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journal or publication title

Reports of the University of Electro-Communications

volume

27

number

1

page range

125-128

year

1976-08

URL

http://hdl.handle.net/2241/119449
A Remark on Asymptotic Sufficiency of Statistics in Non-Regular Cases*

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Abstract

Suppose that $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of independent identically distributed random variables with the density $f(x: \theta)$ with a compact support, where $\theta$ is a real valued parameter. We suppose that a strongly $\{c_n\}$-consistent estimator of $\theta$ exists. Then we show that a statistic $(\min_{1 \leq i \leq n} X_i, \max_{1 \leq i \leq n} X_i)$ is asymptotically sufficient in non-regular cases.

1. Introduction

A consistent estimator with order $\{c_n\}$ (or a $\{c_n\}$-consistent estimator) is defined and discussed in Akahira [1], where the necessary conditions for the existence of such an estimator are established and the bounds of the orders of convergence of consistent estimators are obtained for non-regular cases. Further the asymptotic accuracies of $\{c_n\}$-consistent estimators are discussed in Akahira [2].

Asymptotic sufficiency has been discussed under regularity conditions by LeCam [4]. In this paper we extend a similar approach to non-regular cases.

Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent identically distributed random variables with the density $f(x: \theta)$ with a compact support, where $\theta$ is a real valued parameter. We suppose that a strongly $\{c_n\}$-consistent estimator of $\theta$ exists. Then we shall obtain that a statistic $(\min_{1 \leq i \leq n} X_i, \max_{1 \leq i \leq n} X_i)$ is asymptotically sufficient in non-regular cases.

2. Notations and definitions

Let $\mathcal{X}$ be an abstract sample space whose generic point is denoted by $x$, $\mathcal{B}$ a $\sigma$-field of subsets of $\mathcal{X}$ and $\{P_\theta: \theta \in \Theta\}$ a set of probability measures on $\mathcal{B}$, where $\Theta$ is called a parameter space. We suppose that $\Theta$ is an open set in a Euclidean 1-space $\mathbb{R}^1$. Consider $n$-fold direct products $(\mathcal{X}(n), \mathcal{B}(n))$ of $(\mathcal{X}, \mathcal{B})$ and the corresponding product measure $P_\theta^{(n)}$ of $P_\theta$. For each $n=1, 2, \ldots$, the points of $\mathcal{X}^{(n)}$ will be denoted by $\bar{x}_n = (x_1, \ldots, x_n)$ and the corresponding random variable by $\bar{x}_n$. An estimator of $\theta$ is defined to be a sequence $\{\hat{\theta}_n\}$ of $\mathcal{B}^{(n)}$-measurable functions $\hat{\theta}_n$ on $\mathcal{X}^{(n)}$ into $\Theta$. For a sequence of positive numbers $\{c_n\}$ ($c_n$ tending to infinity) an estimator $\{\hat{\theta}_n\}$ is called strongly consistent with order $\{c_n\}$ (or strongly $\{c_n\}$-consistent for short) if for every $\varepsilon > 0$ and for every compact subset $K$ of $\Theta$, there exists a sufficiently large positive number $L$ satisfying the following:

$$\lim_{n \to \infty} \sup_{\theta \in K} P_\theta^{(n)}(\{c_n|\hat{\theta}_n - \theta| \geq L\}) < \varepsilon.$$

* Received on June 9, 1976
This research was supported by Japan Ministry of Education.
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A weaker definition of a \( \{ c_n \} \)-consistent estimator than that of the above form has been given in Akahira [1].

We suppose that every \( P_\theta(\cdot) (\theta \in \Theta) \) is absolutely continuous with respect to a \( \sigma \)-finite measure \( \mu \). Then we denote the density \( dP_\theta/d\mu \) by \( f(\cdot : \theta) \). If the distribution of \( x_n \) is the product measure \( P_\theta^{(n)} \), then the corresponding density with respect to the product measure \( \mu^{(n)} \) will be denoted by \( \prod f(x_i : \theta) \). A statistic \( T_n(\bar{x}_n) \) is called asymptotically sufficient if there exist a nonnegative function \( p_n(x_n : \theta) \), each the product of a function of \( x_n \) only by a function of \( T_n \) and \( \theta \) only such that
\[
\lim_{n \to \infty} \sup_{\theta \in K, \bar{x} \in \mathcal{X}} \left| \prod f(x_i : \theta) - p_n(x_n : \theta) \right| d\mu^{(n)} = 0
\]
for any compact subset \( K \) of \( \Theta \) (LeCam [4]).

3. Asymptotically sufficient statistics

Before discussing the asymptotic sufficiency in detail we shall give a definition and a lemma.

Definition. (Generalized from Gnedenko and Kolmogorov [3]) For each \( \theta \in \Theta \) the sums
\[
Y_n(\theta) = X_1(\theta) + X_2(\theta) + \cdots + X_n(\theta)
\]
of positive independent random variables \( X_1(\theta), X_2(\theta), \ldots, X_n(\theta), \ldots \) are said to be uniformly relatively stable for constants \( B_n(\theta) \) if there exist positive constants \( B_1, B_2, \ldots \) such that for any \( \varepsilon > 0 \)
\[
P_\theta^{(n)} \left( \left| \frac{Y_n(\theta)}{B_n(\theta)} - 1 \right| > \varepsilon \right) \to 0
\]
as \( n \to \infty \) uniformly in any compact subset of \( \Theta \).

In the subsequent lemma we use the notation that for each \( k \) and each \( \theta \in \Theta \), \( F_{\theta k}(x) \) is the distribution function of \( X_k(\theta) \).

Lemma. (Gnedenko and Kolmogorov [3]).

For each \( \theta \in \Theta \), let \( X_1(\theta), X_2(\theta), \ldots, X_n(\theta), \ldots \) be a sequence of positive independent random variables. The sums
\[
Y_n(\theta) = X_1(\theta) + X_2(\theta) + \cdots + X_n(\theta)
\]
are uniformly relatively stable for constants \( B_n(\theta) \) if there exists a sequence of positive constants \( B_1, B_2, \ldots, B_n, \ldots \) such that for any \( \varepsilon > 0 \)
\[
\sum_{k=1}^{n} \sum_{x B_n} dF_{\theta k}(x) \to 0
\]
as \( n \to \infty \) uniformly in any compact subset of \( \Theta \),
\[
\frac{1}{B_n(\theta)} \sum_{k=1}^{n} e^{B_k(\theta)} \sum_{x B_n} x dF_{\theta k}(x) \to 1
\]
as \( n \to \infty \) uniformly in any compact subset of \( \Theta \).

Let \( \mathcal{X} = \mathbb{R}^1 \). Now we suppose that every \( P_\theta(\cdot) (\theta \in \Theta) \) is absolutely continuous with respect to a Lebesgue measure \( m \). Then we denote the density \( dP_\theta/dm \) by \( f(\cdot : \theta) \) and by \( A(\theta) \subseteq \mathcal{X} \) the set of points in the space of \( \mathcal{X} \) for which \( f(x : \theta) > 0 \) and suppose \( f(x : \theta) = f(x - \theta) \).

We make the following assumptions (A), (B) and (C).

Assumption (A). \( f(x) > 0 \) for \( a \leq x \leq b \);
\( f(x) = 0 \) for \( x < a, x > b \),
and \( f(a) \) and \( f(b) \) are finite.
Assumption (B). \( f(x) \) is twice continuously differentiable in the interval \((a, b)\).

Define
\[
\varphi(\theta) = \int_0^\infty w \text{d}F(w : \theta)
\]
where \( F(w : \theta) \) is the distribution function of
\[
W(X : \theta) = \chi(a, b \cap A(\theta))(X) \log \frac{f(x - \theta)}{f(x)}
\]
\((\chi(a, b \cap A(\theta))(\cdot)) \) denotes the indicator of \((a, b) \cap A(\theta))\).

Let \( T_n = (Y, Z) \), where \( Y = \min X_i \) and \( Z = \max X_i \). We suppose that \( \{\hat{\theta}_n(T_n)\} \) is a \( \{c_n\} \)-consistent estimator. The existence of the estimator is guaranteed (See Theorem 4.1 of [1]). Then for any \( \delta > 0 \) and any compact subset \( K \) of \( \Theta \) there exists a sufficiently large positive number \( L \) satisfying the following:

\[
\lim \sup_{n \to \infty} P_{\theta} \left( \{||\hat{\theta}_n(T_n) - \theta| > Lc_n^{-1}\} \right) < \delta. \tag{3.1}
\]

Assumption (C). The following \((3.2) \sim (3.4)\) hold:
\[
\lim_{n \to \infty} n \varphi(Lc_n^{-1}) = 0 \tag{3.2}
\]
\[
\lim_{n \to \infty} \sup_{\theta \in K} n \int_{\varphi(Lc_n^{-1})}^\infty \text{d}F(w : \theta) = 0 \tag{3.3}
\]
for any \( \varepsilon > 0 \) and any compact subset \( K \) of \( \Theta \);
\[
\lim_{n \to \infty} \sup_{\theta \in K} \frac{1}{\varphi(Lc_n^{-1})} \int_{\varphi(Lc_n^{-1})}^\infty \text{d}F(w : \theta) = 1 \tag{3.4}
\]
for any \( \varepsilon > 0 \) and any compact subset \( K \) of \( \Theta \).

**Theorem.** Under Assumptions (A), (B) and (C), the statistic \( T_n \), i.e. \((\min X_i, \max X_i)\), is asymptotically sufficient.

**Proof.** Let \( \varepsilon \) be an arbitrary positive number. We define \( h(T_n, \theta) \) and \( g(\bar{x}_n, \hat{\theta}_n(T_n)) \) as follows:
\[
h(T_n, \theta) = \chi(\theta, x) = \begin{cases} 1, & \text{if } x - b < \theta < y - a; \\ 0, & \text{otherwise} \end{cases} \tag{3.5}
\]
\[
g(\bar{x}_n, \hat{\theta}_n(T_n)) = \prod_{i=1}^n f(x_i - \theta_n(T_n)). \tag{3.6}
\]

It follows from \((3.3), (3.4)\) and Lemma that \( \sum_{i=1}^n W(X_i : \hat{\theta}_n(T_n) - \theta) \) is uniformly relatively stable for \( n \varphi(Lc_n^{-1}) \). Hence we have for any compact subset \( K \) of \( \Theta \)
\[
\lim_{n \to \infty} \inf_{\theta \in K} P_{\theta} \left( A_n(\hat{\theta}_n(T_n) - \theta : \varepsilon) \right) = 1,
\]
where \( A_n(\hat{\theta}_n(T_n) - \theta : \varepsilon) = \left[ \bar{x}_n : \left[ \frac{1}{n \varphi(Lc_n^{-1})} \sum_{i=1}^n W(x_i : \hat{\theta}_n(T_n) - \theta) \right] - 1 \right] < \varepsilon \).

It follows from \((3.1), (3.2)\) and \((3.5) \sim (3.7)\) that for any compact subset \( K \) of \( \Theta \)
\[
\limsup_{n \to \infty} \sup_{\theta \in K} \left( \prod_{i=1}^n f(x_i - \theta) - h(T_n, \theta)g(\bar{x}_n, \hat{\theta}_n(T_n)) \right) \leq \sum_{\theta \in K} \left( \int_{|\hat{\theta}_n(T_n) - \theta| \leq Lc_n^{-1}} A_n(\hat{\theta}_n(T_n) - \theta : \varepsilon) \right) \cdot \left( \prod_{i=1}^n f(x_i - \theta) - h(T_n, \theta)g(\bar{x}_n, \hat{\theta}_n(T_n)) \right) \prod_{i=1}^n \text{d}x_i
\]
\[
\leq \sum_{\theta \in K} \left( \int_{|\hat{\theta}_n(T_n) - \theta| > Lc_n^{-1}} A_n(\hat{\theta}_n(T_n) - \theta : \varepsilon) \right) \cdot \left( \prod_{i=1}^n f(x_i - \theta) - h(T_n, \theta)g(\bar{x}_n, \hat{\theta}_n(T_n)) \right) \prod_{i=1}^n \text{d}x_i
\]
\[
\leq \sum_{\theta \in K} \left( \int_{|\hat{\theta}_n(T_n) - \theta| \leq Lc_n^{-1}} A_n(\hat{\theta}_n(T_n) - \theta : \varepsilon) \right) \cdot \left( \prod_{i=1}^n f(x_i - \theta) - h(T_n, \theta)g(\bar{x}_n, \hat{\theta}_n(T_n)) \right) \prod_{i=1}^n \text{d}x_i
\]
\[
\leq \sum_{\theta \in K} \left( \int_{|\hat{\theta}_n(T_n) - \theta| > Lc_n^{-1}} A_n(\hat{\theta}_n(T_n) - \theta : \varepsilon) \right) \cdot \left( \prod_{i=1}^n f(x_i - \theta) - h(T_n, \theta)g(\bar{x}_n, \hat{\theta}_n(T_n)) \right) \prod_{i=1}^n \text{d}x_i
\]
\[= \lim_{n \to \infty} \sup_{\theta \in K} \left\{ \prod_{i=1}^{n} f(x_i - \theta) - \prod_{i=1}^{n} f(x_i - \hat{\theta}_n(T_n)) \right\} d(x_i) \]

\[+ \lim_{n \to \infty} \sup_{\theta \in K} 2 \sum_{i=1}^{n} f(x_i - \theta) d(x_i) \]

\[+ \lim_{n \to \infty} \sup_{\theta \in K} \left\{ 1 - P_{\theta}^{(n)}(A_n(\theta) - \theta : \varepsilon) \right\} \]

\[\leq \lim_{n \to \infty} \sup_{\theta \in K} \left\{ \prod_{i=1}^{n} f(x_i - \hat{\theta}_n(T_n)) - 1 \right\} \prod_{i=1}^{n} f(x_i - \theta) d(x_i) + 2\delta \]

\[\leq \lim_{n \to \infty} \left\{ \exp \sum_{i=1}^{n} W(x_i : Lc_n^{-1}) \right\} - 1 \prod_{i=1}^{n} f(x_i) d(x_i) + 2\delta \]

\[= 2\delta \]

Letting \(\delta \to 0\), we complete the proof of the theorem.

**Acknowledgements**

The author wishes to thank Professor K. Takeuchi of Tokyo University for valuable suggestions and Professor T. Homma for his encouragement.

**References**


