{a,A}, \hat{G}_{m,A}),$$

$$H_0^2(\hat{G}_{a,A}, \hat{G}_{m,A}) = Z_0^2(\hat{G}_{a,A}, \hat{G}_{m,A})/B^2(\hat{G}_{a,A}, \hat{G}_{m,A}).$$

We define also the additive formal groups $Z^2(\hat{G}_{a,A}, \hat{G}_{a,A}), Z^2_0(\hat{G}_{a,A}, \hat{G}_{a,A})$ and $B^2(\hat{G}_{a,A}, \hat{G}_{a,A})$ by

$$Z^2(\hat{G}_{a,A}, \hat{G}_{a,A}) = \left\{ F(X, Y) \in A[[X, Y]] ;
\begin{align*}
F(X, Y) &\equiv 0 \mod \deg 1, \\
F(X, Y) + F(X + Y, Z) &\equiv F(X, Y + Z) + F(Y, Z).
\end{align*}
\right\}.$$  

$$Z^2_0(\hat{G}_{a,A}, \hat{G}_{a,A}) = \left\{ F(X, Y) \in A[[X, Y]] ;
\begin{align*}
F(X, Y) &\equiv 0 \mod \deg 1, \\
F(X, Y) + F(X + Y, Z) &\equiv F(X, Y + Z) + F(Y, Z), \\
F(X, Y) &\equiv F(Y, X)
\end{align*}
\right\},$$

$$B^2(\hat{G}_{a,A}, \hat{G}_{a,A}) = \{ F(X) + F(Y) - F(X + Y) ;
\begin{align*}
F(T) &\in A[[T]], F(T) \equiv 0 \mod \deg 1 \}.
\right\}.$$  

Then we have

$$B^2(\hat{G}_{a,A}, \hat{G}_{a,A}) \subset Z^2_0(\hat{G}_{a,A}, \hat{G}_{a,A}) \subset Z^2(\hat{G}_{a,A}, \hat{G}_{a,A}).$$

We put

$$H^2(\hat{G}_{a,A}, \hat{G}_{a,A}) = Z^2(\hat{G}_{a,A}, \hat{G}_{a,A})/B^2(\hat{G}_{a,A}, \hat{G}_{a,A}),$$

$$H_0^2(\hat{G}_{a,A}, \hat{G}_{a,A}) = Z_0^2(\hat{G}_{a,A}, \hat{G}_{a,A})/B^2(\hat{G}_{a,A}, \hat{G}_{a,A}).$$

It is well known that:

1) $H^2(\hat{G}_{a,A}, \hat{G}_{m,A})$ (resp. $H_0^2(\hat{G}_{a,A}, \hat{G}_{m,A})$) is isomorphic to the group of classes of central (resp. commutative) extensions of $\hat{G}_{a,A}$ by $\hat{G}_{m,A}$, which split as extensions of formal $A$-schemes.
2) $H^2(\hat{G}_{a,A}, \hat{G}_{a,A})$ (resp. $H^2_0(\hat{G}_{a,A}, \hat{G}_{a,A})$) is isomorphic to the group of classes of central (resp. commutative) extensions of $\hat{G}_{a,A}$ by $\hat{G}_{a,A}$.

Proposition 2.3. Let $A$ be an $F_p$-algebra. If $P(X, Y) \in Z^2(G_{a,A}, G_{a,A})$, then $P(X, Y)$ is cohomologous to a cycle of the form:

$$\sum_{r \geq 1} a_r \frac{(X + Y)^{p^r} - X^{p^r} - Y^{p^r}}{p} + \sum_{0 \leq i < j} b_{ij} X^{p^i} Y^{p^j}, \quad a_r, b_{ij} \in A.$$

Proof. The statement is proved in [1, Chap. II.3] when $A$ is a field of characteristic $p$. However the argument works well for an arbitrary ring of characteristic $p$. We reproduce the proof presented in [loc.cit.] with a slight modification for the reader's convenience. For simplicity, we put

$$W(X, Y) = \sum_{i=1}^{p-1} \binom{p}{i} X^{p^{-i}} Y^i = \frac{(X + Y)^p - X^p - Y^p}{p}. $$

Let $P(X, Y) \in Z^2(G_{a,A}, G_{a,A}) \subset A[X, Y]$. We may assume that $P(X, Y)$ is homogeneous. Put

$$P(X, Y) = \sum_{n=0}^{n} a_i X^{n-i} Y^i, \quad n > 0. \quad (1)$$

By the assumption, we have

$$P(X, Y) + P(X + Y, Z) = P(X, Y + Z) + P(Y, Z). \quad (2)$$

Derivating (2) by $X$ and substituting 0 for $X$, we obtain

$$\frac{\partial P}{\partial X}(0, Y) + \frac{\partial P}{\partial X}(Y, Z) = \frac{\partial P}{\partial X}(0, Y + Z),$$

and therefore

$$\frac{\partial P}{\partial X}(X, Y) = a_{n-1} \{(X + Y)^{n-1} - X^{n-1}\}. $$

Derivating (2) by $Z$ and substituting 0 for $Z$, we obtain

$$\frac{\partial P}{\partial Z}(X + Y, 0) = \frac{\partial P}{\partial Z}(X, Y) + \frac{\partial P}{\partial Z}(Y, 0),$$

and therefore
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By Euler's formula, we obtain

$$nP(X, Y) = X \frac{\partial P}{\partial X}(X, Y) + Y \frac{\partial P}{\partial Y}(X, Y)$$

$$= a_1\{(X + Y)^n - X^n - Y^n\} + (a_{n-1} - a_1)(X + Y)^{n-1} - X^n.$$  

Now we distinguish several cases.

Case 1: $a_{n-1} \neq a_1$. Put $c = a_{n-1} - a_1$ and $Q(X, Y) = c\{(X + Y)^n - X^n\}$. Then

$$nP(X, Y) - Q(X, Y) \in B^2(G_{a, A}, G_{a, A}), \quad Q(X, Y) \in Z^2(G_{a, A}, G_{a, A}).$$

Replacing $X$ by $-Y$ in

$$Q(X, Y) + Q(X + Y, Z) = Q(X, Y + Z) + Q(Y, Z),$$

we obtain

$$cY\{(Y + Z)^n - Y^n - Z^n\} = 0. \quad (3)$$

Therefore

$$\binom{n-1}{k} \equiv 0 \text{ mod } p \quad \text{for each } k \text{ with } 0 < k < n-1$$

since $c \neq 0$. Hence we can conclude that $n - 1$ is a power of $p$. Put $n = 1 + p'$. Then

$$Q(X, Y) = cXY^{p'}$$

and therefore $P(X, Y)$ is cohomologous to $Q(X, Y)$.

Case 2: $a_{n-1} = a_1 \neq 0$. If $n \equiv 0 \text{ mod } p$, then

$$P(X, Y) = \frac{a_1\{(X + Y)^n - X^n - Y^n\}}{n} \in B^2(G_{a, A}, G_{a, A}).$$

On the other hand, assume that $n \equiv 0 \text{ mod } p$. Then we have a congruence

$$\binom{n-1}{p-1} = \frac{(n-1)(n-2) \cdots (n-p+1)}{1 \cdot 2 \cdots (p-1)} \equiv 1 \text{ mod } p.$$

If $n \neq p$, it follows that $a_1\{(X + Y)^{n-1} - Y^{n-1}\}$ contains the term $a_1X^{n-p}Y^{p-1}$, which is a contradiction to $a_1 \neq 0$ since
\begin{equation}
\frac{\partial P}{\partial Y}(X, Y) = a_1 \{(X + Y)^{n-1} - Y^{n-1}\}.
\end{equation}

Therefore \( n = p \), we obtain
\begin{equation}
\frac{\partial P}{\partial X}(X, Y) = a_1 \{(X + Y)^{p-1} - Y^{p-1}\} = a_1 \frac{\partial W}{\partial X}(X, Y)
\end{equation}
and
\begin{equation}
\frac{\partial P}{\partial Y}(X, Y) = a_1 \{(X + Y)^{p-1} - Y^{p-1}\} = a_1 \frac{\partial W}{\partial Y}(X, Y).
\end{equation}

Derivating \( P(X, Y) - a_1 W(X, Y) \) by \( X \) and by \( Y \) respectively, we obtain
\begin{equation}
\frac{\partial P}{\partial X}(X, Y) - a_1 \frac{\partial W}{\partial X}(X, Y) = \frac{\partial P}{\partial Y}(X, Y) - a_1 \frac{\partial W}{\partial Y}(X, Y) = 0.
\end{equation}

Hence we obtain
\begin{equation}
P(X, Y) = a_1 W(X, Y) + a_0 X^p + a_p Y^p,
\end{equation}
and \( a_0 = a_p = 0 \) since \( a_0 X^p + a_p Y^p \in Z^2(G_{a, A}, G_{a, A}) \).

Case 3: \( a_{n-1} = a_1 = 0 \). Then we have
\begin{equation}
\frac{\partial P}{\partial X}(X, Y) = \frac{\partial P}{\partial Y}(X, Y) = 0.
\end{equation}

Hence we obtain \( P(X, Y) = P_1(X^p, Y^p) \), where \( P_1(X, Y) \) is a 2-cocycle of degree \( n/p < n \) if \( P(X, Y) \neq 0 \).

Replacing \( P(X, Y) \) by \( P_1(X, Y) \) and repeating the same argument as above, we can obtain the required result.

Now we define symmetric 2-cocycles of \( \hat{G}_{a, A} \) with coefficients in \( \hat{G}_{m, A} \), using the Artin-Hasse exponential series. For details, see [3, 2.2].

2.4. A formal power series

\begin{equation}
F_p(U; X, Y) = \exp \left( \sum_{i \geq 1} U^{p^{-1}} \frac{X^{p^i} + Y^{p^i} - (X + Y)^{p^i}}{p^i} \right) \in Z_{(p)}[[X, Y]].
\end{equation}
is defined in [3, 2.2].

For \( U = (U_r)_{r \geq 0} \), we put
\begin{equation}
F_p(U; X, Y) = \prod_{r \geq 0} F_p(U_r; X^{p^r}, Y^{p^r}) \in Z_{(p)}[[X, Y]].
\end{equation}
It is readily seen that
\[ F_p(S(U, V); X, Y) = F_p(U; X, Y)F_p(V; X, Y). \]

2.5. Assume now that \( A \) is an \( F_p \)-algebra. Let \( a = (a_r)_{r \geq 0} \in W(A) \). Define a formal power series by
\[ F_p(a; X, Y) = \prod_{r \geq 0} F_p(a_r; X^{p^r}, Y^{p^r}) \in A[[X, Y]]. \]

The following assertion was proved in [3, 3.4]:
Let \( A \) be an \( F_p \)-algebra. Then the correspondence \( a \mapsto F_p(a; X, Y) \) gives rise to isomorphisms
\[ \text{Coker}[F : W(A) \to W(A)] \cong H^3_0(\hat{G}_{a,A}, \hat{G}_{m,A}) \]
and
\[ \text{Coker}[F : \hat{W}(A) \to \hat{W}(A)] \cong H^3_0(G_{a,A}, G_{m,A}). \]

Remark 2.6. Let \( A \) be an \( F_p \)-algebra. If \( G(X, Y) \in Z^2_0(G_{a,A}, G_{a,A}) \) and \( G(X, Y) \) is a homogeneous polynomial of degree \( l \), then there exists \( F(X, Y) \in Z^2_0(G_{a,A}, G_{m,A}) \) such that
\[ F(X, Y) \equiv 1 + G(X, Y) \mod \deg(l + 1). \]
(cf. [3, Proof of Lemma 3.1])

2.7. Now we observe the following facts:
If \( F(T) \in \text{Hom}_{A-gr}(\hat{G}_{a,A}, \hat{G}_{m,A}) \) and \( G(X, Y) \in Z^2(\hat{G}_{a,A}, \hat{G}_{a,A}) \), then
\[ F(G(X, Y)) \in Z^2(\hat{G}_{a,A}, \hat{G}_{m,A}). \]
For example, \( E_p(a; T) \in \text{Hom}_{A-gr}(\hat{G}_{a,A}, \hat{G}_{m,A}) \) for \( a = (a_r)_{r \geq 0} \in \text{Ker}[F : W(A) \to W(A)] \) (see 1.5) and
\[ X^{p^r}Y^{p^r} \in Z^2(G_{a,A}, G_{a,A}) \subset Z^2(\hat{G}_{a,A}, \hat{G}_{a,A}) \quad (r > 0). \]

Then
\[ E_p(a; X^{p^r}Y^{p^r}) \in Z^2(\hat{G}_{a,A}, \hat{G}_{m,A}). \]

Any non-symmetric 2-cocycle of \( \hat{G}_{a,A} \) with coefficients in \( \hat{G}_{m,A} \) is obtained in the above way. In fact, we have the following:
THEOREM 2.8. Let $A$ be an $F_p$-algebra. Then the correspondence $(a_r)_{r \geq 1} \mapsto \prod_{r \geq 1} E_p(a_r; XY^r)$ gives rise to isomorphisms

$$(\text{Ker}[F : W(A) \to W(A)])^N \cong H^2(\hat{G}_a, \hat{G}_m, A)/H_0(\hat{G}_a, \hat{G}_m, A)$$

and

$$(\text{Ker}[F : \hat{W}(A) \to \hat{W}(A)])^{(N)} \cong H^2(\hat{G}_a, \hat{G}_m, A)/H_0(\hat{G}_a, \hat{G}_m, A).$$

COROLLARY 2.9. Let $A$ be an $F_p$-algebra. If $P(X, Y) \in Z^2(\hat{G}_a, \hat{G}_m, A)$ (resp. $Z^2(\hat{G}_a, \hat{G}_m, A)$), then $P(X, Y)$ is cohomologous to a 2-cocycle of the form:

$$F_p(b; X, Y) \prod_{r \geq 1} E_p(a_r; XY^r),$$

where $b \in W(A)$ and $(a_r)_{r \geq 1} \in (\text{Ker}[F : W(A) \to W(A)])^N$ (resp. $b \in \hat{W}(A)$ and $(a_r)_{r \geq 1} \in (\text{Ker}[F : \hat{W}(A) \to \hat{W}(A)])^{(N)}$).

3. Proof of the Theorem

Now we start proving necessary lemmas for our proof of Theorem 2.8.

LEMMA 3.1. Let $A$ be an $F_p$-algebra and $F(X, Y) \in Z^2(\hat{G}_a, \hat{G}_m, A)$. If

$$F(X, Y) \equiv 1 \mod \text{deg}(p^r + 1) \quad (r > 0),$$

then there exists $\tilde{F}(X, Y) \in Z_0^2(\hat{G}_a, \hat{G}_m, A)$ and $a_{r,0}, a_{r-1,1}, \ldots, a_{1,r-1} \in A$ such that

$$F(X, Y)\tilde{F}(X, Y)^{-1} \equiv \sum_{k=0}^{p-1} \frac{1}{k!} \left\{a_{r,0}XY^p + a_{r-1,1}X^pY^p + \cdots + a_{1,r-1}X^{p^{r-1}}Y^p\right\}^k \mod \text{deg}(p^{r+1} + 1).$$

PROOF. We shall prove that there exist $\tilde{F}(X, Y) \in Z_0^2(\hat{G}_a, \hat{G}_m, A)$ and $a_{r,0}, a_{r-1,1}, \ldots, a_{1,r-1} \in A$ such that

$$F(X, Y)\tilde{F}(X, Y)^{-1} \equiv \sum_{k=0}^{l+1} \frac{1}{k!} \left\{a_{r,0}XY^p + a_{r-1,1}X^pY^p + \cdots + a_{1,r-1}X^{p^{r-1}}Y^p\right\}^k \mod \text{deg}(l + 2)(p^r + 1)$$

by the induction on $l \ (0 \leq l \leq p - 3)$.

Step 1. Assume that
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\[ F(X, Y) \equiv 1 + H(X, Y) \mod \deg 2(p^r + 1), \]

where

\[ H(X, Y) = \sum_{i=(p^r+1)}^{2(p^r+1)-1} H_i(X, Y), \]

here $H_i(X, Y)$ is the homogeneous part of degree $i$. It is readily seen that $H_i(X, Y)$ satisfies the functional equation

\[ H_i(X, Y) + H_i(X + Y, Z) = H_i(X, Y + Z) + H_i(Y, Z). \]

Hence we obtain that $H(X, Y)$ satisfies the functional equation

\[ H(X, Y) + H(X + Y, Z) = H(X, Y + Z) + H(Y, Z). \]

By Proposition 2.3, there exists $\tilde{H}(X, Y) \in Z_0^2(G_{a, A}, G_{a, A})$ and $a_{r,0}, a_{r-1,1}, \ldots, a_{1,r-1} \in A$ such that

\[ H(X, Y) = \tilde{H}(X, Y) + \{a_{r,0}XY^{p^r} + a_{r-1,1}X^{p^r}Y^{p^r} + \cdots + a_{1,r-1}X^{p^{r-1}}Y^{p^r}\}. \]

Note that $\tilde{H}(X, Y)$ is the sum of homogeneous polynomials. By Remark 2.6, there exists $\tilde{F}(X, Y) \in Z_0^2(G_{a, A}, \hat{G}_{m, A})$ such that

\[ \tilde{F}(X, Y) \equiv 1 + \tilde{H}(X, Y) \mod \deg 2(p^r + 1). \]

Hence, we obtain

\[ F(X, Y)\tilde{F}(X, Y)^{-1} \equiv 1 + \{a_{r,0}XY^{p^r} + a_{r-1,1}X^{p^r}Y^{p^r} + \cdots + a_{1,r-1}X^{p^{r-1}}Y^{p^r}\} \mod \deg 2(p^r + 1). \]

Step 2. By the assumption of the induction, we can put

\[ F(X, Y) \equiv \sum_{k=0}^{l-1} \frac{1}{k!} G(X, Y)^k + H(X, Y) \mod \deg(l + 2)(p^r + 1) \quad (4) \]

for some $l \leq p - 3$, where

\[ H(X, Y) = \sum_{i=(l+1)(p^r+1)}^{(l+2)(p^r+1)-1} H_i(X, Y), \]

here $H_i(X, Y)$ is homogeneous part of degree $i$ and

\[ G(X, Y) = a_{r,0}XY^{p^r} + a_{r-1,1}X^{p^r}Y^{p^r} + \cdots + a_{1,r-1}X^{p^{r-1}}Y^{p^r}. \]
Since
\[
\left( \sum_{k=0}^{l} \frac{1}{k!} X^k \right) \left( \sum_{k=0}^{l} \frac{1}{k!} Y^k \right) = \sum_{k=0}^{l} \frac{1}{k!} (X + Y)^k \\
+ \frac{1}{(l+1)!} \{(X + Y)^{l+1} - X^{l+1} - Y^{l+1}\} \mod \deg(l + 2), \tag{5}
\]
we have
\[
\left\{ \sum_{k=0}^{l} \frac{1}{k!} G(X, Y)^k \right\} \left\{ \sum_{k=0}^{l} \frac{1}{k!} G(X + Y, Z)^k \right\} = \sum_{k=0}^{l} \frac{1}{k!} (G(X, Y) + G(X + Y, Z))^k \\
+ \frac{1}{(l+1)!} \{(G(X, Y) + G(X + Y, Z))^{l+1} - G(X, Y)^{l+1} - G(X + Y, Z)^{l+1}\} \\
\mod \deg(l + 2)(p^r + 1)
\]
and
\[
\left\{ \sum_{k=0}^{l} \frac{1}{k!} G(X, Y + Z)^k \right\} \left\{ \sum_{k=0}^{l} \frac{1}{k!} G(Y, Z)^k \right\} = \sum_{k=0}^{l} \frac{1}{k!} (G(X, Y + Z) + G(Y, Z))^k \\
+ \frac{1}{(l+1)!} \{(G(X, Y + Z) + G(Y, Z))^{l+1} - G(X, Y + Z)^{l+1} - G(Y, Z)^{l+1}\} \\
\mod \deg(l + 2)(p^r + 1).
\]
On the other hand, since
\[
F(X, Y)F(X + Y, Z) = F(X, Y + Z)F(Y, Z),
\]
we have
\[
\left\{ \sum_{k=0}^{l} \frac{1}{k!} G(X, Y)^k + H(X, Y) \right\} \left\{ \sum_{k=0}^{l} \frac{1}{k!} G(X + Y, Z)^k + H(X + Y, Z) \right\} \\
= \left\{ \sum_{k=0}^{l} \frac{1}{k!} G(X, Y + Z)^k + H(X, Y + Z) \right\} \left\{ \sum_{k=0}^{l} \frac{1}{k!} G(Y, Z)^k + H(Y, Z) \right\} \\
\mod \deg(l + 2)(p^r + 1).
\]
Comparing the terms of degree $i$ with $(l+1)(p^r + 1) \leq i \leq (l + 2)(p^r + 1) - 1$, we have
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\[ H(X, Y) + H(X + Y, Z) \]
\[ + \frac{1}{(l+1)!} \left\{ (G(X, Y) + G(X + Y, Z))^{l+1} - G(X, Y)^{l+1} - G(X + Y, Z)^{l+1} \right\} \]
\[ = H(X, Y + Z) + H(Y, Z) \]
\[ + \frac{1}{(l+1)!} \left\{ (G(X, Y + Z) + G(Y, Z))^{l+1} - G(X, Y + Z)^{l+1} - G(Y, Z)^{l+1} \right\}. \]

Since we see that $G(X, Y) \in Z^2(G_a, A, G_a, A)$, and so

\[ H(X, Y) + H(X + Y, Z) - \frac{1}{(l+1)!} G(X, Y)^{l+1} - \frac{1}{(l+1)!} G(X + Y, Z)^{l+1} \]
\[ = H(X, Y + Z) + H(Y, Z) - \frac{1}{(l+1)!} G(X, Y + Z)^{l+1} - \frac{1}{(l+1)!} G(Y, Z)^{l+1}, \]

it follows that

\[ \tilde{H}(X, Y) := H(X, Y) - \frac{1}{(l+1)!} G(X, Y)^{l+1} \in Z^2(G_a, A, G_a, A). \quad (6) \]

Noting that $\tilde{H}(X, Y)$ has only terms of degree $i$ with $(l+1)(p' + 1) \leq i \leq (l + 2)(p' + 1) - 1$, we can conclude by Proposition 2.3 that

\[ \tilde{H}(X, Y) \in Z^2_0(G_a, A, G_a, A). \]

By Remark 2.6, there exist $\tilde{F}(X, Y) \in Z^3_0(G_a, A, G_m, A)$ such that

\[ \tilde{F}(X, Y) \equiv 1 + \tilde{H}(X, Y) \mod \deg((l + 2)(p' + 1)). \]

Hence we have

\[ F(X, Y)\tilde{F}(X, Y)^{-1} = \left\{ \sum_{k=0}^{l} \frac{1}{k!} G(X, Y)^{k} + H(X, Y) \right\}\{1 - \tilde{H}(X, Y)\} \]
\[ = \sum_{k=0}^{l} \frac{1}{k!} G(X, Y)^{k} + \frac{1}{(l+1)!} G(X, Y)^{l+1} \]
\[ \mod \deg((l + 2)(p' + 1)) \]
by (4) and (6).

Step 3. Assume that

\[ F(X, Y) \equiv \sum_{k=0}^{p-2} \frac{1}{k!} G(X, Y)^{k} + H(X, Y) \mod \deg(p^{r+1} + 1), \quad (7) \]
where
\[ H(X, Y) = \sum_{i=(p-1)(p^r+1)}^{p^r+1} H_i(X, Y), \]
here \( H_i(X, Y) \) is homogeneous part of degree \( i \) and
\[ G(X, Y) = a_{r,0}XY^{p^r} + a_{r-1,1}X^{p}Y^{p^r} + \cdots + a_{1, r-1}X^{p-1}Y^{p^r}. \]
By (5), we obtain that
\[
\left\{ \sum_{k=0}^{p-2} \frac{1}{k!} G(X, Y)^k \right\} \left\{ \sum_{k=0}^{p-2} \frac{1}{k!} G(X + Y, Z)^k \right\} = \sum_{k=0}^{p-2} \frac{1}{k!} (G(X, Y) + G(X + Y, Z))^k
\]
\[ + \frac{1}{(p-1)!} ((G(X, Y) + G(X + Y, Z))^{p-1} - G(X, Y)^{p-1} - G(X + Y, Z)^{p-1}) \]
mod deg \( p(p^r+1) \)
and
\[
\left\{ \sum_{k=0}^{p-2} \frac{1}{k!} G(X + Z)^k \right\} \left\{ \sum_{k=0}^{p-2} \frac{1}{k!} G(Y, Z)^k \right\} = \sum_{k=0}^{p-2} \frac{1}{k!} (G(X + Z) + G(Y, Z))^k
\]
\[ + \frac{1}{(p-1)!} ((G(X + Z) + G(Y, Z))^{p-1} - G(X + Z)^{p-1} - G(Y, Z)^{p-1}) \]
mod deg \( p(p^r+1) \).
On the other hand, since
\[ F(X, Y)F(X + Y, Z) = F(X, Y + Z)F(Y, Z), \]
we have
\[
\left\{ \sum_{k=0}^{p-2} \frac{1}{k!} G(X, Y)^k + H(X, Y) \right\} \left\{ \sum_{k=0}^{p-2} \frac{1}{k!} G(X + Y, Z)^k + H(X + Y, Z) \right\}
\]
\[ \equiv \left\{ \sum_{k=0}^{p-2} \frac{1}{k!} G(X + Z)^k + H(X, Y + Z) \right\} \left\{ \sum_{k=0}^{p-2} \frac{1}{k!} G(Y, Z)^k + H(Y, Z) \right\} \]
mod deg\((p^{r+1}+1)\).
Comparing the terms of degree \( i \) with \((p-1)(p^r+1) \leq i \leq p^{r+1}\), we have
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\[ H(X, Y) + H(X + Y, Z) \]
\[ + \frac{1}{(p-1)!} \{(G(X, Y) + G(X + Y, Z))^{p-1} - G(X, Y)^{p-1} - G(X + Y, Z)^{p-1}\} \]
\[ = H(X, Y + Z) + H(Y, Z) \]
\[ + \frac{1}{(p-1)!} \{(G(X, Y + Z) + G(Y, Z))^{p-1} - G(X, Y + Z)^{p-1} - G(Y, Z)^{p-1}\}. \]

Since we see that $G(X, Y) \in Z^2(G_a, A, G_m, A)$, and so

\[ H(X, Y) + H(X + Y, Z) - \frac{1}{(p-1)!} G(X, Y)^{p-1} - \frac{1}{(p-1)!} G(X + Y, Z)^{p-1} \]
\[ = H(X, Y + Z) + H(Y, Z) - \frac{1}{(p-1)!} G(X, Y + Z)^{p-1} - \frac{1}{(p-1)!} G(Y, Z)^{p-1}, \]

it follows that

\[ \tilde{H}(X, Y) := H(X, Y) - \frac{1}{(p-1)!} G(X, Y)^{p-1} \in Z^2(G_a, A, G_a, A). \] (8)

Noting that $\tilde{H}(X, Y)$ has only terms of degree $i$ with $(p-1)(p^r+1) \leq i \leq p^{r+1}$, we can conclude by Proposition 2.3 that

\[ \tilde{H}(X, Y) \in Z^2_0(G_a, A, G_a, A). \]

By Remark 2.6, there exist $\tilde{F}(X, Y) \in Z^3_0(\hat{G}_a, A, \hat{G}_m, A)$ such that

\[ \tilde{F}(X, Y) \equiv 1 + \tilde{H}(X, Y) \mod \text{deg}(p^{r+1} + 1). \]

Hence we have

\[ F(X, Y)\tilde{F}(X, Y)^{-1} \equiv \left\{ \sum_{k=0}^{p-2} \frac{1}{k!} G(X, Y)^k + H(X, Y) \right\} \{1 - \tilde{H}(X, Y)\} \]
\[ \equiv \sum_{k=0}^{p-2} \frac{1}{k!} G(X, Y)^k + \frac{1}{(p-1)!} G(X, Y)^{p-1} \mod \text{deg}(p^{r+1} + 1) \]

by (7) and (8).

**Lemma 3.2.** Let $A$ be an $F_p$-algebra and $F(X, Y) \in Z^2(\hat{G}_a, A, \hat{G}_m, A)$. If

\[ F(X, Y) \equiv \sum_{k=0}^{p-1} \frac{1}{k!} \left\{ a_{r,0}XY^{p^r} + a_{r-1,1}X^pY^{p^r} + \cdots + a_{1, r-1}X^{p^{r-1}}Y^{p^r} \right\}^k \]
\[ \mod \text{deg}(p^{r+1} + 1) \]
with $r > 0$ and $a_{r,0}, a_{r-1,1}, \ldots, a_{1,r-1} \in A$, then

$$a_{r,0}^p = a_{r-1,1}^p = \cdots = a_{1,r-1}^p = 0.$$ 

**Proof.** We shall prove $a_{r-l,l}^p = 0$ by the induction on $l$ ($0 \leq l \leq r - 1$).

Step 1. Assume that

$$F(X, Y) = \sum_{k=0}^{p-1} \frac{1}{k!} G(X, Y)^k + H(X, Y) \mod \deg(p(p' + 1) + 1),$$

where

$$H(X, Y) = \sum_{i=p+1}^{p(p'+1)} H_i(X, Y),$$

here $H_i(X, Y)$ is homogeneous part of degree $i$ and

$$G(X, Y) = a_{r,0}XY^{p'} + a_{r-1,1}X^{p'}Y + \cdots + a_{1,r-1}X^{p'-1}Y^{p'}.$$ 

Now put

$$W(X, Y) = \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} X^{p-i}Y^i = \frac{(X + Y)^p - X^p - Y^p}{p}$$

as in the proof of Proposition 2.3. By (5), we obtain that

$$\left\{ \sum_{k=0}^{p-1} \frac{1}{k!} G(X, Y)^k \right\} \left\{ \sum_{k=0}^{p-1} \frac{1}{k!} G(X + Y, Z)^k \right\}
= \sum_{k=0}^{p-1} \frac{1}{k!} \{G(X, Y) + G(X + Y, Z)\}^k - W(G(X, Y), G(X + Y, Z))
\mod \deg(p + 1)(p' + 1)$$

and

$$\left\{ \sum_{k=0}^{p-1} \frac{1}{k!} G(X, Y + Z)^k \right\} \left\{ \sum_{k=0}^{p-1} \frac{1}{k!} G(Y, Z)^k \right\}
= \sum_{k=0}^{p-1} \frac{1}{k!} \{G(X, Y + Z) + G(Y, Z)\}^k - W(G(X, Y + Z), G(Y, Z))
\mod \deg(p + 1)(p' + 1)$$
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since $(p - 1)! \equiv -1 \mod p$. On the other hand, since

$$F(X, Y)F(X + Y, Z) = F(X, Y + Z)F(Y, Z),$$

we have

$$\left\{ \sum_{k=0}^{p-1} \frac{1}{k!} G(X, Y)^k + H(X, Y) \right\} \left\{ \sum_{k=0}^{p-1} \frac{1}{k!} G(X + Y, Z)^k + H(X + Y, Z) \right\}$$

$$\equiv \left\{ \sum_{k=0}^{p-1} \frac{1}{k!} G(X, Y + Z)^k + H(X, Y + Z) \right\} \left\{ \sum_{k=0}^{p-1} \frac{1}{k!} G(Y, Z)^k + H(Y, Z) \right\}$$

$$\mod \deg(p(p^r + 1) + 1).$$

Hence we obtain

$$\sum_{k=0}^{p-1} \frac{1}{k!} \{G(X, Y) + G(X + Y, Z)\}^k - W(G(X, Y), G(X + Y, Z))$$

$$+ H(X, Y) + H(X + Y, Z)$$

$$\equiv \sum_{k=0}^{p-1} \frac{1}{k!} \{G(X, Y + Z) + G(Y, Z)\}^k - W(G(X, Y + Z), G(Y, Z))$$

$$+ H(X, Y + Z) + H(Y, Z) \mod \deg(p(p^r + 1) + 1).$$

Since we see that $G(X, Y) \in Z^2(G_{0, A}, G_{n, A})$, we obtain

$$-W(G(X, Y), G(X + Y, Z)) + H(X, Y) + H(X + Y, Z)$$

$$\equiv -W(G(X, Y + Z), G(Y, Z)) + H(X, Y + Z) + H(Y, Z)$$

$$\mod \deg(p(p^r + 1) + 1).$$

Now noting that

$$W(G(X, Y), G(X + Y, Z))$$

$$\equiv W(a_{r, 0}XY^{p^r}, a_{r, 0}(X + Y)Z^{p^r}) \mod \deg(p(p^r + 1) + 1)$$

and

$$W(G(X, Y + Z), G(Y, Z))$$

$$\equiv W(a_{r, 0}X(Y + Z)^{p^r}, a_{r, 0}YZ^{p^r}) \mod \deg(p(p^r + 1) + 1),$$
we obtain
\[ H_{p(p'+1)}(X, Y) + H_{p(p'+1)}(X + Y, Z) - a_{r,0}^p W(XY^{p'}, XZ^{p'} + YZ^{p'}) \]
\[ = H_{p(p'+1)}(X, Y + Z) + H_{p(p'+1)}(Y, Z) - a_{r,0}^p W(XY^{p'} + XZ^{p'}, YZ^{p'}). \]  
(9)

Put now
\[ H_{p(p'+1)}(X, Y) = \sum_{i+j=p(p'+1)} c_{ij} X^i Y^j. \]

It is easily verified that
\[ W(XY^{p'}, XZ^{p'} + YZ^{p'}) = \frac{1}{p} \sum_{l=1}^{p-1} \binom{p}{l} (XY^{p'})^p - l (XZ^{p'} + YZ^{p'})^l \]
\[ = \sum_{\substack{i+j+k=p \atop i \geq 1, j+k \geq 1}} \frac{(p-1)!}{i!j!k!} X^{i+j} Y^{p'+k} Z^{(j+k)p}. \]

and
\[ W(XY^{p'} + XZ^{p'}, YZ^{p'}) = \frac{1}{p} \sum_{l=1}^{p-1} \binom{p}{l} (XY^{p'} + XZ^{p'})^{p-l} (YZ^{p'})^l \]
\[ = \sum_{\substack{i+j+k=p \atop k \geq 1, i+j \geq 1}} \frac{(p-1)!}{i!j!k!} X^{i+j} Y^{p'+k} Z^{(j+k)p}. \]

Equating coefficients of \( XY^{p-1} Z^{p'+1}, XY^{p'+1} Z^{p-1}, X^{p'+1} Y Z^{p-1} \) on (9) gives
\[
\begin{cases}
0 = c_{1,p(p'+1)-1} - a_{r,0}^p, \\
c_{p+1+1,p-1} = c_{1,p(p'+1)-1}, \\
c_{p+1+1,p-1} = 0.
\end{cases}
\]

Hence we obtain
\[ a_{r,0}^p = 0. \]

Step 2. Let \( r \geq 2 \). Assume that
\[ a_{r,0}^p = a_{r-1,1}^p = \cdots = a_{r-l,1}^p = 0 \quad (l < r - 1). \]

Then
\[ E_p(a_{r,0} XY^{p'}) \cdots E_p(a_{r-l,1} X^{p'} Y^{p'}) \in Z^2(G_{a,A}, G_{m,A}) = Z^2(\hat{G}_{a,A}, \hat{G}_{m,A}) \]
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and then

$$F(X, Y)E_p(a_{r,0}XY^{p'})^{-1}E_p(a_{r-1,1}X^pY^{p'})^{-1} \cdots E_p(a_{r-l,1}X^{p_l}Y^{p'})^{-1} \in \mathbb{Z}^2(\hat{G}_a, \hat{G}_m, \hat{G}_a).$$

We have also

$$F(X, Y)E_p(a_{r,0}XY^{p'})^{-1}E_p(a_{r-1,1}X^pY^{p'})^{-1} \cdots E_p(a_{r-l,1}X^{p_l}Y^{p'})^{-1}$$

$$= \sum_{k=0}^{p-1} \frac{1}{k!} \left( a_{r-(l+1),l+1}X^{p_{l+1}}Y^{p'} + \cdots + a_{1,r-1}X^{p_{r-1}}Y^{p'} \right)^k \text{ mod deg}(p^{r+1} + 1).$$

Replacing $F(X, Y)E_p(a_{r,0}XY^{p'})^{-1}E_p(a_{r-1,1}X^pY^{p'})^{-1} \cdots E_p(a_{r-l,1}X^{p_l}Y^{p'})^{-1}$ by $F(X, Y)$, we may assume that

$$F(X, Y) = \sum_{k=0}^{p-1} \frac{1}{k!} \left( a_{r-(l+1),l+1}X^{p_{l+1}}Y^{p'} + \cdots + a_{1,r-1}X^{p_{r-1}}Y^{p'} \right)^k \text{ mod deg}(p^{r+1} + 1).$$

Assume that

$$F(X, Y) = \sum_{k=0}^{p-1} \frac{1}{k!} G(X, Y)^k + H(X, Y) \text{ mod deg}(p(p^r + p^l) + 1),$$

where

$$H(X, Y) = \sum_{i=p^{r+1}+1}^{p(p^r+p^l)} H_i(X, Y),$$

here $H_i(X, Y)$ is homogeneous part of degree $i$ and

$$G(X, Y) = a_{r-(l+1),l+1}X^{p_{l+1}}Y^{p'} + \cdots + a_{1,r-1}X^{p_{r-1}}Y^{p'}.$$ 

Now put $W(X, Y)$ as in the proof of Proposition 2.3. By (5), we obtain that

$$\left\{ \sum_{k=0}^{p-1} \frac{1}{k!} G(X, Y)^k \right\} \left\{ \sum_{k=0}^{p-1} \frac{1}{k!} G(X + Y, Z)^k \right\}$$

$$= \sum_{k=0}^{p-1} \frac{1}{k!} \left( G(X, Y) + G(X + Y, Z) \right)^k - W(G(X, Y), G(X + Y, Z))$$

$$\text{ mod deg}(p + 1)(p^r + 1).$$
and
\[
\left\{ \sum_{k=0}^{p-1} \frac{1}{k!} G(X, Y + Z)^k \right\} \left\{ \sum_{k=0}^{p-1} \frac{1}{k!} G(Y, Z)^k \right\}
\equiv \sum_{k=0}^{p-1} \frac{1}{k!} \{G(X, Y + Z) + G(Y, Z)\}^k - W(G(X, Y + Z), G(Y, Z))
\mod \deg(p + 1)(p' + 1).
\]

On the other hand, since
\[
F(X, Y)F(X + Y, Z) = F(X, Y + Z)F(Y, Z),
\]
we have
\[
\left\{ \sum_{k=0}^{p-1} \frac{1}{k!} G(X, Y)^k + H(X, Y) \right\} \left\{ \sum_{k=0}^{p-1} \frac{1}{k!} G(X + Y, Z)^k + H(X + Y, Z) \right\}
\equiv \left\{ \sum_{k=0}^{p-1} \frac{1}{k!} G(X, Y + Z)^k + H(X, Y + Z) \right\} \left\{ \sum_{k=0}^{p-1} \frac{1}{k!} G(Y, Z)^k + H(Y, Z) \right\}
\mod \deg(p(p' + p') + 1).
\]
Hence we obtain
\[
\sum_{k=0}^{p-1} \frac{1}{k!} \{G(X, Y) + G(X + Y, Z)\}^k - W(G(X, Y), G(X + Y, Z))
+ H(X, Y) + H(X + Y, Z)
\equiv \sum_{k=0}^{p-1} \frac{1}{k!} \{G(X, Y + Z) + G(Y, Z)\}^k - W(G(X, Y + Z), G(Y, Z))
+ H(X, Y + Z) + H(Y, Z) \mod \deg(p(p' + p') + 1).
\]
Since we see that \(G(X, Y) \in Z^2(G_{a,a}, G_{a,a})\), we obtain
\[
-W(G(X, Y), G(X + Y, Z)) + H(X, Y) + H(X + Y, Z)
\equiv -W(G(X, Y + Z), G(Y, Z)) + H(X, Y + Z) + H(Y, Z)
\mod \deg(p(p' + p') + 1).
\]
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Now noting that
\[ W(G(X, Y), G(X + Y, Z)) \equiv W(a_{r-(l+1), l+1}X^{p^{l+1}}Y^{p^r}, a_{r-(l+1), l+1}(X + Y)^{p^{l+1}}Z^{p^r}) \mod \deg(p(p^r + p^l) + 1) \]
and
\[ W(G(X, Y + Z), G(Y, Z)) \equiv W(a_{r-(l+1), l+1}X^{p^{l+1}}(Y + Z)^{p^r}, a_{r-(l+1), l+1}Y^{p^{l+1}}Z^{p^r}) \mod \deg(p(p^r + p^l) + 1), \]
we obtain
\[
H_{p(p^r + p^{l+1})}(X, Y) + H_{p(p^r + p^{l+1})}(X + Y, Z) \\
- a_{r-(l+1), l+1}W(X^{p^{l+1}}Y^{p^r}, X^{p^{l+1}}Z^{p^r} + Y^{p^{l+1}}Z^{p^r}) \\
= H_{p(p^r + p^{l+1})}(X, Y + Z) + H_{p(p^r + p^{l+1})}(Y, Z) \\
- a_{r-(l+1), l+1}W(X^{p^{l+1}}Y^{p^r} + X^{p^{l+1}}Z^{p^r}, Y^{p^{l+1}}Z^{p^r}).
\]
Note that $X^{p^{l+1}}Y^{p^r} = (XY^{p^r-l-1})^{p^{l+1}}$. Replacing $X^{p^{l+1}}Y^{p^r}$ by $XY^{p^r-l-1}$, we see
\[
H_{p(p^r-l-1+1)}(X, Y) + H_{p(p^r-l-1+1)}(X + Y, Z) \\
- a_{r-(l+1), l+1}W(XY^{p^r-l-1}, XZ^{p^r-l-1} + YZ^{p^r-l-1}) \\
= H_{p(p^r-l-1+1)}(X, Y + Z) + H_{p(p^r-l-1+1)}(Y, Z) \\
- a_{r-(l+1), l+1}W(XY^{p^r-l-1} + XZ^{p^r-l-1}, YZ^{p^r-l-1}).
\]

Put now
\[
H_{p(p^r-l-1+1)}(X, Y) = \sum_{i+j=p(p^r-l-1+1)} c_{ij}X^iY^j.
\]
It is easily verified that
\[
W(XY^{p^r-l-1}, XZ^{p^r-l-1} + YZ^{p^r-l-1}) = \frac{1}{p-1} \sum_{u=1}^{p-1} \binom{p}{u} (XY^{p^r-l-1})^{p-u} (XZ^{p^r-l-1} + YZ^{p^r-l-1})^u \\
= \sum_{i+j+k=p, i \geq 1, j+k \geq 1} \frac{(p-1)!}{i!j!k!} X^{i+j}Y^{p^r-l-1+k}Z^{(j+k)p^r-l-1}.
\]
and
\[ W(XY^{p^{r+1}} + XZ^{p^{r+1}}, YZ^{p^{r+1}}) = \frac{1}{p} \sum_{u=1}^{p-1} \binom{p}{u} (XY^{p^{r+1}} + XZ^{p^{r+1}})^{p-u}(YZ^{p^{r+1}})^u \]
\[ = \sum_{i+j+k=p \atop k \geq 1, i+j \geq 1} \frac{(p-1)!}{i!j!k!} X^{i+j} Y^k Z^{(j+k)p^{r+1}}. \]

Equating coefficients of \( XY^{p-1}Z^{p^{r+1}}, XY^{p^{r+1}}Z^{p-1}, X^{p^{r+1}}YZ^{p-1} \) on (10) gives
\[
\begin{cases}
0 = c_{1, p^{r+1}+p-1} - a_{r-(i+1), i+1}, \\
c_{p^{r+1}+1, p-1} = c_{1, p^{r+1}+p-1}, \\
c_{p^{r+1}+1, p-1} = 0.
\end{cases}
\]

Hence we obtain
\[ a_{r-(i+1), i+1} = 0. \]

**COROLLARY 3.3.** Under the assumption of Lemma 3.2, we have
\[
F(X, Y) E_p(a_{r,0}XY^{p^r})^{-1} \cdots E_p(a_{1,r-1}X^{p^{r-1}}Y^{p^r})^{-1} \in Z^2(\hat{G}_a, \hat{G}_m, A)
\]
and
\[
F(X, Y) E_p(a_{r,0}XY^{p^r})^{-1} \cdots E_p(a_{1,r-1}X^{p^{r-1}}Y^{p^r})^{-1} \equiv 1 \mod \mathrm{deg}(p^{r+1} + 1).
\]

3.4. Now we prove the first result of Theorem 2.8 for formal group schemes, that is, the bijectivity of the homomorphism
\[
(Ker[F : W(A) \to W(A)])^N \to H^2(\hat{G}_a, \hat{G}_m, A)/H^2_0(\hat{G}_a, \hat{G}_m, A)
\]
which was explicitly given in the theorem. It is enough to prove the surjectivity since the injectivity is obvious.

Let \( F(X, Y) \in Z^2(\hat{G}_a, \hat{G}_m, A) \). By 2.2, \( F(X, Y) \equiv 1 \mod \mathrm{deg} 1 \). Assume that
\[
F(X, Y) \equiv 1 + H(X, Y) \mod \mathrm{deg} 2,
\]
where
\[
H(X, Y) = c_{1,0}X + c_{0,1}Y.
\]

Since
\[
F(X, Y)F(X + Y, Z) = F(X, Y + Z)F(Y, Z),
\]
we have
\[ c_{1,0} = c_{0,1} = 0. \]
Moreover, we obtain the following fact by the same argument as in Lemma 3.1.

If \( F(X, Y) \equiv 1 \mod \deg 2 \), then there exists \( \tilde{F}(X, Y) \in Z_0^2(\hat{G}_n, \hat{G}_m, A) \) such that
\[
F(X, Y)\tilde{F}(X, Y)^{-1} \equiv 1 \mod \deg(p + 1).
\]
Replacing \( F(X, Y)\tilde{F}(X, Y)^{-1} \) by \( F(X, Y) \), we may assume that
\[
F(X, Y) \equiv 1 \mod \deg(p + 1).
\]
By Lemma 3.1, there exists \( \tilde{F}(X, Y) \in Z_0^2(\hat{G}_n, \hat{G}_m, A) \) and \( a_{1,0} \in A \) such that
\[
F(X, Y)\tilde{F}(X, Y)^{-1} \equiv \sum_{k=0}^{p-1} \frac{1}{k!} \{ a_{1,0}XY^p \}^k \mod \deg(p^2 + 1).
\]
By Lemma 3.2,
\[
a_{1,0}F(X, Y)F(X, Y)^{-1} = 0.
\]
Hence
\[
F(X, Y)\tilde{F}(X, Y)^{-1} \equiv E_p(a_{1,0}XY^p) \mod \deg(p^2 + 1),
\]
and therefore
\[
F(X, Y)\tilde{F}(X, Y)^{-1}E_p(a_{1,0}XY^p)^{-1} \equiv 1 \mod \deg(p^2 + 1).
\]
Note that
\[
F(X, Y)\tilde{F}(X, Y)^{-1}E_p(a_{1,0}XY^p)^{-1} \in Z^2(\hat{G}_n, \hat{G}_m, A)
\]
by Corollary 3.3. Replacing \( F(X, Y)\tilde{F}(X, Y)^{-1}E_p(a_{1,0}XY^p)^{-1} \) by \( F(X, Y) \), we may assume that
\[
F(X, Y) \equiv 1 \mod \deg(p^2 + 1).
\]
Continuing this process, we find \( \tilde{F}(X, Y) \in Z_0^2(\hat{G}_n, \hat{G}_m, A) \) such that
\[
F(X, Y)\tilde{F}(X, Y)^{-1} = \prod_{r=1}^{\infty} \prod_{j=0}^{r-1} E_p(a_{r-j,0}X^{p^j}Y^p) = \prod_{r=1}^{\infty} \prod_{j=0}^{\infty} E_p(a_{r,j}X^{p^j}Y^{p^j})
\]
\[
= \prod_{r=1}^{\infty} E_p(a_{r,0}XY^p).
\]
This proves the desired surjectivity. The second result of Theorem 2.8 for group schemes follows by the next:

**Lemma 3.5.** Let \( A \) be an \( F_p \)-algebra and \( F(X, Y) \in \mathbb{Z}^2(G_a, A, G_m, A) \subset A[X, Y]^\times \). Then there exists \( \tilde{F}(X, Y) \in \mathbb{Z}^2(G_a, G_m, A) \) and \( (a_r)_{r \geq 1} \in (\text{Ker}[F : \tilde{W}(A) \to \tilde{W}(A)])^{(N)} \) such that

\[
F(X, Y)\tilde{F}(X, Y)^{-1} = \prod_{r \geq 1} E_p(a_r; XY^p^r).
\]

**Proof.** Let \( P = \{ p^r; r \geq 0 \} \). Dividing \( F(X, Y) \) by its constant term, we may assume that \( F(X, Y) \equiv 1 \mod \text{deg} 1 \). By 3.4, there exists \( \tilde{F}(X, Y) \in \mathbb{Z}^2(\tilde{G}_a, \tilde{G}_m, A) \subset A[[X, Y]]^\times \) and \( a_r, j \in A \ (0 \leq j < r) \) such that

\[
F(X, Y)\tilde{F}(X, Y)^{-1} = \prod_{r \geq 1} E_p(a_r; XY^r).
\]

By a result of [3, 3.4], there exist \( a_k \in A \ (k \notin P) \), \( b = (b_l)_{l \geq 0} \in W(A) \) such that

\[
\tilde{F}(X, Y) = \prod_{k \notin P} \{ E_p(a_k X^k)E_p(a_k Y^k)E_p(a_k(X + Y)^k)^{-1} \} F_p(b; X, Y).
\]

Hence we obtain a factorization:

\[
F(X, Y) = \prod_{k \notin P} \{ E_p(a_k X^k)E_p(a_k Y^k)E_p(a_k(X + Y)^k)^{-1} \}
\]

\[
\times \prod_{l \geq 0} E_p(b_l; X^p^l, Y^p^l) \prod_{r=1}^{\infty} \prod_{j=0}^{r-1} E_p(a_r-j, j X^p^j Y^p^j).
\]

Now we prove that \( a_k \) is nilpotent for all \( k \ (k \notin P) \) and is zero for all but finite number of \( k \), \( b = (b_l)_{l \geq 0} \in W(A) \) and \( (a_r)_{r \geq 1} \in (\text{Ker}[F : \tilde{W}(A) \to \tilde{W}(A)])^{(N)} \). We now observe that:

1. Putting

\[
E_p(X^k)E_p(Y^k)E_p((X + Y)^k)^{-1} = 1 + \sum_{l=1}^{\infty} \sum_{i+j=l} c_{ij} X^{ki} Y^{kj} \in \mathbb{Z}(p)[[X, Y]],
\]

we have, for \( a \in A \),

\[
E_p(a X^k)E_p(a Y^k)E_p(a(X + Y)^k)^{-1} = 1 + \sum_{l=1}^{\infty} a^l \left( \sum_{i+j=l} c_{ij} X^{ki} Y^{kj} \right) \in A[[X, Y]].
\]
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and

$$E_p(aX^k)E_p(aY^k)E_p(a(X + Y)^k)^{-1} \equiv 1 + a\{X^k + Y^k - (X + Y)^k\} \mod \deg(k + 1).$$

(2) Putting

$$F_p(1; X^{p^r}, Y^{p^r}) = 1 + \sum_{l>0, \ p^{r+1}|l} \sum_{i+j=l} c_{ij}X^iY^j \in \mathbb{Z}_p[[X, Y]],$$

we have, for $a \in A$,

$$F_p(a; X^{p^r}, Y^{p^r}) = 1 + \sum_{l>0, \ p^{r+1}|l} a^{l/p^{r+1}} \left( \sum_{i+j=l} c_{ij}X^iY^j \right) \in A[[X, Y]]$$

and

$$F_p(a; X^{p^r}, Y^{p^r}) \equiv 1 + aX^{p^{r+1}} + Y^{p^{r+1}} - (X + Y)^{p^{r+1}} \mod \deg(p^{r+1} + 1).$$

(3) Putting

$$E_p(X^{p^j}Y^{p^r}) = 1 + \sum_{l=1}^{\infty} c_l(X^{p^j}Y^{p^r})^l \in \mathbb{Z}_p[[X, Y]],$$

we have, for $a \in A$,

$$E_p(aX^{p^j}Y^{p^r}) = 1 + \sum_{l=1}^{\infty} c_la^l(X^{p^j}Y^{p^r})^l \in A[[X, Y]]$$

and

$$E_p(aX^{p^j}Y^{p^r}) \equiv 1 + aX^{p^j}Y^{p^r} \mod \deg(p^j + p^r + 1).$$

Let $N$ be the degree of $F(X, Y)$ and let $a$ denote the ideal of $A$ generated by the coefficients of terms of degree $\geq 1$ in $F(X, Y)$. Since the polynomial $F(X, Y)$ is invertible, $a$ is nilpotent.

For the simplicity, we put $a_{p^{i+1}} = b_i$ and

$$F_k(X, Y) = \begin{cases} 
F_p(a_{p^{i+1}}; X^{p^i}, Y^{p^i}) & \text{if } k = \p^{i+1} (i \geq 0) \\
\frac{E_p(a_{p^j}X^k)}{E_p(a_{p^j}X^k)^k} \frac{E_p(a_{p^j}Y^k)}{E_p(a_{p^j}Y^k)^k} E_p(a_{p^j}X^k + Y^k)^{p^k} & \text{if } k = p^i + p^r (0 \leq j < r) \\
\frac{E_p(a_{p^j}X^k)}{E_p(a_{p^j}X^k)^k} & \text{otherwise.}
\end{cases}$$
Then we have

$$F(X, Y) = \prod_{k=2}^{\infty} F_k(X, Y)$$

and, up to deg($k+1$),

$$F_k(X, Y) \equiv \begin{cases} 
1 + a_{p^{l+1}} \frac{X^{p^{l+1}} + Y^{p^{l+1}} - (X + Y)^{p^{l+1}}}{p} & \text{if } k = p^{l+1} \ (l \geq 0) \\
1 + a_k \{X^k + Y^k - (X + Y)^k\} + a_{r-j,j} X^{p^j} Y^{p^r} & \text{if } k = p^j + p^r \\
1 + a_k \{X^k + Y^k - (X + Y)^k\} & \text{otherwise.}
\end{cases}$$

Furthermore, let

$$F_k(X, Y) = 1 + \sum_{i \geq k} \sum_{i+j=l} b_{ij} X^i Y^j, \quad b_{ij} \in A.$$ 

Then we can conclude that if $b_{ij} \in a^s$ for all $(i, j)$ with $i + j = k$, then $b_{ij} \in a^{s+[(i+j)/k]-1}$ for all $(i, j)$ with $i + j > k$.

Step 1. We shall prove that

$$a_k \in a \quad \text{if } k \leq N$$

and

$$a_{r-j,j} \in a \quad \text{if } p^j + p^r \leq N$$

by the induction on $k$ and $(r, j)$ with $0 \leq j < r$.

Let $k$ be an integer $< N$. Assume that

$$a_i \in a \quad \text{if } i < k$$

and

$$a_{r-j,j} \in a \quad \text{if } p^j + p^r < k.$$ 

Then we obtain

$$F(X, Y) \equiv F_k(X, Y) \mod(a, \deg(k+1)).$$

Case 1: When $k = p^{l+1} \ (l \geq 0)$,

$$\frac{1}{p} \left( \frac{p^{l+1}}{p^h} \right) a_{p^{l+1}} \in a \quad \text{for } 1 \leq h \leq l.$$
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Since $\frac{1}{p} \left( \frac{p^{l+1}}{p^h} \right) \neq 0 \mod p$, we obtain $a_{p^{l+1}} \in a$.

Case 2: When $k = p^j + p^r \ (0 \leq j < r)$,

$$\left( \frac{p^j + p^r}{p^j} \right) a_k + a_{r-j,j} \in a.$$  

Since $\left( \frac{p^j + p^r}{p^j} \right) \neq 0 \mod p$, we obtain $a_k \in a$ and $a_{r-j,j} \in a$.

Case 3: Otherwise,

$$ka_k \in a.$$  

Since $(k, p) = 1$, we obtain $a_k \in a$.

Step 2. We shall prove that

$$a_k \in a^i \text{ if } (s - 1)N < k \leq sN$$

and

$$a_{r-j,j} \in a^i \text{ if } (s - 1)N < p^j + p^r \leq sN$$

by the induction on $k$ and $(r, j)$ with $0 \leq j < r$.

Let $k$ be an integer $< sN$. Assume that

$$a_i \in a^i \text{ if } (s - 1)N < i < k$$

and

$$a_{r-j,j} \in a^i \text{ if } (s - 1)N < p^j + p^r < k.$$  

Then we obtain

$$F(X, Y) \left\{ \prod_{i < k} F_i(X, Y) \right\}^{-1} \equiv F_k(X, Y) \equiv 1 + \sum_{i+j=k} c_{ij}X^iY^j \mod (a^i, \deg(k + 1)).$$

Now we put

$$F(X, Y) = 1 + \sum \alpha_{ij}X^iY^j,$$

$$\left\{ \prod_{i < k} F_i(X, Y) \right\}^{-1} = 1 + \sum \beta_{ij}X^iY^j$$

and

$$F(X, Y) \left\{ \prod_{i < k} F_i(X, Y) \right\}^{-1} = 1 + \sum \gamma_{ij}X^iY^j.$$
By the assumption, 
\[ \alpha_{ij}, \beta_{ij} \in \alpha^s \] if \( (s-1)N < i + j \leq sN \).

Hence, we obtain 
\[ \gamma_{ij} \in \alpha^s \] if \( (s-1)N < i + j \leq sN \).

Since \( (s-1)N < k < sN \), we obtain \( c_{ij} \in \alpha^s \).

Hence, \( a_k \) and \( a_{r-j,j} \) are nilpotent for all \( k \) and \( (r,j) \) with \( 0 \leq j < r \), and are zero for all but a finite number of \( k \) and \( (r,j) \) with \( 0 \leq j < r \).

**Remark 3.6.** We establish some functorialities. For example,

1. The diagrams 
\[
\begin{array}{ccc}
\left( \text{Ker}[F : W(A) \to W(A)] \right)^N & \xrightarrow{\nu} & \left( \text{Ker}[F : W(A) \to W(A)] \right)^N \\
\downarrow & & \downarrow \\
H^2(\hat{G}_a, \hat{G}_m) / H_0^2(\hat{G}_a, \hat{G}_m) & \xrightarrow{F^*} & H^2(\hat{G}_a, \hat{G}_m) / H_0^2(\hat{G}_a, \hat{G}_m)
\end{array}
\]

and
\[
\begin{array}{ccc}
\left( \text{Ker}[F : \tilde{W}(A) \to \tilde{W}(A)] \right)^{(N)} & \xrightarrow{\nu} & \left( \text{Ker}[F : \tilde{W}(A) \to \tilde{W}(A)] \right)^{(N)} \\
\downarrow & & \downarrow \\
H^2(\hat{G}_a, \hat{G}_m) / H_0^2(\hat{G}_a, \hat{G}_m) & \xrightarrow{F^*} & H^2(\hat{G}_a, \hat{G}_m) / H_0^2(\hat{G}_a, \hat{G}_m)
\end{array}
\]

are commutative.

2. Let \( a \in A \). Then the diagrams 
\[
\begin{array}{ccc}
\left( \text{Ker}[F : W(A) \to W(A)] \right)^N & \xrightarrow{[\alpha^{r+1}]} & \left( \text{Ker}[F : W(A) \to W(A)] \right)^N \\
\downarrow & & \downarrow \\
H^2(\hat{G}_a, \hat{G}_m) / H_0^2(\hat{G}_a, \hat{G}_m) & \xrightarrow{[\alpha^{r+1}]} & H^2(\hat{G}_a, \hat{G}_m) / H_0^2(\hat{G}_a, \hat{G}_m)
\end{array}
\]

and
\[
\begin{array}{ccc}
\left( \text{Ker}[F : \tilde{W}(A) \to \tilde{W}(A)] \right)^{(N)} & \xrightarrow{[\alpha^{r+1}]} & \left( \text{Ker}[F : \tilde{W}(A) \to \tilde{W}(A)] \right)^{(N)} \\
\downarrow & & \downarrow \\
H^2(\hat{G}_a, \hat{G}_m) / H_0^2(\hat{G}_a, \hat{G}_m) & \xrightarrow{[\alpha^{r+1}]} & H^2(\hat{G}_a, \hat{G}_m) / H_0^2(\hat{G}_a, \hat{G}_m)
\end{array}
\]
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are commutative. Here $[a^{p+1}] = (a^{p+1}, 0, 0, \ldots)$ and $[a]^*$ denotes the maps induced by the endomorphism of $\hat{G}_{a,A}$ or of $G_{a,A}$, defined by $T \mapsto aT$.

REMARK 3.7. Let $A$ be a $Q$-algebra. It is well known that $H^2(\hat{G}_{0,A}, \hat{G}_{0,A}) = 0$ and $H^2(G_{a,A}, G_{a,A}) = 0$ (cf. [1, Chap. II]), from which we can deduce that $H^2(\hat{G}_{a,A}, \hat{G}_{m,A}) = 0$ and $H^2(G_{a,A}, G_{m,A}) = 0$ by the same argument as Theorem 2.8 from Proposition 2.3.

References


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