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INVERSE PROBLEMS FOR TIME-DEPENDENT SINGULAR HEAT CONDUCTIVITIES—ONE-DIMENSIONAL CASE*

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Abstract. We consider an inverse boundary value problem for the heat equation on the interval $(0, 1)$, where the heat conductivity $\gamma(t, x)$ is piecewise constant and the point of discontinuity depends on time: $\gamma(t, x) = k^2$ $(0 < x < s(t))$, $\gamma(t, x) = 1$ $(s(t) < x < 1)$. First, we show that k and $s(t)$ on the time interval $[0, T]$ are determined from a partial Dirichlet-to-Neumann map: $u(t, 1) \to \partial_x u(t, 1)$, $0 < t < T$, $u(t, x)$ being the solution to the heat equation such that $u(t, 0) = 0$, independently of the initial data $u(0, x)$. Second, we show that another partial Dirichlet-to-Neumann map: $u(t, 0) \to \partial_x u(t, 1)$, $0 < t < T$, $u(t, x)$ being the solution to the heat equation such that $u(t, 1) = 0$, restricts the pair $(k, s(t))$ to, at most, two cases on the time interval $[0, T]$, independently of the initial data $u(0, x)$.

Key words. inverse problem, Dirichlet-to-Neumann map, heat probing

AMS subject classifications. 35R30, 35K05

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1. Introduction.

1.1. Inverse heat conductivity problem. Let $\Omega = (0, 1)$, and consider the following initial boundary value problem:

$$\begin{cases} \partial_t u(t,x) - \partial_x (\gamma(t,x)\partial_x u(t,x)) = 0 \quad \text{in} \quad (0,T)\times\Omega, \\ u(t,0) = f_0(t), \quad u(t,1) = f_1(t) \quad \text{for} \quad 0<t<T, \\ u(0,x) = u_0(x) \quad \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $\gamma(t, x) \in L^\infty((0, T) \times \Omega)$ has the following properties: There exist a constant $k > 0$, $k \neq 1$, and $s(t) \in C^2([0, T])$ such that

$$0 < \inf_{0<t<T} s(t) \le \sup_{0<t<T} s(t) < 1, \tag{1.2}$$

$$\gamma(t,x) = \begin{cases} k^2 & \text{if} \quad 0 < x < s(t), \\ 1 & \text{if} \quad s(t) < x < 1. \end{cases}$$

Let $u(t, x)$ be the solution to (1.1). The problem we address in this paper is to detect the region $D(t) = (0, s(t))$ and to determine $\gamma(t, x)$ *from the inputs* (f_0, f_1) *on the boundary and local measurements at* $x = 1$ of the heat flux $\partial_x u(t, x)$, *without taking into account the information of the initial data* u_0. We consider two cases for the inputs:

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Case 1 : $f_0 = 0$.
Case 2 : $f_1 = 0$.

Physically, the region $D(t)$ in the domain Ω corresponds to some inclusion in the medium with heat conductivity different from the one in the background. The first case, $f_0 = 0$ (respectively, the second one, $f_1 = 0$), corresponds to maintaining zero temperature at the point $x = 0$ (respectively, at $x = 1$) for the finite time $0 < t < T$, and measuring the resulting heat flux at the point $x = 1$. We shall show that, in the first case, $\gamma(t, x)$ can be determined by suitable choice of the input of the temperature $u(t, 1)$. The second case is more complicated: We can determine the value $(1 - \frac{1}{k})s(t)$ and we show that k (and so, $s(t)$) can take two values at most.

Theoretically, the infinite-precision measurement needs to be repeated infinitely many times to recover $D(t)$ and $\gamma(t, x)$ perfectly. However, approximate recovery should be possible from a finite number of finite-precision measurements similarly to [8, 6, 7], but this is outside the scope of the present paper.

1.2. Main theorems. Take a large parameter $\lambda > 0$, and put

$$h_{\mathrm{fw}}(t, x; \lambda) = e^{\lambda^2 t + \lambda x}, \quad h_{\mathrm{bw}}(t, x; \lambda) = e^{-\lambda^2 t + \lambda x},$$

which solve the forward and the backward heat equation, respectively,

$$\left(\partial_t - \partial_x^2\right) h_{\mathrm{fw}} = 0, \quad \left(\partial_t + \partial_x^2\right) h_{\mathrm{bw}} = 0.$$

THEOREM 1.1. *Let $u(t, x; \lambda)$ be the solution to* (1.1) *with $f_0 = 0$, $f_1 = h_{\mathrm{fw}}(t, 1; \lambda)$, and define*

$$I_{\mathrm{ind}}(T; \lambda) = \int_0^T e^{\lambda\nu t} h_{\mathrm{bw}}(t, 1; \lambda)\partial_x \left(u(t, x; \lambda) - h_{\mathrm{fw}}(t, x; \lambda)\right)\Big|_{x=1} dt.$$

Fix ν such that

$$\nu > \max\left(\frac{2}{T}, \max\left(2, |1 - k|, \left|1 - \frac{1}{k}\right|\right) \sup_{0<t<T} \left|\dot{s}(t)\right|\right). \tag{1.3}$$

Then for $\lambda \to \infty$, we have

$$I_{\mathrm{ind}}(T; \lambda) \simeq \frac{2(k-1)}{(k+1)(\nu + 2\dot{s}(T))} e^{\lambda\nu T + 2\lambda s(T) - \dot{s}(T)(1 - s(T))}.$$

COROLLARY 1.2. *For any initial data $u_0(x) \in L^2(0, 1)$, one can determine k and $s(t)$, $0 < t < T$, from the partial Dirichlet-to-Neumann map*

$$\Lambda_{\mathrm{partial}}^{u_0} : f_1 \to \partial_x u\Big|_{x=1},$$

with $u(t, 0) \equiv 0$.

THEOREM 1.3. *Let $u(t, x; \lambda)$ be the solution to* (1.1) *with $f_1 = 0$, $f_0(t) = h_{\mathrm{fw}}(t, 0; \lambda)$, and define*

$$\widetilde{I_{\mathrm{ind}}}(T; \lambda) = \int_0^T e^{\lambda\nu t} h_{\mathrm{bw}}(t, 1; \lambda)\partial_x u(t, x; \lambda)\Big|_{x=1} dt.$$

Set $L_1(t) = (1 - \frac{1}{k})s(t)$. Fix ν such that

$$\nu > \max\left(\left(3 + \left|1 - \frac{1}{k}\right|\right)/T, \sup_{0<t<T} \left|\dot{L}_1(t)\right|\right). \tag{1.4}$$

Then for $\lambda \to \infty$, we have

$$\widetilde{I_{\text{ind}}}(\lambda; T) \simeq -\frac{4k}{k+1}\frac{1}{\nu + \dot{L}_1(T)} e^{\lambda\nu T + \lambda L_1(T) - \frac{1}{2}\dot{L}_1(T)(1-s(T))}. \tag{1.5}$$

COROLLARY 1.4. *For any initial data $u_0(x) \in L^2(0,1)$, one can determine $L_1(t)$, $\xi(t) = \frac{1}{2}\dot{L}_1(t)L_1(t)$ and $F = \frac{k}{k+1}e^{\frac{k}{k-1}\xi(t)}$, $0 < t < T$, from the partial Dirichlet-to-Neumann map*

$$\widetilde{\Lambda}^{u_0}_{partial} : f_0 \to \partial_x u\Big|_{x=1},$$

with $u(t,1) \equiv 0$. Furthermore, the couple $(k, s(t))$ can take two values at most. More precisely, we have the following cases:

(i) $\xi = 1$: *We have either $0 < F \le (4\sqrt{e})^{-1}$ or $F > e$. If $0 < F < (4\sqrt{e})^{-1}$, then there are two couples $(k_i, s_i(t))$, $i = 1, 2$, such that* (1.5) *holds, and $0 < k_1 < 1/3 < k_2 < 1$. If $F > e$, then $(k, s(t))$ is uniquely determined and $k > 1$. If $F = (4\sqrt{e})^{-1}$, then $k = 1/3$.*

(ii) $\xi = 0$: *The couple $(k, s(t))$ is uniquely determined.*

(iii) $0 < \xi < 1$: *There exist two values F_*, F^* with $0 < F_* < F^* < e^\xi$ such that $F \notin (F_*, F^*)$. If $F \ge e^\xi$ or $F = F^*$ or $F = F_*$, then $(k, s(t))$ is uniquely determined. If $F < F_*$ or $F^* < F < e^\xi$, then there are two couples such that* (1.5) *holds.*

(iv) $\xi > 1$: *There exists a value $F_* \in (0, e^\xi)$ such that $F \notin (F_*, e^\xi]$. If $F > e^\xi$ or $F = F_*$, then $(k, s(t))$ is uniquely determined. If $F < F_*$, then there are two couples such that* (1.5) *holds.*

(v) $\xi < 0$: *If $F \ge e^\xi$, then the couple $(k, s(t))$ is uniquely determined and $0 < k < 1$. If $F < e^\xi$, then there are two couples $(k_i, s_i(t))$, $i = 1, 2$ such that* (1.5) *holds, and $0 < k_1 < 1 < k_2$.*

The issues on uniqueness, stability and reconstruction of the inclusion-identification problem have been centered around the case in which $s(t)$ is independent of t. Bellout [2] proved the local uniqueness and stability. Elayyan and Isakov [5] proved the global uniqueness of the inverse problem using the localized Neumann-to-Dirichlet (N-D) map. In [3], Di Cristo and Vessella gave logarithmic stability estimates of the inclusion from the Dirichlet-to-Neumann map. Ikehata [9], and Ikehata and Kawashita [10] developed the probe method for the heat equation with time-independent inclusions. In [6], the case of time-independent inclusions was treated and a numerical computation result was given. The idea is based on the complex spherical wave given by Ide et al. [8] for the elliptic case. The work of Daido, Kang, and Nakamura [4] may be the closest to the present paper. They studied the case of moving inclusions $D(t) = \{0 < a_0(t) < x < a_1(t) < 1\}$ using the probe method, which is based on the explicit form of the heat kernel, and Runge's approximation theorem, and proved that $a_1(t)$ is obtained from the whole knowledge of the N-D map. Their initial data is assumed to be zero: $u_0 = 0$, and the computation of k was not done. As for the recent works on the inverse problem for the parabolic equation, see Bacchelli et al. [1] for the corrosion problem, Vessella [14] and Kawakami and Tsuchiya [11] for the time-varying domain problem.

We use two main tools in this paper: The approximate solution of the heat equation to be constructed in section 4 for the first case and in section 6 for the second one, and the energy inequality in section 3, common to each case. The details of the construction of the approximate solution will be explained only for the first case. Although it is based on the standard construction of parametrics for the parabolic equation, a delicate choice is necessary for amplitude functions due to the discontinuity of the coefficient. The energy inequality is also a familiar tool; however, we need a careful choice of the auxiliary function to be multiplied by the equation. Our method can be extended to the multidimensional case, which will be discussed elsewhere.

Throughout the paper, we only deal with real-valued functions.

2. Existence theorem.

2.1. A theorem of J. L. Lions. Let $\mathcal{H}$ be a Hilbert space equipped with inner product $(\ ,\ )$ and norm $\|\cdot\|$. Suppose there exists another Hilbert space $\mathcal{H}_1$ with inner product $(\ ,\ )_1$ and norm $\|\cdot\|_1$ such that $\mathcal{H}_1$ is a dense subset of $\mathcal{H}$ and there exists a constant $C > 0$ such that

$$\|u\| \le C\|u\|_1 \quad \forall u \in \mathcal{H}_1.$$

Then we have the following inclusion relations:

$$\mathcal{H}_1 \subset \mathcal{H} \subset \mathcal{H}_1^*.$$

For $t \in [0, T]$, let $a(t, \cdot, \cdot)$ be a symmetric, bilinear form on $\mathcal{H}_1 \times \mathcal{H}_1$. We assume that there exist constants $\delta > 0$, $C_0 > 0$ such that

$$\begin{aligned} |a(t, u, v)| &\le C_0\|u\|_1\|v\|_1 \quad \forall u, v \in \mathcal{H}_1, \quad \forall t \in [0, T], \\ a(t, u, u) &\ge \delta\|u\|_1^2 - C_0\|u\|^2 \quad \forall u \in \mathcal{H}_1, \quad \forall t \in [0, T]. \end{aligned} \tag{2.1}$$

The last assumption is

$$\text{For any } u, v \in \mathcal{H}_1, [0, T] \ni t \mapsto a(t, u, v) \text{ is measurable.} \tag{2.2}$$

These assumptions imply that there exists a unique self-adjoint operator $A(t)$ such that $D(A(t)) \subset \mathcal{H}_1$ and

$$(A(t)u, v) = a(t, u, v) \quad \forall u \in D(A(t)), \quad \forall v \in \mathcal{H}_1.$$

With this operator $A(t)$, we consider the following evolution equation on $\mathcal{H}$:

$$\begin{cases} \partial_t u(t) + A(t)u(t) = f(t) \quad \text{in} \quad (0, T), \\ u(0) = u_0 \in \mathcal{H}. \end{cases} \tag{2.3}$$

The theorem of Lions asserts as follows (see [12], [13]).

THEOREM 2.1. *Let $u_0 \in \mathcal{H}$ and $f \in L^2((0, T); \mathcal{H}_1^*)$. Then there exists a unique $u(t)$ having the following properties.*

(1) $u(t) \in C([0, T]; \mathcal{H}) \cap L^2((0, T); \mathcal{H}_1)$.

(2) *$u(t)$ is $\mathcal{H}_1^*$-valued and absolutely continuous on $[0, T]$, $\partial_t u(t) \in L^2((0, T); \mathcal{H}_1^*)$, and $u(t)$ satisfies* (2.3).

(3) *$u(t)$ satisfies the following (in)equalities:*

$$\frac{1}{2}\|u(t)\|^2 + \int_0^t a(s, u(s), u(s))ds = \frac{1}{2}\|u_0\|^2 + \int_0^t (f(s), u(s))ds,$$

$$\|u(t)\|^2 + \delta \int_0^t \|u(s)\|_1^2 ds \le \|u_0\|^2 + \frac{1}{\delta}\int_0^t \|f(s)\|_{\mathcal{H}_1^*}^2 ds.$$

2.2. Heat equation. We take $\mathcal{H} = L^2((0,1))$ and $\mathcal{H}_1 = H_0^1((0,1))$, the Sobolev space of order 1 with 0 trace on the boundary $\partial\Omega = \{0,1\}$. For $u, v \in H_0^1((0,1))$, we pick

$$a(t,u,v) = k^2 \int_0^{s(t)} \partial_x u(x)\partial_x v(x)dx + \int_{s(t)}^1 \partial_x u(x)\partial_x v(x)dx.$$

Then the assumptions (2.1), (2.2) are satisfied, and the associated $A(t)$ is given by

$$D(A(t)) \ni u \Longleftrightarrow \begin{cases} u \in H_0^1((0,1)) \cap H^2((0,s(t))) \cap H^2((s(t),1)), \\ \partial_x u(s(t)+0) = k^2 \partial_x u(s(t)-0), \end{cases}$$

$$A(t)u = \begin{cases} -k^2\partial_x^2 u \quad \text{on} \quad (0,s(t)), \\ -\partial_x^2 u \quad \text{on} \quad (s(t),1). \end{cases} \tag{2.4}$$

In the following sections, we also use the notation $A(t)$ to denote the *formal* differential operator (2.4).

Remark 2.2. Theorem 2.1 shows that the solution $u(t,x;\lambda)$ to (1.1) is uniquely well-defined, and guarantees the existence of $I_{\mathrm{ind}}(T;\lambda)$.

3. Energy estimates. In the following, we use the notation $\dot{u} = \partial_t u,\ u' = \partial_x u$. Let us first prepare an elementary lemma.

LEMMA 3.1. *Let $0 < \delta < 1$ and let $I = [1-\delta, 1]$. Suppose that $u(x) \in C^1(I)$ satisfies $u(1) = 0$. Then for any nonnegative function $a(x) \in C(I)$, we have*

$$\liminf_{\varepsilon \to 0} \int_{1-\delta}^1 a(x)|u'(x)|^2 \frac{\varepsilon}{\left(\varepsilon + |u(x)|^2\right)^{3/2}}\, dx \geq a(1)|u'(1)|.$$

Proof. By replacing u by $-u$ if necessary, we need only consider the case $u'(1) < 0$. Shrinking I if necessary, we can assume that $u(x) > 0$ on $[1-\delta, 1)$ and adopt $u = u(x)$ as a new variable. Letting $c = u(1-\delta)$ and $\widetilde{a}(u) = a(x(u))$, we have

$$\int_{1-\delta}^1 a(x)\Big|\frac{du}{dx}\Big|^2 \frac{\varepsilon}{\left(\varepsilon + u^2\right)^{3/2}} dx = \int_0^c \widetilde{a}(u)\Big|\frac{dx}{du}\Big|^{-1} \frac{\varepsilon}{\left(\varepsilon + u^2\right)^{3/2}} du.$$

By the change of variable $u = \sqrt{\varepsilon}y$, this is equal to

$$\int_0^{c/\sqrt{\varepsilon}} \widetilde{a}(\sqrt{\varepsilon}y)\Big|\frac{dx}{du}(\sqrt{\varepsilon}y)\Big|^{-1} \frac{dy}{(1+y^2)^{3/2}},$$

which tends to

$$\widetilde{a}(0)\Big|\frac{dx}{du}(0)\Big|^{-1} \int_0^\infty \frac{dy}{(1+y^2)^{3/2}} = a(1)\Big|\frac{du}{dx}(1)\Big|,$$

as ε tends to 0. ∎

We put

$$I_-(t) = (0, s(t)), \quad I_+(t) = (s(t), 1),$$
$$\mathcal{D}_\pm = \{(t,x)\,;\ 0 < t < T,\ x \in I_\pm(t)\},$$

and also

$$\big[f\big]_{s(t)} = f(s(t)+0) - f(s(t)-0).$$

LEMMA 3.2. *Let $U = U(t,x) \in C([0,T]; L^2(\Omega)) \cap L^2((0,T); H_0^1(\Omega))$ be such that*

$$U(t)\big|_{I_\pm(t)} \in H^2(I_\pm(t)), \quad \partial_t(U(t)\big|_{I_\pm(t)}) \in L^2(I_\pm(t)), \quad 0 < t \le T.$$

Assume that U is a solution to the equation

$$\dot{U} + A(t)U = F \quad \text{in} \quad L^2(\mathcal{D}_\pm), \tag{3.1}$$

and satisfies

$$\begin{cases} U(t,s(t)+0) = U(t,s(t)-0) \quad \text{in} \quad L^2(0,T), \\ U(t,0) = U(t,1) = 0 \quad \text{in} \quad L^2(0,T), \end{cases} \tag{3.2}$$

where $F = F(t,x) \in L^2((0,T)\times\Omega)$. Let $H(t,x) \in C^2(\overline{\mathcal{D}_+}) \cap C^2(\overline{\mathcal{D}_-})$ be such that

$$H(t,x) \ge 0 \quad \text{on} \quad [0,T]\times\Omega, \tag{3.3}$$

$$\big[H\big]_{s(t)} = \big[\gamma\partial_x H\big]_{s(t)} = 0, \quad 0 \le t \le T. \tag{3.4}$$

Assume that there exists a real constant K such that

$$-\partial_t H + A(t)H \ge KH \quad \text{in} \quad (0,T)\times\Omega. \tag{3.5}$$

We put

$$E(U,H;t) = \int_0^1 |U(t,x)|H(t,x)\,dx.$$

Then we have the following inequality:

$$\begin{aligned} &E(U,H;T) + \int_0^T e^{K(t-T)}|U'(t,1)|H(t,1)dt \\ &\quad \le e^{-KT}E(U,H;0) + \iint_{(0,T)\times\Omega} e^{K(t-T)}|F(t,x)|H(t,x)\,dtdx \\ &\qquad + \int_0^T e^{K(t-T)}\big|\big[\gamma(t,x)U'(t,x)\big]_{s(t)}\big|\,H(t,s(t))dt. \end{aligned}$$

Remark 3.3. In Lemma 3.2, U corresponds to a solution u of (2.3) (in the sense of Theorem 2.1 with $\mathcal{H}_1 = H_0^1((0,1))$), with $F = f|_{\mathcal{D}_\pm} \in L^2(\mathcal{D}_\pm)$. But $f \in L^2((0,T)\times\Omega)$ if and only if $\partial_x U(t)|_{s(t)+0} = k^2\partial_x U(t)|_{s(t)-0}$, $0 < t \le T$.

Proof of Lemma 3.2. Let $\chi_\varepsilon(x) = x(\varepsilon + x^2)^{-1/2}$ for $\varepsilon > 0$, and note the following properties:

$$|\chi_\varepsilon| \le 1, \quad x\chi_\varepsilon(x) \to |x|, \quad \varepsilon \to 0, \tag{3.6}$$

$$\chi_\varepsilon'(x) > 0, \quad |x\chi_\varepsilon'(x)| \le 1/2, \quad x\chi_\varepsilon'(x) \to 0, \quad \varepsilon \to 0. \tag{3.7}$$

In fact, (3.7) follows from $\chi_\varepsilon'(x) = \varepsilon(\varepsilon+x^2)^{-3/2}$, and $|x\chi_\varepsilon'(x)| = |y(1+y^2)^{-3/2}| \le 1/2$, where $x = \sqrt{\varepsilon}y$.

We pick $\Omega' = \Omega \setminus \{s(t)\}$. Integration by parts using (3.2) and (3.4) yields

$$\begin{aligned}
&\int_{\Omega'} (\gamma U'')\chi_\varepsilon(U)H dx - \int_{\Omega'} U\chi_\varepsilon(U)(\gamma H'')dx \\
&\quad = -\left[\gamma U'(t,x)\right]_{s(t)}\chi_\varepsilon(U(t,s(t)))H(t,s(t)) - \int_0^1 \gamma |U'|^2 \chi'_\varepsilon(U)H dx \\
&\qquad + \int_0^1 U\chi'_\varepsilon(U)\gamma U' H' dx,
\end{aligned} \tag{3.8}$$

where $U = U(x) = U(t,x)$ and $H = H(t,x)$, and we have used $\chi_\varepsilon(U(0)) = \chi_\varepsilon(U(1)) = 0$, since $\chi_\varepsilon(0) = 0$. We pick

$$E_\varepsilon(t) = \int_{\Omega'} U(t,x)\chi_\varepsilon(U(x,t))H(t,x)dx.$$

Then we have

$$\begin{aligned}
\dot{E}_\varepsilon(t) = &\int_{\Omega'} (\gamma U'')\,\chi_\varepsilon(U)H dx - \int_{\Omega'} U\chi_\varepsilon(U)\,(\gamma H'')\,dx \\
&+ \int_0^1 F\chi_\varepsilon(U)H dx + \int_{\Omega'} U\chi_\varepsilon(U)\big(\dot{H} + \gamma H''\big)dx \\
&+ \int_{\Omega'} U\chi'_\varepsilon(U)\dot{U}H dx.
\end{aligned}$$

Plugging this with (3.8), we have

$$\begin{aligned}
\dot{E}_\varepsilon(t) = &- [\gamma U']_{s(t)}\chi_\varepsilon(U(t,s(t)))H(t,s(t)) - \int_0^1 \gamma|U'|^2\chi'_\varepsilon(U)H dx \\
&+ \int_0^1 U\chi'_\varepsilon(U)\gamma U' H'\, dx + \int_0^1 F\chi_\varepsilon(U)H\, dx \\
&+ \int_{\Omega'} U\chi_\varepsilon(U)\big(\dot{H} + \gamma H''\big)dx + \int_{\Omega'} U\chi'_\varepsilon(U)\dot{U}H\, dx.
\end{aligned}$$

By (3.7), the integrals containing the term $U\chi'_\varepsilon(U)$ vanish as $\varepsilon \to 0$. Using (3.5), we then have

$$\begin{aligned}
&\dot{E}_\varepsilon(t) + KE_\varepsilon(t) + \int_0^1 \gamma|U'|^2\chi'_\varepsilon(U)H\, dx \\
&\quad \le -[\gamma U']_{s(t)}\chi_\varepsilon(U(t,s(t)))H(t,s(t)) + \int_0^1 F\chi_\varepsilon(U)H\, dx + o(1).
\end{aligned}$$

We multiply this inequality by $e^{K(t-T)}$ and integrate on the time interval $[0,T]$ to obtain

$$\begin{aligned}
&E_\varepsilon(T) + \int_0^T e^{K(t-T)}dt \int_\Omega \gamma|U'|^2\chi'_\varepsilon(U)H\, dx \\
&\quad \le e^{-KT}E_\varepsilon(0) - \int_0^T e^{K(t-T)}[\gamma U'(t,x)]_{S(t)}\chi_\varepsilon(U(t,s(t)))H(t,s(t))dt \\
&\qquad + \int_0^T e^{K(t-T)}dt \int_\Omega F\chi_\varepsilon(U)H\, dx + o(1).
\end{aligned} \tag{3.9}$$

By Lemma 3.1, we have

$$H(t,1)|U'(t,1)| \le \liminf_{\varepsilon\to 0} \int_0^1 \gamma |U'(t,x)|^2 \chi'_\varepsilon(U(t,x)) H\,dx.$$

Taking the inferior limit in (3.9) and noting (3.6), we obtain the lemma. $\square$

4. Approximate solutions.

4.1. Ansatz. We shall construct an approximate solution of (1.1) in Case 1, $f_0 = 0$, for $\lambda > \sup\{|\dot{s}(t)|,\, 0 < t < T\}$. As can be easily imagined, the first approximation will be

$$v_0(t,x;\lambda) = \begin{cases} h_{\rm fw}(t,x;\lambda), & s(t) < x < 1, \\ e^{\lambda^2 t + \frac{\lambda}{k}(x - s(t))} e^{\lambda s(t)}, & 0 < x < s(t), \end{cases}$$

which satisfies $v_0(t, s(t)+0;\lambda) = v_0(t, s(t)-0;\lambda)$. Although the other conditions are not satisfied, this suggests the introduction of the factor $e^{\lambda(x-s(t))}$. Our ansatz is

$$\begin{aligned} v(t,x;\lambda) &= v_+(t,x;\lambda)\chi_+(t,x) + v_-(t,x;\lambda)\chi_-(t,x), \\ v_+(t,x;\lambda) &= h_{\rm fw}(t,x;\lambda) + h_{\rm fw}(t,s(t);\lambda)[a_+(t;\lambda)\exp\{(\lambda + \dot{s}(t))(x - s(t))\} \\ &\qquad + b_+(t;\lambda)\exp\{-(\lambda+\dot{s}(t))(x-s(t))\}], \\ v_-(t,x;\lambda) &= h_{\rm fw}(t,s(t);\lambda) a_-(t,x;\lambda) \exp\left\{\left(\frac{\lambda + \dot{s}(t)}{k}\right)(x - s(t))\right\}, \end{aligned} \tag{4.1}$$

where $\chi_+(t,x)$, $\chi_-(t,x)$ are the characteristic functions of the sets $\{x\,;\, s(t) < x\}$, $\{x\,;\, x < s(t)\}$, respectively. The functions a_+, b_+, and a_- are to be determined. Note that $a_+(t;\lambda)$ and $b_+(t;\lambda)$ do not depend on x. We use the abbreviations

$$a_+ = a_+(t;\lambda), \quad b_+ = b_+(t;\lambda), \quad a_-(x) = a_-(t,x;\lambda), \quad s = s(t), \quad \dot{s} = \dot{s}(t).$$

The intuition for this ansatz is as follows. The heat flow $h_{\rm fw}$ given on the boundary $x = 1$ is transmitted and reflected at the inner boundary $x = s(t)$, which gives rise to $a_- \exp\left(\frac{\lambda+\dot{s}}{k}(x-s)\right)$ and $b_+\exp(-(\lambda+\dot{s})(x-s))$. This latter is again reflected at the boundary $x = 1$ and produces $a_+\exp((\lambda+\dot{s})(x-s))$.

It must satisfy the following conditions:

$$\begin{cases} v_+(t,1;\lambda) = h_{\rm fw}(t,1;\lambda), \\ v_+(t,s(t);\lambda) = v_-(t,s(t);\lambda), \\ v'_+(t,s(t);\lambda) = k^2 v'_-(t,s(t);\lambda). \end{cases} \tag{4.2}$$

We can rewrite (4.2) as

$$\begin{cases} a_+ e^{(\lambda+\dot{s})(1-s)} + b_+ e^{-(\lambda+\dot{s})(1-s)} = 0, \\ 1 + a_+ + b_+ = a_-(s), \\ \lambda + (\lambda+\dot{s})(a_+ - b_+) = k^2 a'_-(s) + k(\lambda+\dot{s})a_-(s). \end{cases} \tag{4.3}$$

We pick

$$\varphi = \varphi(t,x;\lambda) = (\lambda + \dot{s}(t))(x - s(t)),$$

$$\varphi_1 = \varphi_1(t;\lambda) = (\lambda + \dot{s}(t))(1 - s(t)).$$

By a direct computation, we have for $x > s(t)$ that

$$\begin{aligned}\frac{\dot{v}_+ - v''_+}{h_{\mathrm{fw}}(t,s;\lambda)} &= [-2\lambda\dot{s}a_+ + \dot{a}_+ + \left(\ddot{s}(x-s) - 2\dot{s}^2\right)a_+]e^{\varphi} \\ &\quad + \left[\dot{b}_+ - \ddot{s}(x-s)b_+\right]e^{-\varphi},\end{aligned} \tag{4.4}$$

and for $x < s(t)$

$$\begin{aligned}\frac{\dot{v}_- - k^2 v''_-}{h_{\mathrm{fw}}(t,s;\lambda)} = \Bigg[&-\lambda\left(2ka'_- + \left(1+\frac{1}{k}\right)\dot{s}a_-\right) \\ &+ \left(\frac{\ddot{s}(x-s)}{k} - \left(1+\frac{1}{k}\right)\dot{s}^2\right)a_- + \dot{a_-} - 2k\dot{s}a'_- - k^2a''_-\Bigg]e^{\varphi/k}.\end{aligned} \tag{4.5}$$

4.2. Construction. By the first equation of (4.3), we have

$$a_+ = -b_+e^{-2\varphi_1}. \tag{4.6}$$

By the second equation of (4.3), we have

$$a_-(s) = 1 + b_+(1 - e^{-2\varphi_1}). \tag{4.7}$$

We take a_- to be the solution of the differential equation

$$a'_- + \frac{1}{2k}\left(1+\frac{1}{k}\right)\dot{s}a_- = 0,$$

satisfying (4.7), i.e.,

$$a_-(x) = e^{-\frac{1}{2k}(1+\frac{1}{k})\dot{s}(x-s)}\left(1 + b_+(1-e^{-2\varphi_1})\right).$$

Plugging them into the third equation of (4.3) and noting that $\sup\{s(t),\ 0 < t < T\} < 1$ by our assumption, we have

$$b_+ = \frac{1-k}{1+k} + O(e^{-2\varphi_1}), \tag{4.8}$$

$$a_+ = \frac{k-1}{k+1}e^{-2\varphi_1}\left(1 + O(e^{-2\varphi_1})\right), \tag{4.9}$$

$$a_-(x) = \frac{2}{1+k}e^{-\frac{1}{2k}(1+\frac{1}{k})\dot{s}(x-s)} + O(e^{-2\varphi_1}), \tag{4.10}$$

and these expansions can be differentiated term by term. By (4.4) and (4.6), we have

$$\frac{\dot{v}_+ - v''_+}{h_{\mathrm{fw}}(t,s;\lambda)} = \left(\ddot{s}(2-s-x)b_+ - \dot{b}_+\right)e^{\varphi-2\varphi_1} + \left(\dot{b}_+ - \ddot{s}(x-s)b_+\right)e^{-\varphi}. \tag{4.11}$$

In view of (4.11) and (4.5), we then have

$$\begin{aligned}|\dot{v}_+ - v''_+| &\le Ch_{\mathrm{fw}}(t,s;\lambda)e^{-\varphi}, \quad s < x < 1, \\ |\dot{v}_- - k^2v''_-| &\le Ch_{\mathrm{fw}}(t,s;\lambda)e^{\varphi/k}, \quad 0 < x < s,\end{aligned}$$

where the constant C is independent of $0 < t < T$ and $\lambda > \sup\{|\dot{s}(t)|,\ 0 < t < T\}$.

The above $v(t,x;\lambda)$ does not satisfy $v(t,0;\lambda)=0$. We modify it in the following way. Let

$$s_0 = \inf_{0<t<T} s(t).$$

By our assumption, $0 < s_0 < 1$. Pick $\chi(x) \in C^\infty(\mathbf{R})$ such that

$$\chi(x) = \begin{cases} 0, & x < s_0/4, \\ 1, & s_0/2 < x, \end{cases} \tag{4.12}$$

and then put

$$w(t,x;\lambda) = \chi(x) v(t,x;\lambda). \tag{4.13}$$

Lemma 4.1. *Let $w(t,x;\lambda)$ be defined by* (4.13). *Then w satisfies*

$$w(t,1;\lambda) = h_{\mathrm{fw}}(t,1;\lambda), \quad w(t,0;\lambda) = 0. \tag{4.14}$$

Moreover, we have

$$|\dot{w} + A(t)w| \leq C e^{\lambda^2 t + \lambda s(t)} \begin{cases} e^{-\varphi}, & s(t) < x < 1, \\ e^{\varphi/k} + e^{-\delta_0 \lambda}, & 0 < x < s(t), \end{cases} \tag{4.15}$$

where $C, \delta_0 > 0$ are constants independent of $t, \lambda > 0$.

Proof. Equation (4.14) follows directly from (4.13). Since

$$\dot{w} + A(t)w = \chi(x)(\dot{v} + A(t)v) - 2k^2 \chi'(x) v_-' - k^2 \chi''(x) v_-,$$

and $x < s_0/2$ on the support of $\chi'(x)$, we obtain (4.15). ∎

4.3. The function *H*. We next construct the function H in Lemma 3.2, with parameters $\lambda > \nu > 0$, where ν satisfies (1.3). The idea is the same as above; however, it must be C^1 at $(t, s(t))$ with the trade off that it merely satisfies the differential inequality (3.5). Letting $\chi_\pm(t,x)$ be the characteristic function of $\mathcal{D}_\pm$, we construct $H = H(t,x)$ in the following form:

$$\begin{aligned} H(t,x) &= \chi_+(t,x) e^{-\lambda^2 t} \left(e^{(\lambda-\nu)x} + b_1 e^{(\lambda-\nu)(2s(t)-x)} \right) \\ &\quad + \chi_-(t,x) b_2 e^{-\lambda^2 t + (\lambda-\nu)(\frac{x-s(t)}{k} + s(t))}. \end{aligned}$$

The condition (3.4) is satisfied by setting

$$b_1 = \frac{1-k}{1+k}, \quad b_2 = \frac{2}{1+k}. \tag{4.16}$$

For $s(t) < x$ we have

$$\frac{H(t,x)}{e^{-\lambda^2 t + (\lambda-\nu)(2s(t)-x)}} = e^{(\lambda-\nu)(2x-2s(t))} + \frac{1-k}{1+k} \geq \frac{2}{1+k} > 0.$$

This leads to

$$e^{-\lambda^2 t + (\lambda-\nu)(2s(t)-x)} \leq \frac{1+k}{2} H(t,x), \quad s(t) < x < 1. \tag{4.17}$$

By computing $f \equiv -\dot{H} - \gamma H''$ in $\mathcal{D}_\pm$, we see that

$$f(t,x) = (2\lambda\nu - \nu^2)H(t,x) - 2(\lambda-\nu)\dot{s}(t)b_1 e^{-\lambda^2 t + (\lambda-\nu)(2s(t)-x)}, \quad s(t) < x < 1, \tag{4.18}$$

$$f(t,x) = \left((2\lambda\nu - \nu^2) + \left(\frac{1}{k} - 1\right)(\lambda-\nu)\dot{s}(t)\right)H(t,x), \quad 0 < x < s(t). \tag{4.19}$$

Using (4.17), (4.18), and (4.19) we have

$$f \geq \left((2\lambda\nu - \nu^2) - (\lambda-\nu)|1-k|\,|\dot{s}|\right)H, \quad s(t) < x < 1, \tag{4.20}$$

$$f \geq \left((2\lambda\nu - \nu^2) - (\lambda-\nu)\left|\frac{1}{k} - 1\right|\,|\dot{s}|\right)H, \quad 0 < x < s(t). \tag{4.21}$$

Thanks to (1.3), we have

$$\nu \geq \sup_{0<t<T}\left(\sup\left(\left|\frac{1}{k} - 1\right|, |1-k|\right)|\dot{s}(t)|\right),$$

and so, for $0 < t < T$,

$$2\lambda\nu - \nu^2 \geq \lambda\nu + (\lambda-\nu)\max\left(|1-k|\,|\dot{s}(t)|, \left|\frac{1}{k} - 1\right||\dot{s}(t)|\right). \tag{4.22}$$

Thanks to (4.20), (4.21), and (4.22), we obtain $f \geq \lambda\nu H$ in $L^2(\mathcal{D}_\pm)$, and so, (3.5) is satisfied with $K = \lambda\nu$.

We shall also need the following upper bound for H:

$$H(t,x) \leq 2e^{-\lambda^2 t + \lambda \max(s(t),x)} \quad \text{in} \quad (0,T)\times\Omega. \tag{4.23}$$

4.4. Proof of Theorem 1.1. Using $w(t,x;\lambda)$ from (4.13), we put

$$U(t,x) = u(t,x;\lambda) - w(t,x;\lambda). \tag{4.24}$$

Then we have, using (4.8), (4.9), and (4.10), $|w(0,x;\lambda)| \leq Ce^{\lambda s(0)}$, hence

$$|U(0,x)| \leq |u_0(x)| + Ce^{\lambda s(0)},$$

which implies

$$|U(0,x)|H(0,x) \leq 2(|u_0(x)| + Ce^{\lambda})e^{\lambda}, \tag{4.25}$$

where we have used (4.23).

By the construction of the ansatz we have $[\gamma U']_{s(t)} = 0$ in $L^2(0,T)$. Letting $F(t,x) = \dot{U} + A(t)U$, and in view of (4.15), (4.23), we also have

$$|F(t,x)|H(t,x) \leq Ce^{2\lambda s(t)},$$

Thanks to (1.2) we have

$$h_{\mathrm{bw}}(t,1;\lambda) \leq C(\nu)H(t,1), \quad 0 < t < T,$$

and then, in view of Lemma 3.2,

$$\int_0^T e^{\lambda\nu(t-T)}|U'(t,1)|h_{\text{bw}}(t,1;\lambda)dt$$
$$\leq C\left(e^{\lambda(2-\nu T)}+\int_0^T e^{\lambda\nu(t-T)+2\lambda s(t)}dt\right).$$

Since $\nu > 2\sup_{0<t<T}|\dot{s}(t)|$, we can make the change of variable $y = t + 2s(t)/\nu$ to see that

$$\int_0^T e^{\lambda\nu t+2\lambda s(t)}dt = \int_{2s(0)/\nu}^{T+2s(T)/\nu} e^{\lambda\nu y}\frac{dy}{1+2\dot{s}(t)/\nu}$$
$$\leq C\int_{2s(0)/\nu}^{T+2s(T)/\nu} e^{\lambda\nu y}dy \leq C\frac{e^{\lambda\nu T+2\lambda s(T)}}{\lambda\nu}.$$

Since $\nu > 2/T$ (see (1.3)), we have $2 - \nu T - 2s(T) < 0$, and this yields

$$\int_0^T e^{\lambda\nu t}|U'(t,1)|h_{\text{bw}}(t,1;\lambda)dt \leq \frac{C}{\lambda\nu}e^{\lambda\nu T+2\lambda s(T)}$$

for λ sufficiently large. On the other hand, by using (4.1), (4.8), and (4.9), we have

$$h_{\text{bw}}(t,1;\lambda)\partial_x\big(w(t,x;\lambda)-h_{\text{fw}}(t,x;\lambda)\big)\Big|_{x=1}$$
$$\simeq \frac{2\lambda(k-1)}{k+1}e^{2\lambda s(t)-\dot{s}(t)(1-s(t))}.$$

By integration by parts, and using $\nu > 2|\dot{s}|_\infty$ again, we then have

$$\int_0^T e^{\lambda\nu t}h_{\text{bw}}(t,1;\lambda)\partial_x\big(w(t,x;\lambda)-h_{\text{fw}}(t,x;\lambda)\big)\Big|_{x=1}dt$$
$$\simeq \frac{2\lambda(k-1)}{k+1}\int_0^T e^{\lambda\nu t+2\lambda s(t)-\dot{s}(t)(1-s(t))}dt$$
$$= \frac{2(k-1)}{k+1}\left\{\left[\frac{1}{\nu+2\dot{s}(t)}e^{\lambda(\nu t+2s(t))}e^{-\dot{s}(t)(1-s(t))}\right]_0^T\right.$$
$$\left.-\int_0^T e^{\lambda(\nu t+2s(t))}\frac{d}{dt}\left(\frac{e^{-\dot{s}(t)(1-s(t))}}{\nu+2\dot{s}(t)}\right)dt\right\}$$
$$= \frac{2(k-1)}{k+1}\frac{1}{\nu+2\dot{s}(T)}e^{\lambda(\nu T+2s(T))}e^{-\dot{s}(T)(1-s(T))}+RT1+RT2.$$

Using $\nu T > 2(s(0)-s(T))$, we have

$$RT1 \approx e^{2\lambda s(0)} = o(e^{\lambda(\nu T+2s(T))}),$$
$$|RT2| \approx \left|\int_0^T e^{\lambda(\nu t+2s(t))}\frac{d}{dt}\left(\frac{e^{-\dot{s}(t)(1-s(t))}}{\nu+2\dot{s}(t)}\right)dt\right|$$
$$\leq C\int_0^T e^{\lambda(\nu t+2s(t))}dt = o(e^{\lambda(\nu T+2s(T))}).$$

This proves Theorem 1.1. ∎

5. Proof of Theorem 1.3. We set the following ansatz for (1.1) in Case 2, $f_1 = 0$:

$$\begin{aligned} w(t,x) := {} & \chi_+(t,x)e^{\lambda^2 t}\beta(\lambda;t)(e^{-\lambda(x-s(t))-\theta_+(t)(x-1)} \\ & - e^{\lambda(x+s(t)-2)+\theta_+(t)(x-1)}) \\ & + \chi_-(t,x)e^{\lambda^2 t}(e^{-\frac{\lambda}{k}x} - \chi(x)b_1 e^{\frac{\lambda}{k}(x-2s(t))+\theta_-(t)(x-s(t))}), \end{aligned}$$

where b_1 is defined by (4.16), χ by (4.12),

$$\begin{aligned} \theta_-(t) &= -\frac{\dot{s}(t)}{k^2}, \quad \theta_+(t) = \frac{k-1}{2k}\dot{s}(t), \\ \beta(\lambda;t) &= (1-b_1)e^{-\frac{\lambda}{k}s(t)}\left(e^{-\theta_+(t)(s(t)-1)} - e^{(2\lambda+\theta_+(t))(s(t)-1)}\right)^{-1} \\ &= \frac{2k}{1+k}e^{-\frac{\lambda}{k}s(t)+\theta_+(t)(s(t)-1)}(1+O(e^{-\lambda\delta})) \end{aligned}$$

for some $\delta > 0$. We see that the ansatz satisfies $w(t,0) = h_{\rm fw}(t,0;\lambda)$, $w(t,1) = 0$, and, thanks to the choice of β,

$$\begin{aligned} [w(t,\cdot)]_{s(t)} &= 0, \\ [\gamma w'(t,\cdot)]_{s(t)} &= O(e^{\lambda^2 t - \frac{\lambda}{k}s(t)}). \end{aligned} \tag{5.1}$$

Setting $f = \dot{w} - \gamma w'' \in L^2(\mathcal{D}_\pm)$, we have, in view of $\dot{\beta}(t) = \lambda\dot{s}(1-1/k)(1+O(1))\beta(t)$ and the choice of $\theta_\pm$,

$$|f(t,x)| \le Ce^{\lambda^2 t + \frac{\lambda}{k}(x-2s(t))} \quad \forall x \in (0,s(t)), \tag{5.2}$$

$$|f(t,x)| \le Ce^{\lambda^2 t + \lambda(-x+(1-1/k)s(t))} \quad \forall x \in (s(t),1). \tag{5.3}$$

From (5.1), (5.2), and (5.3) we have the following estimates:

$$\begin{aligned} \int_0^1 |f(t,x)|H(t,x)dx &\le Ce^{\lambda(1-\frac{1}{k})s(t)}, \\ |[\gamma w'(t,\cdot)]_{s(t)}|\, H(t,s(t)) &\le Ce^{\lambda(1-\frac{1}{k})s(t)}. \end{aligned}$$

Observing that $|w(0,x)|H(0,x) \le Ce^{\lambda}$ and defining U again by (4.24), the inequality (4.25) holds again. In view of Lemma 3.2, and thanks to (1.4), we then have

$$\begin{aligned} & \int_0^T e^{\lambda\nu(t-T)}|U'(t,1)|h_{\rm bw}(t,1;\lambda)dt \\ & \quad \le C\lambda e^{\lambda(3-\nu T)} + \frac{C}{\lambda}e^{(1-\frac{1}{k})\lambda s(T)}. \end{aligned}$$

Now, we end the proof by observing that

$$\int_0^T e^{\lambda\nu(t-T)}w'(t,1)h_{\rm bw}(t,1;\lambda)dt = -2\lambda\int_0^T \beta(t)e^{\lambda\nu(t-T)+\lambda s(t)}dt. \qquad \square$$

6. Proof of Corollary 1.4. Set $L_1(t) = (1-\frac{1}{k})s(t)$, $\xi(t) = \frac{1}{2}\dot{L}_1(t)L_1(t)$. The asymptotic behavior of $\widetilde{I_{\rm ind}}(\lambda;t)$, as $\lambda \to \infty$, for $t \in (0,T]$, shows that we can asymptotically determine $L_1(t)$, $0 < t \le T$, and so we can determine $\dot{L}_1(t)$, $\xi(t)$, and, for fixed $t \in (0,T]$, the value of

$$\lim_{\lambda\to\infty}\frac{\nu+\dot{L}_1(t)}{-4}e^{-\lambda(\nu t+L_1(t))+\frac{1}{2}\dot{L}_1(t)}\widetilde{I_{\text{ind}}}(\lambda;t)=\frac{k}{k+1}e^{\frac{k}{k-1}\xi}\equiv f(k),$$

with $\xi=\xi(t)$. The function $f(k)$ is defined for $k\in(0,1)\cup(1,\infty)$. Now, we regard ξ and F as parameters, and consider the following equation with respect to k:

$$f(k)=F,\quad F\in\mathbb{R}. \tag{6.1}$$

We assume that $F\in\ \mathrm{Im}\, f$, since, in fact,

$$F=\lim_{\lambda\to\infty}\frac{\nu+\dot{L}_1(t)}{-4}e^{-\lambda(\nu t+L_1(t))+\frac{1}{2}\dot{L}_1(t)}\widetilde{I_{\text{ind}}}(\lambda;t).$$

We have $f>0$, $\lim f(k)=e^{\xi}$ as $k\to+\infty$, and

$$f'(k)=\frac{1}{(1+k)^2(1-k)^2}p(k)e^{\frac{k}{k-1}\xi},\ \ p(k):=(1-\xi)k^2-(2+\xi)k+1.$$

Let k_3, k_4 be the roots of p. The discriminant of p is $\Delta(k)=\xi(\xi+8)$. We thus have to analyze the following cases for ξ.

(i) $\xi=1$: In this case $p(k)=1-3k$. If $0<F<(4\sqrt{e})^{-1}=f(1/3)$, then (6.1) admits two solutions: $k_1\in(0,1/3)$, $k_2\in(1/3,1)$. If $(4\sqrt{e})^{-1}<F\le e$, then (6.1) does not have a solution (this case is forbidden). If $e<F$, then (6.1) admits only one solution $k_1>1$.

(ii) $\xi=0$: Then f is increasing from 0 to 1 as $k>0$, and so (6.1) admits a unique solution.

(iii) $0<\xi<1$: The roots of p are positive: $0<k_3<k_4$. Since $p(0)=1$, $p(1)=-2\xi<0$, $p(+\infty)=+\infty$, we have $0<k_3<1<k_4$. So f is increasing in $(0,k_3)$ (respectively, in $(k_4,+\infty)$) from 0 to $f(k_3)$ (respectively, from $f(k_4)$ to e^{ξ}), and decreasing in $(k_3,1)$ (respectively, in $(1,k_4)$) from $f(k_3)$ to 0 (respectively, from $+\infty$ to $f(k_4)$). Observing that $\frac{k_3\xi}{k_3-1}<0<\frac{k_4\xi}{k_4-1}$ and that $\frac{k_3}{k_3+1}<\frac{k_4}{k_4+1}$, we then have $f(k_3)<f(k_4)$, hence $F\notin(f(k_3),f(k_4))$. If $F<f(k_3)$ or if $F>f(k_4)$, then (6.1) admits two roots. If $F=f(k_i)$, then $k=k_i$, $i=3,4$.

(iv) $\xi>1$: In this case, p admits one positive root $k_3<1$. So f is increasing in $(0,k_3)$ from 0 to $f(k_3)$, and decreasing in $(k_3,1)$ (respectively, in $(1,+\infty)$) from $f(k_3)$ to 0 (respectively, from $+\infty$ to e^{ξ}). Observing that $f(k_3)<e^{\xi}$, we have $F\notin(f(k_3),e^{\xi})$. So (6.1) admits two roots if $F<f(k_3)$, and only one if $F=f(k_3)$ or $F>e^{\xi}$.

(vi) $\xi<0$: If $\xi>-8$, then $\Delta(k)<0$ and $p>0$. If $\xi\le-8$, then the roots of p are negative and so $p(k)>0$ for $k>0$. Thus, in any case, $f(k)$ is increasing from 0 to ∞ when k varies over $[0,1)$ and $f(k)$ is increasing from 0 to e^{ξ} when k varies over $(1,\infty)$. Hence if $F\ge e^{\xi}$, then (6.1) admits a unique solution $k_1\in(0,1)$, and if $0<F<e^{\xi}$, then (6.1) admits two solutions k,k' such that $0<k<1<k'$.

7. Numerical example. Consider four different conductivities,

$$\gamma_i(t,x)=10 \text{ when } x<s_i(t),\quad i=1,2,3,4,$$

where

$$\begin{aligned}s_1(t)&=0.95,\\ s_2(t)&=0.8,\\ s_3(t)&=0.85+t,\\ s_4(t)&=0.85+0.1\cos(t/0.1\cdot4\pi).\end{aligned}$$

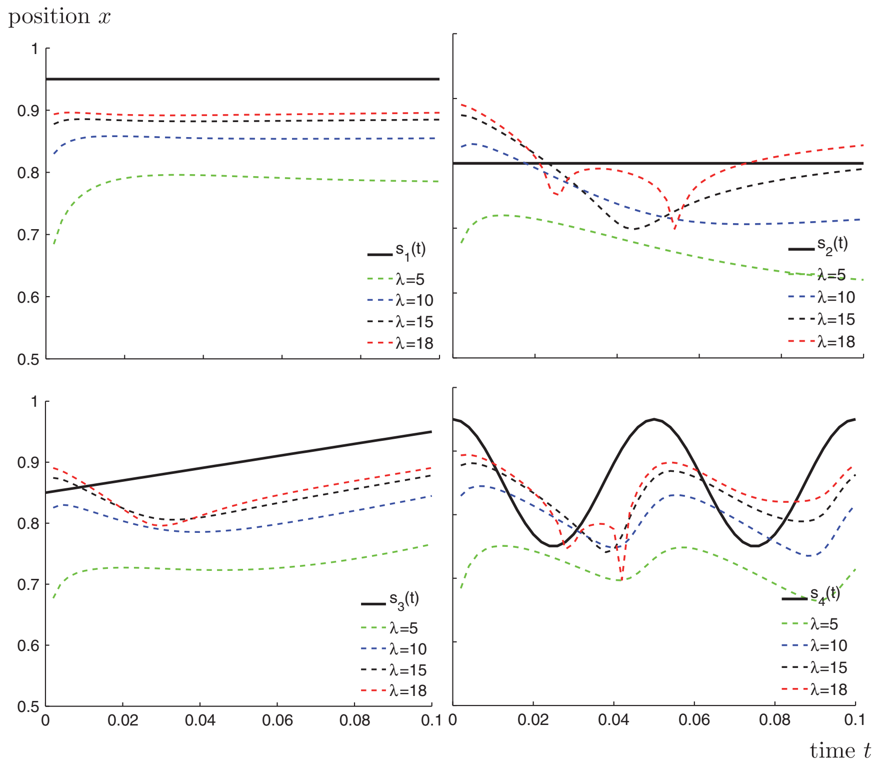

FIG. 7.1. *Test conductivites and their reconstructions; the vertical axis is the variable x, the horizontal axis is the time t. The solid black line is the boundary $s_i(t)$, dashed lines are the reconstructions with $\lambda = 5, 10, 15, 18$. Top left: $s_1(t) = 0.95$. Top right: $s_2(t) = 0.8$. Bottom left: $s_3(t) = 0.85 + t$. Bottom right: $s_4(t) = 0.85 + 0.1\cos(t/0.1 \cdot 4\pi)$.*

The indicator function can be written as

$$I(T;\lambda) = e^{\lambda} \int_0^T e^{(\lambda\nu-\lambda^2)T} \partial_x u(T, x; \lambda)|_{x=1} dT - \frac{1}{\nu} e^{2\lambda}(e^{\lambda\nu T} - 1). \tag{7.1}$$

The behavior of the indicator function can be written as

$$\frac{\log(I(T;\lambda))}{\lambda} \underset{\lambda\to\infty}{\to} \nu T + 2s(T).$$

Using a large fixed λ we get the approximative reconstruction equation

$$s(T) \approx \frac{1}{2}\left[\frac{\log(I(T;\lambda))}{\lambda} - \nu T\right], \tag{7.2}$$

which is the more accurate the larger λ is.

Let $T = 0.002, \dots, 0.1$ be 50 discrete points and $\lambda = 5, 10, 15, 18$, also choose a fixed $\nu = 9$. We use the software MATLAB and finite element method to compute the solution $u(t, x; \lambda)$ for each pair T, λ with 400 nodes of x and 200 nodes of t. Subsequently we compute the indicator (7.1), where the differentiation ∂_x is done via MATLAB's function "pdeval.m." Finally, we compute the reconstruction $s(T)$ using (7.2). The test conductivities and the reconstructions are pictured in Figure 7.1; the solid black line is the boundary $s_i(t)$, dashed lines are the reconstructions with $\lambda = 5, 10, 15, 18$. Note that in the pictures only the part $[0.5, 1]$ of $\Omega = [0, 1]$ is shown.

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