From Lie Algebras to Lie Groups within Synthetic Differential Geometry:Weil Sprouts of Lie's Third Fundamental Theorem

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Abstract

Weil prolongations of a Lie group are naturally Lie groups. It is not known in the theory of infinite-dimensional Lie groups how to construct a Lie group with a given Lie algebra as its Lie algebra or whether there exists such a Lie group at all. We will show in this paper how to construct some Weil prolongations of this mythical Lie group from a given Lie algebra. We will do so within our favorite framework of synthetic differential geometry.

1 Introduction

In the theory of finite-dimensional Lie groups, Lie's third fundamental theorem is usually established via the Levi decomposition. The Levi-Mal'čev theorem asserts the existence of a Levi decomposition for any finite-dimensional Lie algebra. That is to say, any finite-dimensional Lie algebra is the semidirect product of a solvable Lie algebra $\mathfrak{m}$ and a semisimple Lie algebra $\mathfrak{q}$. Since it is easy to establish Lie's third fundamental theorem in both the solvable case and the semisimple one, the desired Lie group is obtained as the semidirect product of the established Lie groups with their respective Lie algebras $\mathfrak{m}$ and $\mathfrak{q}$, for which the reader is referred, e.g., to §3.15 of [8].

This route to Lie's third fundamental theorem does not seem susceptible of any meaningful infinite-dimensional generalization, and, as far as we know, Lie's third fundamental theorem is not available in the theory of infinite-dimensional Lie groups at present. In this sense "a Lie group with a given Lie algebra as its Lie algebra" beyond the finite-dimensional realm is *mythical*. The principal objective in this paper is to show that some Weil prolongations of this mythical Lie group are *real* to our great surprise, which can be regarded as *Weil sprouts* of Lie's third fundamental theorem in a sense. We will do so within our favorite framework of synthetic differential geometry, for which the reader is referred to [4]. Our considerations shall be restricted to lower-dimensional cases because of computational complexity.

The construction of this paper goes as follows. After giving some preliminaries in the coming section, we will study some Weil prolongations of a Lie group, which are again Lie groups, and their Lie algebras in §3. In the succeeding section we will establish a slight generalization of the Baker-Campbell-Hausdorff formula discussed in the concluding two sections of our previous paper [5], which enables us to endow some Weil prolongations of a given Lie algebra with Lie group structures. The concluding two sections are devoted to showing that the derived structures really make some Weil prolongations of a given Lie algebra Lie groups with the desired Lie algebras as their Lie algebras.

2 Preliminaries

We assume the reader to be familiar with the first three chapters of [4] as well as the first five section of [5].

The following theorem is due to Kock [2].

Theorem 1 *Let $\mathbb{E}$ be a Euclidean $\mathbb{R}$-module. Given $f \in \mathbb{E}^{D_n}$, there exist unique $X_0, X_1, ..., X_n \in \mathbb{E}$ such that*

$$f(d) = X_0 + dX_1 + ... + d^n X_n$$

for any $d \in D_n$

A variant of the above theorem is

Theorem 2 *Let $\mathbb{E}$ be a Euclidean $\mathbb{R}$-module. Given $f \in \mathbb{E}^{D_n \times D}$, there exist unique $X_0, X_1, ..., X_n, Y_0, Y_1, ..., Y_n \in \mathbb{E}$ such that*

$$f(d,e) = (X_0 + eY_0) + d(X_1 + eY_1) + ... + d^n(X_n + eY_n)$$

for any $d \in D_n$ and any $e \in D$.

Proof. The exponential law ensures that

$$\mathbb{E}^{D_n \times D} = \left(\mathbb{E}^{D_n}\right)^D$$

Thanks to Theorem 1, we are allowed to think that

$$\mathbb{E}^{D_n} = \mathbb{E}^{n+1}$$

Since $\mathbb{E}$ is Euclidean, $\mathbb{E}^{n+1}$ is also Euclidean. Therefore we have the desired result. ■

We recall that

Definition 3 *A Lie group is a group which is microlinear as a space.*

Similarly, we should be precise in saying a "Lie algebra"

Definition 4 *A Lie algebra is a Euclidean $\mathbb{R}$-module endowed with a bilinear binary operation $[,]$ (called a Lie bracket) abiding by the antisymmetric law and the Jacobi identity which is microlinear as a mere space.*

The unit element of a group is usually denoted by 1, while the unit element of the underlying abelian group of an $\mathbb{R}$-module is usually denoted by 0. Given a Lie group G and a space U, G^U is naturally a Lie group. Similarly, given a Lie algebra $\mathfrak{g}$ and a space U, $\mathfrak{g}^U$ is naturally a Lie algebra.

Notation 5 *We make it a rule that abbreviated Lie brackets should be inserted from the right to the left. Therefore*

$$[X, Y, Z]$$

stands for

$$[X, [Y, Z]]$$

by way of example.

3 Weil Prolongations of Lie Groups and their Lie Algebras

As is the case in the equivalence of the three distinct viewpoints of vector fields (cf. §3.2.1 of [4]), the exponential law plays a significant role in synthetic differential geometry, which is also the case in the considerations to follow. Since each Weil algebra has its counterpart in an adequate model of synthetic differential geometry (called an *infinitesimal object*), Weil prolongations are merely exponentiations by infinitesimal objects in synthetic differential geometry. It is not difficult to externalize Weil prolongations, for which the reader is referred, e.g., to Chapter VIII of [1] or §31 of [3]. Weil prolongations play a significant role in axiomatic differential geometry under construction, for which the reader is referred to [6] and [7].

Let us begin by fixing our notation.

Notation 6 *We denote by*

$$\mathbf{Lie}$$

the functor assigning to each Lie group G its Lie algebra $\mathbf{Lie}(G)$ and to each homomorphism $\varphi : G \to G'$ of Lie groups to its induced homomorphism $\mathbf{Lie}(\varphi) : \mathbf{Lie}(G) \to \mathbf{Lie}(G')$ of their Lie algebras. We will often write $\mathfrak{g}$ for $\mathbf{Lie}(G)$, as is usual.

Notation 7 *Given a Lie group G, we denote by*

$$\left(G^{D_n}\right)_1$$

the subgroup

$$\left\{f \in G^{D_n} \mid f(0) = 1\right\}$$

of the Lie group G^{D_n}.

Notation 8 *Given a Lie algebra $\mathfrak{g}$, we denote by*

$$\left(\mathfrak{g}^{D_n}\right)_0$$

the subalgebra

$$\left\{f \in \mathfrak{g}^{D_n} \mid f(0) = 0\right\}$$

of the Lie algebra $\mathfrak{g}^{D_n}$.

Theorem 9 *Given a Lie group G with its Lie algebra $\mathfrak{g}$, we have*

$$\mathbf{Lie}\left(G^{D_n}\right) = \mathfrak{g}^{D_n}$$

Proof. This follows mainly from the familiar exponential law

$$\left(G^{D_n}\right)^D = G^{D_n \times D} = \left(G^D\right)^{D_n}$$

which naturally gives rise, by restriction, to

$$\begin{aligned}
&\left(\left(G^{D_n}\right)^D\right)_1 \\
&= \left\{f \in G^{D_n \times D} \mid f(d, 0) = 1 \ (\forall d \in D_n)\right\} \\
&= \mathfrak{g}^{D_n}
\end{aligned}$$

■

Corollary 10

$$\mathbf{Lie}\left(\left(G^{D_n}\right)_1\right) = \left(\mathfrak{g}^{D_n}\right)_0$$

Proof. This follows mainly from the familiar exponential law

$$\left(G^{D_n}\right)^D = G^{D_n \times D} = \left(G^D\right)^{D_n}$$

which naturally gives rise, by restriction, to

$$\begin{aligned}
&\left(\left(\left(G^{D_n}\right)_1\right)^D\right)_{1^{D_n}} \\
&= \left\{f \in G^{D_n \times D} \mid f(d, 0) = 1 \ (\forall d \in D_n) \ \text{and} \ f(0, e) = 1 \ (\forall e \in D)\right\} \\
&= \left(\mathfrak{g}^{D_n}\right)_0
\end{aligned}$$

■

Corollary 11 *Given $\sum_{i=0}^n X_i d^i, \sum_{j=0}^n Y_j d^j \in \mathfrak{g}^{D_n} = \mathbf{Lie}\left(G^{D_n}\right)$ with $d \in D_n$, we can easily compute their Lie bracket as follows:*

$$\left[\sum_{i=0}^n X_i d^i, \sum_{j=0}^n Y_j d^j\right] = \sum_{k=0}^n \left(\sum_{i+j=k} [X_i, Y_j]\right) d^k$$

4 Generalized Baker-Campbell-Hausdorff Formulas

In this section, G is assumed to be a regular Lie group with its Lie algebra $\mathfrak{g}$. It should be obvious that

Theorem 12 *With $d_1 \in D$ and $X_1, Y_1 \in \mathfrak{g}$, we have*

$$\begin{aligned} &\exp\, d_1 X_1 . \exp\, d_1 Y_1 \\ &= \exp\, d_1 (X_1 + Y_1) \end{aligned}$$

Theorem 13 *With $d_1, d_2 \in D$ and $X_1, X_2, Y_1, Y_2 \in \mathfrak{g}$, we have*

$$\begin{aligned} &\exp\, (d_1 + d_2) X_1 + \frac{1}{2} (d_1 + d_2)^2 X_2 . \exp\, (d_1 + d_2) Y_1 + \frac{1}{2} (d_1 + d_2)^2 Y_2 \\ &= \exp\, (d_1 + d_2)(X_1 + Y_1) + \frac{1}{2} (d_1 + d_2)^2 (X_2 + Y_2 + [X_1, Y_1]) \end{aligned}$$

Proof. We have

$$\begin{aligned} &\exp\, (d_1 + d_2) X_1 + \frac{1}{2} (d_1 + d_2)^2 X_2 . \exp\, (d_1 + d_2) Y_1 + \frac{1}{2} (d_1 + d_2)^2 Y_2 \\ &= \exp\, d_1 X_1 + d_2 (X_1 + d_1 X_2) . \exp\, d_1 Y_1 + d_2 (Y_1 + d_1 Y_2) \\ &= \exp\, d_2 (X_1 + d_1 X_2) . \exp\, d_1 X_1 . \exp\, d_1 Y_1 . \exp\, d_2 (Y_1 + d_1 Y_2) \\ &\text{)By Lemmas 14 and 15(} \\ &= \exp\, d_2 (X_1 + d_1 X_2) . \exp\, d_1 (X_1 + Y_1) . \exp\, d_2 (Y_1 + d_1 Y_2) \\ &\text{)By Theorem 12(} \\ &= \exp\, -\frac{1}{2} d_1 d_2 [Y_1, X_1] . \exp\, d_1 (X_1 + Y_1) + d_2 (X_1 + d_1 X_2) . \exp\, d_2 (Y_1 + d_1 Y_2) \\ &\text{)By Lemma 16(} \\ &= \exp\, -\frac{1}{2} d_1 d_2 [Y_1, X_1] . \exp\, d_1 (X_1 + Y_1) + d_2 (X_1 + d_1 X_2) + d_2 (Y_1 + d_1 Y_2) . \\ &\exp\, \frac{1}{2} d_1 d_2 [X_1, Y_1] \\ &\text{)By Lemma 17(} \end{aligned}$$

so that we have the desired formula. ■

Lemma 14

$$\begin{aligned} &\exp\, d_1 X_1 + d_2 (X_1 + d_1 X_2) \\ &= \exp\, d_2 (X_1 + d_1 X_2) . \exp\, d_1 X_1 \end{aligned}$$

Proof. Letting $*_1$ denote

$$d_1 X_1$$

and letting $*_2$ denote

$$X_1 + d_1 X_2,$$

we have

$$\begin{aligned}&\exp *_1 + d_2 *_2 \\ &= \exp d_2 *_2 . \exp *_1\end{aligned} \tag{1}$$

by right logarithmic derivation. Therefore the desired formula follows. ■

Lemma 15

$$\begin{aligned}&\exp d_1 Y_1 + d_2 (Y_1 + d_1 Y_2) \\ &= \exp d_1 Y_1 . \exp d_2 (Y_1 + d_1 Y_2)\end{aligned}$$

Proof. Letting $*_1$ denote

$$d_1 Y_1$$

and letting $*_2$ denote

$$Y_1 + d_1 Y_2,$$

we have

$$\begin{aligned}&\exp *_1 + d_2 *_2 \\ &= \exp *_1 . \exp d_2 *_2\end{aligned} \tag{2}$$

by left logarithmic derivation. Therefore the desired formula follows. ■

Lemma 16 *We have*

$$\begin{aligned}&\exp d_1 (X_1 + Y_1) + d_2 (X_1 + d_1 X_2) \\ &= \exp d_2 \left(X_1 + d_1 X_2 + \frac{1}{2} d_1 [Y_1, X_1] \right) . \exp d_1 (X_1 + Y_1)\end{aligned}$$

Proof. Letting $*_1$ denote

$$d_1 (X_1 + Y_1)$$

and letting $*_2$ denote

$$X_1 + d_1 X_2,$$

we have

$$\begin{aligned}&\exp *_1 + d_2 *_2 \\ &= \exp d_2 \left\{ *_2 + \frac{1}{2} [*_1, *_2] \right\} . \exp *_1\end{aligned} \tag{3}$$

by right logarithmic derivation. By the way, we have the following:

$$[*_1, *_2] = d_1 [Y_1, X_1]$$

Therefore the desired formula follows. ■

Lemma 17 *We have*

$$\exp\, d_1\left(X_1+Y_1\right)+d_2\left(X_1+d_1X_2\right)+d_2\left(Y_1+d_1Y_2\right)$$
$$=\exp\, d_1\left(X_1+Y_1\right)+d_2\left(X_1+d_1X_2\right).\exp\, d_2\left(Y_1+d_1Y_2-\frac{1}{2}d_1\left[X_1,Y_1\right]\right)$$

Proof. Letting $*_1$ denote

$$d_1\left(X_1+Y_1\right)+d_2\left(X_1+d_1X_2\right)$$

and letting $*_2$ denote

$$Y_1+d_1Y_2,$$

we have

$$\exp *_1+d_2*_2$$
$$=\exp *_1.\exp\, d_2\left\{*_2-\frac{1}{2}\left[*_1,*_2\right]\right\}. \tag{4}$$

by left logarithmic derivation. By the way, we have the following:

$$\left[*_1,*_2\right]=d_1\left[X_1,Y_1\right]$$

Therefore the desired formula follows. ■

Theorem 18 *With $d_1,d_2,d_3\in D$ and $X_1,X_2,X_3,Y_1,Y_2,Y_3\in\mathfrak{g}$, we have*

$$\exp\left(d_1+d_2+d_3\right)X_1+\frac{1}{2}\left(d_1+d_2+d_3\right)^2X_2+\frac{1}{6}\left(d_1+d_2+d_3\right)^3X_3.$$
$$\exp\left(d_1+d_2+d_3\right)Y_1+\frac{1}{2}\left(d_1+d_2+d_3\right)^2Y_2+\frac{1}{6}\left(d_1+d_2+d_3\right)^3Y_3$$
$$=\exp\left(d_1+d_2+d_3\right)\left(X_1+Y_1\right)+\frac{1}{2}\left(d_1+d_2+d_3\right)^2\left(X_2+Y_2+\left[X_1,Y_1\right]\right)+$$
$$\frac{1}{6}\left(d_1+d_2+d_3\right)^3\left\{\left(X_3+Y_3\right)+\frac{3}{2}\left(\left[X_1,Y_2\right]+\left[X_2,Y_1\right]\right)+\frac{1}{2}\left[X_1-Y_1,X_1,Y_1\right]\right\}$$

Proof. We have

$$\exp\,(d_1+d_2+d_3)\,X_1+\frac{1}{2}\,(d_1+d_2+d_3)^2\,X_2+\frac{1}{6}\,(d_1+d_2+d_3)^3\,X_3.$$

$$\exp\,(d_1+d_2+d_3)\,Y_1+\frac{1}{2}\,(d_1+d_2+d_3)^2\,Y_2+\frac{1}{6}\,(d_1+d_2+d_3)^3\,Y_3$$

$$=\exp\left\{(d_1+d_2)\,X_1+\frac{1}{2}\,(d_1+d_2)^2\,X_2\right\}+d_3\left\{X_1+(d_1+d_2)\,X_2+\frac{1}{2}\,(d_1+d_2)^2\,X_3\right\}.$$

$$\exp\left\{(d_1+d_2)\,Y_1+\frac{1}{2}\,(d_1+d_2)^2\,Y_2\right\}+d_3\left\{Y_1+(d_1+d_2)\,Y_2+\frac{1}{2}\,(d_1+d_2)^2\,Y_3\right\}$$

$$=\exp\,d_3\left\{X_1+(d_1+d_2)\,X_2+(d_1+d_2)^2\left(\frac{1}{2}X_3+\frac{1}{4}\,[X_1,X_2]\right)\right\}.$$

$$\exp\,(d_1+d_2)\,X_1+\frac{1}{2}\,(d_1+d_2)^2\,X_2.\exp\,(d_1+d_2)\,Y_1+\frac{1}{2}\,(d_1+d_2)^2\,Y_2.$$

$$\exp\,d_3\left\{Y_1+(d_1+d_2)\,Y_2+\frac{1}{2}\,(d_1+d_2)^2\left(Y_3-\frac{1}{2}\,[Y_1,Y_2]\right)\right\}$$

)By Lemmas 19 and 20(

$$=\exp\,d_3\left\{X_1+(d_1+d_2)\,X_2+\frac{1}{2}\,(d_1+d_2)^2\,X_3+\frac{1}{4}\,(d_1+d_2)^2\,[X_1,X_2]\right\}.$$

$$\exp\,(d_1+d_2)\,(X_1+Y_1)+\frac{1}{2}\,(d_1+d_2)^2\,(X_2+Y_2+[X_1,Y_1])\,.$$

$$\exp\,d_3\left\{Y_1+(d_1+d_2)\,Y_2+\frac{1}{2}\,(d_1+d_2)^2\,Y_3-\frac{1}{4}\,(d_1+d_2)^2\,[Y_1,Y_2]\right\}$$

)By Theorem 13(

$$=\exp\,-d_3\left\{\begin{array}{c}\frac{1}{2}\,(d_1+d_2)\,[Y_1,X_1]+\\ \frac{1}{2}\,(d_1+d_2)^2\left(\begin{array}{c}[X_1+Y_1,X_2]+\\ \frac{1}{2}\,[X_2+Y_2+[X_1,Y_1]\,,X_1]+\\ \frac{1}{3}\,[X_1+Y_1,Y_1,X_1]\end{array}\right)\end{array}\right\}.$$

$$\exp\,(d_1+d_2)\,(X_1+Y_1)+\frac{1}{2}\,(d_1+d_2)^2\,(X_2+Y_2+[X_1,Y_1])+$$

$$d_3\left\{X_1+(d_1+d_2)\,X_2+\frac{1}{2}\,(d_1+d_2)^2\left(X_3+\frac{1}{2}\,[X_1,X_2]\right)\right\}.$$

$$\exp\,d_3\left\{Y_1+(d_1+d_2)\,Y_2+\frac{1}{2}\,(d_1+d_2)^2\left(Y_3-\frac{1}{2}\,[Y_1,Y_2]\right)\right\}$$

)By Lemma 21(

We keep on.

$$= \exp \frac{1}{4}(d_1+d_2)^2 d_3 [X_1+Y_1, Y_1, X_1] .$$

$$\exp (d_1+d_2)(X_1+Y_1) + \frac{1}{2}(d_1+d_2)^2 (X_2+Y_2+[X_1,Y_1]) +$$

$$d_3 \left\{ X_1 + (d_1+d_2) X_2 + \frac{1}{2}(d_1+d_2)^2 \left(X_3 + \frac{1}{2}[X_1,X_2] \right) \right\} -$$

$$d_3 \left\{ \begin{array}{c} \frac{1}{2}(d_1+d_2)[Y_1,X_1] + \\ \frac{1}{2}(d_1+d_2)^2 \left(\begin{array}{c} [X_1+Y_1,X_2] + \frac{1}{2}[X_2+Y_2+[X_1,Y_1],X_1] + \\ \frac{1}{3}[X_1+Y_1,Y_1,X_1] \end{array} \right) \end{array} \right\} .$$

$$\exp d_3 \left\{ Y_1 + (d_1+d_2) Y_2 + \frac{1}{2}(d_1+d_2)^2 \left(Y_3 - \frac{1}{2}[Y_1,Y_2] \right) \right\}$$

)By Lemma 22(

$$= \exp \frac{1}{4}(d_1+d_2)^2 d_3 [X_1+Y_1, Y_1, X_1] .$$

$$\exp (d_1+d_2)(X_1+Y_1) + \frac{1}{2}(d_1+d_2)^2 (X_2+Y_2+[X_1,Y_1]) +$$

$$d_3 \left\{ X_1 + (d_1+d_2) X_2 + \frac{1}{2}(d_1+d_2)^2 \left(X_3 + \frac{1}{2}[X_1,X_2] \right) \right\} -$$

$$d_3 \left\{ \begin{array}{c} \frac{1}{2}(d_1+d_2)[Y_1,X_1] + \\ \frac{1}{2}(d_1+d_2)^2 \left(\begin{array}{c} [X_1+Y_1,X_2] + \frac{1}{2}[X_2+Y_2+[X_1,Y_1],X_1] + \\ \frac{1}{3}[X_1+Y_1,Y_1,X_1] \end{array} \right) \end{array} \right\} +$$

$$d_3 \left\{ Y_1 + (d_1+d_2) Y_2 + \frac{1}{2}(d_1+d_2)^2 \left(Y_3 - \frac{1}{2}[Y_1,Y_2] \right) \right\} .$$

$$\exp d_3 \left\{ \begin{array}{c} \frac{1}{2}(d_1+d_2)(d_1+d_2)[X_1,Y_1] \\ \frac{1}{2}(d_1+d_2)^2 \left(\begin{array}{c} [X_1+Y_1,Y_2] + \frac{1}{2}[X_2+Y_2+[X_1,Y_1],Y_1] - \\ \frac{1}{3}[X_1+Y_1,Y_1,X_1] \end{array} \right) \end{array} \right\}$$

)By Lemma 23(

We keep on again.

$$
\begin{aligned}
&= \exp \frac{1}{4}(d_1+d_2)^2 d_3 [X_1+Y_1, Y_1, X_1] . \\
&\exp\ (d_1+d_2)(X_1+Y_1) + \frac{1}{2}(d_1+d_2)^2 (X_2+Y_2+[X_1,Y_1]) + \\
&d_3 \left\{ X_1 + (d_1+d_2) X_2 + \frac{1}{2}(d_1+d_2)^2 \left(X_3 + \frac{1}{2}[X_1,X_2] \right) \right\} - \\
&d_3 \left\{ \begin{array}{c} \frac{1}{2}(d_1+d_2)[Y_1,X_1] + \\ \frac{1}{2}(d_1+d_2)^2 \left(\begin{array}{c} [X_1+Y_1,X_2] + \frac{1}{2}[X_2+Y_2+[X_1,Y_1],X_1] + \\ \frac{1}{3}[X_1+Y_1,Y_1,X_1] \end{array} \right) \end{array} \right\} + \\
&d_3 \left\{ Y_1 + (d_1+d_2) Y_2 + \frac{1}{2}(d_1+d_2)^2 \left(Y_3 - \frac{1}{2}[Y_1,Y_2] \right) \right\} + \\
&d_3 \left\{ \begin{array}{c} \frac{1}{2}(d_1+d_2)[X_1,Y_1] + \\ \frac{1}{2}(d_1+d_2)^2 \left(\begin{array}{c} [X_1+Y_1,Y_2] + \frac{1}{2}[X_2+Y_2+[X_1,Y_1],Y_1] - \\ \frac{1}{3}[X_1+Y_1,Y_1,X_1] \end{array} \right) \end{array} \right\} . \\
&\exp \frac{1}{4}(d_1+d_2)^2 d_3 [X_1+Y_1, X_1, Y_1]
\end{aligned}
$$

Therefore the desired formula follows at once. ■

Lemma 19

$$
\begin{aligned}
&\exp \left\{ (d_1+d_2) X_1 + \frac{1}{2}(d_1+d_2)^2 X_2 \right\} + d_3 \left\{ X_1 + (d_1+d_2) X_2 + \frac{1}{2}(d_1+d_2)^2 X_3 \right\} \\
&= \exp\ d_3 \left\{ X_1 + (d_1+d_2) X_2 + \frac{1}{2}(d_1+d_2)^2 \left(X_3 + \frac{1}{2}[X_1,X_2] \right) \right\} . \\
&\exp \left\{ (d_1+d_2) X_1 + \frac{1}{2}(d_1+d_2)^2 X_2 \right\}
\end{aligned}
$$

Proof. Letting $*_1$ denote

$$(d_1+d_2) X_1 + \frac{1}{2}(d_1+d_2)^2 X_2$$

and letting $*_2$ denote

$$X_1 + (d_1+d_2) X_2 + \frac{1}{2}(d_1+d_2)^2 X_3,$$

we have

$$
\begin{aligned}
&\exp *_1 + d_3 *_2 \\
&= \exp\ d_3 \left(*_2 + \frac{1}{2}[*_1, *_2] \right) . \exp *_1
\end{aligned}
\tag{5}
$$

by right logarithmic derivation. By the way, we have the following:

$$\begin{aligned}[*_1, *_2] &= (d_1 + d_2)^2 \left([X_1, X_2] + \frac{1}{2} [X_2, X_1] \right) \\ &= \frac{1}{2} (d_1 + d_2)^2 [X_1, X_2]\end{aligned}$$

Therefore the desired formula follows. ■

Lemma 20

$$\begin{aligned}&\exp \left\{ (d_1 + d_2) Y_1 + \frac{1}{2} (d_1 + d_2)^2 Y_2 \right\} + d_3 \left\{ Y_1 + (d_1 + d_2) Y_2 + \frac{1}{2} (d_1 + d_2)^2 Y_3 \right\} \\ &= \exp\ (d_1 + d_2) Y_1 + \frac{1}{2} (d_1 + d_2)^2 Y_2. \\ &\exp\ d_3 \left\{ Y_1 + (d_1 + d_2) Y_2 + \frac{1}{2} (d_1 + d_2)^2 \left(Y_3 - \frac{1}{2} [Y_1, Y_2] \right) \right\}\end{aligned}$$

Proof. Letting $*_1$ denote

$$(d_1 + d_2) Y_1 + \frac{1}{2} (d_1 + d_2)^2 Y_2$$

and letting $*_2$ denote

$$Y_1 + (d_1 + d_2) Y_2 + \frac{1}{2} (d_1 + d_2)^2 Y_3,$$

we have

$$\begin{aligned}&\exp *_1 + d_3 *_2 \\ &= \exp *_1 . \exp\ d_3 \left(*_2 - \frac{1}{2} [*_1, *_2] \right)\end{aligned} \tag{6}$$

by left logarithmic derivation. By the way, we have the following:

$$\begin{aligned}[*_1, *_2] &= (d_1 + d_2)^2 \left([Y_1, Y_2] + \frac{1}{2} [Y_2, Y_1] \right) \\ &= \frac{1}{2} (d_1 + d_2)^2 [Y_1, Y_2]\end{aligned}$$

Therefore the desired formula follows. ■

Lemma 21

$$\exp\left\{(d_1+d_2)(X_1+Y_1)+\frac{1}{2}(d_1+d_2)^2(X_2+Y_2+[X_1,Y_1])\right\}+$$
$$d_3\left\{X_1+(d_1+d_2)X_2+\frac{1}{2}(d_1+d_2)^2\left(X_3+\frac{1}{2}[X_1,X_2]\right)\right\}$$
$$=\exp d_3\left\{\begin{array}{c} X_1+(d_1+d_2)X_2+\frac{1}{2}(d_1+d_2)^2\left(X_3+\frac{1}{2}[X_1,X_2]\right)+\\ \frac{1}{2}(d_1+d_2)[Y_1,X_1]+\\ (d_1+d_2)^2\left(\begin{array}{c}\frac{1}{2}[X_1+Y_1,X_2]+\frac{1}{4}[X_2+Y_2+[X_1,Y_1],X_1]+\\ \frac{1}{6}[X_1+Y_1,Y_1,X_1]\end{array}\right)\end{array}\right\}.$$
$$\exp\,(d_1+d_2)(X_1+Y_1)+\frac{1}{2}(d_1+d_2)^2(X_2+Y_2+[X_1,Y_1])$$

Proof. Letting $*_1$ denote

$$(d_1+d_2)(X_1+Y_1)+\frac{1}{2}(d_1+d_2)^2(X_2+Y_2+[X_1,Y_1])$$

and letting $*_2$ denote

$$X_1+(d_1+d_2)X_2+\frac{1}{2}(d_1+d_2)^2\left(X_3+\frac{1}{2}[X_1,X_2]\right),$$

we have

$$\begin{aligned}&\exp *_1+d_3*_2\\ &=\exp d_3\left(*_2+\frac{1}{2}[*_1,*_2]+\frac{1}{6}[*_1,*_1,*_2]\right).\exp *_1\end{aligned} \tag{7}$$

by right logarithmic derivation. By the way, we have the following:

$$[*_1,*_2]=(d_1+d_2)[Y_1,X_1]+(d_1+d_2)^2\left([X_1+Y_1,X_2]+\frac{1}{2}[X_2+Y_2+[X_1,Y_1],X_1]\right)$$

$$[*_1,*_1,*_2]=(d_1+d_2)^2[X_1+Y_1,Y_1,X_1]$$

$$\frac{1}{2}[*_1,*_2]+\frac{1}{6}[*_1,*_1,*_2]$$
$$=\frac{1}{2}(d_1+d_2)[Y_1,X_1]+$$
$$(d_1+d_2)^2\left(\frac{1}{2}[X_1+Y_1,X_2]+\frac{1}{4}[X_2+Y_2+[X_1,Y_1],X_1]+\frac{1}{6}[X_1+Y_1,Y_1,X_1]\right)$$

Therefore the desired formula follows. ■

Lemma 22

$$\exp\, (d_1+d_2)(X_1+Y_1)+\frac{1}{2}(d_1+d_2)^2(X_2+Y_2+[X_1,Y_1])+$$
$$d_3\left\{X_1+(d_1+d_2)X_2+\frac{1}{2}(d_1+d_2)^2\left(X_3+\frac{1}{2}[X_1,X_2]\right)\right\}-$$
$$d_3\left\{\begin{array}{c}\frac{1}{2}(d_1+d_2)[Y_1,X_1]+\\ \frac{1}{2}(d_1+d_2)^2\left(\begin{array}{c}[X_1+Y_1,X_2]+\frac{1}{2}[X_2+Y_2+[X_1,Y_1],X_1]+\\ \frac{1}{3}[X_1+Y_1,Y_1,X_1]\end{array}\right)\end{array}\right\}$$
$$=\exp\, -d_3\left\{\begin{array}{c}\frac{1}{2}(d_1+d_2)[Y_1,X_1]+\\ (d_1+d_2)^2\left(\begin{array}{c}\frac{1}{2}[X_1+Y_1,X_2]+\frac{1}{4}[X_2+Y_2+[X_1,Y_1],X_1]+\\ \frac{1}{6}[X_1+Y_1,Y_1,X_1]\end{array}\right)+\\ \frac{1}{4}(d_1+d_2)^2[X_1+Y_1,Y_1,X_1]\end{array}\right\}.$$
$$\exp\, (d_1+d_2)(X_1+Y_1)+\frac{1}{2}(d_1+d_2)^2(X_2+Y_2+[X_1,Y_1])+$$
$$d_3\left\{X_1+(d_1+d_2)X_2+\frac{1}{2}(d_1+d_2)^2\left(X_3+\frac{1}{2}[X_1,X_2]\right)\right\}.$$

Proof. Letting $*_1$ denote

$$(d_1+d_2)(X_1+Y_1)+\frac{1}{2}(d_1+d_2)^2(X_2+Y_2+[X_1,Y_1])+$$
$$d_3\left\{X_1+(d_1+d_2)X_2+\frac{1}{2}(d_1+d_2)^2\left(X_3+\frac{1}{2}[X_1,X_2]\right)\right\}-$$
$$d_3\left\{\begin{array}{c}\frac{1}{2}(d_1+d_2)[Y_1,X_1]+\\ \frac{1}{2}(d_1+d_2)^2\left(\begin{array}{c}[X_1+Y_1,X_2]+\frac{1}{2}[X_2+Y_2+[X_1,Y_1],X_1]+\\ \frac{1}{3}[X_1+Y_1,Y_1,X_1]\end{array}\right)\end{array}\right\}$$

and letting $*_2$ denote

$$-\frac{1}{2}(d_1+d_2)[Y_1,X_1]-$$
$$\frac{1}{2}(d_1+d_2)^2\left(\begin{array}{c}[X_1+Y_1,X_2]+\frac{1}{2}[X_2+Y_2+[X_1,Y_1],X_1]+\\ \frac{1}{3}[X_1+Y_1,Y_1,X_1]\end{array}\right)$$

we have (5). By the way, we have the following:

$$[*_1,*_2]=-\frac{1}{2}(d_1+d_2)^2[X_1+Y_1,Y_1,X_1]$$

Therefore the desired formula follows at once. ■

Lemma 23

$$\exp\left\{(d_1+d_2)(X_1+Y_1)+\frac{1}{2}(d_1+d_2)^2(X_2+Y_2+[X_1,Y_1])\right\}+$$
$$d_3\left\{X_1+(d_1+d_2)X_2+\frac{1}{2}(d_1+d_2)^2\left(X_3+\frac{1}{2}[X_1,X_2]\right)\right\}-$$
$$d_3\left\{\begin{array}{c}\frac{1}{2}(d_1+d_2)[Y_1,X_1]+\\ \frac{1}{2}(d_1+d_2)^2\left(\begin{array}{c}[X_1+Y_1,X_2]+\frac{1}{2}[X_2+Y_2+[X_1,Y_1],X_1]+\\ \frac{1}{3}[X_1+Y_1,Y_1,X_1]\end{array}\right)\end{array}\right\}+$$
$$d_3\left\{Y_1+(d_1+d_2)Y_2+\frac{1}{2}(d_1+d_2)^2\left(Y_3-\frac{1}{2}[Y_1,Y_2]\right)\right\}$$
$$=\exp\left\{(d_1+d_2)(X_1+Y_1)+\frac{1}{2}(d_1+d_2)^2(X_2+Y_2+[X_1,Y_1])\right\}+$$
$$d_3\left\{X_1+(d_1+d_2)X_2+\frac{1}{2}(d_1+d_2)^2\left(X_3+\frac{1}{2}[X_1,X_2]\right)\right\}-$$
$$d_3\left\{\begin{array}{c}\frac{1}{2}(d_1+d_2)[X_1,Y_1]\\ \frac{1}{2}(d_1+d_2)^2\left(\begin{array}{c}[X_1+Y_1,Y_2]+\frac{1}{2}[X_2+Y_2+[X_1,Y_1],Y_1]-\\ \frac{1}{3}[X_1+Y_1,Y_1,X_1]\end{array}\right)\end{array}\right\}$$

Proof. Letting $*_1$ denote

$$(d_1+d_2)(X_1+Y_1)+\frac{1}{2}(d_1+d_2)^2(X_2+Y_2+[X_1,Y_1])+$$
$$d_3\left\{X_1+(d_1+d_2)X_2+\frac{1}{2}(d_1+d_2)^2\left(X_3+\frac{1}{2}[X_1,X_2]\right)\right\}-$$
$$d_3\left\{\begin{array}{c}\frac{1}{2}(d_1+d_2)[Y_1,X_1]+\\ \frac{1}{2}(d_1+d_2)^2\left(\begin{array}{c}[X_1+Y_1,X_2]+\frac{1}{2}[X_2+Y_2+[X_1,Y_1],X_1]+\\ \frac{1}{3}[X_1+Y_1,Y_1,X_1]\end{array}\right)\end{array}\right\}$$

and letting $*_2$ denote

$$Y_1+(d_1+d_2)Y_2+\frac{1}{2}(d_1+d_2)^2\left(Y_3-\frac{1}{2}[Y_1,Y_2]\right),$$

we have

$$\begin{aligned}&\exp *_1+d_3*_2\\ &=\exp *_1.\exp d_3\left(*_2-\frac{1}{2}[*_1,*_2]+\frac{1}{6}[*_1,*_1,*_2]\right)\end{aligned}\tag{8}$$

by left logarithmic derivation. By the way, we have the following:

$$[*_1,*_2]=(d_1+d_2)[X_1,Y_1]+(d_1+d_2)^2\left([X_1+Y_1,Y_2]+\frac{1}{2}[X_2+Y_2+[X_1,Y_1],Y_1]\right)$$

$$[*_1,*_1,*_2]=(d_1+d_2)^2[X_1+Y_1,X_1,Y_1]$$

$$-\frac{1}{2}[*_1,*_2]+\frac{1}{6}[*_1,*_1,*_2]$$
$$=-\frac{1}{2}(d_1+d_2)[X_1,Y_1]-$$
$$\frac{1}{2}(d_1+d_2)^2\left([X_1+Y_1,Y_2]+\frac{1}{2}[X_2+Y_2+[X_1,Y_1],Y_1]-\frac{1}{3}[X_1+Y_1,Y_1,X_1]\right)$$

Therefore the desired formula follows. ■

Lemma 24

$$\exp\ (d_1+d_2)(X_1+Y_1)+\frac{1}{2}(d_1+d_2)^2(X_2+Y_2+[X_1,Y_1])+$$
$$d_3\left\{X_1+(d_1+d_2)X_2+(d_1+d_2)^2\left(\frac{1}{2}X_3+\frac{1}{4}[X_1,X_2]\right)\right\}-$$
$$d_3\left\{\begin{array}{c}\frac{1}{2}(d_1+d_2)[Y_1,X_1]+\\ \frac{1}{2}(d_1+d_2)^2\left(\begin{array}{c}[X_1+Y_1,X_2]+\frac{1}{2}[X_2+Y_2+[X_1,Y_1],X_1]+\\ \frac{1}{3}[X_1+Y_1,Y_1,X_1]\end{array}\right)\end{array}\right\}+$$
$$d_3\left\{Y_1+(d_1+d_2)Y_2+\frac{1}{2}(d_1+d_2)^2\left(Y_3-\frac{1}{2}[Y_1,Y_2]\right)\right\}+$$
$$d_3\left\{\begin{array}{c}\frac{1}{2}(d_1+d_2)[X_1,Y_1]+\\ \frac{1}{2}(d_1+d_2)^2\left(\begin{array}{c}[X_1+Y_1,Y_2]+\frac{1}{2}[X_2+Y_2+[X_1,Y_1],Y_1]-\\ \frac{1}{3}[X_1+Y_1,Y_1,X_1]\end{array}\right)\end{array}\right\}$$
$$=\exp\ (d_1+d_2)(X_1+Y_1)+\frac{1}{2}(d_1+d_2)^2(X_2+Y_2+[X_1,Y_1])+$$
$$d_3\left\{X_1+(d_1+d_2)X_2+\frac{1}{2}(d_1+d_2)^2\left(X_3+\frac{1}{2}[X_1,X_2]\right)\right\}-$$
$$d_3\left\{\begin{array}{c}\frac{1}{2}(d_1+d_2)[Y_1,X_1]+\\ \frac{1}{2}(d_1+d_2)^2\left(\begin{array}{c}[X_1+Y_1,X_2]+\frac{1}{2}[X_2+Y_2+[X_1,Y_1],X_1]+\\ \frac{1}{3}[X_1+Y_1,Y_1,X_1]\end{array}\right)\end{array}\right\}+$$
$$d_3\left\{Y_1+(d_1+d_2)Y_2+\frac{1}{2}(d_1+d_2)^2\left(Y_3-\frac{1}{2}[Y_1,Y_2]\right)\right\}.$$
$$\exp\ d_3\left\{\begin{array}{c}\frac{1}{2}(d_1+d_2)[X_1,Y_1]+\\ \frac{1}{2}(d_1+d_2)^2\left(\begin{array}{c}[X_1+Y_1,Y_2]+\frac{1}{2}[X_2+Y_2+[X_1,Y_1],Y_1]-\\ \frac{1}{3}[X_1+Y_1,Y_1,X_1]-\\ \frac{1}{2}[X_1+Y_1,X_1,Y_1]\end{array}\right)\end{array}\right\}$$

Proof. Letting $*_1$ denote

$$(d_1+d_2)(X_1+Y_1)+\frac{1}{2}(d_1+d_2)^2(X_2+Y_2+[X_1,Y_1])+$$
$$d_3\left\{X_1+(d_1+d_2)X_2+\frac{1}{2}(d_1+d_2)^2\left(X_3+\frac{1}{2}[X_1,X_2]\right)\right\}-$$
$$d_3\left\{\begin{array}{c}\frac{1}{2}(d_1+d_2)[Y_1,X_1]+\\ \frac{1}{2}(d_1+d_2)^2\left(\begin{array}{c}[X_1+Y_1,X_2]+\frac{1}{2}[X_2+Y_2+[X_1,Y_1],X_1]+\\ \frac{1}{3}[X_1+Y_1,Y_1,X_1]\end{array}\right)\end{array}\right\}+$$
$$d_3\left\{Y_1+(d_1+d_2)Y_2+\frac{1}{2}(d_1+d_2)^2\left(Y_3-\frac{1}{2}[Y_1,Y_2]\right)\right\}$$

and letting $*_2$ denote

$$\frac{1}{2}(d_1+d_2)[X_1,Y_1]+$$
$$\frac{1}{2}(d_1+d_2)^2\left([X_1+Y_1,Y_2]+\frac{1}{2}[X_2+Y_2+[X_1,Y_1],Y_1]-\frac{1}{3}[X_1+Y_1,Y_1,X_1]\right)$$

we have (6). By the way, we have the following:

$$[*_1,*_2]=\frac{1}{2}(d_1+d_2)^2[X_1+Y_1,X_1,Y_1]$$

Therefore the desired formula follows at once. ■

5 Associativity

From now on, $\mathfrak{g}$ shall be an arbitrary Lie algebra not necessarily coming from a Lie group as its Lie algebra. The principal objective in the rest of this paper is to show that the spaces $\left(\mathfrak{g}^{D_n}\right)_0$ $(n=1,2,3)$ are naturally endowed with Lie group structures, which can be regarded as the Weil prolongations of a mythical (i.e., not necessarily existing) Lie group whose Lie algebra is supposed to be $\mathfrak{g}$. This section aims to demonstrate that the spaces $\left(\mathfrak{g}^{D_n}\right)_0$ $(n=1,2,3)$ are naturally endowed with *associative* binary operations. First of all, let us define binary operations on them.

Definition 25 *Inspired by Theorems 12, 13 and 18, we will define a binary operation on* $\left(\mathfrak{g}^{D_n}\right)_0$ $(n=1,2,3)$ *as follows:*

1. *Given* $dX_1,dY_1\in\left(\mathfrak{g}^{D_1}\right)_0$, *we define*

$$dX_1.dY_1$$

to be

$$d(X_1+Y_1)$$

2. *Given* $dX_1 + \frac{1}{2}d^2X_2, dY_1 + \frac{1}{2}d^2Y_2 \in \left(\mathfrak{g}^{D_2}\right)_0$, *we define*

$$dX_1 + \frac{1}{2}d^2X_2.dY_1 + \frac{1}{2}d^2Y_2$$

to be

$$d\left(X_1 + Y_1\right) + \frac{1}{2}d^2\left(X_2 + Y_2 + [X_1, Y_1]\right)$$

3. *Given* $dX_1 + \frac{1}{2}d^2X_2 + \frac{1}{6}d^3X_3, dY_1 + \frac{1}{2}d^2Y_2 + \frac{1}{6}d^3Y_3 \in \left(\mathfrak{g}^{D_3}\right)_0$, *we define*

$$dX_1 + \frac{1}{2}d^2X_2 + \frac{1}{6}d^3X_3.dY_1 + \frac{1}{2}d^2Y_2 + \frac{1}{6}d^3Y_3$$

to be

$$\begin{aligned}&d\left(X_1 + Y_1\right) + \frac{1}{2}d^2\left(X_2 + Y_2 + [X_1, Y_1]\right) + \\ &\frac{1}{6}d^3\left\{\left(X_3 + Y_3\right) + \frac{3}{2}\left([X_1, Y_2] + [X_2, Y_1]\right) + \frac{1}{2}\left[X_1 - Y_1, X_1, Y_1\right]\right\}\end{aligned}$$

The principal objective in this section is to show that the above binary operations are all associative. It should be obvious that

Theorem 26

$$\left(dX_1.dY_1\right).dZ_1 = dX_1.\left(dY_1.dZ_1\right)$$

Theorem 27

$$\begin{aligned}&\left(dX_1 + \frac{1}{2}d^2X_2.dY_1 + \frac{1}{2}d^2Y_2\right).dZ_1 + \frac{1}{2}d^2Z_2 \\ &= dX_1 + \frac{1}{2}d^2X_2.\left(dY_1 + \frac{1}{2}d^2Y_2.dZ_1 + \frac{1}{2}d^2Z_2\right)\end{aligned}$$

Proof. We have

$$\begin{aligned}&\left(dX_1 + \frac{1}{2}d^2X_2.dY_1 + \frac{1}{2}d^2Y_2\right).dZ_1 + \frac{1}{2}d^2Z_2 \\ &= d\left(X_1 + Y_1\right) + \frac{1}{2}d^2\left(X_2 + Y_2 + [X_1, Y_1]\right).dZ_1 + \frac{1}{2}d^2Z_2 \\ &= d\left(X_1 + Y_1 + Z_1\right) + \frac{1}{2}d^2\left(X_2 + Y_2 + [X_1, Y_1] + Z_2 + [X_1 + Y_1, Z_1]\right) \\ &= d\left(X_1 + Y_1 + Z_1\right) + \frac{1}{2}d^2\left(X_2 + Y_2 + Z_2 + [X_1, Y_1] + [X_1, Z_1] + [Y_1, Z_1]\right)\end{aligned}$$

on the one hand, while we have

$$
\begin{aligned}
& dX_1 + \frac{1}{2}d^2X_2.\left(dY_1 + \frac{1}{2}d^2Y_2.dZ_1 + \frac{1}{2}d^2Z_2\right) \\
& = dX_1 + \frac{1}{2}d^2X_2.d\left(Y_1 + Z_1\right) + \frac{1}{2}d^2\left(Y_2 + Z_2 + [Y_1, Z_1]\right) \\
& = d\left(X_1 + Y_1 + Z_1\right) + \frac{1}{2}d^2\left(X_2 + Y_2 + Z_2 + [Y_1, Z_1] + [X_1, Y_1 + Z_1]\right) \\
& = d\left(X_1 + Y_1 + Z_1\right) + \frac{1}{2}d^2\left(X_2 + Y_2 + Z_2 + [X_1, Y_1] + [X_1, Z_1] + [Y_1, Z_1]\right)
\end{aligned}
$$

on the other. ■

Theorem 28

$$
\begin{aligned}
& \left(dX_1 + \frac{1}{2}d^2X_2 + \frac{1}{6}d^3X_3.dY_1 + \frac{1}{2}d^2Y_2 + \frac{1}{6}d^3Y_3\right).dZ_1 + \frac{1}{2}d^2Z_2 + \frac{1}{6}d^3Z_3 \\
& = dX_1 + \frac{1}{2}d^2X_2 + \frac{1}{6}d^3X_3.\left(dY_1 + \frac{1}{2}d^2Y_2 + \frac{1}{6}d^3Y_3.dZ_1 + \frac{1}{2}d^2Z_2 + \frac{1}{6}d^3Z_3\right)
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
& \left(dX_1 + \frac{1}{2}d^2X_2 + \frac{1}{6}d^3X_3.dY_1 + \frac{1}{2}d^2Y_2 + \frac{1}{6}d^3Y_3\right).dZ_1 + \frac{1}{2}d^2Z_2 + \frac{1}{6}d^3Z_3 \\
& = d\left(X_1 + Y_1\right) + \frac{1}{2}d^2\left(X_2 + Y_2 + [X_1, Y_1]\right) + \\
& \frac{1}{6}d^3\left\{\left(X_3 + Y_3\right) + \frac{3}{2}\left([X_1, Y_2] + [X_2, Y_1]\right) + \frac{1}{2}\left[X_1 - Y_1, X_1, Y_1\right]\right\}. \\
& dZ_1 + \frac{1}{2}d^2Z_2 + \frac{1}{6}d^3Z_3 \\
& = d\left(X_1 + Y_1 + Z_1\right) + \frac{1}{2}d^2\left(X_2 + Y_2 + [X_1, Y_1] + Z_2 + [X_1 + Y_1, Z_1]\right) + \\
& \frac{1}{6}d^3\left\{\begin{array}{c} \left(X_3 + Y_3\right) + \frac{3}{2}\left([X_1, Y_2] + [X_2, Y_1]\right) + \frac{1}{2}\left[X_1 - Y_1, X_1, Y_1\right] + Z_3 + \\ \frac{3}{2}\left([X_1 + Y_1, Z_2] + \left[X_2 + Y_2 + [X_1, Y_1], Z_1\right]\right) + \\ \frac{1}{2}\left[X_1 + Y_1 - Z_1, [X_1 + Y_1, Z_1]\right] \end{array}\right\} \\
& = d\left(X_1 + Y_1 + Z_1\right) + \frac{1}{2}d^2\left(X_2 + Y_2 + Z_2 + [X_1, Y_1] + [X_1, Z_1] + [Y_1, Z_1]\right) + \\
& \frac{1}{6}d^3\left(\begin{array}{c} X_3 + Y_3 + Z_3 + \\ \frac{3}{2}\left([X_1, Y_2] + [X_1, Z_2] + [Y_1, Z_2] + [X_2, Y_1] + [X_2, Z_1] + [Y_2, Z_1]\right) + \\ \frac{1}{2}\left(\begin{array}{c} [X_1, X_1, Y_1] + [Y_1, Y_1, X_1] + \\ [X_1, X_1, Z_1] + [Z_1, Z_1, X_1] + \\ [Y_1, Y_1, Z_1] + [Z_1, Z_1, Y_1] \end{array}\right) + \\ \frac{3}{2}\left[[X_1, Y_1], Z_1\right] + \frac{1}{2}\left([X_1, Y_1, Z_1] + [Y_1, X_1, Z_1]\right) \end{array}\right)
\end{aligned}
$$

on the one hand, while we have

$$
\begin{aligned}
& dX_1+\frac{1}{2}d^2X_2+\frac{1}{6}d^3X_3.\left(dY_1+\frac{1}{2}d^2Y_2+\frac{1}{6}d^3Y_3.dZ_1+\frac{1}{2}d^2Z_2+\frac{1}{6}d^3Z_3\right) \\
& =dX_1+\frac{1}{2}d^2X_2+\frac{1}{6}d^3X_3. \\
& d\left(Y_1+Z_1\right)+\frac{1}{2}d^2\left(Y_2+Z_2+[Y_1,Z_1]\right)+ \\
& \frac{1}{6}d^3\left\{\left(Y_3+Z_3\right)+\frac{3}{2}\left([Y_1,Z_2]+[Y_2,Z_1]\right)+\frac{1}{2}\left[Y_1-Z_1,Y_1,Z_1\right]\right\} \\
& =d\left(X_1+Y_1+Z_1\right)+\frac{1}{2}d^2\left(X_2+Y_2+Z_2+[Y_1,Z_1]+[X_1,Y_1+Z_1]\right)+ \\
& \frac{1}{6}d^3\left\{\begin{array}{c} X_3+\left(Y_3+Z_3\right)+\frac{3}{2}\left([Y_1,Z_2]+[Y_2,Z_1]\right)+\frac{1}{2}\left[Y_1-Z_1,Y_1,Z_1\right]+ \\ \frac{3}{2}\left([X_1,Y_2+Z_2+[Y_1,Z_1]]+[X_2,Y_1+Z_1]\right)+ \\ \frac{1}{2}\left[X_1-\left(Y_1+Z_1\right),X_1,Y_1+Z_1\right] \end{array}\right\} \\
& =d\left(X_1+Y_1+Z_1\right)+\frac{1}{2}d^2\left(X_2+Y_2+Z_2+[X_1,Y_1]+[X_1,Z_1]+[Y_1,Z_1]\right)+ \\
& \frac{1}{6}d^3\left\{\begin{array}{c} X_3+Y_3+Z_3+ \\ \frac{3}{2}\left([X_1,Y_2]+[X_1,Z_2]+[Y_1,Z_2]+[X_2,Y_1]+[X_2,Z_1]+[Y_2,Z_1]\right)+ \\ \frac{1}{2}\left(\begin{array}{c} [X_1,X_1,Y_1]+[Y_1,Y_1,X_1]+ \\ [X_1,X_1,Z_1]+[Z_1,Z_1,X_1]+ \\ [Y_1,Y_1,Z_1]+[Z_1,Z_1,Y_1] \end{array}\right)+ \\ \frac{3}{2}\left[X_1,Y_1,Z_1\right]-\frac{1}{2}\left([Y_1,X_1,Z_1]+[Z_1,X_1,Y_1]\right) \end{array}\right\}
\end{aligned}
$$

on the other hand. Therefore we are well done by the following lemma. ■

Lemma 29 *We have*

$$
\begin{aligned}
& \frac{3}{2}\left[[X_1,Y_1],Z_1\right]+\frac{1}{2}\left([X_1,Y_1,Z_1]+[Y_1,X_1,Z_1]\right) \\
& =\frac{3}{2}\left[X_1,Y_1,Z_1\right]-\frac{1}{2}\left([Y_1,X_1,Z_1]+[Z_1,X_1,Y_1]\right)
\end{aligned}
$$

Proof. As is expected, this follows easily from the Jacobi identity. We have

$$\left\{\frac{3}{2}\left[\left[X_1,Y_1\right],Z_1\right]+\frac{1}{2}\left(\left[X_1,Y_1,Z_1\right]+\left[Y_1,X_1,Z_1\right]\right)\right\}-$$
$$\left\{\frac{3}{2}\left[X_1,Y_1,Z_1\right]-\frac{1}{2}\left(\left[Y_1,X_1,Z_1\right]+\left[Z_1,X_1,Y_1\right]\right)\right\}$$
$$=\frac{3}{2}\left(\left[\left[X_1,Y_1\right],Z_1\right]-\left[X_1,Y_1,Z_1\right]\right)+$$
$$\frac{1}{2}\left(\left[X_1,Y_1,Z_1\right]+\left[Z_1,X_1,Y_1\right]\right)+\left[Y_1,X_1,Z_1\right]$$
$$=-\frac{3}{2}\left[\left[Z_1,X_1\right],Y_1\right]-\frac{1}{2}\left[Y_1,Z_1,X_1\right]+\left[Y_1,X_1,Z_1\right]$$
$$\left)\begin{array}{c}\left[\left[X_1,Y_1\right],Z_1\right]-\left[X_1,Y_1,Z_1\right]=-\left[\left[Z_1,X_1\right],Y_1\right]\\ \text{and}\\ \left[X_1,Y_1,Z_1\right]+\left[Z_1,X_1,Y_1\right]=-\left[Y_1,Z_1,X_1\right]\\ \text{by the Jacobi identity}\end{array}\right($$
$$=0$$

■

6 From Lie Algebras to Lie Groups

Theorem 30 *The spaces* $\left(\mathfrak{g}^{D_n}\right)_0$ $(n=1,2,3)$ *are Lie groups with respect to the binary operations in Definition 25.*

Proof. The microlinearity of $\left(\mathfrak{g}^{D_n}\right)_0$ follows from that of $\mathfrak{g}$. We have already seen that the binary operations are associative. In order to let us done, we have only to note, in case of $n=3$ by way of example, that 0 is the unit element, while the inverse element of

$$dX_1+\frac{1}{2}d^2X_2+\frac{1}{6}d^3X_3\in\left(\mathfrak{g}^{D_3}\right)_0$$

is

$$d\left(-X_1\right)+\frac{1}{2}d^2\left(-X_2\right)+\frac{1}{6}d^3\left(-X_3\right)$$

■

In order to be sure whether the Lie group structure on $\left(\mathfrak{g}^{D_n}\right)_0$ in Theorem 30 is indeed that of an appropriate Weil prolongation $\left(G^{D_n}\right)_1$ of a *mythical* Lie group G whose Lie algebra is supposed to be $\mathfrak{g}$.Lie algebra is supposed to be $\mathfrak{g}$, we need to see its Lie algebra in computation.

Theorem 31 *With* $d,e_1,e_2\in D$*, we have*

$$\left[de_1X_1,de_2Y_1\right]=0,$$

as is expected in Corollary 11.

Theorem 32 *With $d \in D_2$ and $e_1, e_2 \in D$, we have*

$$\left[de_1X_1 + \frac{1}{2}d^2e_1X_2, de_2Y_1 + \frac{1}{2}d^2e_2Y_2\right]$$
$$= d^2e\left[X_1, Y_1\right],$$

as is expected in Corollary 11.

Proof. We have

$$de_1X_1 + \frac{1}{2}d^2e_1X_2.de_2Y_1 + \frac{1}{2}d^2e_2Y_2.d\left(-e_1\right)X_1 +$$
$$\frac{1}{2}d^2\left(-e_2\right)X_2.d\left(-e_1\right)Y_1 + \frac{1}{2}d^2\left(-e_2\right)Y_2$$
$$= d\left(e_1X_1 + e_2Y_1\right) + \frac{1}{2}d^2\left(e_1X_2 + e_2Y_2 + e_1e_2\left[X_1, Y_1\right]\right).$$
$$d\left(-e_1X_1 - e_2Y_1\right) + \frac{1}{2}d^2\left(-e_1X_2 - e_2Y_2 + e_1e_2\left[X_1, Y_1\right]\right)$$
$$= \frac{1}{2}d^2 2e_1e_2\left[X_1, Y_1\right]$$

■

Theorem 33 *With $d \in D_3$ and $e_1, e_2 \in D$, we have*

$$\left[de_1X_1 + \frac{1}{2}d^2e_1X_2 + \frac{1}{6}d^3e_1X_3, de_2Y_1 + \frac{1}{2}d^2e_2Y_2 + \frac{1}{6}d^3e_2Y_3\right]$$
$$= d^2e\left[X_1, Y_1\right] + \frac{1}{2}d^3e\left(\left[X_1, Y_2\right] + \left[X_2, Y_1\right]\right),$$

as is expected in Corollary 11.

Proof.

$$de_1X_1 + \frac{1}{2}d^2e_1X_2 + \frac{1}{6}d^3e_1X_3.de_2Y_1 + \frac{1}{2}d^2e_2Y_2 + \frac{1}{6}d^3e_2Y_3.$$
$$d\left(-e_1\right)X_1 + \frac{1}{2}d^2\left(-e_1\right)X_2 + \frac{1}{6}d^3\left(-e_1\right)X_3.d\left(-e_2\right)Y_1 + \frac{1}{2}d^2\left(-e_2\right)Y_2 + \frac{1}{6}d^3\left(-e_2\right)Y_3$$
$$= d\left(e_1X_1 + e_2Y_1\right) + \frac{1}{2}d^2\left(e_1X_2 + e_2Y_2 + e_1e_2\left[X_1, Y_1\right]\right) +$$
$$\frac{1}{6}d^3\left\{\left(e_1X_3 + e_2Y_3\right) + \frac{3}{2}e_1e_2\left(\left[X_1, Y_2\right] + \left[X_2, Y_1\right]\right)\right\}.$$
$$d\left(\left(-e_1\right)X_1 + \left(-e_2\right)Y_1\right) + \frac{1}{2}d^2\left(\left(-e_1\right)X_2 + \left(-e_2\right)Y_2 + e_1e_2\left[X_1, Y_1\right]\right) +$$
$$\frac{1}{6}d^3\left\{\left(\left(-e_1\right)X_3 + \left(-e_2\right)Y_3\right) + \frac{3}{2}e_1e_2\left(\left[X_1, Y_2\right] + \left[X_2, Y_1\right]\right)\right\}$$
$$= d^2e_1e_2\left[X_1, Y_1\right] + \frac{1}{2}d^3e_1e_2\left(\left[X_1, Y_2\right] + \left[X_2, Y_1\right]\right)$$

■

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