GENERALIZED TATE COHOMOLOGY

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Abstract. We consider two classes of left R-modules, $\mathcal{P}$ and $\mathcal{C}$, such that $\mathcal{P} \subset \mathcal{C}$. If the module $M$ has a $\mathcal{P}$-resolution and a $\mathcal{C}$-resolution then for any module $N$ and $n \geq 0$ we define generalized Tate cohomology modules $\widehat{\text{Ext}}^n_{\mathcal{P}}(M, N)$ and show that we get a long exact sequence connecting these modules and the modules $\text{Ext}^n_{\mathcal{C}}(M, N)$ and $\text{Ext}^n_{\mathcal{P}}(M, N)$. When $\mathcal{C}$ is the class of Gorenstein projective modules, $\mathcal{P}$ is the class of projective modules and when $M$ has a complete resolution we show that the modules $\widehat{\text{Ext}}^n_{\mathcal{P}}(M, N)$ for $n \geq 1$ are the usual Tate cohomology modules and prove that our exact sequence gives an exact sequence provided by Avramov and Martsinkovsky. Then we show that there is a dual result. We also prove that over Gorenstein rings Tate cohomology $\widehat{\text{Ext}}^n_R(M, N)$ can be computed using either a complete resolution of $M$ or a complete injective resolution of $N$. And so, using our dual result, we obtain Avramov and Martsinkovsky's exact sequence under hypotheses different from theirs.

1. Introduction

We consider two classes of left $R$-modules $\mathcal{P}$, $\mathcal{C}$ such that $\text{Proj} \subset \mathcal{P} \subset \mathcal{C}$, where $\text{Proj}$ is the class of projective modules. Let $M$ be a left $R$-module. Let $P.$ be a deleted $\mathcal{P}$-resolution of $M$, $C.$ a deleted $\mathcal{C}$-resolution of $M$ (see Section 2 for definitions), let $u : P. \to C.$ be a chain map induced by $\text{Id}_M$, and $M(u)$ the associated mapping cone. We define the generalized Tate cohomology module $\widehat{\text{Ext}}^n_{\mathcal{P}}(M, N)$ by the equality $\widehat{\text{Ext}}^n_{\mathcal{P}}(M, N) = H^{n+1}(\text{Hom}(M(u), N))$, for any $n \geq 0$ and any left $R$-module $N$. We show that $\widehat{\text{Ext}}^n_{\mathcal{P}}(M, -)$ is well-defined. We
also show that there is an exact sequence connecting these modules and the modules $\text{Ext}_{q}^{r}(M,N)$ and $\text{Ext}_{q}^{f}(M,N)$:

$$(1) \quad 0 \to \text{Ext}_{q}^{1}(M,N) \to \text{Ext}_{q}^{f}(M,N) \to \widehat{\text{Ext}}_{q}(M,N) \to \cdots$$

We prove (Proposition 1) that when we apply this procedure to $\mathcal{C} = \text{Gor Proj}$, $\mathcal{P} = \text{Proj}$, over a left noetherian ring $R$, for an $R$-module $M$ with $\text{Gor proj dim } M = g < \infty$, the modules $\widehat{\text{Ext}}_{q}^{n}(M,N)$ are the usual Tate cohomology modules for any $n \geq 1$. In this case our exact sequence (1) becomes L. L. Avramov and A. Martsinkovsky’s exact sequence ([1], th. 7.1):

$$0 \to \text{Ext}_{q}^{1}(M,N) \to \text{Ext}_{q}^{f}(M,N) \to \widehat{\text{Ext}}_{q}(M,N) \to \cdots$$

$$\to \text{Ext}_{q}^{d}(M,N) \to \text{Ext}_{q}^{R}(M,N) \to \widehat{\text{Ext}}_{q}(M,N) \to 0$$

Our proof works in a more general case, for any module $M$ of finite Gorenstein projective dimension, whether finitely generated or not.

There is also a dual result (Theorem 1). If $\text{Gor inj dim } N = d < \infty$ then the $d$th cosyzygy $H$ of an injective resolution of $N$ is a Gorenstein injective module. So there exists an exact sequence $\mathcal{E} : \cdots \to E_{1} \to E_{0} \to E_{-1} \to E_{-2} \to \cdots$ of injective modules such that $\text{Hom}(I, \mathcal{E})$ is exact for any injective left $R$-module $I$ and $H = \text{Ker}(E_{0} \to E_{-1})$. We call such sequence a complete injective resolution of $N$. We show that a complete injective resolution of $N$ is unique up to homotopy. For each left $R$-module $M$ and for each $n \in \mathbb{Z}$ let $\overline{\text{Ext}}_{R}^{n}(M,N) \overset{\text{def}}{=} H^{n}(\text{Hom}(M, \mathcal{E}))$. A dual argument of the proof of Proposition 1 shows the existence of an exact sequence $0 \to \text{Ext}_{q}^{1}(M,N) \to \text{Ext}_{q}^{f}(M,N) \to \widehat{\text{Ext}}_{q}(M,N) \to \text{Ext}_{q}^{d}(M,N) \to \cdots \to \text{Ext}_{q}^{d}(M,N) \to \text{Ext}_{q}^{R}(M,N) \to \overline{\text{Ext}}_{R}^{n}(M,N) \to 0$

where $\text{Ext}_{q}^{d}(M,N)$ are the right derived functors of $\text{Hom}(M,N)$, computed using a right Gorenstein injective resolution of $N$. If $\text{Gor proj dim } M < \infty$ then $\text{Ext}_{q}^{d}(M,N) \simeq \text{Ext}_{q}^{d}(M,N)$, for all $i \geq 0$ ([4], Theorem 3.6). So in this case we obtain an exact sequence

$$0 \to \text{Ext}_{q}^{1}(M,N) \to \text{Ext}_{q}^{f}(M,N) \to \widehat{\text{Ext}}_{q}(M,N) \to \cdots$$

We prove (Theorem 2) that over Gorenstein rings we have $\overline{\text{Ext}}_{R}^{n}(M,N) \simeq \widehat{\text{Ext}}_{q}^{n}(M,N)$ for all left $R$-modules $M$, $N$, for any $n \in \mathbb{Z}$. Thus, over Gorenstein rings there is a new way of computing the Tate cohomology.

2. Preliminaries

Let $R$ be an associative ring with 1 and let $\mathcal{P}$ be a class of left $R$-modules.
DEFINITION 1 [3]. For a left R-module $M$ a morphism $\phi : P \to M$ where $P \in \mathcal{P}$ is a $\mathcal{P}$-precover of $M$ if $\text{Hom}(P', P) \to \text{Hom}(P', M)$ $\to 0$ is exact for any $P' \in \mathcal{P}$.

DEFINITION 2. A $\mathcal{P}$-resolution of a left R-module $M$ is a complex $P : \cdots \to P_1 \to P_0 \to M \to 0$ (not necessarily exact) with each $P_i \in \mathcal{P}$ and such that for any $P' \in \mathcal{P}$ the complex $\text{Hom}(P', P)$ is exact.

Throughout the paper we refer to the complex $P : \cdots \to P_1 \to P_0 \to 0$ as a deleted $\mathcal{P}$ resolution of $M$.

We note that a complex $P$ as in Definition 2 is a $\mathcal{P}$-resolution if and only if $P_0 \to M$, $P_1 \to \text{Ker}(P_0 \to M)$ and $P_i \to \text{Ker}(P_{i-1} \to P_{i-2})$ for $i \geq 2$ are $\mathcal{P}$-precovers. If $\mathcal{P}$ contains all the projective left R-modules then any $\mathcal{P}$-precover is a surjective map and therefore any $\mathcal{P}$-resolution is an exact complex.

A $\mathcal{P}$-resolution of a left R-module $M$ is unique up to homotopy ([3], pg. 169) and so it can be used to compute derived functors.

DEFINITION 3. Let $M$ be a left R-module that has a $\mathcal{P}$-resolution $P : \cdots \to P_1 \to P_0 \to M \to 0$. Then $\text{Ext}^n_{\mathcal{P}}(M, N) = H^n(\text{Hom}(P, N))$ for any left R-module $N$ and any $n \geq 0$, where $P$ is the deleted resolution.

We prove the existence of the exact sequence (1).

Let $\mathcal{P}$, $\mathcal{C}$ be two classes of left $R$-modules such that $\text{Proj} \subset \mathcal{P} \subset \mathcal{C}$ where $\text{Proj}$ is the class of projective modules. Let $M$ be a left $R$-module that has both a $\mathcal{P}$-resolution $P : \cdots \to P_1 \to P_0 \to M \to 0$ and a $\mathcal{C}$-resolution $C : \cdots \to C_1 \to C_0 \to M \to 0$.

$P_i \in \mathcal{P} \subset \mathcal{C}$ so $\text{Hom}(P_i, C)$ is an exact complex for any $i \geq 0$. It follows that there are morphisms $P_i \to C_i$ making

$$
\begin{array}{ccccccccc}
P : \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
\downarrow \Phi & & \downarrow \Psi & & \downarrow \Omega & & \downarrow \iota & & \\
C : \cdots & \longrightarrow & C_1 & \longrightarrow & C_0 & \longrightarrow & M & \longrightarrow & 0 
\end{array}
$$

into a commutative diagram.

Let $u : P \to C$, $u = (u_i)_{i \geq 0}$ be such a chain map induced by $\text{Id}_M$ and let $\overline{M(u)}$ be the associated mapping cone. Since $0 \to C \to \overline{M(u)} \to P[1] \to 0$ is exact and both $P$ and $C$ are exact complexes, the exactness of $\overline{M(u)}$ follows. $\overline{M(u)}$ has the exact subcomplex $0 \to M \to \text{Id} M \to 0$. Forming the quotient, we get an exact
complex, $M(u)$, which is the mapping cone of the chain map $u : P \to C$. ($P$ and $C$ being the deleted $\mathcal{P}, \mathcal{Q}$-resolutions). The sequence $0 \to C \to M(u) \to P[1] \to 0$ is split exact in each degree, so for any left $R$-module $N$ we have an exact sequence of complexes $0 \to \text{Hom}(P[1], N) \to \text{Hom}(M(u), N) \to \text{Hom}(C, N) \to 0$ and therefore an associated cohomology exact sequence: $\cdots \to H^n(\text{Hom}(M(u), N)) \to H^{n+1}(\text{Hom}(P[1], N)) \to H^{n+1}(\text{Hom}(M(u), N)) \to H^{n+2}(\text{Hom}(C, N)) \to \cdots$. Since $M(u)$ is exact and the functor $\text{Hom}(-, N)$ is left exact, it follows that $H^0(\text{Hom}(M(u), N)) = H^1(\text{Hom}(M(u), N)) = 0$. We have $H^0(\text{Hom}(C, N)) \cong \text{Hom}(M, N)$ and $H^1(\text{Hom}(P[1], N)) \cong \text{Hom}(M, N)$.

The sequence $0 \to \text{Hom}(P[1], N) \to \text{Hom}(M(u), N) \to \text{Hom}(C, N) \to 0$ is split exact in each degree, so for any left $R$-module $N$ we have an exact sequence of complexes $0 \to \text{Hom}(P[1], N) \to \text{Hom}(M(u), N) \to \text{Hom}(C, N) \to 0$ and therefore an associated cohomology exact sequence: $\cdots \to H^n(\text{Hom}(M(u), N)) \to H^{n+1}(\text{Hom}(P[1], N)) \to H^{n+1}(\text{Hom}(M(u), N)) \to H^{n+2}(\text{Hom}(C, N)) \to \cdots$. After factoring out the exact sequence $0 \to \text{Hom}(M(u), N) \sim \text{Hom}(M, N)$ we obtain the exact sequence (1):

$$0 \to \text{Ext}^1_{\mathcal{Q}}(M, N) \to \text{Ext}^1_{\mathcal{Q}}(M, N) \to \text{Ext}^1_{\mathcal{Q}}(M, N) \to 0$$

We prove that the generalized Tate cohomology $\widetilde{\text{Ext}}_{\mathcal{Q}}(M, -)$ is well defined.

Let $\mathcal{P}, \mathcal{Q}$ be two classes of left $R$-modules such that $\mathcal{P} \subset \mathcal{Q}$.

Let $P, P'$ be two $\mathcal{P}$-resolutions of $M$ and let $C, C'$ be two $\mathcal{Q}$-resolutions of $M$.

$$P : \cdots \to P_2 \to P_1 \to P_0 \to M \to 0, \quad P' : \cdots \to P'_2 \to P'_1 \to P'_0 \to M \to 0$$

$$C : \cdots \to C_2 \to C_1 \to C_0 \to M \to 0, \quad C' : \cdots \to C'_2 \to C'_1 \to C'_0 \to M \to 0$$

There exist maps of complexes $u : P \to C$ and $v : P' \to C'$, both induced by $\text{Id}_M$. $M(u) : \cdots \to C_2 \oplus P_2 \to C_2 \oplus P_1 \to C_1 \oplus P_0 \to C_0 \oplus M \to 0$ and $M(v) : \cdots \to C'_2 \oplus P'_2 \to C'_2 \oplus P'_1 \to C'_1 \oplus P'_0 \to C'_0 \oplus M \to 0$ (with $\delta_n(x, y) = (g_n(x) + u_{n-1}(y), -f_{n-1}(y))$ for $n \geq 1$, $\delta_0(x, y) = g_0(x) + y$, $\delta'_n(x, y) = (g'_n(x) + v_{n-1}(y), -f'_{n-1}(y))$ for $n \geq 1$, $\delta'_0(x, y) = g'_0(x) + y$) are the associated mapping cones.

$$M(u) : \cdots \to C_3 \oplus P_2 \delta_1 \to C_2 \oplus P_1 \delta_2 \to C_1 \oplus P_0 \delta_3 \to C_0 \to 0 \quad \text{(with } \delta_3(x, y) = g_1(x) + u_0(y) \text{)}$$

and $M(v) : \cdots \to C'_3 \oplus P'_2 \delta'_1 \to C'_2 \oplus P'_1 \delta'_2 \to C'_1 \oplus P'_0 \delta'_3 \to C'_0 \to 0 \quad \text{(with } \delta'_3(x, y) = g'_1(x) + v_0(y) \text{)}$ are the mapping cones of $u : P \to C$ and $v : P' \to C'$.

Since the exact sequence of complexes $0 \to C \to M(u) \to P[1] \to 0$ is split exact in each degree, for each $R_F$ we have an exact sequence: $0 \to \text{Hom}(F, C) \to \text{Hom}(F, M(u)) \to \text{Hom}(F, P[1]) \to 0$. If $F \in \mathcal{P} \subset \mathcal{Q}$ then both complexes $\text{Hom}(F, C)$ and $\text{Hom}(F, P[1])$ are exact, so the exactness of $\text{Hom}(F, M(u))$ follows.

Each $P_i \in \mathcal{P}$, so by the above, the complex $\text{Hom}(P_i, M(u))$ is exact.
Let $\overline{M}$ denote the complex $0 \to M \xrightarrow{Id} M \to 0$. The exact sequence of complexes $0 \to \overline{M} \xrightarrow{Id} \overline{M(u)} \to M(u) \to 0$ is split exact in each degree. Consequently the sequence $0 \to \text{Hom}(P_i, \overline{M}) \to \text{Hom}(P_i, \overline{M(u)}) \to \text{Hom}(P_i, M(u)) \to 0$ is exact for any $i \geq 0$. Since both $\text{Hom}(P_i, \overline{M(u)})$ and $\text{Hom}(P_i, \overline{M})$ are exact complexes, it follows that

\begin{equation}
\text{Hom}(P_i, M(u)) \text{ is an exact complex, for any } i \geq 0.
\end{equation}

The identity map $Id_M$ induces maps of complexes $h : P \to P'$, and $k : C \to C'$. Both $v \circ h : P \to C'$, and $k \circ u : P \to C'$, are maps of complexes induced by $Id_M$, so $v \circ h$ and $k \circ u$ are homotopic. Hence there exists $s_i \in \text{Hom}(P_i, C_{i+1})$, $i \geq 0$ such that $v_0 \circ h_0 - k_0 \circ u_0 = g'_0 \circ s_0$ and $v_n \circ h_n - k_n \circ u_n = g'_{n+1} \circ s_n + s_{n-1} \circ f_n$ for any $n \geq 1$.

Then $\omega : M(u) \to M(v)$ defined by $\tilde{\omega} : C_0 \to C'_0$, $\omega = k_0$, $\omega_n : C_{n+1} \oplus P_n \to C'_{n+1} \oplus P'_n$, $\omega_n(x, y) = (k_{n+1}(x) - s_n(y), h_n(y))$ for any $n \geq 0$, is a map of complexes.

The identity map $Id_M$ also induces maps of complexes $l : P' \to P$, $t : C'. \to C$. Then $t \circ v : P' \to C$ and $u \circ l : P' \to C$ are homotopic.

We have a map of complexes $\psi : M(v) \to M(u)$ where $\psi_n : C'_{n+1} \oplus P'_n \to C_{n+1} \oplus P_n$ is defined by $\psi_n(x, y) = (t_{n+1}(x) - s_n(y), l_n(y))$, $n \geq 0$ (with $s_n : P'_n \to C_{n+1}$ such that $u_n \circ l_n - t_n \circ v_n = s_{n-1} \circ f'_n + g'_{n+1} \circ s_n$, $\forall n \geq 1$, $u_0 \circ l_0 - t_0 \circ v_0 = g_1 \circ s_0$) and $\tilde{\psi} : C'_0 \to C_0$, $\tilde{\psi} = t_0$.

We prove that $\psi \circ \omega$ is homotopic to $Id_M(u)$.

Since $t \circ k : C. \to C$ is a chain map induced by $Id_M$, we have $t \circ k \sim Id_C$. So there exist maps $\gamma_i \in \text{Hom}(C_i, C_{i+1})$, $i \geq 0$ such that $t_0 \circ k_0 - Id = g_0 \circ \beta_0$ and $t_i \circ k_i - Id = \beta_{i-1} \circ g_i + g_{i+1} \circ \beta_i$, $\forall i \geq 1$.

Let $\chi_0 : C_0 \to C_1 \oplus P_0$, $\chi_0(x) = (\beta_0(x), 0)$, $\forall x \in C_0$. Then $\bar{\delta}_1 \circ \chi_0(x) = \bar{\delta}_1(\beta_0(x), 0) = g_1(\bar{\delta}_1(\beta_0(x))) + u_0(0) = (t_0 \circ k_0 - Id)(x) = (\tilde{\psi} \circ \tilde{\omega} - Id)(x)$, $\forall x \in C_0$.

We have $\bar{\delta}_1 \circ (\tilde{\psi} \circ \omega_0 - \chi_0 \circ \bar{\delta}_1 - Id) = \bar{\delta}_1 \circ \psi_0 \circ \omega_0 - (\bar{\delta}_1 \circ \chi_0) \circ \bar{\delta}_1 - \bar{\delta}_1 = t_0 \circ k_0 \circ \bar{\delta}_1 - (t_0 \circ k_0 - Id) \circ \bar{\delta}_1 - \bar{\delta}_1 = 0$.

Let $r_0 : P_0 \to C_1 \oplus P_0$, $r_0 = (\psi_0 \circ \omega_0 - Id - \chi_0 \circ \bar{\delta}_1) \circ e_0$ with $e_0 : P_0 \to C_1 \oplus P_0$, $e_0(y) = (0, y)$. We have $\bar{\delta}_1 \circ r_0 = \bar{\delta}_1 \circ (\tilde{\psi} \circ \omega_0 - Id - \chi_0 \circ \bar{\delta}_1) \circ e_0 = 0$. Since $r_0 \in \text{Ker Hom}(P_0, \bar{\delta}_1) = \text{Im Hom}(P_0, \delta_2)$ (by (2)) it follows that $r_0 \circ \gamma_1 \circ t_0$ for some $\gamma_1 \in \text{Hom}(P_0, C_2 \oplus P_1)$. Hence $(\psi_0 \circ \omega_0 - Id - \chi_0 \circ \bar{\delta}_1)(0, y) = \delta_2(\gamma_1(y))$.

Also we have $(\tilde{\psi} \circ \omega_0 - Id - \chi_0 \circ \bar{\delta}_1)(x, 0) = \psi_0(\omega_0(x, 0)) - (x, 0) - \chi_0(\bar{\delta}_1(x, 0)) = \psi_0(k_1(x), 0) - (x, 0) - \chi_0(g_1(x)) = ((t_1 \circ k_1 - Id - \beta_0 \circ g_1)(x), 0) = ((g_2 \circ \beta_1)(x), 0) = \delta_2(\beta_1(x), 0)$.
So \((\psi_0 \circ \omega_0 - Id - \chi_0 \circ \delta_1)(x, y) = \delta_2 \circ \chi_1(x, y)\) where \(\chi_1 : C_1 \oplus P_0 \to C_2 \oplus P_1\), \(\chi_1(x, y) = (\beta_1(x), 0) + \gamma_1(y)\). Hence \(\psi_0 \circ \omega_0 - Id = \chi_0 \circ \delta_1 + \delta_2 \circ \chi_1\).

Similarly, there exists \(\chi_l \in \Hom(C_l \oplus P_{l-1}, C_{l+1} \oplus P_l)\) such that \(\psi_l \circ \omega_l - Id = \chi_l \circ \delta_{l+1} + \delta_{l+2} \circ \chi_{l+1}, \forall i \geq 1\).

Thus \(\psi \circ \omega \sim Id_{M(u)}\). Similarly, \(\omega \circ \psi \sim Id_{M(v)}\). Then \(H^n(Hom(M(v), N)) \simeq H^n(Hom(M(u), N))\) for any \(R\mathcal{N}\), for any \(n \geq 0\).

**Remark 1.** The proof above does not depend on \(\mathcal{P}, \mathcal{C}\) containing all the projective \(R\)-modules. It works for any two classes \(\mathcal{P}, \mathcal{C}\) of left \(R\)-modules such that \(\mathcal{P} \subset \mathcal{C}\). And even without assuming that \(\mathcal{P}, \mathcal{C}\) contain the projectives we still get an Avramov-Martsinkovsky type sequence. Let \(\mathcal{P}, \mathcal{C}\) be two classes of left \(R\)-modules such that \(\mathcal{P} \subset \mathcal{C}\). If the \(R\)-module \(M\) has a \(\mathcal{P}\)-resolution \(P\) and a \(\mathcal{C}\)-resolution \(C\) then \(Id_M\) induces a chain map \(u : P \to C\) and we have an exact sequence of complexes \(0 \to C \to M(u) \to P[1] \to 0\) which is split exact in each degree, so \(0 \to \Hom(P[1], N) \to \Hom(M(u), N) \to \Hom(C, N) \to 0\) is still exact for any \(R\)-module \(N\). Its associated long exact sequence is: \(0 \to H^0(\Hom(M(u), N)) \to \Ext^0_{\mathcal{P}}(M, N) \to \Ext^0_{\mathcal{C}}(M, N) \to \Ext^1_{\mathcal{P}}(M, N) \to \cdots\) (with \(\Ext^i_{\mathcal{P}}(M, N) = H^{i+1}(\Hom(M(u), N))\) for any \(n \geq 0\)).

**Example 1.** Let \(R = \mathbb{Z}\), \(\mathcal{P}\) be the class of projective \(\mathbb{Z}\)-modules, \(\mathcal{F}\) the class of torsion free modules (so \(\mathcal{P} \subset \mathcal{F}\)), \(M = \mathbb{Z}/2\mathbb{Z}\), \(N = \mathbb{Z}/2\mathbb{Z}'\). A \(\mathcal{P}\)-resolution of \(M\) is \(0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \to 0\). A \(\mathcal{F}\)-resolution of \(M\) is \(0 \to 2\hat{\mathbb{Z}}_2 \to \hat{\mathbb{Z}}_2 \xrightarrow{\varphi} \mathbb{Z}/2\mathbb{Z} \to 0\), with \(\varphi \left( \sum_{i=0}^{\infty} \alpha_i \cdot 2^i \right) = a_0\). There is a map of complexes \(u : P \to T\). \((P, T, \mathcal{F}\)-resolutions) and the mapping cone \(M(u) : 0 \to \mathbb{Z} \to 2\hat{\mathbb{Z}}_2 \oplus \mathbb{Z} \to \hat{\mathbb{Z}}_2 \to 0\) is exact. Since the class \(\mathcal{F}\) of torsion free \(\mathbb{Z}\)-modules coincides with the class of flat \(\mathbb{Z}\)-modules and \(\mathcal{P} \subset \mathcal{F}\), \(M(u)\) is an exact sequence of flat \(\mathbb{Z}\)-modules. We have \(\Hom(\mathbb{Z}/2\mathbb{Z}, \mathcal{Q}/\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}\). So \(\mathbb{Z}/2\mathbb{Z}\) is pure injective and therefore cotorsion. It follows that \(\Hom(M(u), \mathbb{Z}/2\mathbb{Z})\) is an exact complex and therefore \(\tilde{\Ext}^i_{\mathcal{F}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = 0\) for all \(n\). So, in this case, the exact sequence \(0 \to \Ext^1_{\mathcal{F}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \to \Ext^1_{\mathcal{F}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \to \Ext^1_{\mathcal{F}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \to \cdots\) is \(0 \to \Ext^1_{\mathcal{F}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \to \Ext^2_{\mathcal{F}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \to 0\) with \(\Ext^1_{\mathcal{F}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}\).

3. **Avramov-Martsinkovsky's Exact Sequence**

For the rest of the article \(R\) denotes a left noetherian ring (unless otherwise specified) and \(R\)-module means left \(R\)-module. For unexplained terminology and notation please see [1] and [3].
Proposition 1 below shows that when \( \mathcal{P} \) is the class of projective \( R \)-modules, \( \mathcal{G} \) is the class of Gorenstein projective \( R \)-modules and \( M \) is an \( R \)-module of finite Gorenstein projective dimension, the modules \( \widehat{\text{Ext}}^n_{\mathcal{G},\mathcal{P}}(M,N) \) are the usual Tate cohomology modules for any \( n \geq 1 \).

We recall first the following:

**Definition 4** ([1]). A complete resolution of an \( R \)-module \( M \) is a diagram \( T \xrightarrow{u} P \xrightarrow{\pi} M \) where \( P \xrightarrow{\pi} M \) is a projective resolution of \( M \), \( T \) is a totally acyclic complex, \( u \) is a morphism of complexes and \( u_n \) is bijective for all \( n \gg 0 \). If \( T \xrightarrow{u} P \xrightarrow{\pi} M \) is such a complete resolution of \( M \) then for each left \( R \)-module \( N \) and for each \( n \in \mathbb{Z} \) the usual Tate cohomology module \( \widehat{\text{Ext}}^n_R(M,N) \) is defined by the equality \( \widehat{\text{Ext}}^n_R(M,N) = H^n(\text{Hom}(T,N)) \).

**Proposition 1.** If \( M \) is an \( R \)-module with \( \text{Gor proj dim } M < \infty \) then for each \( R \)-module \( N \) we have \( \widehat{\text{Ext}}^n_{\mathcal{G},\mathcal{P}}(M,N) \simeq \widehat{\text{Ext}}^n_R(M,N) \) for any \( n \geq 1 \).

**Proof.** Let \( g = \text{Gor proj dim } M \).

We start by constructing a complete resolution of \( M \).

If \( 0 \to C \xrightarrow{i} P_{g-1} \xrightarrow{f_{g-1}} P_{g-2} \xrightarrow{f_{g-2}} \cdots \xrightarrow{f_1} P_0 \xrightarrow{\pi} M \to 0 \) is a partial projective resolution of \( M \) then \( C \) is a Gorenstein projective module ([5], Theorem 2.20). Hence there exists an exact sequence \( T: \cdots \to P^{-2} \xrightarrow{d_{-2}} P^{-1} \xrightarrow{d_{-1}} P^{-0} \xrightarrow{d_{-0}} P^{-1} \xrightarrow{d_{-1}} \cdots \) of projective modules such that \( C = \text{Ker } d_0 \) and \( \text{Hom}(T,P) \) is an exact complex for any projective \( R \)-module \( P \). In particular \( \text{Hom}(T,R) \) is exact. Since each \( P^n \) is a projective module and \( H_n(T) = 0 = H_n(T^*) \) for any integer \( n \), the complex \( T \) is totally acyclic.

Since \( C = \text{Im } d_{-1} = \text{Ker } f_{g-1} \) and \( \cdots \to P^{-2} \xrightarrow{d_{-2}} P^{-1} \xrightarrow{d_{-1}} C \to 0 \) is exact, the complex \( P: \cdots \to P^{-2} \xrightarrow{d_{-2}} P^{-1} \xrightarrow{\text{id}_{-1}} P^{-1} \xrightarrow{f_{g-1}} P_{g-1} \xrightarrow{f_{g-2}} P_{g-2} \cdots \xrightarrow{f_1} P_0 \xrightarrow{\pi} M \to 0 \) is a projective resolution of \( M \).

\[
\begin{array}{cccccccccc}
T: & \cdots & \to & P^{-1} & \xrightarrow{d_{-1}} & P^0 & \xrightarrow{d_0} & P^1 & \xrightarrow{d_1} & \cdots & \to & P_{g-2} & \xrightarrow{d_{g-2}} & P_{g-1} & \xrightarrow{d_{g-1}} & P_{g} & \cdots \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
P: & \cdots & \to & P^{-1} & \xrightarrow{i \text{d}_{-1}} & P_{g-1} & \xrightarrow{f_{g-1}} & P_{g-2} & \xrightarrow{f_{g-2}} & \cdots & \to & P_1 & \xrightarrow{f_1} & P_0 & \to & 0 & \to \cdots \\
& & & & & & \downarrow{u_{g-1}} & & & & & \downarrow{u_{g-2}} & & \downarrow{u_1} & & \downarrow{u_0} & & \\
& & & & & & & & & & & & & & & & \\
\end{array}
\]

Since \( P_{g-1} \) is projective, the complex \( \text{Hom}(T,P_{g-1}) \) is exact. We have \( i \circ d_{-1} \in \text{Ker} \text{Hom}(d_{-2},P_{g-1}) = \text{Im} \text{Hom}(d_{-1},P_{g-1}) \). So there exists \( u_{g-1} \in \text{Hom}(P^0,P_{g-1}) \) such that \( i \circ d_{-1} = u_{g-1} \circ d_{-1} \). Similarly there exist \( u_{g-2}, \ldots, u_0 \) that make the diagram commutative. Since \( u: T \to P \) (with \( u_0, u_1, \ldots, u_{g-1} \) as above and \( u_n = \)}
\( \text{id}_{p_{g-1}} \) for \( n \geq g \) is a morphism of complexes, \( u_n \) is bijective for \( n \geq g \), \( T \) is a totally acyclic complex and \( P \to M \) is a projective resolution of \( M \), it follows that \( T \xrightarrow{u} P \xrightarrow{\pi} M \) is a complete resolution of \( M \).

We use now the projective resolution \( P \) and the complete resolution \( T \) to construct a Gorenstein projective resolution of \( M \).

Let \( D = \text{Im} \, d_{g-1} \). Then \( D \) is a Gorenstein projective module (\cite{5}, Obs. 2.2) and there is a commutative diagram:

\[
\begin{array}{cccccccccccccc}
0 & \longrightarrow & C & \longrightarrow & P^0 & \xrightarrow{d_0} & P^1 & \xrightarrow{d_1} & P^2 & \longrightarrow & \cdots & \longrightarrow & P^{g-2} & \xrightarrow{d_{g-2}} & P^{g-1} & \xrightarrow{d_{g-1}} & D & \longrightarrow & 0 \\
& & & \downarrow{u_{g-1}} & & \downarrow{u_{g-2}} & & \downarrow{u_{g-3}} & & \cdots & \downarrow{u_1} & & \downarrow{u_0} & & \downarrow{u} & & \\
0 & \longrightarrow & C & \longrightarrow & P_{g-1} & \xrightarrow{f_{g-1}} & P_{g-2} & \xrightarrow{f_{g-2}} & P_{g-3} & \longrightarrow & \cdots & \longrightarrow & P_1 & \xrightarrow{f_1} & P_0 & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
\end{array}
\]

with \( u \) defined by: \( u(d_{g-1}(x)) = \pi(u_0(x)) \).

Since both rows are exact complexes, the associated mapping cone
\[
\varrho : 0 \to C \xrightarrow{\Delta} C \oplus P^0 \xrightarrow{\delta_0} P_{g-1} \oplus P^1 \xrightarrow{\delta_1} P_{g-2} \oplus P^2 \longrightarrow \cdots \longrightarrow P_1 \oplus P^{g-1} \xrightarrow{\delta_{g-1}} P_0 \oplus D \xrightarrow{\beta} M \longrightarrow 0
\]
is also an exact complex.

\( \varrho \) has the exact subcomplex: \( 0 \to C \xrightarrow{\Delta} C \to 0 \). Forming the quotient complex, we get an exact complex: \( 0 \to 0 \to P^0 \xrightarrow{\delta_0} P_{g-1} \oplus P^1 \xrightarrow{\delta_1} P_{g-2} \oplus P^2 \longrightarrow \cdots \longrightarrow P_1 \oplus P^{g-1} \xrightarrow{\delta_{g-1}} P_0 \oplus D \xrightarrow{\beta} M \longrightarrow 0 \).

Let \( L \) be a Gorenstein projective module. Since \( \text{proj dim Ker } \beta < \infty \), we have \( \text{Ext}^1_K(L, \text{Ker } \beta) = 0 \) (\cite{5}, Proposition 2.3). The sequence \( 0 \to \text{Ker } \beta \to P_0 \oplus D \to M \to 0 \) is exact, so we have the associated exact sequence: \( 0 \to \text{Hom}(L, \text{Ker } \beta) \to \text{Hom}(L, P_0 \oplus D) \to \text{Hom}(L, M) \to \text{Ext}^1_K(L, \text{Ker } \beta) = 0 \). Thus \( P_0 \oplus D \to M \) is a Gorenstein projective precover. Similarly \( P_1 \oplus P^{g-1} \to \text{Ker } \beta \) is a Gorenstein projective precover, \( \ldots, P^0 \to \text{Ker } \delta_{g-1} \) is a Gorenstein projective precover, so \( G : 0 \to P^0 \to P_{g-1} \oplus P^1 \to P_{g-2} \oplus P^2 \to \cdots \to P_0 \oplus D \to M \to 0 \) is a Gorenstein projective resolution of \( M \).

There is a map of complexes \( e : P \to G \)

\[
\begin{array}{cccccccccccccc}
\ldots & \longrightarrow & P_{-2} & \xrightarrow{d_{-3}} & P_{-1} & \xrightarrow{d_{-2}} & P_{g-1} & \xrightarrow{f_{g-1}} & \cdots & \longrightarrow & P_1 & \xrightarrow{f_1} & P_0 & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
\downarrow{d_{-3}} & & \downarrow{d_{-2}} & & \downarrow{f_{g-1}} & & \cdots & \downarrow{f_1} & & \downarrow{f_0} & & \cdots & \downarrow{e_0} & & \\
\ldots & \longrightarrow & 0 & \longrightarrow & P^0 & \xrightarrow{\delta_0} & P_{g-1} \oplus P^1 & \xrightarrow{\delta_1} & \cdots & \longrightarrow & P_1 \oplus P^{g-1} & \xrightarrow{\delta_{g-1}} & P_0 \oplus D & \xrightarrow{\beta} & M & \longrightarrow & 0 \\
\end{array}
\]

with

\[
\begin{align*}
e_0 &: P_0 \to P_0 \oplus D, \quad e_0(x) = (x, 0) \\
e_j &: P_j \to P_j \oplus P^{g-j}, \quad e_j(x) = (x, 0) \quad 1 \leq j \leq g - 1
\end{align*}
\]
\( P \) is a projective resolution of \( M \), \( G \) is a Gorenstein projective resolution of \( M \) and \( e : P \to G \) is a chain map induced by \( Id_M \), so \( \widetilde{\text{Ext}}^n_{\mathcal{I}_R}(M, N) = H^{n+1}(\text{Hom}(M(e), N)), \forall n \geq 0 \), where \( M(e) \) is the mapping cone of \( e : P \to G \).

Let
\[
\mathbf{T} : \ldots \xrightarrow{d_{-2}} P_{-1} \xrightarrow{d_{-1}} P_0 \xrightarrow{d_0} \cdots \xrightarrow{d_{g-2}} P_{g-2} \xrightarrow{d_{g-1}} P_{g-1} \xrightarrow{d_{g}} D \xrightarrow{} 0.
\]

We prove that \( M(e) \) and \( \mathbf{T}[1] \) are homotopically equivalent.

There is a map of complexes \( \alpha : \mathbf{T}[1] \to M(e) \) with
\[
\alpha_0 : P^0 \to P^0 \oplus P_{g-1}, \quad \alpha_0(x) = (x, -u_{g-1}(x)) \quad \forall x \in P^0,
\]
\[
\alpha_j : P^j \to P_{g-j} \oplus P^j \oplus P_{g-j-1}, \quad \alpha_j(x) = (0, x, -u_{g-j-1}(x)), \quad \forall x \in P^j, \quad 1 \leq j \leq g-1
\]
\[
\alpha' : D \to P_0 \oplus D, \quad \alpha'(x) = (0, x) \quad \forall x \in D; \quad \alpha_j = -Id_{P^j} \quad \text{if} \quad j \leq -1 \quad \text{is odd}; \quad \alpha_j = Id_{P^j} \quad \text{if} \quad j \leq -1 \quad \text{is even.}
\]

There is also a map of complexes \( l : M(e) \to \mathbf{T}[1] \):
\[
l_0 : P^0 \oplus P_{g-1} \to P^0, \quad l_0(x, y) = x \quad \forall (x, y) \in P^0 \oplus P_{g-1}
\]
\[
l_j : P_{g-j} \oplus P^j \oplus P_{g-j-1} \to P^j, \quad l_j(x, y, z) = y \quad \forall (x, y, z) \in P_{g-j} \oplus P^j \oplus P_{g-j-1}, \quad 1 \leq j \leq g-1
\]
\[
l' : P^0 \oplus D \to D, \quad l'(x, y) = y \quad \forall (x, y) \in P^0 \oplus D
\]
\[
l_j = -Id_{P^j} \quad \text{if} \quad j \leq -1 \quad \text{is odd}; \quad l_j = Id_{P^j} \quad \text{if} \quad j \leq -1 \quad \text{is even.}
\]

We have
\[
(3) \quad l \circ \alpha = Id_{\mathbf{T}[1]} \quad \text{and} \quad \alpha \circ l \sim Id_{M(e)}
\]

(a chain homotopy between \( \alpha \circ l \) and \( Id_M \) is given by the maps:
\[
\chi_0 : P_0 \oplus D \to P_1 \oplus P_{g-1} \oplus P_0, \quad \chi_0(x, y) = (0, 0, -x)
\]
\[
\chi_j : P_j \oplus P^j \oplus P_{g-j-1} \to P_{j+1} \oplus P^j \oplus P_{g-j}, \quad \chi_j(x, y, z) = (0, 0, -x), \quad 1 \leq j \leq g-2
\]
\[
\chi_{g-1} : P_{g-1} \oplus P^1 \oplus P_{g-2} \to P^0 \oplus P_{g-1}, \quad \chi_{g-1}(x, y, z) = (0, 0, -x)
\]

By (3) we have \( H^{n+1}(\text{Hom}(M(e), N)) \simeq H^{n+1}(\text{Hom}(\mathbf{T}[1], N)) \) that is \( \widetilde{\text{Ext}}^n_{\mathcal{I}_R}(M, N) = \widetilde{\text{Ext}}^n_R(M, N) \), for any \( R \text{N} \), for all \( n \geq 1 \).

\[\square\]

**Corollary 1** (Avramov-Martsinkovsky). Let \( M \) be an \( R \)-module with \( \text{Gor proj dim} \ M = g < \infty \). For each \( R \)-module \( N \) there is an exact sequence: \( 0 \to \text{Ext}^1_{\mathcal{I}_R}(M, N) \to \text{Ext}^1_R(M, N) \to \widetilde{\text{Ext}}^1_R(M, N) \to \cdots \to \text{Ext}^g_R(M, N) \to \text{Ext}^g_{\mathcal{I}_R}(M, N) \to 0 \).

**Proof.** By (1) there is an exact sequence: \( 0 \to \text{Ext}^1_{\mathcal{I}_R}(M, N) \to \text{Ext}^1_R(M, N) \to \widetilde{\text{Ext}}^1_{\mathcal{I}_R}(M, N) \to \cdots \).

By Proposition 1 we have \( \widetilde{\text{Ext}}^i_{\mathcal{I}_R}(M, N) \simeq \widetilde{\text{Ext}}^i_R(M, N), \forall i \geq 1 \).
Since $\text{Ext}^{i}_{R}(M,N) = 0$, for all $i \geq 1$ the exact sequence above gives us: $0 \to \text{Ext}^{i}_{R}(M,N) \to \text{Ext}^{i+1}_{R}(M,N) \to \cdots \to \text{Ext}^{d}_{R}(M,N) \to \text{Ext}^{d+1}_{R}(M,N) \to \cdots \to \text{Ext}^{\infty}_{R}(M,N) \to \text{Ext}^{\infty}_{R}(M,N) \to 0$.

4. Computing the Tate Cohomology Using Complete Injective Resolutions

The classical groups $\text{Ext}^{i}_{R}(M,N)$ can be computed using either a projective resolution of $M$ or an injective resolution of $N$. In this section we want to prove an analogous result for the groups $\tilde{\text{Ext}}^{i}_{R}(M,N)$. We note that we cannot use a straightforward modification of the proof in classical case. This is basically because the associated double complex in our case is not a first (or third) quadrant one and so we cannot use the usual machinery of spectral sequences.

We start by defining a complete injective resolution.

Let $N$ be an $R$-module with $\text{Gor inj dim } N = d < \infty$.

If $0 \longrightarrow N \longrightarrow E^{0} \longrightarrow E^{1} \longrightarrow \cdots \longrightarrow E^{d-1} \longrightarrow H \longrightarrow 0$ is a partial injective resolution of $N$, then $H$ is a Gorenstein injective module ([5], Theorem 2.22). Hence there exists a $\text{Hom} (\text{Inj}, -)$ exact sequence

$$\mathcal{E} : \cdots \longrightarrow E_{2} \xrightarrow{d_{2}} E_{1} \xrightarrow{d_{1}} E_{0} \xrightarrow{d_{0}} E_{-1} \xrightarrow{d_{-1}} E_{-2} \xrightarrow{d_{-2}} \cdots$$

of injective modules such that $\mathcal{E}$ is exact and $H = \text{Ker } d_{0}$ ([3], 10.1.1).

We say that $\mathcal{E}$ is a complete injective resolution of $N$.

For each module $R_{M}$ and each $i \in \mathbb{Z}$ let $\tilde{\text{Ext}}^{i}_{R}(M,N) \overset{\text{def}}{=} H^{i}(\text{Hom}(M, \mathcal{E}))$.

We prove that any two complete injective resolutions of $N$ are homotopically equivalent.

Let $\mathcal{E} : \cdots \longrightarrow I^{-1} \xrightarrow{g_{-1}} I^{0} \xrightarrow{g_{0}} I^{1} \xrightarrow{g_{1}} I^{2} \longrightarrow \cdots$ and $\tilde{\mathcal{E}} : \cdots \longrightarrow \tilde{I}^{-1} \xrightarrow{\tilde{g}_{-1}} \tilde{I}^{0} \xrightarrow{\tilde{g}_{0}} \tilde{I}^{1} \xrightarrow{\tilde{g}_{1}} \tilde{I}^{2} \longrightarrow \cdots$ be two complete injective resolutions of $N$ corresponding to two injective resolutions, $\mathcal{N}$ and $\tilde{\mathcal{N}}$, of $N$ ($H = \text{Ker } g_{0} = \text{Im } g_{-1}$ is the $d$th cosyzygy of $\mathcal{N}$ and $\tilde{H} = \text{Ker } \tilde{g}_{0} = \text{Im } \tilde{g}_{-1}$ is the $d$th cosyzygy of $\tilde{\mathcal{N}}$).

If $\mathcal{H}$ is the injective resolution of $H$ obtained from $\mathcal{N}$ and $\tilde{\mathcal{H}}$ is the injective resolution of $\tilde{H}$ obtained from $\tilde{\mathcal{N}}$ then $\mathcal{H}$ and $\tilde{\mathcal{H}}$ are homotopically equivalent (since the two injective resolutions of $N$, $\mathcal{N}$ and $\tilde{\mathcal{N}}$, are homotopically equivalent).

Since $\mathcal{E}' : 0 \to H \to I^{0} \to I^{1} \to \cdots$ is an injective resolution of $H$ it follows that $\mathcal{E}'$ and $\mathcal{H}$ are homotopically equivalent. Similarly $\tilde{\mathcal{E}}' : 0 \to \tilde{H} \to \tilde{I}^{0} \to \tilde{I}^{1} \to \cdots$ is homotopically equivalent to $\tilde{\mathcal{H}}$. Then, by the above, $\mathcal{E}'$ and $\tilde{\mathcal{E}}'$ are homotopically equivalent. So there exist chain maps $u : \mathcal{E}' \to \tilde{\mathcal{E}}'$ and $v : \tilde{\mathcal{E}}' \to \mathcal{E}'$ ($u$ defined by $\bar{u} \in \text{Hom}(H, \tilde{H})$, $u_{j} \in \text{Hom}(I^{j}, \tilde{I}^{j})$, $j \geq 0$ and $v$ defined by $\bar{v} \in \text{Hom}(\tilde{H}, H)$, $v_{j} \in \text{Hom}(\tilde{I}^{j}, I^{j})$, $j \geq 0$).
Generalized Tate cohomology

\( \text{Hom}(\overline{H}, H) \) and \( v_j \in \text{Hom}(\overline{I^j}, I^j) \), there exist \( \beta \in \text{Hom}(I^0, H) \), \( \beta_j \in \text{Hom}(I^j, I^{j-1}) \), \( j \geq 1 \) such that \( \overline{u} \circ \overline{u} - \text{Id} = \beta \circ i \) (where \( i : H \to I^0 \) is the inclusion map), and

\[
\begin{align*}
v_0 \circ u_0 - \text{Id} &= \beta_1 \circ g_0 + i \circ \beta, \\
v_j \circ u_j - \text{Id} &= g_{j-1} \circ \beta_j + \beta_{j+1} \circ g_j, & \quad \forall j \geq 1.
\end{align*}
\]

Since \( \delta'' : \cdots \to I^{-2} \to I^{-1} \to H \to 0 \) is an injective resolvent of \( H \) ([2], 1.3) and \( \delta'' : \cdots \to \overline{I^{-2}} \to \overline{I^{-1}} \to \overline{H} \to 0 \) is an injective resolvent of \( \overline{H} \), \( \bar{u} \in \text{Hom}(H, \overline{H}) \) induces a map of complexes \( u : \delta'' \to \overline{\delta''}, u = (u_j) \leq -1 \). Similarly, there is a map of complexes \( \nu : \overline{\delta''} \to \delta'' \), \( \nu = (v_j) \leq -1 \), induced by \( \bar{v} \in \text{Hom}(\overline{H}, H) \).

Since \( I^0 \) is injective and \( g_{-1} : I^{-1} \to H \) is an injective precover, there exists \( \beta_0 \in \text{Hom}(I^0, I^{-1}) \) such that \( \beta = g_{-1} \circ \beta_0 \). So \( v_0 \circ u_0 - \text{Id} = \beta_1 \circ g_0 + i \circ \beta = \beta_1 \circ g_0 + g_{-1} \circ \beta_0 \).

We have \( g_{-1} \circ (v_{-1} \circ u_{-1} - \text{Id} - \beta_0 \circ g_{-1}) = 0 \iff \text{Im}(v_{-1} \circ u_{-1} - \text{Id} - \beta_0 \circ g_{-1}) \subset \ker g_{-1} \). Since \( I^{-1} \) is injective and \( \overline{I^{-2}} \overset{g_{-2}}{\to} \ker g_{-1} \) is an injective precover, there is \( \beta_{-1} \in \text{Hom}(I^{-1}, I^{-2}) \) such that \( v_{-1} \circ u_{-1} - \text{Id} - \beta_0 \circ g_{-1} = g_{-2} \circ \beta_{-1} \).

Similarly, there exist \( \beta_j \in \text{Hom}(I^j, I^{j-1}) \), \( \forall j \leq -1 \) such that \( v_j \circ u_j - \text{Id} = \beta_{j+1} \circ g_j + g_{j-1} \circ \beta_j, \forall j \leq -1 \). Thus \( u \circ v \sim Id_{\mathcal{G}} \). Similarly \( u \circ v \sim Id_{\overline{\mathcal{G}}} \).

Hence \( H^i(Hom(M, \mathcal{G})) \simeq H^i(Hom(M, \overline{\mathcal{G}})) \) for any \( RM \), for all \( i \in \mathbb{Z} \).

So \( Ext^n_R(-, N) \) is well-defined.

If \( \mathcal{N} \) is a deleted injective resolution of \( N \), \( \mathcal{G} \) is a deleted Gorenstein injective resolution of \( N \) and \( v : \mathcal{G} \to \mathcal{N} \) is a chain map induced by \( Id_N \) then a dual argument of the proof of Theorem 1 shows that the cohomology of \( Hom(M, M(v)) \) gives us the functor \( Ext^n_R(M, N) \) and that there is an exact sequence

\[
0 \to Ext_{\mathcal{G}}^i(M, N) \to Ext_R^i(M, N) \to Ext_{\overline{\mathcal{G}}}^i(M, N) \to Ext_R^i(M, N) \to 0
\]

where \( Ext_{\mathcal{G}}^i(M, N) = H^i(Hom(M, \mathcal{G})) \) for any \( i \geq 0 \).

If \( Gor \text{ proj dim } M < \infty \) then \( Ext_{\mathcal{G}}^i(M, N) \simeq Ext_{\overline{\mathcal{G}}}^i(M, N) \) for any \( i \geq 0 \) ([4], Theorem 3.6).

Thus we have:

**Theorem 1.** Let \( N \) be an \( R \)-module with \( Gor \text{ inj dim } N = d < \infty \). For each \( R \)-module \( M \) with \( Gor \text{ proj dim } M < \infty \) there is an exact sequence:

\[
0 \to Ext_R^1(M, N) \to Ext_R^1(M, N) \to Ext_R^1(M, N) \to \cdots
\]
Theorem 2 shows that over Gorenstein rings $\widehat{\text{Ext}}^n_R(M, N) \simeq \text{Ext}^n_R(M, N)$ for any left $R$-modules $M$ and $N$, for any $n \in \mathbb{Z}$.

**Theorem 2.** If $R$ is a Gorenstein ring then $\widehat{\text{Ext}}^n_R(M, N) \simeq \text{Ext}^n_R(M, N)$ for any $R$-modules $M$, $N$ for any $n \in \mathbb{Z}$.

**Proof.** Let $g = \text{Gor proj dim } M$ and $d = \text{Gor inj dim } N$. $R$ is a Gorenstein ring, so $g < \infty$ ([3], Corollary 11.5.8) and $d < \infty$ (this follows from [3], Theorem 11.2.1).

We are using the notations of Proposition 1 and Theorem 1.

• We prove first that if $M$ is Gorenstein projective then $\widehat{\text{Ext}}^n_R(M, N) \simeq \text{Ext}^n_R(M, N)$ for any $n \in \mathbb{Z}$.

Since $M$ is Gorenstein projective we have a complete resolution $T \xrightarrow{u} P \xrightarrow{n} M$ with $T^n = P^n, \forall n \geq 0$ and $u_n = id_{P^n}, \forall n \geq 0.$

So

$$\widehat{\text{Ext}}^n_R(M, N) \simeq \text{Ext}^n_R(M, N) \quad \forall n \geq 1$$

We have the exact sequence (by Theorem 1):

$$0 \to \text{Ext}^1_g(M, N) \to \text{Ext}^1_R(M, N) \to \widehat{\text{Ext}}^1_R(M, N) \to \text{Ext}^2_g(M, N) \to \cdots$$

Since $\text{Ext}^1_g(M, N) = 0, \forall i \geq 1$ it follows that

$$\widehat{\text{Ext}}^1_R(M, N) \simeq \text{Ext}^1_R(M, N), \quad \forall i \geq 1$$

By (4) and (5) we have $\widehat{\text{Ext}}^i_R(M, N) \simeq \text{Ext}^i_R(M, N)$, for all $i \geq 1$.

• Case $n \leq 0$

Let $n = -k, \ k \geq 0$.

Let $\mathcal{E}$ be a complete injective resolution of $N$.

Since $T: \cdots \to P^{-2} \xrightarrow{d_{-2}} P^{-1} \xrightarrow{d_{-1}} P^0 \xrightarrow{d_0} P^1 \xrightarrow{d_1} P^2 \to \cdots$ is exact with each $P^i$ projective and such that $\text{Hom}(T, Q)$ is exact for any projective module $Q$, it follows that $M^i = \text{Im} \ d_i$ is a Gorenstein projective module for any $i \in \mathbb{Z}$ ([5], Obs. 2.2).

Let $M^1 = \text{Im} \ d_1$. Since $0 \to M \to P^1 \to M^1 \to 0$ is exact and all the terms of $\mathcal{E}$ are injective modules, we have an exact sequence of complexes $0 \to \text{Hom}(M^1, \mathcal{E}) \to \text{Hom}(P^1, \mathcal{E}) \to \text{Hom}(M, \mathcal{E}) \to 0$ and therefore an associated long exact sequence:
(6) \[ \cdots \rightarrow H^i(Hom(P^1, \mathcal{E})) \rightarrow H^i(Hom(M, \mathcal{E})) \rightarrow H^{i+1}(Hom(M^1, \mathcal{E})) \rightarrow \cdots \]

Since a complete injective resolution \( \mathcal{E} \) of \( N \) is exact and \( P^1 \) is projective, the complex \( Hom(P^1, \mathcal{E}) \) is exact. Then, by (6), we have \( H^i(Hom(M, \mathcal{E})) \cong H^{i+1}(Hom(M^1, \mathcal{E})) \approx Ext^i_R(M, N) \cong Ext^{i+1}_R(M^1, N) \) for any \( R N \), for any \( i \in \mathbb{Z} \).

Similarly,

\[ Ext^i_R(M, N) \cong Ext^{i+k+1}_R(M^{k+1}, N) \]

for any \( R N \) for all \( i \in \mathbb{Z} \) where \( M^{k+1} = \text{Im} \ g_{k+1} \in \text{Gor Proj} \).

Since \( R \) is a Gorenstein ring there is an exact sequence \( 0 \rightarrow G' \rightarrow L' \rightarrow N \rightarrow 0 \) with \( \text{proj dim \ } L' < \infty \) and \( G' \) a Gorenstein injective module ([3], Exercise 6, pp. 277).

Since each term of a complete resolution \( T \) is a projective module, we have an exact sequence of complexes \( 0 \rightarrow Hom(T, G') \rightarrow Hom(T, L') \rightarrow Hom(T, N) \rightarrow 0 \) and therefore an associated long exact sequence:

(8) \[ \cdots \rightarrow H^i(Hom(T, G')) \rightarrow H^i(Hom(T, L')) \rightarrow H^i(Hom(T, N)) \rightarrow \cdots \]

Since \( \text{proj dim \ } L' < \infty \) it follows that \( Hom(T, L') \) is an exact complex ([5], Proposition 2.3). Then, by (8), we have \( H^i(Hom(T, N)) \cong H^{i+1}(Hom(T, G')) \) that is

\[ Ext^i_R(M, N) \cong Ext^{i+1}_R(M, G') \]

for any \( i \in \mathbb{Z} \) and for any \( R M \).

Let \( \mathcal{E} : \cdots \rightarrow E_2 \xrightarrow{g_2} E_1 \xrightarrow{g_1} E_0 \xrightarrow{g_0} E_1 \xrightarrow{g_1} E_2 \rightarrow \cdots \) be a complete injective resolution of the Gorenstein injective module \( G' \) (\( G' = \text{Ker} \ g_0 = \text{Im} \ g_{-1} \)) and let \( G_i = \text{Ker} \ g_i \).

We have (same argument as above)

(10) \[ Ext^i_R(M, N) \cong Ext^{i+k+1}_R(M, G_{-k}), \quad \forall i \in \mathbb{Z} \]

for any \( R M \), where \( G_{-k} = \text{Ker} \ g_{-k} \).

By (7), \( Ext^k_R(M, N) \cong Ext^1_R(M^{k+1}, N) \cong Ext^1_R(M^{k+1}, N) \cong Ext^1_R(M^{k+1}, N) \cong Ext^{k+2}_R(M^{k+1}, G_{-k}) \)

(since \( M^{k+1} \) is Gorenstein projective). Then, by (10), \( Ext^1_R(M^{k+1}, N) \cong Ext^{k+2}_R(M^{k+1}, G_{-k}) \).

So \( Ext^k_R(M, N) \approx Ext^{k+2}_R(M^{k+1}, G_{-k}) \).

By (10), \( Ext^k_R(M, N) \cong Ext^1_R(M, G_{-k}) \approx Ext^1_R(M, G_{-k}) \approx Ext^1_R(M, G_{-k}) \).
(since $M$ is Gorenstein projective). Then, by (7), $\widetilde{\text{Ext}}^{k+2}_R(M, G_{-k}) \simeq \widetilde{\text{Ext}}^{k+2}_R(M^{k+1}, G_{-k})$.

So $\widetilde{\text{Ext}}^{k}_R(M, N) \simeq \text{Ext}^{k+2}_R(M^{k+1}, G_{-k}) \simeq \widetilde{\text{Ext}}^{-k}_R(M, N)$ for any $k \in \mathbb{Z}$, $k \geq 0$.

Hence $\widetilde{\text{Ext}}^{n}_R(M, N) \simeq \text{Ext}^{n}_R(M, N)$ for any $n \in \mathbb{Z}$, if $M$ is Gorenstein projective.

Similarly, $\widetilde{\text{Ext}}^{n}_R(M, N) \simeq \text{Ext}^{n}_R(M, N)$ for any $n \in \mathbb{Z}$, if $N$ is Gorenstein injective.

• Case $g = \text{Gor proj dim } M \geq 1$

$R$ is a Gorenstein ring, so there is an exact sequence $0 \to M \to L \to C' \to 0$ with $\text{proj dim } L < \infty$ and $C'$ a Gorenstein projective module (the same argument used in [6], Corollary 3.3.7, gives this result for $R$-modules).

Since $\text{proj dim } L < \infty$ it follows that

\begin{equation}
\text{(11)}
\text{Hom}(L, \mathcal{E}) \text{ is an exact complex.}
\end{equation}

Since $0 \to M \to L \to C' \to 0$ is exact and each term of $\mathcal{E}$ is an injective module we have an exact sequence of complexes $0 \to \text{Hom}(C', \mathcal{E}) \to \text{Hom}(L, \mathcal{E}) \to \text{Hom}(M, \mathcal{E}) \to 0$ and therefore an associated long exact sequence: \( \cdots \to H^n(\text{Hom}(C', \mathcal{E})) \to H^n(\text{Hom}(L, \mathcal{E})) \to H^n(\text{Hom}(M, \mathcal{E})) \to H^{n+1}(\text{Hom}(C', \mathcal{E})) \to H^{n+1}(\text{Hom}(L, \mathcal{E})) \to \cdots \).

By (11) we have $H^n(\text{Hom}(L, \mathcal{E})) = 0 \ \forall n \in \mathbb{Z}$. So

\begin{equation}
\text{(12)}
H^n(\text{Hom}(M, \mathcal{E})) \simeq H^{n+1}(\text{Hom}(C', \mathcal{E}))
\end{equation}

\[ \Leftrightarrow \widetilde{\text{Ext}}^{n}_R(M, N) \simeq \widetilde{\text{Ext}}^{n+1}_R(C', N) \]

for any $R N$, for any $n \in \mathbb{Z}$.

So $\widetilde{\text{Ext}}^{n}_R(M, N) \simeq \widetilde{\text{Ext}}^{n+1}_R(C', N) \simeq \widetilde{\text{Ext}}^{n+1}_R(C', N)$ (since $C' \in \text{Gor Proj}$) for any $R N$, for all $n \in \mathbb{Z}$.

By (9) $\widetilde{\text{Ext}}^{n+1}_R(C', N) \simeq \widetilde{\text{Ext}}^{n+2}_R(C', G') \ \forall n \in \mathbb{Z}$. (where $0 \to G' \to L' \to N \to 0$ is exact, $G' \in \text{Gor Inj}$, $L \in \mathcal{L}$)

Hence $\widetilde{\text{Ext}}^{n}_R(M, N) \simeq \widetilde{\text{Ext}}^{n+2}_R(C', G') \ \forall n \in \mathbb{Z}$.

By (9) $\widetilde{\text{Ext}}^{n}_R(M, N) \simeq \widetilde{\text{Ext}}^{n+1}_R(M, G') \simeq \widetilde{\text{Ext}}^{n+1}_R(M, G')$ (since $G'$ is Gorenstein injective), for all $n \in \mathbb{Z}$. Then, by (12) $\widetilde{\text{Ext}}^{n+1}_R(M, G') \simeq \widetilde{\text{Ext}}^{n+2}_R(C', G') \simeq \widetilde{\text{Ext}}^{n+2}_R(C', G')$ (since $C'$ is Gorenstein projective) for all $n \in \mathbb{Z}$.

Hence $\widetilde{\text{Ext}}^{n}_R(M, N) \simeq \widetilde{\text{Ext}}^{n+2}_R(C', G') \simeq \widetilde{\text{Ext}}^{n}_R(M, N) \ \forall n \in \mathbb{Z}$. \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ 

**Remark 2.** Theorem 2 shows that over Gorenstein rings there is a new way of computing the Tate cohomology, i.e. by using a complete injective resolution of $N$. 


In a subsequent publication we hope to show how we can exploit this procedure to gain new information about Tate cohomology modules.

Theorem 1 together with Theorem 2 give us the following result:

Let $R$ be a Gorenstein ring, let $N$ be an $R$-module with $Gor \text{ inj} \ dim N = d < \infty$. For each $R$-module $M$ there is an exact sequence:

$$
0 \rightarrow \text{Ext}^1_R(M',N) \rightarrow \text{Ext}^1_R(M,N) \rightarrow \text{Ext}_R^1(M,N) \rightarrow \cdots \\
\rightarrow \text{Ext}^d_R(M,N) \rightarrow \text{Ext}_R^d(M,N) \rightarrow 0.
$$

Theorem 2 allows us to give an easy proof of the existence of a long exact sequence of Tate cohomology associated with any short exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$.

\textbf{Theorem 3.} Let $R$ be a Gorenstein ring. Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be an exact sequence of $R$-modules. For any $R$-module $N$ there exists a long exact sequence of Tate cohomology modules $\cdots \rightarrow \text{Ext}^a_R(M'',N) \rightarrow \text{Ext}^a_R(M,N) \rightarrow \text{Ext}^a_R(M',N) \rightarrow \text{Ext}^a_R(M'',N) \rightarrow \cdots$

\textbf{Proof.} Let $\mathcal{E}$ be a complete injective resolution of $N$. Then, by Theorem 2, $\text{Ext}^a_R(M,N) \simeq H^n(\text{Hom}(M, \mathcal{E}))$ for any $R$ and any $n \in \mathbb{Z}$.

Since $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is exact and each term of $\mathcal{E}$ is an injective module, we have an exact sequence of complexes: $0 \rightarrow \text{Hom}(M'', \mathcal{E}) \rightarrow \text{Hom}(M, \mathcal{E}) \rightarrow \text{Hom}(M', \mathcal{E}) \rightarrow 0$.

Its associated cohomology exact sequence is the desired long exact sequence. \hfill $\square$

\textbf{Remark 3.} J. Asadollahi and Sh. Salarian also have a proof of the claim of Theorem 2 in a recent preprint (Gorenstein Local Cohomology Modules) of theirs.

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\section*{References}


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