A remark on comparison with a maximum likelihood estimator in asymptotic variances in a non-regular case.
A Remark on Comparison with a Maximum Likelihood Estimator in Asymptotic Variances in a Non-Regular Case*

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Abstract

Let \( X_1, X_2, \ldots, X_n, \ldots \) be a sequence of independent and identically distributed random variables with a truncated exponential density. Then it is shown that there exists an estimator whose asymptotic variance is smaller than that of a maximum likelihood estimator.

1. Introduction

Suppose that \( X_1, X_2, \ldots, X_n, \ldots \) is a sequence of independent and identically distributed random variables with a density \( f(x - \theta) \) satisfying

\[
  f(x) = \begin{cases} 
    c e^{-x} & \text{for } 0 < x < 1; \\
    0 & \text{otherwise},
  \end{cases}
\]

(1)

where \( c = 1/(1 - e^{-1}) \).

It is easily seen that the maximum likelihood estimator (MLE) \( \hat{\theta}_{ML} \) is given by \( \hat{\theta}_{ML} = \min_{1 \leq i \leq n} X_i \). Then it is shown that there exists an estimator whose asymptotic variance is smaller than that of \( \hat{\theta}_{ML} \).

Although the density of only the form (1) is treated in this paper, it is possible to extend it to the density \( f(x) \) such that \( f(x) \) is continuously differentiable in the interval \((\alpha, \beta)\),

\[
  f(x) > 0 \quad \text{for} \quad \alpha < x < \beta;
\]

\[
  f(x) = 0 \quad \text{otherwise},
\]

and \( 0 < \lim_{x \to \alpha^+} f(x) = \lim_{x \to \beta^-} f(x) < \infty \).

2. Results

Let \( \mathcal{X} \) be an abstract sample space whose generic point is denoted by \( x \), \( \mathcal{B} \) a \( \sigma \)-field of subsets of \( \mathcal{X} \) and \( \{ P_\theta : \theta \in \Theta \} \) a set of probability measures on \( \mathcal{B} \), where \( \Theta \) is called a parameter space. We suppose that \( \mathcal{X} = \Theta = R^1 \) and \( \mathcal{B} \) is a Borel \( \sigma \)-field and for each \( \theta \) \( P_\theta \) has the density \( f(x - \theta) \) of the form (1). Consider \( n \)-fold direct products \( (R^n, \mathcal{B}^n) \) of \( (R^1, \mathcal{B}) \) and the corresponding product measure \( P_\theta^n \) of \( P_\theta \). An estimator of \( \theta \) is defined to be a sequence \( \{ \hat{\theta}_n \} \) of \( \mathcal{B}^n \)-measurable functions \( \hat{\theta}_n \) on \( R^n \) into \( \Theta \). For simplicity we denote \( \{ \hat{\theta}_n \} \) by \( \hat{\theta} \). A distribution function \( F_{\theta, \hat{\theta}}(\cdot) \) is called to be the asymptotic distribution function of an estimator \( \hat{\theta} \) of order \( C = \{ c_n \} \) if for each real number \( y \), \( F_{\theta, \hat{\theta}}(y) \) is continuous in \( \theta \) and for

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* Received on December 7, 1976.
This research was supported by Japan Ministry of Education.
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any $\theta \in \Theta$ there exists a positive number $d$ such that for any continuity point $y$ of $F_{\theta, \theta}(y)$,
\[
\lim_{n \to \infty} \sup_{\theta: |\theta - \bar{\theta}| < d} |P_{n, \theta} \{ c_n(\theta_n - \theta) \leq d\} - P_{\theta, \theta}(y)| = 0
\]
(Akahira [1]).

Since
\[
\lim_{n \to \infty} P_{n, \theta} \{ n(\theta_{\text{ML}} - \theta) \leq y \}
= \lim_{n \to \infty} P_{n, \theta} \{ \min_{1 \leq i \leq n} x_i \leq \theta + yn^{-1} \}
= \begin{cases} 
1 - e^{-cy} & \text{for } y > 0; \\
0 & \text{for } y \leq 0,
\end{cases}
\]
it follows that the density $f_{\theta_{\text{ML}}}(y)$ of the asymptotic distribution of $\hat{\theta}_{\text{ML}}$ of order \{n\} is given by
\[
f_{\theta_{\text{ML}}}(y) = \begin{cases} 
 ce^{-cy} & \text{for } y > 0; \\
0 & \text{for } y \leq 0.
\end{cases}
\] (2)

Let $\hat{\theta}^* = \max_{1 \leq i \leq n} X_i - 1$. Then we have
\[
\lim_{n \to \infty} P_{n, \theta} \{ n(\theta^* - \theta) \leq y \}
= \lim_{n \to \infty} P_{n, \theta} \{ \max_{1 \leq i \leq n} x_i \leq \theta + yn^{-1} \}
= \begin{cases} 
1 & \text{for } y \geq 0; \\
c e^{-cy} & \text{for } y < 0.
\end{cases}
\]
Hence the density $g_{\theta^*}(y)$ of the asymptotic distribution of $\hat{\theta}^*$ of order \{n\} is given by
\[
g_{\theta^*}(y) = \begin{cases} 
0 & \text{for } y \geq 0; \\
e^{-cy} & \text{for } y < 0.
\end{cases}
\] (3)

We define an estimator $\theta_a$ by $a\hat{\theta}_{\text{ML}} + (1-a)\hat{\theta}^*$, where $4/5 \leq a < 1$.

We remark that $(\min_{1 \leq i \leq n} X_i, \max_{1 \leq i \leq n} X_i)$ is asymptotically sufficient (Akahira [2]).

Since $\hat{\theta}_{\text{ML}}$ and $\hat{\theta}^*$ are asymptotically independent, the density $h_{\theta_a}(y)$ of the asymptotic distribution of $\theta_a$ of order \{n\} is a convolution of $f_{\theta_{\text{ML}}}(y)$ and $g_{\theta^*}(y)$. It follows from (2) and (3) that
\[
h_{\theta_a}(y) = \begin{cases} 
K_a \exp \left( \frac{c}{(1-a)c} y \right) & \text{for } y \leq 0; \\
K_a \exp \left( -\frac{c}{a} y \right) & \text{for } y < 0,
\end{cases}
\]
where $K_a = c/(a + (1-a)c)$.

Next we shall calculate the asymptotic variances $V_a(Y)$ and $V_{\text{ML}}(Y)$ of $\theta_a$ and $\hat{\theta}_{\text{ML}}$, respectively.

Since
\[
E_a(Y^2) = K_a \frac{a^2 - e^{2(1-a)^2}}{c^2};
E_a(Y^2) = 2K_a \frac{a^3 + a^2(1-a)^2}{c^3},
\]
it follows that
\[
V_a(Y) = \frac{a^2 + 4e^{2(1-a)^2}}{c^2}.
\]

Since $e < 3$ and $4/5 \leq a < 1$, we have
\[
V_a(Y) < \frac{a^2 + 9(1 - a)^2}{c^2} \leq \frac{1}{c^2}.
\]
On the other hand it is easily seen that
\[
V_{ML}(Y) = \frac{1}{c^2}.
\]
Hence
\[
V_a(Y) < V_{ML}(Y).
\]
Therefore it is shown that the asymptotic variance of $\hat{\theta}_a$ is smaller than that of $\hat{\theta}_{ML}$.

References