A remark on comparison with a maximum likelihood estimator in asymptotic variances in a non-regular case

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A Remark on Comparison with a Maximum Likelihood Estimator in Asymptotic Variances in a Non-Regular Case*

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Abstract

Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent and identically distributed random variables with a truncated exponential density. Then it is shown that there exists an estimator whose asymptotic variance is smaller than that of a maximum likelihood estimator.

1. Introduction

Suppose that $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of independent and identically distributed random variables with a density $f(x-\theta)$ satisfying

$$f(x) = \begin{cases} ce^{-x} & \text{for } 0 < x < 1; \\ 0 & \text{otherwise}, \end{cases}$$

where $c = 1/(1-e^{-1})$.

It is easily seen that the maximum likelihood estimator (MLE) $\hat{\theta}_{ML}$ is given by $\hat{\theta}_{ML} = \min X_i$. Then it is shown that there exists an estimator whose asymptotic variance is smaller than that of $\hat{\theta}_{ML}$.

Although the density of only the form (1) is treated in this paper, it is possible to extend it to the density $f(x)$ such that $f(x)$ is continuously differentiable in the interval $(\alpha, \beta)$,

$$f(x) > 0 \text{ for } \alpha < x < \beta;$$
$$f(x) = 0 \text{ otherwise},$$

and $0 < \lim_{x \to \alpha^+} f(x) = \lim_{x \to \beta^-} f(x) < \infty$.

2. Results

Let $\mathcal{X}$ be an abstract sample space whose generic point is denoted by $x$, $\mathcal{B}$ a $\sigma$-field of subsets of $\mathcal{X}$ and $\{P_\theta : \theta \in \Theta\}$ a set of probability measures on $\mathcal{B}$, where $\Theta$ is called a parameter space. We suppose that $\mathcal{X}=\Theta=R^1$ and $\mathcal{B}$ is a Borel $\sigma$-field and for each $\theta$ $P_\theta$ has the density $f(x-\theta)$ of the form (1). Consider $n$-fold direct products $(R^n, \mathcal{B}^n)$ of $(R^1, \mathcal{B})$ and the corresponding product measure $P_\theta^n$ of $P_\theta$. An estimator of $\theta$ is defined to be a sequence $\{\hat{\theta}_n\}$ of $\mathcal{B}^n$-measurable functions $\hat{\theta}_n$ on $R^n$ into $\Theta$. For simplicity we denote $\{\hat{\theta}_n\}$ by $\hat{\theta}$. A distribution function $F_{\theta, \hat{\theta}}(\cdot)$ is called to be the asymptotic distribution function of an estimator $\hat{\theta}$ of order $C=\{c_n\}$ if for each real number $y$, $F_{\theta, \hat{\theta}}(y)$ is continuous in $\theta$ and for

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any $\theta \in \Theta$ there exists a positive number $d$ such that for any continuity point $y$ of $F_\theta, \bar{g}(y)$,
\[
\lim_{n \to \infty} \sup_{\theta : |\theta - \bar{\theta}| < d} |P_\theta^n(\{c_n(\bar{\theta} - \theta) \leq d\}) - P_\theta, \bar{g}(y)| = 0
\]
(Akahira [1]).

Since
\[
\lim_{n \to \infty} P_\theta^n(\{|n(\hat{\theta}_{ML} - \theta) \leq y\}) = \lim_{n \to \infty} P_\theta^n(\{ \max_{1 \leq i \leq n} x_i \leq \bar{\theta} + yn^{-1}\})
\]
\[
= \begin{cases} 
1 - e^{-cy} & \text{for } y > 0; \\
0 & \text{for } y \leq 0,
\end{cases}
\]
it follows that the density $f_{\theta_{ML}}(y)$ of the asymptotic distribution of $\hat{\theta}_{ML}$ of order \{n\} is given by
\[
f_{\theta_{ML}}(y) = \begin{cases} 
c e^{-cy} & \text{for } y > 0; \\
0 & \text{for } y \leq 0.
\end{cases}
\]

Let $\bar{\theta} = \max_{1 \leq i \leq n} X_i - 1$. Then we have
\[
\lim_{n \to \infty} P_\theta^n(\{|n(\hat{\theta}^* - \theta) \leq y\}) = \lim_{n \to \infty} P_\theta^n(\{ \min_{1 \leq i \leq n} x_i \leq \bar{\theta} + yn^{-1}\})
\]
\[
= \begin{cases} 
1 & \text{for } y \geq 0; \\
e^{-cy} & \text{for } y < 0.
\end{cases}
\]
Hence the density $g_{\hat{\theta}^*}(y)$ of the asymptotic distribution of $\hat{\theta}^*$ of order \{n\} is given by
\[
g_{\hat{\theta}^*}(y) = \begin{cases} 
0 & \text{for } y \geq 0; \\
c e^{-cy} & \text{for } y < 0.
\end{cases}
\]

We define an estimator $\hat{\theta}_a$ by $a\hat{\theta}_{ML} + (1 - a)\hat{\theta}^*$, where $4/5 \leq a < 1$.

We remark that $\{ \min_{1 \leq i \leq n} X_i, \max_{1 \leq i \leq n} X_i \}$ is asymptotically sufficient (Akahira [2]).

Since $\hat{\theta}_{ML}$ and $\hat{\theta}^*$ are asymptotically independent, the density $h_{\hat{\theta}_a}(y)$ of the asymptotic distribution of $\hat{\theta}_a$ of order \{n\} is a convolution of $f_{\theta_{ML}}(y)$ and $g_{\hat{\theta}^*}(y)$. It follows from (2) and (3) that
\[
h_{\hat{\theta}_a}(y) = \begin{cases} 
K_a \exp\left(\frac{c}{(1-a)e}y\right) & \text{for } y \leq 0; \\
K_a \exp\left(-\frac{c}{a}y\right) & \text{for } y < 0,
\end{cases}
\]
where $K_a = c/(a + (1-a)e)$.

Next we shall calculate the asymptotic variances $V_a(Y)$ and $V_{ML}(Y)$ of $\hat{\theta}_a$ and $\hat{\theta}_{ML}$, respectively.

Since
\[
E_a(Y) = K_a \left[ a^2 - e^2(1-a)^2 \right] / c^2
\]
\[
E_a(Y^2) = 2K_a \left[ a^3 + e^2(1-a)^3 \right] / c^3,
\]
it follows that
\[
V_a(Y) = \frac{a^2 + e^2(1-a)^2}{c^2} - 2.
\]
Since $e<3$ and $4/5 \leq a < 1$, we have

$$V_a(Y) < \frac{a^2 + 9(1-a)^2}{c^2} \leq \frac{1}{c^2}.$$ 

On the other hand it is easily seen that

$$V_{ML}(Y) = \frac{1}{c^2}.$$ 

Hence

$$V_a(Y) < V_{ML}(Y).$$

Therefore it is shown that the asymptotic variance of $\hat{\theta}_a$ is smaller than that of $\hat{\theta}_{ML}$.

References
