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AN INFORMATION INEQUALITY FOR THE
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Abstract: For a family of uniform distribution on the interval \([\theta - (1/2), \theta - (1/2)]\), the information inequality for the bayes risk of any estimator of \(\theta\) is given under the quadratic loss and the uniform prior distribution on an interval \([-c, c]\). The lower bound for the Bayes risk is shown to be sharp. And also the lower bound for the limit inferior of Bayes risk as \(c \to \infty\) is seen to be attained by the mid-range estimator.

Key words: Cramér-Rao inequality; Bayes Estimator; lower bound; mid-range

1. Introduction: In the paper, Vincze (1979) obtained Cramer-Rao type inequality in the non-regular case, and for the uniform distribution on the interval \([\theta - (1/2), \theta - (1/2)]\) got the lower bound for the variance of unbiased estimator with the right order of magnitude, but it was not sharp. Following ideas of Vincze (1979), Khatri (1980) gave a simple general approach to the non-regular Cramer-Rao bound. In the relation to Vincze (1979), Mórı (1983) also obtained the lower bound for the limit inferior of the expected quadratic risk of unbiased estimators of \(\theta\) under the uniform distribution on the interval \([-c, c]\) as \(c \to \infty\) and showed that it was sharp. In this paper, for a family of uniform distributions on \([\theta - (1/2), \theta - (1/2)]\), we obtain the information inequality for the Bayes risk of any estimator of \(\theta\) under the quadratic loss and the uniform prior distribution on an interval \([-c, c]\) by a somewhat different way of Mori (1983). We also show that the lower bound for the Bayes risk of any estimator of \(\theta\) is sharp, and that the lower bound for the limit inferior of Bayes risk of any estimator of \(\theta\) as \(c \to \infty\) is attained by the mid-range, which involves the result for unbiased estimators of \(\theta\) by Mori (1983). The related results to the above are found in Akahira and Takeuchi (1995).

2. An information inequality for the Bayes risk of any estimator: Suppose that \(X_1, X_2, \ldots, X_n\) are independent and identically distributed random variables according to the uniform distribution with a density \(p(x, \theta)\) on the interval \([\theta - (1/2), \theta - (1/2)]\), where \(-\infty < \theta < \infty\). Let \(n\) be fixed, and let \(\widehat{\theta} = \widehat{\theta}(X)\) be an estimator of \(\theta\) based on the sample \(X = (X_1, X_2, \ldots, X_n)\). Then we consider the Bayes risk \(r_c(\widehat{\theta})\) of any estimator \(\widehat{\theta}\) of \(\theta\) under the quadratic loss and the uniform prior distribution on an interval \([-c, c]\), where \(-\infty < c < \infty\), i.e.

\[
r_c(\widehat{\theta}) := \frac{1}{2c} \int_{-c}^{c} E_\theta \left[ (\widehat{\theta} - \theta)^2 \right] d\theta.
\]

Let \(f(x, \theta) := \prod_{i=1}^{n} p(x_i, \theta)\) with \(x = (x_1, x_2, \ldots, x_n)\). In order to get the Bayes estimator, i.e. to minimize \(r_c(\widehat{\theta})\), it is enough to obtain the estimator minimizing

\[
\int_{-c}^{c} \left\{ \widehat{\theta}(x) - \theta \right\}^2 f(x, \theta) d\theta
\]
for almost all $x$. Such an estimator is easily given by

$$
\hat{\theta}_c(X) = \int_{-\infty}^{\infty} \theta f(X, \theta) d\theta \Bigg/ \int_{-\infty}^{\infty} f(X, \theta) d\theta.
$$

(1)

Here, we have

$$
f(x, \theta) = \begin{cases} 
1 & \text{for } x_{(n)} - (1/2) \leq \theta \leq x_{(1)} + (1/2) \\
0 & \text{otherwise},
\end{cases}
$$

(2)

where $x_{(1)} := \min_{1 \leq i \leq n} x_i$ and $x_{(n)} := \max_{1 \leq i \leq n} x_i$. Let $\theta := X_{(n)} - (1/2), \bar{\theta} := X_{(1)} + (1/2)$. From (1) and (2) we have

$$
\hat{\theta}_c(X) = \begin{cases} 
\frac{1}{2} (\theta + c) & \text{for } -c < \theta, \theta \leq c \leq \bar{\theta} \\
\frac{1}{2} (\theta + \bar{\theta}) & \text{for } -c < \theta, \bar{\theta} < c \\
\frac{1}{2} (\bar{\theta} - c) & \text{for } \theta \leq -c \leq \bar{\theta}, \theta < c \\
0 & \text{otherwise},
\end{cases}
$$

(3)

where $0/0 = 0$ and $c > 1/2$. Then we have following.

**THEOREM 1.** The information inequality for the Bayes risk of any estimator $\hat{\theta}$ of $\theta$ is given by

$$_c(\hat{\theta}) = \frac{1}{2c} \int_{-\infty}^{\infty} E_{\theta} \left[ (\hat{\theta} - \theta)^2 \right] d\theta
$$

$$
= \frac{1}{2 (n+1) (n+2)} - \frac{1}{2c(n+1)(n+2)(n+3)} = A_0(c) \quad \text{(say)},
$$

(4)

where $c > 1/2$, and the lower bound is sharp, that is, $\hat{\theta}_c$ attains the bound.

**Proof 1.** The joint density function $f_{\theta, \bar{\theta}}$ of $(\theta, \bar{\theta})$ is given by

$$
f(x, \theta) = \begin{cases} 
n(n-1)(y-z+1)^{n-2} & \text{for } y \leq \theta \leq z, 0 \leq z - y \leq 1, \\
0 & \text{otherwise}.
\end{cases}
$$

Then we have

$$
r_c = \int_{-\infty}^{\infty} E_{\theta} \left[ \left( \hat{\theta}_c - \theta \right)^2 \right] d\theta
$$

$$
= \int_{y \leq \theta \leq z} \int_{y \leq z \leq y+1} \int_{-\infty}^{\infty} \left( \hat{\theta}_c(y, z) - \theta \right)^2 f_{\theta, \bar{\theta}}(y, z) d\theta dy dz.
$$

Since

$$
\int_{|\theta| \leq c, y \leq \theta \leq z} \left( \hat{\theta}_c - \theta \right)^2 f_{\theta, \bar{\theta}}(y, z) d\theta
$$

$$
= \hat{\theta}_c^2 \int_{|\theta| \leq c, y \leq \theta \leq z} f_{\theta, \bar{\theta}}(y, z) d\theta - 2\hat{\theta}_c \int_{|\theta| \leq c, y \leq \theta \leq z} \theta f_{\theta, \bar{\theta}}(y, z) d\theta
$$

$$
+ \int_{|\theta| \leq c, y \leq \theta \leq z} \theta^2 f_{\theta, \bar{\theta}}(y, z) d\theta
$$

$$
= n(n-1)(y-z+1)^{n-2} \left( \hat{\theta}_c^2 \int_{|\theta| \leq c, y \leq \theta \leq z} d\theta - 2\hat{\theta}_c \int_{|\theta| \leq c, y \leq \theta \leq z} \theta d\theta
$$

$$
+ \int_{|\theta| \leq c, y \leq \theta \leq z} \theta^2 d\theta \right)
$$

$$
= n(n-1)(y-z+1)^{n-2} \left( I_1 \hat{\theta}_c^2 - 2I_2 \hat{\theta}_c + I_3 \right)
$$

$$
= G_n(y, z) \quad \text{(say)},
$$

where

$$
I_1 := \min \{c, z\} - \max \{-c, y\},
$$

$$
I_2 := \frac{1}{2} \left[ \left( \min \{c, z\} \right)^2 - \left( \max \{-c, y\} \right)^2 \right],
$$

$$
I_3 := \frac{1}{3} \left[ \left( \min \{c, z\} \right)^3 - \left( \max \{-c, y\} \right)^3 \right].
$$
Next we obtain

\[ r^* = \left( \int_{-c}^{1-c} \int_{z-1}^{c} \int_{y}^{z} \int_{y}^{c} G_n(y, z) dy dz + \int_{1-c}^{c} \int_{y}^{z} \int_{y}^{c} G_n(y, z) dy dz \right) \]

Repeating integration by parts we have

\[ J_1 = \int_{-c}^{1-c} \int_{z-1}^{c} n(n-1)(y-z+1)^{n-2} \left\{ \frac{1}{4} (z-c)^2 (z+c) - \frac{1}{2} (z-c) (z^2-c^2) + \frac{1}{3} (z^3-c^3) \right\} dy dz \]

\[ = \frac{1}{2 (n+1)(n+2)(n+3)} \]

\[ J_2 = \int_{-c}^{1-c} \int_{1}^{z} n(n-1)(y-z+1)^{n-2} \left\{ \frac{1}{4} (y+z)^2 (y-z) - \frac{1}{2} (y+z) (y^2-z^2) + \frac{1}{3} (y^3+z^3) \right\} dy dz \]

\[ = \frac{1}{2 (n+1)(n+2)} \]

\[ J_3 = \int_{1}^{c} \int_{z-1}^{c} n(n-1)(y-z+1)^{n-2} \left\{ \frac{1}{4} (y+z)^2 (y-z) - \frac{1}{2} (y+z) (y^2-z^2) + \frac{1}{3} (y^3+z^3) \right\} dy dz \]

\[ = \frac{1}{2 (n+1)(n+2)} \]

\[ J_4 = \int_{c}^{1+c} \int_{z-1}^{1} n(n-1)(y-z+1)^{n-2} \left\{ \frac{1}{4} (c+y)^2 (c-y) - \frac{1}{2} (c+y) (c^2-y^2) + \frac{1}{3} (c^3+y^3) \right\} dy dz \]

\[ = \frac{1}{2 (n+1)(n+2)(n+3)} \]

From (5) to (9) we have

\[ r^* = \frac{c}{(n+1)(n+2)} - \frac{1}{(n+1)(n+2)(n+3)} \]

Since \( \hat{\theta}_c^* \) minimize the Bayes risk \( r_c(\hat{\theta}) \), it follows from (10) that for any estimator \( \hat{\theta} \) of \( \theta \)

\[ r_c(\hat{\theta}) = \frac{1}{2c} \int_{-c}^{c} E_{\theta} \left[ (\hat{\theta} - \theta)^2 \right] d\theta \geq \frac{1}{2c} r^* = \frac{1}{2 (n+1)(n+2)} - \frac{1}{2c (n+1)(n+2)(n+3)} \]

Thus we complete the proof.

**Corollary 1.** For any estimator \( \hat{\theta} \) of \( \theta \)

\[ \lim_{c \to \infty} r_c(\hat{\theta}) \geq \frac{1}{2 (n+1)(n+2)} \]

The proof of Corollary is straightforward from the Theorem. The lower bound (11) is easily seen to be attained by mid-range \( \hat{\theta}_0 := (X_{(1)} + X_{(2)}) / 2 \).

**Remark 1.** The inequality of the Corollary is same as one for any unbiased estimator given by Móri (1983).
3. Comparison of the lower bounds: In this section we compare the lower bound $A_0(c)$ with Móri's one. Let

$$I_c := -A_0(c) + \frac{c^2}{3}. \quad (12)$$

In the proof of the Theorem in the paper, Móri (1983) showed that for any unbiased estimator $\hat{\theta}$ of $\theta$

$$r_c(\hat{\theta}) = \frac{1}{2c} \int_{-c}^{c} V_\theta(\hat{\theta}) \, d\theta \geq \frac{c^4}{9I_c} - \frac{c^2}{3} = M(c) \text{ (say)}, \quad (13)$$

where $c > 1/2$. But the lower bound $M(c)$ is not sharp, as is mentioned in the paper. From (12) and (13) it seen that

$$M(c) > A_0(c) \text{ for } c > 1/2.$$ 

here, note that $A_0(c)$ is the lower bound for the Bayes risk for any estimator and $M(c)$ is one for any unbiased estimator. And also we have

$$M(c) = A_0(c) + \frac{3}{4c^2(n+1)^2(n+2)^2} + O\left(\frac{1}{c^2}\right) \quad c \to \infty,$$

hence

$$\lim_{c \to \infty} M(c) = \lim_{c \to \infty} A_0(c) = \frac{1}{2(n+1)(n+2)}.$$ 

For a family of uniform distribution on $[\theta - (\tau/2), \theta + (\tau/2)]$ with a scale $\tau$ as a nuisance parameter, we also have a similar information inequality to (4) as follows. For any estimator $\hat{\theta}$ of $\theta$

$$R_c(\hat{\theta}) = \int_{-c}^{c} E_\theta \left[ \left( \hat{\theta} - \theta \right)^2 \right] \, d\theta \geq \frac{\tau}{2(n+1)(n+2) - 2c(n+1)(n+2)(n+3)} \quad (14)$$

and

$$\lim_{c \to \infty} R_c(\hat{\theta}) \geq \frac{1}{2(n+1)(n+2)}. \quad (15)$$

In particular, letting $\tau = 1$, we have the inequality (4) from (14). When $c$ tends to infinity, from (15) we have the same lower bound as (11).

4. Comments: In the previous section we obtain the lower bound for the Bayes risk of estimators under the quadratic loss and the uniform prior distribution on an interval $[-c, c]$, where $c > 1/2$, and show that the bound is sharp. Recently Akahira and Takeuchi (2001) shows that for small $c > 0$ the Bayes risk of any estimator in the interval of $\theta$ values of length $2c$ and centered at $\theta_0$ can not be smaller than that of $\hat{\theta}_0 = (X_1 + X_n)/2$. More precisely they prove that for any estimator $\hat{\theta} = \hat{\theta}(X)$ based on the sample $X$ of size $n$

$$\lim_{c \to 0} \lim_{n \to \infty} \frac{n^2}{2c} \int_{\theta_0 - c}^{\theta_0 + c} E_\theta \left[ \left( \hat{\theta} - \theta \right)^2 \right] \, d\theta \geq \frac{1}{2},$$

and the lower bound is attained by $\hat{\theta}_0$. This means that in a sense asymptotically the estimator $\hat{\theta}_0$ can be regarded as uniformly best one.
References


