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著者別名	赤平 昌文, 大谷内 奈穂
journal or publication title	Communications in statistics. Theory and methods
volume	36
number	11
page range	2049-2059
year	2007-08
権利	(C) Taylor & Francis Group, LLC This is an Author's Accepted Manuscript of an article published in M. Akahira & N. Ohyauchi (2007) The Asymptotic Bound by the Kiefer-Type Information Inequality and Its Attainment, Communications in Statistics - Theory and Methods, 36:11, 2049-2059, available at http://www.tandfonline.com/10.1080/03610920601144103.
URL	http://hdl.handle.net/2241/119470

doi: 10.1080/03610920601144103

THE ASYMPTOTIC BOUND BY THE KIEFER TYPE INFORMATION INEQUALITY AND ITS ATTAINMENT

M. Akahira and N. Ohyauchi
Institute of Mathematics
University of Tsukuba
Ibaraki, Japan 305-8571
naopu@math.tsukuba.ac.jp

Key Words: information inequality; lower bound; prior distribution; non-regular case; unbiased estimator; truncated distribution; inverse problem.

AMS 2000 Mathematics Subject Classification: Primary 62F12, Secondary 62F15; 62F30.

ABSTRACT

In non-regular cases when the regularity conditions does not hold, the Chapman-Robbins (1951) inequality for the variance of unbiased estimators is well known, but the lower bound by the inequality is not attainable. In this paper we extend the Kiefer type information inequality applicable to the non-regular case to the asymptotic situation. And we apply it to the case of a family of truncated distributions, in which the lower bound by the Kiefer type inequality derived from an appropriate prior distribution is attained by the asymptotically unbiased estimator. It also follows from the completeness of the sufficient statistic that the lower bound is asymptotically best. Some examples are also given.

1. INTRODUCTION

Under suitable regularity conditions, the Cramér-Rao inequality is well known as the fact that the variance of unbiased estimators can not be smaller than the lower bound. And also the lower bound by the inequality is attainable. On the other hand, in the non-regular cases when the regularity conditions do not always hold, some information inequalities like the Chapman-Robbins inequality are known, but they are not generally attainable. For onedirectional family of distributions with a parameter for which the support moves in the one direction, the existence of zero variance unbiased estimator is also shown (see, e.g. Akahira and Takeuchi (1995)). Related results are found in Akahira (1991, 1993), Barranco-Chamorro et al. (2000, 2001). Further, from the Bayesian viewpoint, the information inequalities are discussed by Vincze (1992), Akahira and Ohyauchi (2003, 2006), Ohyauchi (2004), Ohyauchi and Akahira (2005), and others.

In this paper, from the Bayesian viewpoint we consider the information inequality for the variance of asymptotically unbiased estimators, where the result of Kiefer (1952) is extended to the asymptotic case. For a family of truncated distributions, the lower bound for the variance by the Kiefer type inequality derived from an appropriate prior distribution is attained by the asymptotically unbiased estimator. From the completeness of the sufficient statistic it follows that the lower bound is asymptotically best. This is also regarded as a solution of the inverse problem on the lower bound by the information inequality. Some examples are also given.

2. THE ASYMPTOTIC BOUND BY THE KIEFER TYPE INFORMATION INEQUALITY

Suppose that $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of i.i.d. random variables with p.d.f. $p(x, \theta)$ (with a σ-finite measure μ), where $x \in \mathcal{X}$ and $\theta \in \Omega \subset \boldsymbol{R}^1$ in which $\mathcal{X}$ is a sample space and Ω is a parameter space. Let $f_{\boldsymbol{X}}(\boldsymbol{x}, \theta) = \prod_{i=1}^n p(x_i, \theta)$, $\mathcal{X}^n$ be a n-fold direct product of $\mathcal{X}$ and μ^n be a direct product of μ. For each $\theta \in \Omega$, let

$$\Omega_{\theta,n} := \left\{ \omega \,\middle|\, \theta + \frac{\omega}{n} \in \Omega \right\},$$

and λ_{in} $(i = 1, 2)$ be prior probability measures on $\Omega_{\theta,n}$. We define the prior mean w.r.t. λ_{in} as

$$E_{in}(\omega) = \int_{\Omega_{\theta,n}} \omega d\lambda_{in}(\omega)$$

for $i = 1, 2$. Let $\hat{\theta}_n = \hat{\theta}_n(\boldsymbol{X})$ be an estimator of θ based on the sample $\boldsymbol{X} := (X_1, \cdots, X_n)$.

Let $\hat{\theta}_n$ be an asymptotically unbiased estimator of θ, i.e.

$$E_\theta(\hat{\theta}_n) = \theta + b_n(\theta), \quad \theta \in \Omega,$$

where $b_n(\theta) = o(1/n)$. Here we assume the following condition.

(A1) There exist a positive number α and a function $a(\cdot)$ on Ω independent of n such that

$$\left| b_n\left(\theta + \frac{\omega}{n}\right)\right| \leq \frac{1}{n^{1+\alpha}} a\left(\theta + \frac{\omega}{n}\right)$$

for all $\omega \in \Omega_{\theta,n}$, and also there exists a constant $M_\theta^{(1)}$ independent of n such that

$$\int_{\Omega_{\theta,n}} a\left(\theta + \frac{\omega}{n}\right) d\lambda_{in}(\omega) \leq M_\theta^{(1)} \quad (i = 1, 2).$$

In a similar way to Kiefer(1952), we have the following.

Theorem 2.1 Let $\hat{\theta}_n$ be any asymptotically unbiased estimator of θ satisfying the condition (A1). Then

$$E_\theta\left[\{\hat{\theta}_n(\boldsymbol{X}) - \theta\}^2\right] \geq \frac{\frac{1}{n^2}\{E_{1n}(\omega) - E_{2n}(\omega)\}^2 + O\left(\frac{1}{n^{2+\alpha}}\right)}{\int_{\mathcal{X}^n} \frac{1}{f_{\boldsymbol{X}}(\boldsymbol{x},\theta)} \left\{\int_{\Omega_{\theta,n}} f_{\boldsymbol{X}}\left(\boldsymbol{x}, \theta + \frac{\omega}{n}\right) d\lambda_{1n}(\omega) - \int_{\Omega_{\theta,n}} f_{\boldsymbol{X}}\left(\boldsymbol{x}, \theta + \frac{\omega}{n}\right) d\lambda_{2n}(\omega)\right\}^2 d\mu^n(\boldsymbol{x})} \tag{2.1}$$

for large n.

Proof. First we have

$$\begin{aligned}
&\int_{\mathcal{X}^n} \{\hat{\theta}_n(\boldsymbol{x}) - \theta\}\sqrt{f_{\boldsymbol{X}}(\boldsymbol{x},\theta)} \left\{\frac{\int_{\Omega_{\theta,n}} f_{\boldsymbol{X}}\left(\boldsymbol{x}, \theta + \frac{\omega}{n}\right) d\lambda_{1n}(\omega) - \int_{\Omega_{\theta,n}} f_{\boldsymbol{X}}\left(\boldsymbol{x}, \theta + \frac{\omega}{n}\right) d\lambda_{2n}(\omega)}{f_{\boldsymbol{X}}(\boldsymbol{x},\theta)}\right\} \\
&\qquad \cdot \sqrt{f_{\boldsymbol{X}}(\boldsymbol{x},\theta)} d\mu^n(\boldsymbol{x}) \\
&= \int_{\mathcal{X}^n} \{\hat{\theta}_n(\boldsymbol{x}) - \theta\} \int_{\Omega_{\theta,n}} f_{\boldsymbol{X}}\left(\boldsymbol{x}, \theta + \frac{\omega}{n}\right) d\lambda_{1n}(\omega) d\mu^n(\boldsymbol{x}) \\
&\qquad - \int_{\mathcal{X}^n} \{\hat{\theta}_n(\boldsymbol{x}) - \theta\} \int_{\Omega_{\theta,n}} f_{\boldsymbol{X}}\left(\boldsymbol{x}, \theta + \frac{\omega}{n}\right) d\lambda_{2n}(\omega) d\mu^n(\boldsymbol{x}) \\
&= \int_{\Omega_{\theta,n}} \left\{b_n\left(\theta + \frac{\omega}{n}\right) + \frac{\omega}{n}\right\} d\lambda_{1n}(\omega) - \int_{\Omega_{\theta,n}} \left\{b_n\left(\theta + \frac{\omega}{n}\right) + \frac{\omega}{n}\right\} d\lambda_{2n}(\omega) \\
&= \frac{1}{n}\{E_{1n}(\omega) - E_{2n}(\omega)\} + \int_{\Omega_{\theta,n}} b_n\left(\theta + \frac{\omega}{n}\right) d\lambda_{1n}(\omega) - \int_{\Omega_{\theta,n}} b_n\left(\theta + \frac{\omega}{n}\right) d\lambda_{2n}(\omega).
\end{aligned} \tag{2.2}$$

We also obtain by the condition (A1)

$$\left|\int_{\Omega_{\theta,n}} b_n\left(\theta+\frac{\omega}{n}\right)d\lambda_{in}(\omega)\right| \leq \int_{\Omega_{\theta,n}}\left|b_n\left(\theta+\frac{\omega}{n}\right)\right|d\lambda_{in}(\omega) \leq \int_{\Omega_{\theta,n}}\frac{a\left(\theta+\frac{\omega}{n}\right)}{n^{1+\alpha}}d\lambda_{in}(\omega)$$
$$\leq \frac{M_\theta^{(1)}}{n^{1+\alpha}} \qquad (2.3)$$

for $i=1,2$. By the Schwarz inequality we have from (2.2) and (2.3)

$$\left[\frac{1}{n}\{E_{1n}(\omega)-E_{2n}(\omega)\}+O\left(\frac{1}{n^{1+\alpha}}\right)\right]^2$$
$$\leq \int_{\mathcal{X}^n}\{\hat{\theta}(\boldsymbol{x})-\theta\}^2 f_{\boldsymbol{X}}(\boldsymbol{x},\theta)d\mu^n(\boldsymbol{x})$$
$$\cdot\int_{\mathcal{X}^n}\frac{1}{f_{\boldsymbol{X}}(\boldsymbol{x},\theta)}\left\{\int_{\Omega_{\theta,n}} f_{\boldsymbol{X}}\left(\boldsymbol{x},\theta+\frac{\omega}{n}\right)d\lambda_{1n}(\omega)-\int_{\Omega_{\theta,n}} f_{\boldsymbol{X}}\left(\boldsymbol{x},\theta+\frac{\omega}{n}\right)d\lambda_{2n}(\omega)\right\}^2 d\mu^n(\boldsymbol{x}),$$

which yields the inequality (2.1). Thus we complete the proof.

In particular, letting $\theta\in\Omega_{\theta,n}$ and $\lambda_{2n}(\{0\})=1$, we have from (2.1)

$$E_\theta\left[\{\hat{\theta}_n(\boldsymbol{X})-\theta\}^2\right] \geq \sup_{\lambda_{1n}}\frac{\frac{1}{n^2}\{E_{1n}(\omega)\}^2+O\left(\frac{1}{n^{2+\alpha}}\right)}{J_{\lambda_{1n}}(\theta)} \qquad (2.4)$$

for large n, where

$$J_{\lambda_{1n}}(\theta) := E_\theta\left[\left\{\frac{h^\theta_{\lambda_{1n}}(\boldsymbol{X})}{f_{\boldsymbol{X}}(\boldsymbol{X},\theta)}\right\}^2\right]-1 \qquad (2.5)$$

with

$$h^\theta_{\lambda_{1n}}(\boldsymbol{x}) := \int_{\Omega_{\theta,n}} f_{\boldsymbol{X}}\left(\boldsymbol{x},\theta+\frac{\omega}{n}\right)d\lambda_{1n}(\omega).$$

The inequality (2.4) is an extension of the Kiefer inequality. Since

$$E_\theta\left[\{\hat{\theta}_n(\boldsymbol{X})-\theta\}^2\right] = V_\theta(\hat{\theta}_n)+b_n^2(\theta) = V_\theta(\hat{\theta}_n)+O\left(\frac{1}{n^{2(1+\alpha)}}\right),$$

it follows from (2.4) that

$$V_\theta(\hat{\theta}_n) \geq \sup_{\lambda_{1n}}\frac{\frac{1}{n^2}\{E_{1n}(\omega)\}^2}{J_{\lambda_{1n}}(\theta)}+O\left(\frac{1}{n^{2+\alpha}}\right) \qquad (2.6)$$

for large n, where $V_\theta(\cdot)$ denotes the variance.

3. APPLICATIONS TO A FAMILY OF TRUNCATED DISTRIBUTIONS

Suppose that $X_1, X_2, \cdots, X_n, \cdots$ be i.i.d. random variables according to the left-truncated distribution with a p.d.f. (w.r.t. the Lebesgue measure)

$$p(x, \theta) = \begin{cases} C(\theta)e^{S(x)} & \text{for } x > \theta, \\ 0 & \text{for } x \leq \theta, \end{cases}$$

where $\theta \in \Omega \subset \boldsymbol{R}^1$, and $C(\theta)$ is the normalizing constant. Assume that $S(x)$ is differentiable in x on $\boldsymbol{R}^1$. Then the joint p.d.f. of $\boldsymbol{X}$ is given by

$$f_{\boldsymbol{X}}(\boldsymbol{x}, \theta) = \begin{cases} C^n(\theta)e^{\sum_{i=1}^n S(x_i)} & \text{for } x_{(1)} > \theta, \\ 0 & \text{for } x_{(1)} \leq \theta, \end{cases}$$

where $x_{(1)} := \min_{1 \leq i \leq n} x_i$. Suppose that $\Omega = (0, \infty)$. Then

$$\Omega_{\theta,n} = \{\omega | \omega > -n\theta\}.$$

Since

$$\frac{f_{\boldsymbol{X}}(\boldsymbol{x}, \theta)}{f_{\boldsymbol{X}}\left(\boldsymbol{x}, \theta + \frac{\omega}{n}\right)} = \left\{\frac{C(\theta)}{C\left(\theta + \frac{\omega}{n}\right)}\right\}^n$$

for $\omega > 0$, we take

$$\frac{d\lambda_{1n}}{d\omega} = k_n(\theta)\left\{\frac{C(\theta)}{C\left(\theta + \frac{\omega}{n}\right)}\right\}^n \tag{3.1}$$

for $\omega > 0$ as a prior density for λ_{1n}, where $k_n(\theta)$ is the normalizing constant. Here we put

$$D_{in}(\theta) := \int_0^\infty \omega^i \left\{\frac{C(\theta)}{C\left(\theta + \frac{\omega}{n}\right)}\right\}^n d\omega \tag{3.2}$$

for $i = 0, 1$. Then

$$E_{1n}(\omega) = \int_0^\infty \omega d\lambda_{1n}(\omega) = \{D_{0n}(\theta)\}^{-1} \int_0^\infty \omega \left\{\frac{C(\theta)}{C\left(\theta + \frac{\omega}{n}\right)}\right\}^n d\omega = \{D_{0n}(\theta)\}^{-1} D_{1n}(\theta), \tag{3.3}$$

$$h^\theta_{\lambda_{1n}}(\boldsymbol{x}) := \int_0^\infty f_{\boldsymbol{X}}\left(\boldsymbol{x}, \theta + \frac{\omega}{n}\right) d\lambda_{1n}(\omega) = C^n(\theta)\{D_{0n}(\theta)\}^{-1} n(x_{(1)} - \theta)e^{\sum_{i=1}^n S(x_i)}. \tag{3.4}$$

From (2.5) and (3.4) we obtain

$$\begin{aligned} J_{\lambda_{1n}}(\theta)+1 &= E_\theta\left[\left\{\frac{h^\theta_{\lambda_{1n}}(\boldsymbol{X})}{f_{\boldsymbol{X}}(\boldsymbol{X},\theta)}\right\}^2\right] \\ &= \{D_{0n}(\theta)\}^{-2}\int_0^\infty\cdots\int_0^\infty C^n(\theta)e^{\sum_{i=1}^n S(x_i)}n^2(x_{(1)}-\theta)^2dx_1\cdots dx_n \\ &= \{D_{0n}(\theta)\}^{-2}E_\theta\left[\left\{n(X_{(1)}-\theta)\right\}^2\right], \end{aligned} \tag{3.5}$$

where $X_{(1)} := \min_{1\le i\le n} X_i$. From (2.6), (3.3) and (3.5) we have for any estimator satisfying the condition (A1)

$$\begin{aligned} V_\theta(\hat{\theta}_n) &\ge \frac{\{D_{0n}(\theta)\}^{-2}D^2_{1n}(\theta)}{n^2\left\{(D_{0n}(\theta))^{-2}E_\theta\left[n^2(X_{(1)}-\theta)^2\right]-1\right\}}+O\left(\frac{1}{n^{2+\alpha}}\right) \\ &=: \frac{1}{n^2}B_n(\theta)+O\left(\frac{1}{n^{2+\alpha}}\right) \end{aligned} \tag{3.6}$$

for large n.

Now, we obtain by the mean value theorem

$$\begin{aligned} \left\{\frac{C(\theta)}{C\left(\theta+\frac{\omega}{n}\right)}\right\}^n &= \exp\left[n\left\{\log C(\theta)-\log C\left(\theta+\frac{\omega}{n}\right)\right\}\right] \\ &= e^{-C'(\xi)/C(\xi)}, \end{aligned}$$

where $\theta<\xi<\theta+(\omega/n)$. Here, we assume the following condition.

(A2) There exists a positive constant $M^{(2)}_\theta$ such that

$$\exp\left\{-\frac{C'(\xi)}{C(\xi)}\omega\right\}\le\exp\left(-M^{(2)}_\theta\omega\right)$$

for all $\omega>0$.

Then it follows from (3.2) and the Lebesgue convergence theorem that for large n

$$D_{0n}(\theta)=\int_0^\infty e^{-\frac{C'(\xi)}{C(\xi)}\omega}d\omega=\frac{C(\theta)}{C'(\theta)}+o(1), \tag{3.7}$$

$$D_{1n}(\theta)=\int_0^\infty \omega e^{-\frac{C'(\theta)}{C(\theta)}\omega}d\omega=\left\{\frac{C(\theta)}{C'(\theta)}\right\}^2+o(1). \tag{3.8}$$

On the other hand, we have for $t > 0$

$$P_\theta\left\{n(X_{(1)} - \theta) \leq t\right\} = 1 - \left[1 - \left\{C(\theta)\int_\theta^{\theta+\frac{t}{n}} e^{S(x)}dx\right\}\right]^n. \tag{3.9}$$

Since

$$\int_\theta^{\theta+\frac{t}{n}} e^{S(x)}dx = \frac{1}{C(\theta)} - \frac{1}{C\left(\theta + \frac{t}{n}\right)},$$

it follows from (3.9) that

$$P_\theta\left\{n(X_{(1)} - \theta) \leq t\right\} = 1 - \left\{\frac{C(\theta)}{C\left(\theta + \frac{t}{n}\right)}\right\}^n,$$

for $t > 0$, which implies that the p.d.f. of $T_n := n(X_{(1)} - \theta)$ is

$$f_{T_n}(t, \theta) = \begin{cases} C(\theta)e^{S\left(\theta+\frac{t}{n}\right)}\left\{\frac{C(\theta)}{C\left(\theta+\frac{t}{n}\right)}\right\}^{n-1} & \text{for } t > 0, \\ 0 & \text{for } t \leq 0. \end{cases} \tag{3.10}$$

By the mean value theorem we have

$$f_{T_n}(t, \theta) = \begin{cases} C(\theta)e^{S\left(\theta+\frac{t}{n}\right)} \exp\left\{-\frac{n-1}{n} \cdot \frac{C'(\xi)}{C(\xi)}t\right\} & \text{for } t > 0, \\ 0 & \text{for } t \leq 0, \end{cases} \tag{3.11}$$

where $\theta < \xi < \theta + (t/n)$. Then it follows from the condition (A2) that $f_{T_n}(t, \theta)$ converges pointwise to

$$f_T(t, \theta) = \begin{cases} C(\theta)e^{S(\theta)} \exp\left\{-\frac{C'(\theta)}{C(\theta)}t\right\} & \text{for } t > 0, \\ 0 & \text{for } t \leq 0, \end{cases}$$

which is the p.d.f. of a random variable T, as $n \to \infty$. Further, we assume the following condition.

(A3) For any $t > 0$, there exists $K_\theta(t)$ such that

$$e^{S\left(\theta+\frac{t}{n}\right)} \le K_\theta(t)$$

and

$$\int_0^\infty K_\theta(t) t^i \exp\left(-\frac{1}{2} M_\theta^{(2)} t\right) dt < \infty$$

for $i = 0, 2$.

From the conditions (A2), (A3) and (3.11) it follows that $f_{T_n}(t, \theta)$ is dominated by the function

$$f_0(t, \theta) := C(\theta) K_\theta(t) \exp\left(-\frac{1}{2} M_\theta^{(2)} t\right)$$

for $t > 0$. Hence

$$\lim_{n \to \infty} E_\theta \left[n(X_{(1)} - \theta) \right] = E_\theta(T) = \frac{C(\theta)}{C'(\theta)}, \tag{3.12}$$

$$\lim_{n \to \infty} E_\theta \left[\left\{ n(X_{(1)} - \theta) \right\}^2 \right] = E_\theta(T^2) = 2 \left\{ \frac{C(\theta)}{C'(\theta)} \right\}^2. \tag{3.13}$$

It is seen from (3.12) that $X_{(1)}$ is not asymptotically unbiased estimator. So, we consider a bias-adjusted estimator

$$\hat{\theta}_n^*(\boldsymbol{X}) := X_{(1)} - \frac{C(X_{(1)})}{nC'(X_{(1)})},$$

which yields

$$\hat{\theta}_n^*(\boldsymbol{X}) = X_{(1)} - \frac{C(\theta)}{nC'(\theta)} - \frac{1}{n^2} \left\{ 1 - \frac{C(\xi) C''(\xi)}{(C'(\xi))^2} \right\} n(X_{(1)} - \theta), \tag{3.14}$$

where $|\xi - \theta| < |X_{(1)} - \theta|$. Since $T_n = n(X_{(1)} - \theta)$, it follows that

$$E_\theta \left[n(\hat{\theta}_n^* - \theta) \right] = E_\theta(T_n) - \frac{C(\theta)}{C'(\theta)} - \frac{1}{n} E_\theta \left[\left\{ 1 - \frac{C(\xi) C''(\xi)}{(C'(\xi))^2} \right\} T_n \right]. \tag{3.15}$$

In order to evaluate $E_\theta(\hat{\theta}_n^*)$ up to the order $o(1/n)$, we have from (3.10)

$$f_{T_n}(t, \theta) = C(\theta) e^{S(\theta)} e^{-\frac{C'(\theta)}{C(\theta)} t} \left\{ 1 + \frac{\alpha(\theta)}{n} t - \frac{\beta(\theta)}{n} t^2 + o\left(\frac{1}{n}\right) \right\},$$

where

$$\alpha(\theta) = S'(\theta) + \frac{C'(\theta)}{C(\theta)}, \quad \beta(\theta) = \frac{1}{2}\left(\log C(\theta)\right)''.$$

Then

$$E_\theta(T_n) = \frac{C(\theta)}{C'(\theta)} + \frac{2}{n}\left(\frac{C(\theta)}{C'(\theta)}\right)^2 \left\{\alpha(\theta) - 3\beta(\theta)\left(\frac{C(\theta)}{C'(\theta)}\right)\right\} + o\left(\frac{1}{n}\right).$$

From (3.15) we obtain

$$\begin{aligned} E_\theta\left[n(\hat{\theta}_n^* - \theta)\right] &= \frac{2}{n}\left(\frac{C(\theta)}{C'(\theta)}\right)^2 \left\{\alpha(\theta) - 3\beta(\theta)\left(\frac{C(\theta)}{C'(\theta)}\right)\right\} \\ &\qquad - \frac{1}{n}E_\theta\left[\left\{1 - \frac{C(\xi)C''(\xi)}{(C'(\xi))^2}\right\}T_n\right] + o\left(\frac{1}{n}\right) \\ &= \frac{1}{n}a_0(\theta) + o\left(\frac{1}{n}\right) \quad \text{(say)}. \end{aligned} \tag{3.16}$$

Here, by the Schwarz inequality we have

$$\begin{aligned} |a_0(\theta)| \le 2&\left(\frac{C(\theta)}{C'(\theta)}\right)^2 \left|\alpha(\theta) - 3\beta(\theta)\left(\frac{C(\theta)}{C'(\theta)}\right)\right| \\ &+ \left(E_\theta\left[\left\{1 - \frac{C(\xi)C''(\xi)}{(C'(\xi))^2}\right\}^2\right]\right)^{1/2} \left(E_\theta(T_n^2)\right)^{1/2}. \end{aligned} \tag{3.17}$$

From (3.13) it follows that for given $\varepsilon > 0$ and large n

$$E_\theta\left(T_n^2\right) \le 2\left(\frac{C(\theta)}{C'(\theta)}\right)^2 + \varepsilon.$$

Put

$$\begin{aligned} a_1(\theta) := 2&\left(\frac{C(\theta)}{C'(\theta)}\right)^2 \left|\alpha(\theta) - 3\beta(\theta)\left(\frac{C(\theta)}{C'(\theta)}\right)\right| \\ &+ \left(E_\theta\left[\left\{1 - \frac{C(\xi)C''(\xi)}{(C'(\xi))^2}\right\}^2\right]\right)^{1/2} \left\{2\left(\frac{C(\theta)}{C'(\theta)}\right)^2 + \varepsilon\right\}^{1/2}. \end{aligned} \tag{3.18}$$

Here, we assume the following condition.

(A1)* There exists a constant M_θ^* such that

$$\int_0^\infty a_1\left(\theta + \frac{\omega}{n}\right) d\lambda_{1n}(\omega) \le M_\theta^*.$$

Then it follows from (3.16) to (3.18) that the condition (A1)* implies (A1), and $\hat{\theta}_n^*$ becomes an asymptotically unbiased for θ. It also follows from (3.13) and (3.14) that for large n

$$\begin{aligned} E_\theta\left[\left\{n(\hat{\theta}_n^* - \theta)\right\}^2\right] &= E_\theta\left(T_n^2\right) - \left\{\frac{C(\theta)}{C'(\theta)}\right\}^2 + o(1) \\ &= \left\{\frac{C(\theta)}{C'(\theta)}\right\}^2 + o(1) \end{aligned} \tag{3.19}$$

From (3.6), (3.7), (3.8) and (3.13) we obtain

$$B_n(\theta) = \frac{1}{n^2}\left\{\frac{C(\theta)}{C'(\theta)}\right\}^2 + o\left(\frac{1}{n^2}\right) \tag{3.20}$$

for large n. On the other hand it follows from (3.19) that

$$V_\theta(\hat{\theta}_n^*) = \frac{1}{n^2}\left\{\frac{C(\theta)}{C'(\theta)}\right\}^2 + o\left(\frac{1}{n^2}\right),$$

hence, under the conditions (A1)*, (A2) and (A3), the asymptotically unbiased estimator $\hat{\theta}_n^*$ is shown to be asymptotically efficient in the sense that it attains the bound $B_n(\theta)$ by the inequality (3.6).

Next we show that the bound (3.20) is asymptotically best. In the case of the left-truncated distribution, it is seen that the uniformly minimum variance unbiased (UMVU) estimator based on the complete sufficient statistic $X_{(1)} := \min_{1\le i\le n} X_i$ is given by $\hat{\theta}_n^*$ (see Lwin (1975), and Voinov and Nikulin (1993)). The variance of the UMVU estimator $\hat{\theta}_n^*$ asymptotically coincides with the bound (3.20) derived from the prior density (3.1). Hence the bound (3.20) is seen to be asymptotically best. The fact is also grasped as a solution of the inverse problem on the lower bound by the information inequality.

In a similar way to the above, it is possible to discuss the case of a family of right-truncated distributions using the statistic $X_{(n)} := \max_{1\le i\le n} X_i$.

4. EXAMPLES

In this section, we give examples on truncated normal and Weibull distributions.

Example 4.1 Suppose that $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of i.i.d. random variables according to the left-truncated normal distribution with a p.d.f.

$$p(x,\theta) = \begin{cases} C(\theta)e^{-x^2/2} & \text{for } x > \theta, \\ 0 & \text{for } x \le \theta, \end{cases}$$

where $\theta \in (0, \infty)$, and

$$C(\theta) = \frac{1}{\sqrt{2\pi}\{1 - \Phi(\theta)\}}$$

with the cumulative distribution function (c.d.f.) Φ of the standard normal distribution $N(0,1)$. Putting

$$g(\theta) := \frac{C'(\theta)}{C(\theta)} = \frac{\phi(\theta)}{1 - \Phi(\theta)}$$

with the p.d.f. ϕ of $N(0,1)$. Since $g(\theta)$ is a monotone increasing function on the interval $(0, \infty)$, letting $M_\theta^{(2)} = g(\theta)$ in the condition (A2), it is satisfied. Since $S(x) = -x^2/2$, it follows that

$$e^{S\left(\theta + \frac{t}{n}\right)} = e^{-\frac{1}{2}\left(\theta + \frac{t}{n}\right)^2} \le 1.$$

Letting $K_\theta(t) \equiv 1$ in the condition (A3), we have

$$\int_0^\infty t^i \exp\left\{-\frac{1}{2}g(\theta)t\right\} dt < \infty$$

for $i = 0, 2$, hence (A3) is satisfied. Since

$$\lim_{\theta\to 0} \frac{C(\theta)}{C'(\theta)} = \sqrt{\frac{\pi}{2}}, \quad \lim_{\theta\to\infty} \frac{C(\theta)}{C'(\theta)} = 0,$$

$$\lim_{\theta\to 0} (\log C(\theta))'' = \lim_{\theta\to 0} \frac{\phi(\theta)\left\{\phi(\theta) - \theta\left(1 - \Phi(\theta)\right)\right\}}{\{1 - \Phi(\theta)\}^2} = \frac{8}{\pi}, \quad \lim_{\theta\to\infty} (\log C(\theta))'' = 1,$$

$$\lim_{\theta\to 0} S'(\theta) = \lim_{\theta\to 0}(-\theta e^{-\theta^2/2}) = 0, \quad \lim_{\theta\to\infty} S'(\theta) = 0$$

it follows that $C(\theta)/C'(\theta)$, $(\log C(\theta))''$ and $S'(\theta)$ are bounded on $(0,\infty)$. Since

$$\left(\frac{C(\theta)}{C'(\theta)}\right)' = 1 - \frac{C(\theta)C''(\theta)}{(C'(\theta))^2} = \frac{-\phi(\theta) + \theta\,(1-\Phi(\theta))}{\phi(\theta)},$$

it follows that

$$\lim_{\theta\to 0}\left(\frac{C(\theta)}{C'(\theta)}\right)' = -1, \quad \lim_{\theta\to\infty}\left(\frac{C(\theta)}{C'(\theta)}\right)' = 0,$$

which implies that $(C(\theta)/C'(\theta))'$ is also bounded on $(0,\infty)$. Then it follows from (3.18) that $a_1(\theta)$ is bounded on $(0,\infty)$, hence the condition (A1)* holds. Therefore

$$\hat{\theta}_n^*(\boldsymbol{X}) = X_{(1)} - \frac{1}{n\phi(X_{(1)})}\left\{1-\Phi(X_{(1)})\right\}$$

has the variance

$$V_\theta(\hat{\theta}_n^*) = \frac{1}{n^2}\left\{\frac{1-\Phi(\theta)}{\phi(\theta)}\right\}^2 + o\left(\frac{1}{n^2}\right)$$

for large n, which attains the bound $B_n(\theta)$ by the inequality (3.6).

Example 4.2 Suppose that $X_1, X_2, \cdots, X_n, \cdots$ is a sequence of i.i.d. random variables according to the left-truncated Weibull distribution with a p.d.f.

$$p(x,\theta) = \begin{cases} C_r(\theta) r x^{r-1} e^{-x^r} & \text{for } x > \theta, \\ 0 & \text{for } x \le \theta, \end{cases}$$

where $\theta \in (0,\infty)$, $r > 1$ and $C_r(\theta) = e^{\theta^r}$. Let r be known. In a similar way to Example 4.1, it is shown that the conditions (A1)*, (A2) and (A3) are satisfied. Hence

$$\hat{\theta}_n^*(\boldsymbol{X}) = X_{(1)} - \frac{1}{nrX_{(1)}^{r-1}}$$

has the variance

$$V_\theta\left(\hat{\theta}_n^*\right) = \frac{1}{n^2\theta^2} + o\left(\frac{1}{n^2}\right)$$

for large n, which attains the bound $B_n(\theta)$ by the inequality (3.6).

In a similar way to the above, the right-truncated normal and Weibull cases can be discussed.

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