A Remark on Asymptotic Sufficiency of Statistics in Non-Regular Cases

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A Remark on Asymptotic Sufficiency of Statistics in Non-Regular Cases*

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Abstract

Suppose that \(X_1, X_2, \ldots, X_n, \ldots\) is a sequence of independent identically distributed random variables with the density \(f(x: \theta)\) with a compact support, where \(\theta\) is a real valued parameter. We suppose that a strongly \(\{c_n\}\)-consistent estimator of \(\theta\) exists. Then we show that a statistic \((\min_{1 \leq i \leq n} X_i, \max_{1 \leq i \leq n} X_i)\) is asymptotically sufficient in non-regular cases.

1. Introduction

A consistent estimator with order \(\{c_n\}\) (or a \(\{c_n\}\)-consistent estimator) is defined and discussed in Akahira [1], where the necessary conditions for the existence of such an estimator are established and the bounds of the orders of convergence of consistent estimators are obtained for non-regular cases. Further the asymptotic accuracies of \(\{c_n\}\)-consistent estimators are discussed in Akahira [2].

Asymptotic sufficiency has been discussed under regularity conditions by LeCam [4]. In this paper we extend a similar approach to non-regular cases.

Let \(X_1, X_2, \ldots, X_n, \ldots\) be a sequence of independent identically distributed random variables with the density \(f(x: \theta)\) with a compact support, where \(\theta\) is a real valued parameter. We suppose that a strongly \(\{c_n\}\)-consistent estimator of \(\theta\) exists. Then we shall obtain that a statistic \((\min_{1 \leq i \leq n} X_i, \max_{1 \leq i \leq n} X_i)\) is asymptotically sufficient in non-regular cases.

2. Notations and definitions

Let \(\mathcal{X}\) be an abstract sample space whose generic point is denoted by \(x\), \(\mathcal{B}\) a \(\sigma\)-field of subsets of \(\mathcal{X}\) and \(\{P_\theta: \theta \in \Theta\}\) a set of probability measures on \(\mathcal{B}\), where \(\Theta\) is called a parameter space. We suppose that \(\Theta\) is an open set in a Euclidean 1-space \(\mathbb{R}\). Consider \(n\)-fold direct products \((\mathcal{X}^{(n)}, \mathcal{B}^{(n)})\) of \((\mathcal{X}, \mathcal{B})\) and the corresponding product measure \(P_\theta^{(n)}\) of \(P_\theta\). For each \(n=1, 2, \ldots\), the points of \(\mathcal{X}^{(n)}\) will be denoted by \(\bar{x}_n=(x_1, \ldots, x_n)\) and the corresponding random variable by \(\bar{X}_n\). An estimator of \(\theta\) is defined to be a sequence \(\{\hat{\theta}_n\}\) of \(\mathcal{B}^{(n)}\)-measurable functions \(\hat{\theta}_n\) on \(\mathcal{X}^{(n)}\) into \(\Theta\). For a sequence of positive numbers \(\{c_n\}\) (\(c_n\) tending to infinity) an estimator \(\{\hat{\theta}_n\}\) is called strongly consistent with order \(\{c_n\}\) (or strongly \(\{c_n\}\)-consistent for short) if for every \(\epsilon > 0\) and for every compact subset \(K\) of \(\Theta\), there exists a sufficiently large positive number \(L\) satisfying the following:

\[
\lim_{n \to \infty} \sup_{\theta \in K} P_\theta^{(n)}(\{c_n | \hat{\theta}_n - \theta | \geq L\}) < \epsilon.
\]

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A weaker definition of a \(\{e_n\}\)-consistent estimator than that of the above form has been given in Akahira [1].

We suppose that every \(P_\theta(\cdot) (\theta \in \Theta)\) is absolutely continuous with respect to a \(\sigma\)-finite measure \(\mu\). Then we denote the density \(dP_\theta/d\mu\) by \(f(\cdot : \theta)\). If the distribution of \(\bar{x}_n\) is the product measure \(P^{(n)}_\theta\), then the corresponding density with respect to the product measure \(\mu^{(n)}\) will be denoted by \(\prod_{i=1}^n f(x_i : \theta)\). A statistic \(T_n(\bar{X}_n)\) is called asymptotically sufficient if there exist a nonnegative function \(\rho_n(\bar{x}_n : \theta)\), each the product of a function of \(\bar{x}_n\) only by a function of \(T_n\) and \(\theta\) only such that

\[
\lim_{n \to \infty} \sup_{\theta \in K} \left| \prod_{i=1}^n f(x_i : \theta) - \rho_n(\bar{x}_n : \theta) \right| d\mu^{(n)}(\theta) = 0
\]

for any compact subset \(K\) of \(\Theta\) (LeCam [4]).

3. Asymptotically sufficient statistics

Before discussing the asymptotic sufficiency in detail we shall give a definition and a lemma.

Definition. (Generalized from Gnedenko and Kolmogorov [3]) For each \(\theta \in \Theta\) the sums

\[
Y_n(\theta) = X_1(\theta) + X_2(\theta) + \cdots + X_n(\theta)
\]

of positive independent random variables \(X_1(\theta), X_2(\theta), \cdots, X_n(\theta), \cdots\) are said to be uniformly relatively stable for constants \(B_n(\theta)\) if there exist positive constants \(B_1(\theta), B_2(\theta), \cdots, B_n(\theta), \cdots\) such that for any \(\varepsilon > 0\)

\[
P^{(n)}_\theta\left(\left| \frac{Y_n(\theta)}{B_n(\theta)} - 1 \right| > \varepsilon\right) \to 0
\]

as \(n \to \infty\) uniformly in any compact subset of \(\Theta\).

In the subsequent lemma we use the notation that for each \(k\) and each \(\theta \in \Theta\), \(F_{\theta k}(x)\) is the distribution function of \(X_k(\theta)\).

Lemma. (Gnedenko and Kolmogorov [3]).

For each \(\theta \in \Theta\), let \(X_1(\theta), X_2(\theta), \cdots, X_n(\theta), \cdots\) be a sequence of positive independent random variables. The sums

\[
Y_n(\theta) = X_1(\theta) + X_2(\theta) + \cdots + X_n(\theta)
\]

are uniformly relatively stable for constants \(B_n(\theta)\), if there exists a sequence of positive constants \(B_1(\theta), B_2(\theta), \cdots, B_n(\theta), \cdots\) such that for any \(\varepsilon > 0\)

\[
\sum_{k=1}^n \int_{x \in \mathcal{B}_k} dF_{\theta k}(x) \to 0
\]

as \(n \to \infty\) uniformly in any compact subset of \(\Theta\),

\[
\frac{1}{B_n(\theta)} \sum_{k=1}^n e^{B_n(\theta)} \int_0^{x \in \mathcal{B}_k} x dF_{\theta k}(x) \to 1
\]

as \(n \to \infty\) uniformly in any compact subset of \(\Theta\).

Let \(\mathcal{X} = \mathbb{R}^1\). Now we suppose that every \(P_\theta(\cdot) (\theta \in \Theta)\) is absolutely continuous with respect to a Lebesgue measure \(m\). Then we denote the density \(dP_\theta/dm\) by \(f(\cdot : \theta)\) and by \(A(\theta) \subset \mathcal{X}\) the set of points in the space of \(\mathcal{X}\) for which \(f(x : \theta) > 0\) and suppose \(f(a : \theta) = f(x - \theta)\). We make the following assumptions (A), (B) and (C).

Assumption (A).

\[
f(x) > 0 \quad \text{for} \quad a \leq x \leq b ;
\]

\[
f(x) = 0 \quad \text{for} \quad x < a, \ x > b,
\]

and \(f(a)\) and \(f(b)\) are finite.
Assumption (B). \( f(x) \) is twice continuously differentiable in the interval \((a, b)\).

Define

\[
\varphi(\theta) = \int_0^\infty w d F(w : \theta),
\]

where \( F(w : \theta) \) is the distribution function of

\[
W(X : \theta) = \chi_{(a,b)\cap A(\theta)}(X) \left| \frac{\log f(x - \theta)}{f(x)} \right|
\]

\((\chi_{(a,b)\cap A(\theta)}(\cdot))\) denotes the indicator of \((a, b)\cap A(\theta))\).

Let \( T_n = (Y, Z) \), where \( Y = \min_{1 \leq i \leq n} X_i \) and \( Z = \max_{1 \leq i \leq n} X_i \). We suppose that \( \{\hat{\theta}_n(T_n)\} \) is a \( \{c_n\} \)-consistent estimator. The existence of the estimator is guaranteed (See Theorem 4.1 of \([1]\)).

Then for any \( \delta > 0 \) and any compact subset \( K \) of \( \Theta \) there exists a sufficiently large positive number \( L \) satisfying the following:

\[
\limsup_{n \to \infty} \frac{1}{\{c_n\}} P_{\theta_n(T_n)}(\{\hat{\theta}_n(T_n) - \theta > Lc_n^{-1}\}) < \delta. \tag{3.1}
\]

Assumption (C). The following \((3.2) \sim (3.4)\) hold:

\[
\lim_{n \to \infty} n \varphi(Lc_n^{-1}) = 0 \tag{3.2}
\]

\[
\limsup_{n \to \infty} n \int_{\{x \in K\}} d F(w : \theta) = 0 \tag{3.3}
\]

for any \( \varepsilon > 0 \) and any compact subset \( K \) of \( \Theta \);

\[
\lim_{n \to \infty} \frac{1}{n \varphi(Lc_n^{-1})} \int_{\{x \in K\}} w d F(w : \theta) = 1 \tag{3.4}
\]

for any \( \varepsilon > 0 \) and any compact subset \( K \) of \( \Theta \).

Theorem. Under Assumptions (A), (B) and (C), the statistic \( T_n \), i.e. \( (\min_{1 \leq i \leq n} X_i, \max_{1 \leq i \leq n} X_i) \), is asymptotically sufficient.

Proof. Let \( \varepsilon \) be an arbitrary positive number. We define \( h(T_n, \theta) \) and \( g(\bar{x}_n, \hat{\theta}_n(T_n)) \) as follows:

\[
h(T_n, \theta) = \chi_\theta(y, z) = \begin{cases} 1, & \text{if } z - b < \theta < y - a; \\ 0, & \text{otherwise} \end{cases} \tag{3.5}
\]

\[
g(\bar{x}_n, \hat{\theta}_n(T_n)) = \prod_{i=1}^n f(x_i - \hat{\theta}_n(T_n)). \tag{3.6}
\]

It follows from \((3.3), (3.4) \) and Lemma that \( \sum_{i=1}^n W(X_i : \hat{\theta}_n(T_n) - \theta) \) is uniformly relatively stable for \( n \varphi(Lc_n^{-1}) \). Hence we have for any compact subset \( K \) of \( \Theta \)

\[
\liminf_{n \to \infty} P_{\theta_n(T_n)}(\{A_n(\hat{\theta}_n(T_n) - \theta : \varepsilon) = 1,
\]

where \( A_n(\hat{\theta}_n(T_n) - \theta : \varepsilon) = \left\{ \bar{x}_n : \left| \frac{1}{n \varphi(Lc_n^{-1})} \sum_{i=1}^n W(x_i : \hat{\theta}_n(T_n) - \theta) - 1 \right| < \varepsilon \right\} \).
Letting \( \delta \to 0 \), we complete the proof of the theorem.

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**References**


