On a braid monoid analogue of a theorem of Tits

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<td>本論文では、ブレッドモノイドにおけるティッツの定理の類似物を導入し、その性質を研究した。</td>
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On a braid monoid analogue of a theorem of Tits

Takeo Fukushi

Submitted to Graduate School of Pure and Applied Sciences in Partial Fulfillment of Requirements for Degree of Doctor of Philosophy in Mathematics at University of Tsukuba
Abstract

In the first part of this thesis, I solve a long-standing problem concerning homotopies of braid monoids. More precisely, I show that for a graph of a braid monoid related to Coxeter group of type $A$, every self-homotopy of a word decomposes into self-homotopies each of which is inessential, a cube, a prism or a permutohedron. Next I extend this result to braid monoids corresponding to more general finite Coxeter groups. In the latter case, it turns out that the so-called Coxeterhedra are needed in addition to the self-homotopies listed above. Using the result above, I then prove the coherence theorem for braided Gray monoids. In the last chapter I give a conjecture about the relationship between our theorem and a work of Deligne.

The second part of this thesis presents my old work about Yang-Baxter operators for crossed group-categories which arose in the context of homotopy quantum field theory. Graphical calculus is used a lot, which I like most in mathematics for the last two decades.

In the last part I review my initial piece about braided crossed modules. The extension theory of braided crossed modules and its mod-$q$ analogues will be given.
Acknowledgements

This paper is dedicated to the memory of my teacher Hiroshi Yanagihara who taught me the joy of expressing myself through mathematics.

Life is very short and there’s no time
for fussing and fighting my friend
I have always thought that it’s a crime
so I will ask you once again

— The Beatles
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Part I
Positive braids and an analogue of Tits’s theorem
Chapter 1

Introduction

Part I of this thesis grew out of an attempt to prove a coherence theorem for braided monoidal 2-categories. This theorem is categorical in nature but the essential part is combinatorial and can be viewed as a theorem about homotopies of words defined by braid relations. In the context of Coxeter groups, Tits [18] showed that every self-homotopy decomposes into self-homotopies each of which is inessential or lies in a rank 3 residue. This means that nontrivial self-homotopies of galleries in Coxeter complexes only occur in finite stars of simplices of codimension 3. To obtain a similar result for braid monoids, we first consider Coxeter groups of type $A$, and prove a variant of a result in [9] which asserts that every positive braid has a unique factorization with respect to a given set of generators. Using this factorization we then show that every self-homotopy decomposes into self-homotopies each of which is inessential, a cube, a prism or a permutohedron. This result is an important step toward the coherence theorem, and it seems to be of independent interest as well.

Next we generalize this result to braid monoids corresponding to the other finite Coxeter groups. In the course, we find certain Coxeterhedra in dimension 3 arise as geometric objects, and our theorem says that Coxeterhedra with higher dimensions are not needed at all. Therefore we can use, in some sense, the classification result of semi-regular convex polytopes in dimension 3. In fact, the Coxeterhedron of type $B$ (resp. type $H$) coincides with what is known as the rhombitruncated cuboctahedron (resp. rhombitrun-
cated icosidodecahedron) in the literature. In this paper we use an algebraic

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interpretation of these polytopes, which we hope to represent the interplay between geometry, algebra, and combinatorics.

In Chapter 8 we give a detailed description of our coherence theorem for braided Gray monoids. This result is a 2-categorical version of Joyal and Street’s coherence theorem for braided monoidal categories. In Chapter 9 we give a conjecture about the relationship between our theorem and the work of Deligne [8].
Chapter 2

Coxeter groups and Coxeterhedra

In this chapter, we review Coxeter groups and Coxeterhedra.

2.1 Coxeter groups

Let $S$ be a set. A matrix $m : S \times S \rightarrow \{1, 2, ..., \infty\}$ is called a Coxeter matrix if it satisfies

\[
\begin{align*}
    m(s, s') &= m(s', s); \\
    m(s, s') &= 1 \iff s = s'.
\end{align*}
\]

Equivalently, $m$ can be represented by a Coxeter graph whose node set is $S$ and whose edges are the unordered pairs $\{s, s'\}$ such that $m(s, s') \geq 3$. If $m(s, s') \geq 4$, the edge is labeled by that number.

Let $S_{fin}^2 = \{(s, s') \in S^2 : m(s, s') \neq \infty\}$. A Coxeter matrix $m$ determines a group $W$ with the presentation given by generators $S$ and relations

\[
(ss')^{m(s, s')} = e, \quad \text{for all } (s, s') \in S_{fin}^2
\]

where $e$ denotes the unit in $W$. When a group $W$ has a presentation above, then the pair $(W, S)$ is called a Coxeter system, and the group $W$ is called a Coxeter group.

For example, take the following Coxeter graph.
Then, the corresponding Coxeter group is given by generators $a, b, c$ and the relations:

$$\begin{align*}
    a^2 &= b^2 = c^2 = e; \\
    (ab)^2 &= e; \\
    (ac)^3 &= (bc)^3 = e.
\end{align*}$$

Coxeter classified all finite Coxeter groups; the corresponding Coxeter graphs are listed below:

- $A_n (n \geq 1)$
- $B_n (n \geq 2)$
- $D_n (n \geq 4)$
- $E_6$
- $E_7$
- $E_8$
- $F_4$
- $H_3$
- $H_4$
- $I_2(m) (m \geq 5)$

The classification above was completed in 1935, and every finite Coxeter
group is known to be a reflection group. Since Coxeter groups are defined by generators and relations, one can easily construct non-finite Coxeter groups. In this paper, we deal with only finite Coxeter groups.

2.2 Coxeterhedra

Definition 2.2.1 For any Coxeter system $(W, S)$, we denote by $W_J$ the subgroups generated by subsets $J \subset S$. The subgroups $W_J$ are called parabolic subgroups of $W$. The Coxeterhedron $PW$ is defined as the finite posets of all cosets

$$\{wW_J\}$$

of all parabolic subgroups of $W$, ordered by inclusion.

This definition is combinatorial, but for any Coxeterhedron, there is a geometric realization of it. In particular, its geometric realization becomes particularly simple in dimension 3.

The following figure presents Coxeterhedra of type $A$, $B$ and $D$.

![Coxeterhedra of type A, B, and D](image)

**Figure 1:** The Coxeterhedra $\text{PA}_3$, $\text{PB}_3$, $\text{PD}_3$

Interestingly, the faces of dimension 0 and 1 of these Coxeterhedra coincide with the graphs of weak order of the related Coxeter systems. In other words, these faces form the Cayley graphs of the Coxeter systems. This is true in general, and this algebraic point of view is the one we adopt in the next chapter.
2.3 Permutohedra

A Coxterhedron of type $A$ is called a *permutohedron*. In particular, the permutohedron $PA_3$ can be depicted as follows:

This polytope plays an important role in various areas of mathematics. In my attempt to prove a coherence theorem for braided monoidal 2-categories, this polytope was inevitable, and the main theorem of Part I says that this polytope is almost enough to prove the coherence theorem. This means, in particular, that we do not need permutohedra with higher dimensions. This allows us to concentrate on polytopes with dimension 3.

We now explain how to interpret this permutohederon in algebraic terms. For example, let us consider a path from the vertex labeled $abcd$ at the top to the vertex labeled $dcba$ at the bottom. In the permutohedron, there is an edge connecting the vertices $abcd$ and $bacd$. To this edge, we can assign the permutation $s_1$ since this element exchanges the left-most part $ab$ and $ba$, and as a result, $s_1$ exchanges $abcd$ and $bacd$. Next we look at the edge connecting the vertices $bacd$ and $bcad$. To this edge we assign the permutation $s_2$ since this element exchanges $ac$ and $ca$. Now we take as a sequence of edges which
connects $abcd$ and $dcba$ the following one:

$$ablc \sim bacd \sim bcad \sim cbda \sim cdba \sim dcba.$$  

Then, using the assignments above we can associate to this sequence the word 121321 which corresponds to the permutation $s_1 s_2 s_1 s_3 s_2$. In this way, to each sequence of edges from $abcd$ to $dcba$, we can associate a word of 1, 2, 3.

As a next step, we consider the following sequence

$$abcl \sim bacd \sim bcda \sim cdba \sim cdba.$$  

Then we find that we can associate to this sequence the word 123121, and this replacement can be viewed as an arrow

$$121321 \to 123121.$$  

Similarly, if we replace the sequence of edges by the following one

$$abcd \sim bacd \sim bcad \sim bcda \sim bda \sim dbca \sim dcba,$$

then this replacement can be viewed as an arrow

$$123121 \to 123212.$$  

Thus, if we start with a sequence of edges from $abcd$ to $dcba$ and iterate this process going around the polytope until we come back to the original sequence, then we obtain a sequence of arrows. For example, we obtain the following one:

$$121321 \to 123121 \to 123212 \to 132312 \to 312312$$

$$\uparrow \quad \quad \quad \quad \quad \quad \quad \downarrow$$

$$212321 \quad \quad \quad \quad \quad 312132$$

$$\uparrow \quad \quad \quad \quad \quad \quad \quad \downarrow$$

$$213231 \quad \quad \quad \quad \quad 321232$$

$$\uparrow \quad \quad \quad \quad \quad \quad \quad \downarrow$$

$$213213 \leftarrow 231213 \leftarrow 232123 \leftarrow 323123 \leftarrow 321323.$$  

This is the algebraic form of a permutohedron, which we use in the next chapter.
Chapter 3

Type A

In this chapter we begin our study of this thesis. The Coxeter groups of type $A$ are the symmetric groups, and the related braid groups consist of the usual braids.

3.1 Positive braids

In this section, we consider positive braids and show that every positive braid has a unique factorization with respect to a given subset of $\{1, 2, ..., n - 1\}$. This is a variant of a result in [9].

For $n \geq 1$, denote by $B_n^+$ the monoid generated by $n - 1$ generators $\sigma_1, \sigma_2, ..., \sigma_{n-1}$ and the relations

\[
\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i - j| \geq 2,
\]

\[
\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{if } |i - j| = 1,
\]

where $i, j = 1, 2, ..., n - 1$. The elements of $B_n^+$ are called positive braids on $n$ strings. Throughout this paper, $e$ denotes the unit in $B_n^+$ and $l$ denotes the length function on $B_n^+$.

**Definition 3.1.1** For a positive braid $P$, an element $i \in \{1, 2, ..., n - 1\}$ is called a starting element of $P$ if there exists $Q \in B_n^+$ such that $P = \sigma_i Q$. Similarly, an element $i \in \{1, 2, ..., n - 1\}$ is called a finishing element of $P$ if there exists $Q \in B_n^+$ such that $P = Q \sigma_i$. For a positive braid $P$, we denote by $S(P)$ the set of starting elements of $P$. Similarly, we denote by $F(P)$ the set of finishing elements of $P$. 

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Definition 3.1.2 A positive braid is called a positive permutation braid if it can be drawn as a geometric braid in which every pair of strings crosses at most once.

In other words, positive permutation braids are the image of the map $\rho : S_n \to B_n^+$ defined by $\rho(w) = \sigma_i \sigma_i \ldots \sigma_i$ for some reduced expression $w = s_i s_i s_i \ldots s_i$, in the symmetric group $S_n$.

For a subset $J$ of $\{1, 2, ..., n-1\}$, let $S_J^+$ be the set consisting of the unit and all permutation braids generated by the set $\{\sigma : i \in J\}$ in $B_n^+$.

Definition 3.1.3 Set

$$\sigma_i * \sigma_j = \begin{cases} \sigma_i & \text{if } i = j, \\ \sigma_i \sigma_j \sigma_i & \text{if } |i - j| = 1, \\ \sigma_i \sigma_j & \text{if } |i - j| \geq 2. \end{cases}$$

We frequently use the following lemmas:

Lemma 3.1.1 For elements $i, j$ in $J$ and for $A \in S_J^+$ we have

$$i \notin S(A) \iff \sigma_i A \in S_J^+, \quad i, j \notin S(A) \iff (\sigma_i * \sigma_j) A \in S_J^+.$$

Proof. These follow from the Exchange Property of Coxeter groups and the characterization of the permutation braids by the map $\rho : S_n \to B_n^+$ above.

Lemma 3.1.2 If $P = AB$ with $P \in S_J^+$ then we have $A, B \in S_J^+$.

Proof. Straightforward.

We also use the following lemma of Garside [12].

Lemma 3.1.3 (Garside) Let $P = P_1 \sigma_i = P_2 \sigma_j$ in $B_n^+$. Then $P = P_3 (\sigma_i * \sigma_j)$ for some $P_3$ in $B_n^+$.

Definition 3.1.4 Given a subset $J$ of $\{1, 2, ..., n-1\}$, a factorization $P = AB$ with $A, B \in B_n^+$ is called $J$-weighted if $B \in S_J^+$ and $F(A) \cap J \subseteq S(B)$. For $X, Y$ in $S_J^+$, $X$ is called a subfactor of $Y$ if $Y = QX$ for some $Q$. 

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Proposition 3.1.1 With $J$ above, every positive braid $P$ has a unique $J$-weighted factorization $P = A_1B_1$. If $P = AB$ is another factorization with $B \in S_j^+$, $B$ becomes a subfactor of $B_1$.

Proof. We first show the existence of a $J$-weighted factorization $P = A_1B_1$. Consider all factorizations $P = AB$ with $B \in S_j^+$, and select one in which $l(B)$ is maximal. If $F(A) \cap J \not\subset S(B)$ then we can find $i \in F(A) \cap J$ with $i \not\in S(B)$. Then we can write $A = A'_i\sigma_i$ for some $A'$, and by Lemma 3.1.1 $\sigma_iB$ becomes an element of $S_j^+$. Set $B' = \sigma_iB$. Then $P = AB'$ with $l(B') \geq l(B)$, which is a contradiction.

We now show that every other factorization $P = AB$ with $B \in S_j^+$ satisfies $B_1 = QB$ for some $Q$. Otherwise there exist factorizations

$$P = A'\sigma_iC$$

with $\sigma_iC \in S_j^+$ such that $C$ is a subfactor of $B_1$ but $\sigma_iC$ is not. Choose such a factorization with largest possible length $C$, and write $B_1 = QC$. If $Q = e$ then $P = A\sigma_iB_1$ with $\sigma_iB_1 \in S_j^+$, which contradicts the maximality of $l(B_1)$. Thus $Q \not= e$, and we can choose $j \in F(Q) \cap J$ to write $Q = Q'\sigma_j$ for some $Q'$. Then $P = A_1B_1 = A_1Q'\sigma_jC$, and by setting $A'' = A_1Q'$, we have

$$P = A''\sigma_jC.$$  

From the identity $P = A'\sigma_iC = A''\sigma_jC$, it follows that $A'\sigma_i = A''\sigma_j$, and by Lemma 3.1.3, we have $A'\sigma_i = A'''(\sigma_i * \sigma_j)$ for some $A'''$ in $B_1^+$. As a result we have

$$P = A'''(\sigma_i * \sigma_j)C.$$  

Since $\sigma_iC \in S_j^+$ we have $i \not\in S(C)$. Also, since $B_1 = QC = Q'\sigma_jC$ and $B_1 \in S_j^+$ we have $\sigma_jC \in S_j^+$ by Lemma 3.1.2, and hence $j \not\in S(C)$. Applying Lemma 1 to these facts that $i \not\in S(C)$ and $j \not\in S(C)$, we have $(\sigma_i * \sigma_j)C \in S_j^+$. Since $B_1 = QC = Q'\sigma_jC$, $\sigma_jC$ is a subfactor of $B_1$. On the other hand, $\sigma_iC$ is not a subfactor of $B_1$, so that $i \not= j$. Now suppose $|i - j| \geq 2$. In this case, we have

$$P = A'''(\sigma_i * \sigma_j)C = A''''\sigma_i\sigma_jC.$$  

Since $\sigma_iC$ is not a subfactor of $B_1$, $\sigma_j\sigma_iC = \sigma_i\sigma_jC$ is not a subfactor of $B_1$. So the factor $\sigma_jC$ satisfies the condition of $C$ in the factorization $P = A'\sigma_jC$ but $l(\sigma_jC) \geq l(C) + 1$, which contradicts the maximality of the length of $l(C)$. We next consider the case $|i - j| = 1$. In this case, we have

$$P = A'''(\sigma_i * \sigma_j)C = A''''\sigma_j\sigma_i\sigma_jC.$$  

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Since \( \sigma_i C \) is not a subfactor of \( B_1 \), \( \sigma_j \sigma_i \sigma_j C = \sigma_i \sigma_j \sigma_i C \) is not a subfactor of \( B_1 \). Further, if \( \sigma_i \sigma_j C \) is a subfactor of \( B_1 \), this factor satisfies the condition above but \( l(\sigma_i \sigma_j C) \geq l(C) + 2 \), which contradicts the maximality of the length of \( l(C) \). If \( \sigma_i \sigma_j C \) is not a subfactor of \( B_1 \), this factor satisfies the condition above but \( l(\sigma_j C) \geq l(C) + 1 \), which contradicts the maximality of the length of \( l(C) \). In each of these cases we have a contradiction, so the claim is proved.

We next show the uniqueness of the factorization. Suppose that \( P = AB \) is another \( J \)-weighted factorization. Then we can write \( B_1 = QB \) with \( Q \in B_n^+ \). If \( Q = e \) then \( B_1 = B \), so we can assume \( Q \neq e \). In this case we can find \( i \in F(Q) \cap J \) so that \( Q = Q' \sigma_i \) for some \( Q' \). Since \( B_1 \in S_1^+ \), we have \( \sigma_i B \in S_1^+ \) and hence \( i \notin S(B) \). On the other hand, since \( i \) is an element of \( F(Q) \) and the identity \( A = A_1 Q \) holds, \( i \) is an element of \( F(A) \cap J \). Thus \( F(A) \cap J \not\subset S(B) \), which is a contradiction.

\[ \square \]

### 3.2 Words and homotopies

In this section we consider homotopies between two words and prove the main theorem in this chapter. Given a word \( f = i_1...i_k \) in the free monoid on \( \{1,...,n-1\} \), we set \( r(f) = \sigma_{i_1}...\sigma_{i_k} \) in \( B_n^+ \). Let \( \pi : B_n^+ \to S_n \) be the natural map from \( B_n^+ \) to the symmetric group \( S_n \). A word \( f = i_1...i_k \) is called reduced if \( k \) is minimal among all such expressions for \( \pi \circ r(f) \) in \( S_n \).

Two words \( f \) and \( g \) are called equivalent if \( r(f) = r(g) \). For distinct \( i \) and \( j \) in \( \{1,...,n-1\} \), write

\[
p(i,j) = \begin{cases} 
  jij & \text{if } |i-j| = 1, \\
  ij & \text{if } |i-j| \geq 2.
\end{cases}
\]

An elementary homotopy is an alteration from a word of the form \( f_1 p(i,j) f_2 \) to the word \( f_1 p(j,i) f_2 \) where \( i, j \in \{1,...,n-1\} \) and \( f_1, f_2 \) are some words. We denote by \( f \simeq g \) an elementary homotopy between \( f \) and \( g \).

Two words are called homotopic if there exists a sequence of elementary homotopies between them. Obviously, two words are equivalent if and only if they are homotopic or identical. A self-homotopy is a sequence of elementary homotopies beginning and ending with the same word. In particular, a cube is a self-homotopy of the following form:
A prism is a self-homotopy of the following form:
\[ f_{ijkf_2} \simeq f_{ikjf_2} \simeq f_{ijjf_2} \simeq f_{kijf_2} \simeq f_{kjjf_2}. \]

A permutohedron is a self-homotopy of the following form:
\[ f_{ijkjf_2} \simeq f_{ikjjf_2} \simeq f_{ijkkf_2} \simeq f_{kikjf_2} \simeq f_{kjkjf_2} \simeq f_{kjjkjf_2}. \]

A self-homotopy is inessential if it is of the form
\[ f = f_0 \simeq f_1 \simeq \ldots \simeq f_{k-1} \simeq f_k \simeq f_{k-1} \simeq \ldots \simeq f_1 \simeq f_0 = f; \]
or if it is of the form
\[ f_{ip(i, j)} f_{2p(k, l)} f_3 \simeq f_{ip(p, i)} f_{2p(k, l)} f_3 \simeq f_{ip(i, j)} f_{2p(l, k)} f_3 \simeq f_{ip(p, i)} f_{2p(l, k)} f_3. \]

Given a word \( f \), let \( H(f) \) denote the graph whose vertices are words homotopic to \( f \) and whose edges are elementary homotopies. A self-homotopy
is a circuit in this graph. We shall say that a circuit \( \tau \) in a graph decomposed in two circuits \( \tau_1 \tau_2 \) and \( \tau_1^{-1} \tau_3 \) if \( \tau = \tau_1 \tau_3 \). In the context of Coxeter groups, Tits [18] proved that every self-homotopy decomposes into self-homotopies each of which is inessential or lies in a rank 3 residue.

The main result of this chapter is the following

**Theorem 3.2.1** Every self-homotopy decomposes into self-homotopies each of which is inessential, a cube, a prism or a permutohedron.

**Proof.** We consider everything modulo inessential self-homotopies of the first type, and use induction on the length of the words appearing in a self-homotopy. If all the vertices in a self-homotopy end in \( i \) for some \( i \), then we can use the induction hypothesis to conclude that the self-homotopy decomposes as required. Otherwise, we can find a sequence of elementary homotopies of the form

\[
fi \simeq f'j \simeq ...j \simeq ........ \simeq ...j \simeq g'j \simeq gk,
\]

where \( i, j, k \in \{1, 2, ..., n-1\} \) with \( i \neq j, j \neq k \), and \( f, f', g, g' \) are some words. Let \( w = r(fi) = r(gk) \) in \( B_n^+ \). By applying Proposition 3.1.1 to \( w \) and \( J = \{i, j, k\} \) we obtain a unique factorization \( w = w_1w_2 \) such that \( w_2 \) has maximal length in \( S_j^+ \). Choose words \( h \) and \( h' \) so that \( r(h) = w_1 \) and \( r(h') = w_2 \). The word \( h' \) can be chosen to be reduced and to end in \( i, j \) or \( k \). Since \( S_j^+ \) can be identified with the symmetric group generated by \( \{\pi(\sigma_i); i \in J\} \), we can apply a technique used in [17] to see that there are suitable words \( h_k, h_i, h_j \) such that \( h' \) becomes \( h_kp(j, i) \), \( h_ip(j, k) \), and \( h_jp(k, i) \). This means, in particular, that \( fi \) is homotopic to \( hh_kp(j, i) \). The word \( fi \) can be written as \( fi = \varphi p(j, i) \) with a word \( \varphi \), and we can take as a sequence of elementary homotopies from \( fi \) to \( hh_k p(j, i) \) a sequence which increases the length of reduced words containing \( p(j, i) \). Thus, we have a sequence of elementary homotopies from \( \varphi \) to \( hh_kp(j, i) \) so that the original sequence from \( fi \) to \( hh_kp(j, i) \) is obtained from the sequence by putting \( p(j, i) \) to all the vertices in the sequence. The word \( gk \) is homotopic to \( hh_kp(j, k) \) with the word \( h \) used in common with \( fi \). As a result, we obtain a circuit of the following form:
In the circuit $A$, $f_i = \varphi_p(j,i)$ and $f'_j = \varphi_p(i,j)$, and we can use the sequence from $\varphi$ to $hh_k$ to obtain a sequence from $f'_j$ to $hh_kp(i,j)$. Hence the circuit $A$ decomposes into inessential ones. The same is true for the circuit $C$. In the circuit $B$, all the vertices end in $j$, so we can use the induction hypothesis to conclude that $B$ decomposes as required. If $i = k$ then $D$ reduces to a point modulo inessential self-homotopies of the first type. If $|i - j| \geq 2$, $|j - k| \geq 2$, and $|k - i| \geq 2$, then $D$ becomes a cube. If $\{i,j,k\} = \{a,a+1,b\}$ for some $a,b$ with $b \leq a - 2$ or $b \geq a + 3$, then $D$ becomes a prism. Finally if $\{i,j,k\} = \{a,a+1,a+2\}$ for some $a$, then $D$ becomes a permutohedron. Besides, all the vertices in the altered sequence end in $i$ or $k$, so we can repeat this process until we obtain a circuit whose all vertices end in $i$ for some $i$. This completes the proof.

\textbf{Remark.} The proof above owes a lot to the idea of Ronan [17] (Theorem 2.17). However, the key point is that the (minimum) self-homotopy containing all the maximal reduced words on the set $\{i,j,k\}$ becomes what we called a cube, a prism or a permutohedron depending on the value of $\{i,j,k\}$, which I found by my direct inspection!

\section{3.3 Examples}

In this section we illustrate by examples how the theorem holds. The next example shows a case where $i = 5$, $j = 4$, $k = 3$, $h = 44$ and $h' = 543545, 543454, 345343$, etc.
In this case, we obtain a permutohedron in $D$. The next is a case where $i = 5$, $j = 3$, $k = 2$, $h = 5$ and $h' = 3235, 3253, 3523$, etc.
In this case, we obtain a prism in $D$. We hope these examples convince the reader that the theorem is true.

The theorem itself does not require any knowledge of college math nor high school math. It’s almost a puzzle. That’s why I like this theorem. That’s why I chose this as my thesis. This theorem reminds me the summer of 1979 when I was fourteen and mathematics was only a game!
Chapter 4

Type $B$

In this chapter, we consider Coxeter groups of type $B$ and the related braid monoids.

4.1 Coxeter groups of type $B$

For the Coxeter graph of type $B_n$, we label the vertices as follows:

$$B_n \quad c_1 \quad c_2 \quad c_3 \quad \ldots \quad c_n$$

Then the Coxeter group of type $B_n$ is defined by $n$ generators $c_1, c_2, \ldots, c_n$ and the relations

$$\begin{cases}
  c_i^2 = e, & \\
  c_i c_j = c_j c_i & \text{if } |i - j| \geq 2, \\
  c_i c_j c_i c_j = c_j c_i c_j c_i & \text{if } \{i, j\} = \{1, 2\}, \\
  c_i c_j c_i = c_j c_i c_j & \text{if } \{i, j\} = \{a, a + 1\} \text{ for some } 2 \leq a \leq n - 1,
\end{cases}$$

where $i, j = 1, 2, \ldots, n$. The corresponding braid monoid of type $B_n$ should be defined by $n$ generators $\sigma_1, \sigma_2, \ldots, \sigma_n$ and the relations

$$\begin{cases}
  \sigma_i \sigma_j = \sigma_j \sigma_i & \text{if } |i - j| \geq 2, \\
  \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j & \text{if } \{i, j\} = \{1, 2\}, \\
  \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j & \text{if } \{i, j\} = \{a, a + 1\} \text{ for some } 2 \leq a \leq n - 1,
\end{cases}$$

where $i, j = 1, 2, \ldots, n$. 

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4.2 Homotopies of type $B$

In this section, we consider homotopies between two words with respect to the relations described in the previous section.

In Chapter 2, we have seen the Coxeterhedron of type $PB_3$. This polytope is also known as the rhombitruncated cuboctahedron in the literature, which belongs to Archimedean polytopes. If we appropriately interpret each sequence of edges as a word, the Coxeterhedron becomes the following diagram:

\[
\begin{array}{cccc}
  f_{312312132} & f_{132312132} & f_{123212132} \\
  f_{312312312} & f_{123123123} \\
  f_{31213212} & f_{123123212} \\
  f_{32132122} & f_{123213213} \\
  f_{32132132} & f_{123213231} \\
  f_{32132123} & f_{123213231} \\
  f_{32132131} & f_{123213231} \\
  f_{32132132} & f_{123213231} \\
  f_{23123121} & f_{213231231} \\
  f_{23123121} & f_{213231231} \\
  f_{23123121} & f_{213231231} \\
  f_{23123121} & f_{213231231} \\
  f_{23123121} & f_{213231231} \\
  f_{23123121} & f_{213231231} \\
\end{array}
\]

where $f$ and $g$ are some words. We call this a Coxeterhedron of type $B$. In order to state an analog of Theorem 3.2.1, we need one other kind of polytope,
which we call a prism of type $B$:

\[
\begin{align*}
  f_{1212}k & \quad f_{2121}k \\
  f_{1212}k2 & \quad f_{2121}k2 \\
  f_{1221}k21 & \quad f_{2121}k12 \\
  f_{1221}k12 & \quad f_{2121}k12 \\
  f_{1221}k121 & \quad f_{2121}k121 \\
  f_{1221}k1211 & \quad f_{2121}k1211
\end{align*}
\]

where $k \geq 4$ and $f, g$ are some words.

The following theorem is a complete analogy with Theorem 3.2.1

**Theorem 4.2.1** With the obvious sense of inessential self-homotopies for this setting, every self-homotopy decomposes into self-homotopies each of which is inessential, a cube, a prism, a prism of type $B$, a permutohedron or a Coxeterhedron of type $B$.

*Proof.* One can easily modify the definitions of $S^+_j$ and $\sigma_i * \sigma_j$ so that all the lemmas in Chapter 3 and Proposition 3.1.1 are valid for the braid monoids corresponding to finite Coxeter groups of type $B$. Hence the result follows. \(\square\)
Chapter 5

Type $D$

In this chapter, we consider Coxeter groups of type $D$ and the related braid monoids.

5.1 Coxeter groups of type $D$

For the Coxeter graph of type $D_n$, we label the vertices as follows:

$$D_n$$

Then the Coxeter group of type $D_n$ is defined by $n$ generators $c_1, c_2, \ldots, c_n$ and the relations

\[
\begin{align*}
&\begin{cases}
  c_i^2 = e, \\
  c_ic_j = c_jc_i & \text{if } c_i \text{ and } c_j \text{ are not connected}, \\
  c_ic_jc_i = c_jc_ic_j & \text{if } c_i \text{ and } c_j \text{ are connected},
\end{cases}
\end{align*}
\]

where $i, j = 1, 2, \ldots, n$. The corresponding braid monoid of type $D_n$ should be defined by $n$ generators $\sigma_1, \sigma_2, \ldots, \sigma_n$ and the relations

\[
\begin{align*}
&\begin{cases}
  \sigma_i\sigma_j = \sigma_j\sigma_i & \text{if } c_i \text{ and } c_j \text{ are not connected}, \\
  \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{if } c_i \text{ and } c_j \text{ are connected},
\end{cases}
\end{align*}
\]

where $i, j = 1, 2, \ldots, n$. 

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5.2 Homotopies of type $D$

In Chapter 2, we have seen the Coxeterhedron of type $PD_3$. If we appropriately interpret each sequence of edges as a word, the Coxeterhedron becomes the following diagram

\[
\begin{align*}
&f_{123123}g \quad f_{123213}g \quad f_{132312}g \quad f_{132131}g \quad f_{131231}g \\
&\quad f_{213123}g \quad f_{313231}g \\
&\quad f_{231323}g \quad f_{312321}g \\
&\quad f_{231232}g \quad f_{232132}g \quad f_{323132}g \quad f_{321321}g \\
&\quad f_{321321}g \\
\end{align*}
\]

This self-homotopy belongs to permutohedra which we defined in Chapter 3. Similarly, a permutohedron constructed from $J = \{1, 3, 4\}$ can happen here, but it is also included in our permutohedra. In addition, the following self-homotopy

\[
\begin{align*}
&f_{131k}g \quad f_{313k}g \\
&\quad f_{13k1}g \quad f_{31k3}g \\
&\quad f_{1k31}g \quad f_{3k13}g \\
&\quad f_{k131}g \quad f_{k313}g \\
\end{align*}
\]

arises here for each $k \geq 5$ and for words $f, g$. But this is also included in our prisms defined in chapter 3. These results follow since the local structure of the Coxeter graphs of type $D$ is the same as that of the graphs of type $A$.

Thus, we have the following analog of Theorem 3.2.1.

**Theorem 5.2.1** Every self-homotopy decomposes into self-homotopies each of which is inessential, a cube, a prism or a permutohedron.
Chapter 6

Type $E$ and $F$

In this chapter, we briefly survey the Coxeter groups of type $E$ and $F$.

Of course, one can define the Coxeter groups of types $E_6, E_7, E_8$ and the related braid monoids. However, the local structure of the Coxeter graphs of these is the same as that of the graphs of type $D$. Hence, new polytopes do not arise here. The same is true for the Coxeter group of type $F_4$; the local structure of the Coxeter graph of type $F_4$ is the same as that of the graphs of type $B$.

Instead, we record here a Gossett polytope just for fun!
Chapter 7

Type $H$

In this chapter, we consider Coxeter groups of type $H$ and the related braid monoids.

### 7.1 Coxeter groups of type $H$

For $n = 3$ or $4$, we label the vertices of the Coxeter graph of type $H_n$ as follows:

$$H_3 \quad \begin{array}{ccc} c_1 & 5 & c_2 \\ c_2 & c_3 \end{array}, \quad H_4 \quad \begin{array}{cccc} c_1 & 5 & c_2 & c_3 \\ c_2 & c_3 & c_4 \end{array}$$

Then, the Coxeter group of type $H_n$ is defined by $n$ generators $c_1, c_2, ..., c_n$ and the relations

$$\begin{align*}
    c_i^2 &= e, \\
    c_i c_j = c_j c_i &\quad \text{if } |i - j| \geq 2, \\
    c_i c_j c_i c_j = c_j c_i c_j c_i &\quad \text{if } \{i, j\} = \{1, 2\}, \\
    c_i c_j c_i c_j &= c_j c_i c_j c_i &\quad \text{if } \{i, j\} = \{a, a + 1\} \text{ for some } 2 \leq a \leq n - 1,
\end{align*}$$

where $i, j = 1, 2, ..., n$. The corresponding braid monoid of type $H_n$ should be defined by $n$ generators $\sigma_1, \sigma_2, ..., \sigma_n$ and the relations

$$\begin{align*}
    \sigma_i \sigma_j &= \sigma_j \sigma_i &\quad \text{if } |i - j| \geq 2, \\
    \sigma_i \sigma_j \sigma_i \sigma_j &= \sigma_j \sigma_i \sigma_j \sigma_i &\quad \text{if } \{i, j\} = \{1, 2\}, \\
    \sigma_i \sigma_j \sigma_i &= \sigma_j \sigma_i \sigma_j &\quad \text{if } \{i, j\} = \{a, a + 1\} \text{ for some } 2 \leq a \leq n - 1,
\end{align*}$$

where $i, j = 1, 2, ..., n$.  

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7.2 Homotopies of type $H$

In this section, we consider homotopies between two words with respect to the relations described in the previous section. If we consider the case of $H_4$, the following self-homotopy will appear.

\[
\begin{align*}
&f_{121214} \quad g \\
&f_{121241} \quad f_{212142} \\
&f_{121421} \quad f_{212412} \\
&f_{124121} \quad f_{214212} \\
&f_{142121} \quad f_{241212} \\
&f_{412121} \quad f_{421212}
\end{align*}
\]

We call this self-homotopy a prism of type $H$. The following is the Coxeter-hedron of type $H$, which contains twelve 10-gons, twenty 6-gons, and thirty 4-gons:

With some words $f$ and $g$, we can give an algebraic description of this polytope as follows:
Finally, we state here the type $H$ analog of Theorem 3.2.1.

**Theorem 7.2.1** With the obvious sense of inessential self-homotopies for this case, every self-homotopy decomposes into self-homotopies each of which is inessential, a prism, a prism of type $H$, a permutohedron or a Coxeterhedron of type $H$.

*Remark.* Notice that cubes are not needed here. Further, if we consider the case of type $H_3$, all we need will be inessential self-homotopies and Coxeter-hedra of type $H$.

A few words are in order for the case of $I_2(m)$. Since there are only two generators, every self-homotopy becomes inessential. This is totally *sans intérêt!*
Chapter 8

Coherence for braided Gray monoids

Coherence theorems play an important role in category theory. The first example is the work of Stasheff and Mac Lane which asserts that all diagrams commute in certain monoidal categories. Joyal and Street [14] generalized the coherence theorem to braided monoidal categories $\mathcal{C}$, which asserts that certain diagrams built from braiding morphisms $A \otimes B \to B \otimes A$ commute in $\mathcal{C}$.

In this chapter, we consider a 2-categorical version of the result of Joyal and Street. For any braided Gray monoid $\mathcal{B}$ we exhibit our strategy to prove that certain diagrams commute in the 2-categorical sense. The notion of braiding for a Gray monoid was first introduced in Kapranov and Voevodsky’s paper [15], and they showed that a braiding on a Gray monoid $\mathcal{K}$ leads to a Zamolodchikov system in $\mathcal{K}$. In this paper we use the modified version of it, which is due to Baez and Neuchl [3]. In fact, we assume that the 2-morphisms $S^+$ and $S^-$, which arise from the Yang-Baxter hexagons to be equal. For our purpose, the assumption is necessary because the equality can be viewed as a special case of our theorem. To formulate this, we introduce a notion of a braiding quiver $Q_n$, whose vertices are words of the form $\sigma_1...\sigma_n$ for some $\sigma \in S_n$. An arrow $\sigma_1...\sigma_n \to \tau_1...\tau_n$ in $Q_n$ exists iff $\tau_1...\tau_n = \sigma_1...\sigma_{i-1}\sigma_{j+1}...\sigma_k\sigma_i...\sigma_j\sigma_{k+1}...\sigma_n$ for some integers $1 \leq i \leq j < k \leq n$. Then it is easy to see an object $(A_1, ..., A_n) \in \mathcal{B} \times ... \times \mathcal{B}$ and a braiding quiver induce a certain diagram in $\mathcal{B}$. In particular, if the quiver is closed, we call such a diagram a braiding diagram. For an object $(A_1, ..., A_n)$, our theorem asserts that a closed braiding quiver induces a
unique 2-morphism $\alpha : f_1 f_2 \ldots f_l \Rightarrow g_1 g_2 \ldots g_m$ in $\mathcal{B}$ if the composites $f_1 f_2 \ldots f_l$ and $g_1 g_2 \ldots g_m$ correspond to the same braid. The existence of such a 2-morphism is an easy consequence of the coherence theorem of Joyal and Street. Hence the novelty is in the uniqueness theorem.

### 8.1 Gray monoids and braided Gray monoids

We begin by recalling from [2], [3], [7], [15] the definition of a Gray monoid.

**Definition 8.1.1** A Gray monoid consists of a 2-category $\mathcal{G}$ together with:

1. An object $I \in \mathcal{G}$, called a unit object,
2. A 2-functor $- \otimes ? : \mathcal{G} \otimes \mathcal{G} \to \mathcal{G}$, such that the following equations hold:

$$
- \otimes (? \otimes !) = (- \otimes ?) \otimes !,
$$

$$
I \otimes - = -,
$$

$$
- \otimes I = -.
$$

In this definition, $\otimes_\mathcal{G}$ is the pseudo-version of the Gray tensor product of 2-categories [7].

Next we proceed to define a braiding on a Gray monoid, following Crans [6].

**Definition 8.1.2** A braiding for a Gray monoid $\mathcal{K}$ consists of the following data:

1. A pseudonatural equivalence $R : \otimes \Rightarrow \otimes^{op}$,
2. Two invertible modifications $R_{-|-}$ and $R_{-|-}$ giving for any objects $A, B, C \in \mathcal{K}$ the 2-isomorphisms

These data must satisfy the following conditions.

1. For any objects $X_1, X_2, X_3, Y$ the tetrahedron
is commutative.

(2) For any objects $X_1, X_2, Y_1, Y_2$ the polytope

is commutative.

(3) For any objects $X, Y_1, Y_2, Y_3$ the tetrahedron

is commutative.

(4) For any objects $A, B, C$ the following 2-morphisms $S^+$ and $S^-$ coincide:
(5) The data \( R_{-,-}, R_{-|,-} \) and \( R_{-,|} \) are the identity whenever one of the objects is the unit object \( I \).

Throughout the following, we denote by \( \mathcal{B} \) a braided Gray monoid. Next, we introduce a notion of a braiding quiver \( Q_n \) for an integer \( n \). The vertices of the quiver \( Q_n \) are words of the form \( \sigma_1 \ldots \sigma_n \) where \( \sigma \in S_n \). An arrow \( \sigma_1 \ldots \sigma_n \to \tau_1 \ldots \tau_n \) exists iff \( \tau_1 \ldots \tau_n = \sigma_1 \ldots \sigma_{i-1} \sigma_{j+1} \ldots \sigma_k \sigma_i \ldots \sigma_j \sigma_{k+1} \ldots \sigma_n \) for some integers \( 1 \leq i \leq j < k \leq n \). To an arrow \( \sigma_1 \ldots \sigma_n \to \sigma_1 \ldots \sigma_{i-1} \sigma_{j+1} \ldots \sigma_k \sigma_i \ldots \sigma_j \sigma_{k+1} \ldots \sigma_n \) in \( Q_n \), we can associate the following braid:

More generally, any path in \( Q_n \) induces a composition of the corresponding braids. The same procedure applies to a braided monoidal category \( \mathcal{C} \) and a braided Gray monoid \( \mathcal{B} \). For example, fix an object \((A_1, \ldots, A_n) \in \mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B} \). Then to a vertex \( \sigma_1 \ldots \sigma_n \) in the quiver, we assign the object \( A_{\sigma_1} \ldots A_{\sigma_n} \) in \( \mathcal{B} \). To an arrow \( \sigma_1 \ldots \sigma_n \to \sigma_1 \ldots \sigma_{i-1} \sigma_{j+1} \ldots \sigma_k \sigma_i \ldots \sigma_j \sigma_{k+1} \ldots \sigma_n \) in the quiver we associate the 1-morphism \( A_{\sigma_1} \ldots A_{\sigma_{i-1}} \otimes R_{A_{\sigma_i} \ldots A_{\sigma_j} \ldots A_{\sigma_{k+1}}} \otimes A_{\sigma_{k+1}} \ldots A_{\sigma_n} \) in \( \mathcal{B} \). Thus, this construction may be thought of as a representation of the quiver in \( \mathcal{B} \).

The braiding quiver \( Q_n \) is called closed if it has the following form:

We denote this by \( Q_n(f_1, f_2, \ldots, f_l \mid g_1, g_2, \ldots, g_m) \). Applying the above procedure, we obtain a diagram of the form
which we denote \((A_1, ..., A_n, f_1, f_2, ..., f_l \mid g_1, g_2, ..., g_m)\) and call a braiding diagram.

Note that when we speak about a braiding diagram, it always means a pair of a closed braiding quiver and the corresponding diagram in \(B\). So when we say that a braiding diagram is closed, it means that the corresponding braiding quiver is closed.

For a closed braiding quiver \(Q_n(f_1, f_2, ..., f_l \mid g_1, g_2, ..., g_m)\), the coherence theorem of Joyal and Street [14] implies that if the two braids corresponding to the paths \(f_1 f_2 ... f_l\) and \(g_1 g_2 ... g_m\) coincide, then it can be decomposed into several closed braiding quivers such that the corresponding diagrams in a braided monoidal category becomes commutative by the assumption on data. We call such a decomposition a pasting scheme. It is easy to see that such a pasting scheme yields a unique pasting of 2-morphisms \(f_1 f_2 ... f_l \Rightarrow g_1 g_2 ... g_m\) in \(B\). In fact, the following three types of 2-morphisms will be used in the pasting.

First, let \(R(A_1, ..., A_n)\) be the set of 2-morphisms of the forms

\[A_{\sigma_1} ... A_{\sigma_{i-1}} \otimes R_{A_{\sigma_i}...A_{\sigma_{j-1}},A_{\sigma_j}...A_{\sigma_{k-1}}|A_{\sigma_k},...,A_{\sigma_l}} \otimes A_{\sigma_{l+1}}...A_{\sigma_n}\]

for some integers \(1 \leq i < j < k \leq l \leq n\) or

\[A_{\sigma_1} ... A_{\sigma_{i-1}} \otimes R_{A_{\sigma_i}...A_{\sigma_j}|A_{\sigma_{j+1}}...A_{\sigma_k},A_{\sigma_{k+1}}...A_{\sigma_l}} \otimes A_{\sigma_{l+1}}...A_{\sigma_n}\]

for some integers \(1 \leq i \leq j < k < l \leq n\). Let \(R^{-1}(A_1, ..., A_n)\) be the set of 2-morphisms obtained from the ones in \(R(A_1, ..., A_n)\) by replacing \(R\) by \(R^{-1}\). The notation \(\tilde{R}(A_1, ..., A_n)\) stands for the union of \(R(A_1, ..., A_n)\) and \(R^{-1}(A_1, ..., A_n)\). Sometimes we write \(R\) for a 2-morphism in \(\tilde{R}(A_1, ..., A_n)\).

Second, we define \(\otimes(A_1, ..., A_n)\) to be the set of 2-morphisms of the form

\[A_{\sigma_1} ... A_{\sigma_{i-1}} \otimes \otimes R_{A_{\sigma_i}...A_{\sigma_j},A_{\sigma_{j+1}}...A_{\sigma_k},A_{\sigma_{k+1}}...A_{\sigma_{l-1}},A_{\sigma_{l+1}}...A_{\sigma_m},A_{\sigma_{m+1}}...A_{\sigma_h}} \otimes A_{\sigma_{h+1}}...A_{\sigma_n}\]

for some integers \(1 \leq i \leq j < k < l \leq m < h \leq n\). As in the case above \(\otimes(A_1, ..., A_n)\) denotes the union of \(\otimes(A_1, ..., A_n)\) and \(\otimes^{-1}(A_1, ..., A_n)\).
For short we write $\otimes$ for a 2-morphism in $\otimes(A_1, \ldots, A_n)$. Third, we define $R_f(A_1, \ldots, A_n)$ to be the set of 2-morphisms of the form $A \otimes R_{f,B} \otimes C$ with $A = A_{\sigma_1} \ldots A_{\sigma_m}$, $B = A_{\sigma_{m+1}} \ldots A_{\sigma_s}$, $C = A_{\sigma_{s+1}} \ldots A_{\sigma_n}$, and

$$f = A_{\sigma_i} \ldots A_{\sigma_j} \otimes R_{A_{\sigma_{j-1}} \ldots A_{\sigma_k+1} \ldots A_{\sigma_l}} \otimes A_{\sigma_{l+1}} \ldots A_{\sigma_m},$$

for some integers $1 \leq i \leq j \leq k \leq l \leq m \leq s \leq n$ or of the form $A \otimes R_{B,f} \otimes C$ with $A = A_{\sigma_1} \ldots A_{\sigma_{l-1}}$, $B = A_{\sigma_{l}} \ldots A_{\sigma_{j-1}}$, $C = A_{\sigma_j} \ldots A_{\sigma_n}$, and

$$f = A_{\sigma_{j-1}} \ldots A_{\sigma_k} \otimes R_{A_{\sigma_{k+1}} \ldots A_{\sigma_l} \ldots A_{\sigma_m}} \otimes A_{\sigma_{m+1}} \ldots A_{\sigma_{n}},$$

for some integers $1 \leq i \leq j \leq k \leq l \leq m \leq s \leq n$. Let $\tilde{R}_f(A_1, \ldots, A_n)$ be the union of $R_f(A_1, \ldots, A_n)$ and $R_f^{-1}(A_1, \ldots, A_n)$. We write $R_f$ for a 2-morphism in $\tilde{R}_f(A_1, \ldots, A_n)$.

**Definition 8.1.3** For a braiding diagram $(A_1, \ldots, A_n, f_1, f_2, \ldots, f_l \mid g_1, g_2, \ldots, g_m)$ a 2-morphism $\alpha : f_1f_2\ldots f_l \Rightarrow g_1g_2\ldots g_m$ is called canonical if it is defined by using a pasting scheme and 2-morphisms of the forms above.

The goal of this chapter is to prove the following theorem.

**Theorem 8.1.1** Let $Q_n(f_1, f_2, \ldots, f_l \mid g_1, g_2, \ldots, g_m)$ be a closed braiding quiver such that the composites $f_1 f_2 \ldots f_l$ and $g_1 g_2 \ldots g_m$ correspond to the same braid. Then for a braiding diagram $(A_1, \ldots, A_n, f_1, f_2, \ldots, f_l \mid g_1, g_2, \ldots, g_m)$ there exists a unique canonical 2-morphism $\alpha : f_1f_2\ldots f_l \Rightarrow g_1g_2\ldots g_m$ in $B$.

The rest of this chapter is devoted to the study of uniqueness theorem. For an object $(A_1, \ldots, A_n) \in B \times B \times \ldots \times B$, let $B(A_1, \ldots, A_n)$ be the set of 1-morphisms in $B$ which have the form $A_{\sigma_1} \ldots A_{\sigma_{l-1}} \otimes R_{A_{\sigma_{l}} \ldots A_{\sigma_{j-1}} \ldots A_{\sigma_{k+1}} \ldots A_{\sigma_{m}}}$ for some integers $1 \leq i \leq j < k \leq n$ and $\sigma \in S_n$, the symmetric group.

A 1-morphism in $B(A_1, \ldots, A_n)$ is called elementary if the 1-morphism has the form

$$A_{\sigma_1} \ldots A_{\sigma_{i-1}} \otimes R_{A_{\sigma_i} \ldots A_{\sigma_{i+1}} \ldots A_{\sigma_{n}}}$$

for some integer $1 \leq i \leq n-1$.

**Definition 8.1.4** A braiding diagram $(A_1, \ldots, A_n, f_1, f_2, \ldots, f_l \mid g_1, g_2, \ldots, g_m)$ is called elementary if all the 1-morphisms in the diagram are elementary.
**Definition 8.1.5** A braiding diagram \((A_1, \ldots, A_n, f_1, f_2, \ldots, f_l \mid g_1, g_2, \ldots, g_m)\) is called a shuffle diagram if there exist integers \(p\) and \(q\) such that \(p + q = n\) and such that the vertices in the corresponding quiver become \((p, q)\)-shuffles.

**Definition 8.1.6** A braiding diagram \((A_1, \ldots, A_n, f_1, f_2, \ldots, f_l \mid g_1, g_2, \ldots, g_m)\) is called fundamental if it has the form \((A_1, \ldots, A_n, h_0 * h_1 * \ldots * h_{k-j-1} \mid g)\) with

\[
g = A_{\sigma_1} \ldots A_{\sigma_{l-1}} \otimes R_{A_{\sigma_1}, \ldots, A_{\sigma_{j+1}}} A_{\sigma_k} \otimes A_{\sigma_{k+1}} \ldots A_{\sigma_n},
\]

\[
h_l = (f_{j,j+l+1}, f_{j-1,j+l+1}, f_{j-2,j+l+1}, \ldots, f_{j-i+l+1})
\]

for some integers \(1 \leq i \leq k \leq n\) and \(0 \leq l \leq k - j - 1\) where

\[
f_{j-m,j+l+1} = A_{\sigma_1} \ldots A_{\sigma_{i-1}} A_{\sigma_{j+1}} \ldots A_{\sigma_{j+i}} A_{\sigma_1} \ldots A_{\sigma_{j+1}} \otimes R_{A_{\sigma_{j+1}}, \ldots, A_{\sigma_{j+l+1}}}
\]

\[
A_{\sigma_{j+1}} A_{\sigma_{j+2}} \ldots A_{\sigma_k} A_{\sigma_{k+1}} \ldots A_{\sigma_n}
\]

for \(0 \leq m \leq j - i\). In the definition above the notation \(h_i * h_j\) denotes the concatenation of two sequences \(h_i\) and \(h_j\).

For example, the following diagram

\[
\begin{array}{cccc}
B \otimes R_{A,C} & BAC \rightarrow & BCA \\
R_{A,B} \otimes C & ABC \\
A \otimes R_{B,C} & ACB \rightarrow & C \otimes R_{A,B} \\
R_{A,C} & B \\
\end{array}
\]

is an elementary braiding diagram, and we have the 2-morphism \(S^+ = S^- : (R_{A,B} \otimes C)(B \otimes R_{A,C})(R_{B,C} \otimes A) \Rightarrow (A \otimes R_{B,C})(R_{A,C} \otimes B)(C \otimes R_{A,B})\), which we denote by \(S(A, B, C)\). More generally, we define \(S(A_1, \ldots, A_n)\) to be the set of 2-morphisms of the form

\[
A_{\sigma_1} \ldots A_{\sigma_{l-1}} \otimes S(A_{\sigma_1}, A_{\sigma_{l+1}}, A_{\sigma_{l+2}}) \otimes A_{\sigma_{l+3}} \ldots A_{\sigma_n}
\]

for some integers \(1 \leq i \leq n-2\). Let \(\tilde{S}(A_1, \ldots, A_n)\) be the union of \(S(A_1, \ldots, A_n)\) and \(S^{-1}(A_1, \ldots, A_n)\) as above. We write \(S\) for a 2-morphism in \(\tilde{S}(A_1, \ldots, A_n)\).

**Definition 8.1.7** Let \((A_1, \ldots, A_n, f_1, f_2, \ldots, f_l \mid g_1, g_2, \ldots, g_m)\) be an elementary braiding diagram. Then a canonical 2-morphism \(f_1 f_2 \ldots f_l \Rightarrow g_1 g_2 \ldots g_m\) is called elementary if it is defined as a pasting of 2-morphisms in \(\tilde{S}(A_1, \ldots, A_n)\) and \(\tilde{S}(A_1, \ldots, A_n)\) such that the 1-morphisms surrounding the 2-morphisms are elementary.
**Definition 8.1.8** In the setting of Definition 8.1.5, a canonical 2-morphism $f_1f_2...f_l \Rightarrow g_1g_2...g_m$ is called a *shuffle composite* if it is defined as a pasting of 2-morphisms in $R(A_1, ..., A_n)$. In particular, if the braiding diagram is a fundamental diagram and a canonical 2-morphism is given by pasting of 2-morphisms in $R(A_1, ..., A_n)$, then a shuffle composite is called a *fundamental composite*. We write the set of fundamental composites as $F(A_1, ..., A_n)$.

**Remark.** For fundamental composites, 2-morphisms in $R^{-1}(A_1, ..., A_n)$ are not used.

**Definition 8.1.9** In the setting above, a canonical 2-morphism $f_1f_2...f_l \Rightarrow g_1g_2...g_m$ is called a *standard composite* if it is defined as a pasting of the form $\mu_1^{-1} \circ ... \circ \mu_s^{-1} \circ \Omega \circ \nu_1 \circ ... \circ \nu_t$ where $\mu_i, \nu_j \in F(A_1, ..., A_n)$ and $\Omega$ is an elementary composite.

For example, the following pasting

```
ABCDE → CABDE → CBADE
\downarrow R \quad \downarrow S
ACBDE \quad CBAED
\downarrow R \quad \downarrow \otimes
BACDE \quad BCAED
\downarrow \otimes
BCADE → BCEAD → CBEAD
```

is a standard composite. Our proof of the uniqueness theorem consists of the following three steps:

Step 1. The 2-morphism $\alpha$ in the theorem can be expressed as a certain standard composite.

Step 2. Let $(A_1, ..., A_n, f_1, f_2, ..., f_m | f)$ be a fundamental diagram, and let $\alpha$ and $\beta$ be fundamental composites of the form $f_1f_2...f_m \Rightarrow f$. Then the 2-morphism $\alpha$ coincides with $\beta$.

Step 3. Let $(A_1, ..., A_n, f_1, f_2, ..., f_l | g_1, g_2, ..., g_m)$ be an elementary diagram, and let $\alpha$ and $\beta$ be elementary composites of the form $f_1f_2...f_l \Rightarrow g_1g_2...g_m$. Then the 2-morphism $\alpha$ coincides with $\beta$.
8.2 Step 2

In this section, we consider Step 2.

**Lemma 8.2.1** Let $\alpha$ and $\beta$ be fundamental composites of the form below

\[
\begin{align*}
&\xymatrix{B_1 AB_2 \ldots B_n \ar[r]^{f_2} & B_1 B_2 A \ldots B_n \ar[r]^{f_3} & B_1 B_2 \ldots AB_n} \\
&\xymatrix{AB_1 B_2 \ldots B_n \ar[r] & B_1 B_2 \ldots B_n A}
\end{align*}
\]

where $f_i = B_1 \ldots B_{i-1} \otimes R_{A,B_i} \otimes B_{i+1} \ldots B_n$ for $1 \leq i \leq n$. Then the 2-morphism $\alpha$ coincides with $\beta$.

**Proof.** We give a proof by induction on $n$. When $n = 3$, the following two pastings coincide by the braiding condition (3) for a braided Gray monoid.

Next we assume that the statement is true for $n \leq r$. For $n = r + 1$, we can find an integer $t, 1 \leq t \leq n - 1$, such that the 2-morphism $\alpha$ has the form

\[
\begin{align*}
&\xymatrix{B_1 AB_2 B_3 \ar[r]^{f_2} & B_1 B_2 AB_3} \\
&\xymatrix{R_{A|B_1 B_2} \otimes B_3 \ar[r] & B_1 B_2 B_3 A}
\end{align*}
\]

where $\gamma$ and $\delta$ are fundamental composites. Then by the assumption of induction, the 2-morphisms $\gamma$ and $\delta$ can be assumed to have the forms

\[
\begin{align*}
&\xymatrix{B_1 AB_2 B_3 \ar[r]^{f_2} & B_1 B_2 AB_3} \\
&\xymatrix{\downarrow R_{A|B_1 B_2 B_3} & B_1 \otimes R_{A|B_1 B_2 B_3} \ar[r] & B_1 B_2 B_3 A}
\end{align*}
\]

\[
\begin{align*}
&\xymatrix{B_1 AB_2 B_3 \ar[r]^{f_2} & B_1 B_2 AB_3} \\
&\xymatrix{\downarrow R_{A|B_1 B_2 B_3} & B_1 \otimes R_{A|B_1 B_2 B_3} \ar[r] & B_1 B_2 B_3 A}
\end{align*}
\]

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and

\[ B_1 \ldots B_t B_{t+1} \ldots B_n f_{t+2} \rightarrow B_1 B_2 \ldots B_n \]

\[ f_{t+1} \downarrow \delta_1 \quad \downarrow \delta_2 \quad \downarrow \delta_i \quad \downarrow \delta_{i-1} \quad f_n \rightarrow B_1 B_2 \ldots B_n A \]

where \( \gamma_i = R_{A|B_1 \ldots B_i, B_{i+1} \ldots B_n} \otimes B_{i+2} \ldots B_n \) and \( \delta_i = B_1 \ldots B_{i-1} \otimes R_{A|B_i, B_{i+1} \ldots B_n} \) for \( 1 \leq i \leq t - 1 \). Recall that we have already seen that the following two pastings

\[ B_1 \ldots B_t A \ldots B_n \]

\[ B_1 \ldots B_t B_{t+1} \ldots B_n \]

\[ \downarrow \phi \quad \downarrow \psi \rightarrow B_1 B_2 \ldots B_n A \]

and

\[ B_1 \ldots B_t A \ldots B_n \]

\[ B_1 \ldots B_t B_{t+1} \ldots B_n \]

\[ \downarrow \mu \quad \downarrow \nu \rightarrow B_1 B_2 \ldots B_n A \]

coincide, where \( \varphi = R_{A|B_1 \ldots B_t, B_{t+1} \ldots B_n} \otimes B_{t+2} \ldots B_n \), \( \psi = R_{A|B_1 \ldots B_t B_{t+1}, B_{i+2} \ldots B_n} \), \( \mu = R_{A|B_1 \ldots B_t, B_{t+1} \ldots B_n} \), \( \nu = B_1 \ldots B_t \otimes R_{A|B_{t+1}, B_{i+2} \ldots B_n} \). Applying the same method to the rest of 2-morphisms in \( \delta \), we see the result follows. \( \square \)

The same is true for fundamental composites \( \alpha, \beta : h_1 h_2 \ldots h_n \Rightarrow h \) with \( h = R_{B_1 \ldots B_n, A} \) and \( h_i = B_1 \ldots B_{n-1} \otimes R_{B_{n-i+1}, A} \otimes B_{n-i+2} \ldots B_n \) for \( 1 \leq i \leq n \).

To complete Step 2, we give the following definition:

**Definition 8.2.1** Starting with an object \((A_1, \ldots, A_n) \in \mathcal{B} \times \ldots \times \mathcal{B}\), let \( F \) be a shuffle diagram. In particular, if \( F \) has the form...
A\sigma_1...A\sigma_{i-1}A\sigma_k+1A\sigma_{k+2}A\sigma_i...A\sigma_kA\sigma_{k+3}...A\sigma_n

A\sigma_1...A\sigma_{i-1}A\sigma_{k+1}A\sigma_{k+2}...A\sigma_n

A\sigma_1...A\sigma_{i-1}A\sigma_{k+1}A\sigma_{k+2}...A\sigma_n

\rightarrow

A\sigma_1...A\sigma_{i-1}A\sigma_k+1A\sigma_{k+1}...A\sigma_n

A\sigma_1...A\sigma_{i-1}A\sigma_k+1A\sigma_{k+1}...A\sigma_n

A\sigma_1...A\sigma_{i-1}A\sigma_k+1A\sigma_{k+1}...A\sigma_n

A\sigma_1...A\sigma_{i-1}A\sigma_k+1A\sigma_{k+1}...A\sigma_n

for some integers 1 \leq i \leq k \leq n - 2 and 2 \leq l \leq n with \( f_j = A\sigma_1...A\sigma_{i-1} \otimes A\sigma_{k+1}...A\sigma_{k+2} \otimes R_{A\sigma_{i-1} A\sigma_k} \otimes A\sigma_{k+1}...A\sigma_n \) for 1 \leq j \leq l, or

\[ A_{\sigma_1}...A_{\sigma_{i-2}}A_{\sigma_i}...A_{\sigma_k}A_{\sigma_{i-1}}A_{\sigma_k+1}...A_{\sigma_n} \]

\[ A_{\sigma_1}...A_{\sigma_{i-2}}A_{\sigma_i}...A_{\sigma_k}A_{\sigma_{i-1}}A_{\sigma_k+1}...A_{\sigma_n} \]

for some integers 1 \leq i \leq k \leq n - 2 and 2 \leq l satisfying 1 \leq i - l with \( h_j = A_{\sigma_1}...A_{\sigma_{i-j-1}} \otimes R_{A_{\sigma_{i-j}} A_{\sigma_{i-j+1}}} A_{\sigma_i}...A_{\sigma_{i-1}} A_{\sigma_k+1}...A_{\sigma_n} \) for 1 \leq j \leq l, then we call \( F \) an \( A_{\sigma_i}...A_{\sigma_k} \)-shuffle diagram. In the same setting, a canonical 2-morphism \( \alpha \) is called an \( A_{\sigma_i}...A_{\sigma_k} \)-composite if it is a pasting of 2-morphisms in \( R(A_{\sigma_1}...A_{\sigma_i}...A_{\sigma_k}...A_{\sigma_n}) \) and it fills an \( A_{\sigma_i}...A_{\sigma_k} \)-shuffle diagram.

**Proposition 8.2.1** Let \((A_1,...,A_n,f_1,f_2,...,f_m \mid f)\) be a fundamental diagram. If \( \alpha \) and \( \beta \) are two fundamental composites of the form \( f_1f_2...f_m \Rightarrow f \), then \( \alpha \) coincides with \( \beta \).

**Proof.** We can find some integers 1 \leq i \leq j < k \leq n such that the 1-morphism \( f \) becomes \( A_{\sigma_1}...A_{\sigma_{i-1}} \otimes R_{A_{\sigma_j}...A_{\sigma_j+1}} A_{\sigma_k}...A_{\sigma_{k+1}}...A_{\sigma_n} \). Then the fundamental composites \( \alpha \) and \( \beta \) must be pastings of \( A_{\sigma_{j+1}} \)-composite, \( A_{\sigma_{j+2}} \)-composite, \( A_{\sigma_k} \)-composite and \( A_{\sigma_j}...A_{\sigma_j} \)-composite. This means that the \( A_{\sigma_i}...A_{\sigma_j} \)-composites are used in common with \( \alpha \) and \( \beta \), and the rest part reduces to the forms treated in Lemma 8.2.1. We can also apply Lemma 8.2.1 to the \( A_{\sigma_i}...A_{\sigma_j} \)-composites by regarding \( A_{\sigma_i}...A_{\sigma_j} \) as \( A \). Hence the result follows. \( \square \)
8.3 Step 1

In this section we consider Step 1.

**Proposition 8.3.1** Any 2-morphism \( \alpha \) in the theorem can be expressed as a certain standard composite.

We start with the following lemma.

**Lemma 8.3.1** Any 2-morphism of the form \( \otimes \) can be expressed as a standard composite.

**Proof.** Consider the 2-morphism \( \otimes f, g \) with \( f = C \otimes R_{A_1...A_i...A_{i+1}...A_m} \) and \( g = D \otimes R_{B_1...B_j...B_{j+1}...B_n} \otimes E \). Then by the definition of a Gray monoid, this 2-morphism can be expressed as follows:

\[
\begin{align*}
CA_1...A_iA_{i+1}...A_mB_1...B_jB_{j+1}B_nE & \mapsto CA_{i+1}...A_mB_1...B_jB_{j+1}B_nE \\
\downarrow \lambda & \quad \downarrow \lambda & \quad \downarrow \lambda \\
\downarrow \Omega_0 & \quad \downarrow \Omega_k & \quad \downarrow \Omega_{m-i-1} \\
Q_1 & \quad Q_k & \quad Q_{m-i-1} \\
\downarrow \xi & \downarrow \xi & \downarrow \xi \\
CA_1...A_iA_{i+1}...A_mB_{j+1}B_nB_1...B_jE & \mapsto DA_{i+1}...A_mB_1...B_jB_{j+1}B_nB_1...B_jE \\
\end{align*}
\]

where \( \lambda \) and \( \xi \) are shuffle composites and

\[
P_k = CA_{i+1}...A_{i+k}A_1...A_iA_{i+k+1}...A_mB_1...B_jB_{j+1}B_nE, \\
Q_k = CA_{i+1}...A_{i+k}A_1...A_iA_{i+k+1}...A_mB_{j+1}...B_nB_1...B_jE.
\]

To prove the assertion we use induction on \( m \) and \( n \). First we fix \( n \) to be 2. If \( m = 2 \), the original 2-morphism is elementary. For a general \( m \), we see that each \( \Omega_k \) has a smaller \( m \) than the original 2-morphism, so that by induction on \( m \) each 2-morphism \( \Omega_k \) can be expressed as a standard composite. Also, the shuffle composite \( \lambda \) and \( \xi \) can be expressed as standard composites by the first part of the next lemma. If we take the pasting of all the above standard
composites, the fundamental composites and the inverses of fundamental composites inside the diagram cancel, so that we obtain another standard composite. For a general $n$ one can apply the same technique since each $\Omega_k$ has a similar decomposition. Hence the result follows from the next lemma.

We now consider 2-morphisms of the form $R$.

**Lemma 8.3.2** Any 2-morphism of the form $R$ can be expressed as a standard composite.

**Proof.** Consider the 2-morphism $R_{A_1\ldots A_m|B_1\ldots B_i B_{i+1} \ldots B_n}$ for some integer $1 \leq m$, $1 < n$ and $1 \leq i < n$. We wish to show that this 2-morphism is expressed as the following standard composite:

![Diagram]

where $\beta$ is a fundamental composite and $\alpha$ and $\gamma$ are the inverses of fundamental composites. Then it is easy to see that the above 2-morphism coincides with the following pasting ($\ast$):
where $\mu$, $\nu$ and $\xi$ are certain shuffle composites. Further, if $i < n - 1$, the above 2-morphism coincides with the following one:

\[
(*)
\]

\[
B_1 \ldots B_i A_1 \ldots A_m B_{i+1} \ldots B_n
\]

\[
B_1 \ldots B_{i-1} A_1 \ldots A_m B_i \ldots B_n
\]

\[
B_1 \ldots B_{i+1} A_1 \ldots A_m \ldots B_n
\]

\[
B_1 \ldots B_{n-1} A_1 \ldots A_m B_n
\]

\[
A_1 \ldots A_m B_1 \ldots B_n \rightarrow B_1 \ldots B_n A_1 \ldots A_m
\]

where $\theta_j = R_{A_1 \ldots A_m | B_1 \ldots B_i, B_{i+1}} \otimes B_{j+2} \ldots B_n$ for $i \leq j \leq n - 1$. Next, look at the following part in the above pasting:

\[
(**)
\]

\[
B_1 \ldots B_i A_1 \ldots A_m B_{i+1} \ldots B_n
\]

\[
B_1 \ldots B_{i+1} A_1 \ldots A_m \ldots B_n
\]

\[
B_1 \ldots B_{n-1} A_1 \ldots A_m B_n
\]

\[
A_1 \ldots A_m B_1 \ldots B_n \rightarrow B_1 \ldots B_n A_1 \ldots A_m
\]
Then we find that the 2-morphism coincides with the following one:

$$B_1 \ldots B_{n-2}A_1 \ldots A_mB_{n-1}B_n$$

Applying the same method, we see the desired result. If $i = n - 1$, the 2-morphism $(\ast)$ can be expressed as the following pasting

$$B_1 \ldots B_{n-1}A_1 \ldots A_mB_n$$

where $\eta_j = B_1 \ldots B_{j-1} \otimes R_{A_1 \ldots A_m|B_j, B_{j+1} \ldots B_n}$ for $i \leq j \leq n - 1$. So we can use the same technique.

We next consider a 2-morphism of the form $R_{B_1 \ldots B_i, B_{i+1} \ldots B_n|A_1 \ldots A_m}$ for
some integer $1 \leq m, 1 < n$ and $1 \leq i < n$. Then by the braiding condition (2) for a braided Gray monoid, we see that the 2-morphism can be expressed as the following pasting

\[
\begin{array}{c}
\downarrow \alpha \\
B_1 \ldots B_i A_1 \ldots A_m B_{i+1} \ldots B_n \\
\downarrow (A) \\
A_1 \ldots A_{m-1} B_1 \ldots B_i A_m B_{i+1} \ldots B_n \\
\downarrow (B) \\
B_1 \ldots B_i A_1 \ldots A_{m-1} B_{i+1} \ldots B_n A_m \\
\downarrow (E) \\
\downarrow (D) \\
A_1 \ldots A_{m-1} B_1 \ldots B_i B_{i+1} \ldots B_n A_m \\
\downarrow (C) \\
B_1 \ldots B_i B_{i+1} \ldots B_n A_1 \ldots A_m \\
\end{array}
\]

where $\alpha$ is a certain 2-morphism of the form $\otimes$. In the diagram above, the 2-morphism $\alpha$ can be expressed as a standard composite by Lemma 8.3.1. Also, the 2-morphisms in $(A), (B), (C)$ can be expressed as standard composites since these 2-morphisms have the previously discussed form. For the rest part, we use induction on $m$. For $m = 1$, the result follows since this diagram can be regarded as a variant of the diagram $(\ast\ast)$ with $A = A_1 \ldots A_m$, and we can trace back to a variant of the pasting $(*)$ which is standard. Thus we see that the 2-morphisms in $(D)$ and $(E)$ can also be expressed as a standard composite. By taking the pasting of all the above standard composites, we obtain another standard composite. Hence the claim. \hfill $\square$

**Lemma 8.3.3** Any 2-morphism of the form $R_f$ can be expressed as a standard composite.

**Proof.** For the following 2-morphism

\[
\begin{array}{c}
U_1 \ldots U_p V_1 \ldots V_q W_1 \ldots W_r X_1 \ldots X_u Y_1 \ldots Y_v \\
\downarrow R_{U_1 \ldots U_p, V_1 \ldots V_q \otimes R_{W_1 \ldots W_r, x_1 \ldots x_u \otimes Y_1 \ldots Y_v}} \\
V_1 \ldots V_q W_1 \ldots W_r X_1 \ldots X_u Y_1 \ldots Y_v U_1 \ldots U_p \\
\end{array}
\]

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we first observe that, by setting $V := V_1...V_q$, $X := X_1...X_u$ and $Y := Y_1...Y_v$, the above 2-morphism can be expressed as the following form:

\[
\begin{align*}
U_1...U_pVW_1...W_rXY & 
\rightarrow U_1...U_pVXW_1...W_rY \\
\downarrow \sigma & \\
U_1...U_pVW_1...W_{r-1}XW_rY & 
\rightarrow U_1...U_pVW_1XW_2...W_rY \\
\downarrow \varphi & \\
U_1...U_{p-1}VW_1...W_rXYU_p & 
\rightarrow U_1...U_{p-1}VXW_1...W_rYU_p \\
\downarrow \psi & \\
U_1...U_{p-2}VW_1...W_rXYU_{p-1}U_p & 
\rightarrow U_1...U_{p-2}VXW_1...W_rYU_{p-1}U_p \\
\downarrow W_{ij} & \\
U_1VW_1...W_rXYU_2...U_p & 
\rightarrow U_1VXW_1...W_rYU_2...U_p \\
\downarrow \tau & \\
VW_1...W_{r-1}XW_rYU_1...U_p & 
\rightarrow VW_1XW_2...W_rYU_1...U_p \\
\downarrow & \\
VW_1...W_rXYU_1...U_p & 
\rightarrow VXW_1...W_rYU_1...U_p
\end{align*}
\]

where $\varphi$, $\psi$, $\sigma$ and $\tau$ are shuffle composites, and

\[
W_{ij} = U_1...U_{p-i} \otimes R_{U_{p-i+1},VW_1...W_{r-j}} \otimes R_{W_{r-j+1},X \otimes W_{r-j+2}...W_r,Y} \otimes U_{p-i+2}...U_p,
\]

for $1 \leq i \leq p$, $1 \leq j \leq r$. Next we decompose the above 2-morphisms $W_{ij}$ into standard composites. To do this, we use the fact that by taking a subset $A, B_i, C, D_j, E_k$, $1 \leq i \leq l$, $1 \leq j \leq n$, $1 \leq k \leq s$ of the set $U_a, V_b, W_c, X_d, E_e$, $1 \leq a \leq p$, $1 \leq b \leq q$, $1 \leq c \leq r$, $1 \leq d \leq u$, $1 \leq e \leq v$, the essential part of the 2-morphisms $W_{ij}$ can be expressed as the following form $\Sigma$:

\[
\begin{align*}
AB_1...B_lCD_1...D_nE_1...E_s & 
\rightarrow AB_1...B_lD_1...D_nCE_1...E_s \\
\downarrow R_{A,B_1,...B_l \otimes R_{C,D_1,...D_n} \otimes E_1,...E_s} & \\
B_1...B_lCD_1...D_nE_1...E_sA & 
\rightarrow B_1...B_lD_1...D_nCE_1...E_sA
\end{align*}
\]
For example, the 2-morphism $W_{11}$ is

$$U_1U_pV_1V_qW_1W_rX_1X_uY_1Y_v\rightarrow U_1U_pV_1V_qW_1W_{r-1}X_1X_uW_rY_1Y_v$$

and we take $A = U_p$, $B_1...B_l = V_1...V_qW_1W_{r-1}$, $C = W_r$, $D_1...D_n = X_1...X_u$, $E_1...E_s = Y_1...Y_v$.

Next, consider the following standard composite $\Lambda$.

$$AB_1...B_lCD_1...D_nE_1...E_s \downarrow \lambda \downarrow \mu \downarrow \nu \downarrow \Omega$$

where $\mu$ and $\xi$ are fundamental composites, $\nu$ and $\lambda$ are the inverses of fundamental composites, and $\Omega$ is the following elementary composite:
\[ AB_1...B_tCD_1...D_nE_1...E_s \quad \rightarrow \quad AB_1...B_tD_1...D_nCE_1...E_s \]

\[ B_1A...B_tCD_1...D_nE_1...E_s \quad \rightarrow \quad B_1A...B_tD_1...D_nCE_1...E_s \]

\[ B_1...B_tACD_1...D_nE_1...E_s \quad \downarrow V_0 \quad \rightarrow \quad B_1...B_tAD_1...D_nCE_1...E_s \quad \downarrow V_{j-1} \quad \rightarrow \quad B_1...B_tAD_1...D_nCE_1...E_s \quad \downarrow V_{n-1} \]

\[ B_1...B_tCD_1...D_nAE_1...E_s \quad \rightarrow \quad B_1...B_tD_1...D_nCAE_1...E_s \]

\[ B_1...B_tCD_1...D_nE_1...AE_s \quad \rightarrow \quad B_1...B_tD_1...D_nCE_1...AE_s \]

\[ B_1...B_tCD_1...D_nE_1...E_sA \quad \rightarrow \quad B_1...B_tD_1...D_nCE_1...E_sA \]

where the 2-morphism \( V_{j-1} \) is the following pasting:
To show that the 2-morphism \( \Sigma \) coincides with \( \Lambda \), we first observe that \( \Sigma \) coincides with the following 2-morphism.
where $\eta$ and $\theta$ are fundamental composites. In particular, $\eta$ can be taken to be the following pasting:
Correspondingly, we consider the following 2-morphism $\rho$:
Using the 2-morphism $\rho$, we can construct the cube which has the 2-morphisms $V_0, \eta$ and $\rho$ as the top side, the left side and the right side, respectively. To show that the cube is commutative, we use the fact that the 2-morphism $V_0$ has the form
Then both pastings \( V_0 \) and \( \eta \) contain the same 2-morphism
In addition, we find the following commutative prisms:

\[
\begin{align*}
B_1 \ldots B_{i-1} A B_i \ldots B_l C D_1 \ldots D_n E_1 \ldots E_s & \longrightarrow B_1 \ldots B_{i-1} A B_i \ldots B_l D_1 C \ldots D_n E_1 \ldots E_s \\
\downarrow R & \quad \downarrow \otimes & \quad \downarrow R \\
B_1 \ldots B_{i-1} B_i A \ldots B_l C D_1 \ldots D_n E_1 \ldots E_s & \longrightarrow B_1 \ldots B_{i-1} B_i D_1 C \ldots D_n E_1 \ldots E_s \\
\downarrow & \quad \downarrow R_f & \quad \downarrow R_f \\
B_1 \ldots B_{i-1} B_i \ldots B_l C D_1 \ldots D_n E_1 \ldots E_s A & \longrightarrow B_1 \ldots B_{i-1} B_i \ldots B_l D_1 C \ldots D_n E_1 \ldots E_s A \\
\downarrow & \quad \downarrow \otimes & \quad \downarrow \otimes & \quad \downarrow R_f
\end{align*}
\]

for some integers \(1 \leq i \leq l\),

\[
\begin{align*}
B_1 \ldots B_i A C D_1 \ldots D_n E_1 \ldots E_s & \longrightarrow B_1 \ldots B_i A D_1 C \ldots D_n E_1 \ldots E_s \\
\downarrow R & \quad \downarrow R_f & \quad \downarrow R \\
B_1 \ldots B_i C D_1 \ldots D_j A D_{j+1} \ldots D_n E_1 \ldots E_s & \longrightarrow B_1 \ldots B_i D_1 C \ldots D_j A D_{j+1} \ldots D_n E_1 \ldots E_s \\
\downarrow & \quad \downarrow \otimes & \quad \downarrow R_f \\
B_1 \ldots B_i C D_1 \ldots D_j D_{j+1} A \ldots D_n E_1 \ldots E_s & \longrightarrow B_1 \ldots B_i D_1 C \ldots D_j D_{j+1} A \ldots D_n E_1 \ldots E_s \\
\downarrow & \quad \downarrow \otimes & \quad \downarrow \otimes & \quad \downarrow R_f
\end{align*}
\]

for some integers \(1 \leq j \leq n-1\) and

\[
\begin{align*}
B_1 \ldots B_i A C D_1 \ldots D_n E_1 \ldots E_s & \longrightarrow B_1 \ldots B_i A D_1 C \ldots D_n E_1 \ldots E_s \\
\downarrow R & \quad \downarrow R_f & \quad \downarrow R \\
B_1 \ldots B_i C D_1 \ldots D_n E_1 \ldots E_{h-1} A E_h \ldots E_s & \longrightarrow B_1 \ldots B_i D_1 C \ldots D_n E_1 \ldots E_{h-1} A E_h \ldots E_s \\
\downarrow & \quad \downarrow \otimes & \quad \downarrow \otimes & \quad \downarrow R_f \\
B_1 \ldots B_i C D_1 \ldots D_n E_1 \ldots E_h A E_{h+1} \ldots E_s & \longrightarrow B_1 \ldots B_i D_1 C \ldots D_n E_1 \ldots E_h A E_{h+1} \ldots E_s \\
\downarrow & \quad \downarrow \otimes & \quad \downarrow \otimes & \quad \downarrow \otimes & \quad \downarrow R_f
\end{align*}
\]

for some integers \(1 \leq h \leq s\). So by putting the above commutative prisms together, we obtain a commutative cube which contains the 2-morphism \(V_0\).

Next we construct a commutative cube which contains the 2-morphism \(V_1\).

For this, we need to replace the pasting defining \(\rho\) by the following \(\rho'\).
Then we find the following commutative prisms:
for some integers $1 \leq i \leq l$, 

\[
\begin{array}{c}
B_1 \ldots B_{i-1}AB_1 \ldots B_iD_1C \ldots D_nE_1 \ldots E_s \rightarrow B_1 \ldots B_{i-1}AB_1 \ldots B_iD_1D_2C \ldots D_nE_1 \ldots E_s \\
\downarrow R \hspace{1cm} \downarrow \otimes \hspace{1cm} \downarrow R \\
B_1 \ldots B_{i-1}B_1A \ldots B_iD_1C \ldots D_nE_1 \ldots E_s \rightarrow B_1 \ldots B_{i-1}B_1A \ldots B_iD_1D_2C \ldots D_nE_1 \ldots E_s \\
\downarrow \downarrow R_f \hspace{1cm} \downarrow \\
B_1 \ldots B_{i-1}B_1 \ldots B_iD_1C \ldots D_nE_1 \ldots E_sA \rightarrow B_1 \ldots B_{i-1}B_1 \ldots B_iD_1D_2C \ldots D_nE_1 \ldots E_sA
\end{array}
\]

for some integers $1 \leq i \leq l$.

for some integers $1 \leq j \leq n-2$ and 

\[
\begin{array}{c}
B_1 \ldots B_iD_1A\ldots D_nE_1 \ldots E_s \rightarrow B_1 \ldots B_iD_1AD_2D_3 \ldots D_nE_1 \ldots E_s \\
\downarrow R \hspace{1cm} \downarrow R_f \hspace{1cm} \downarrow R \\
B_1 \ldots B_iD_1A\ldots D_nE_1 \ldots E_s \rightarrow B_1 \ldots B_iD_1D_2C \ldots D_{j+1}A \ldots D_nE_1 \ldots E_s \\
\downarrow \downarrow \downarrow R_f \\
B_1 \ldots B_iD_1A\ldots D_nE_1 \ldots E_s \rightarrow B_1 \ldots B_iD_1D_2C \ldots D_{j+2}A \ldots D_nE_1 \ldots E_s
\end{array}
\]

for some integers $1 \leq j \leq n-2$ and 

\[
\begin{array}{c}
B_1 \ldots B_iA\ldots D_nE_1 \ldots E_s \rightarrow B_1 \ldots B_iAD_1\ldots D_nE_1 \ldots E_s \\
\downarrow R \hspace{1cm} \downarrow R_f \hspace{1cm} \downarrow R \\
B_1 \ldots B_iA\ldots D_nE_1 \ldots E_s \rightarrow B_1 \ldots B_iD_1D_2C \ldots D_nE_1 \ldots E_{h-1}AE_h \ldots E_s \\
\downarrow \downarrow \downarrow R_f \\
B_1 \ldots B_iA\ldots D_nE_1 \ldots E_s \rightarrow B_1 \ldots B_iD_1D_2C \ldots D_nE_1 \ldots E_{h+1}AE_{h+1} \ldots E_s
\end{array}
\]

for some integers $1 \leq h \leq s$. Using these commutative prisms, we can construct a commutative cube which contains the 2-morphism $V_1$. Besides, the 2-morphisms $\rho$ and $\rho'$ coincide, so that we can combine the above two commutative cubes to make a bigger one which contains both $V_0$ and $V_1$. For more general $V_{j-1}$, the following pasting gives the similar commutative cube which contains $V_{j-1}$. 

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Let \((A_1, \ldots, A_n, f_1, f_2, \ldots, f_l \mid g_1, g_2, \ldots, g_m)\) be a braiding diagram and let \(\alpha : f_1f_2\ldots f_l \Rightarrow g_1g_2\ldots g_m\) be a canonical 2-morphism. Then using the above technique, we can replace each canonical 2-morphism in the pasting defining \(\alpha\) by a standard composite. If we take the pasting of all the above standard composites, we obtain another standard composite whose remaining fundamental diagram have their non-elementary 1-morphisms in the set \(\{f_1, f_2, \ldots, f_l, g_1, g_2, \ldots, g_m\}\). This finishes Step 1.
8.4 Step 3

Finally, we consider Step 3.

**Proposition 8.4.1** Let \((A_1, \ldots, A_n, f_1, f_2, \ldots, f_l \mid g_1, g_2, \ldots, g_m)\) be an elementary diagram, and let \(\alpha\) and \(\beta\) be elementary composites of the form \(f_1 f_2 \ldots f_l \Rightarrow g_1 g_2 \ldots g_m\). Then the 2-morphism \(\alpha\) coincides with \(\beta\).

**Proof.** Since each 1-morphism \(f_i\) is elementary it corresponds to a natural number. Therefore, the sequence of 1-morphisms \(f_1, f_2, \ldots, f_l\) corresponds to a word of natural numbers. Let us denote this word by \(X\). Likewise the sequence of 1-morphisms \(g_1, g_2, \ldots, g_m\) corresponds to another word \(W\). Let \(\tau_1 \tau_2 \tau_3 \ldots \tau_s\) and \(\eta_1 \eta_2 \eta_3 \ldots \eta_t\) be the pastings \(\alpha\) and \(\beta\) respectively. Then the step 3 is equivalent to the commutativity (in the 2-categorical sense) of the diagram below:

\[
\begin{array}{c}
X \\
\eta_1 \\
Y \\
\eta_2 \\
Z \\
\eta_3 \\
\Rightarrow \\
\Rightarrow \\
\Rightarrow \\
\Rightarrow \\
\Rightarrow
\end{array}
\]

At this point, Theorem 3.2.1 will do the job. In the diagram above, the characters \(X, Y, Z, \ldots\) represent words, and the 2-morphisms of the form \(S\) (resp. \(\otimes\)) can be viewed as self-homotopies of the form \(iji \simeq jij\) (\(ij \simeq ji\)). Thus, if we ignore the direction of the 2-morphisms, we obtain by Theorem 3.2.1 a decomposition of the diagram each of which is inessential, a cube, a prism or a permutohedron. For cubes, we can use the following commutative diagram:

\[
\begin{array}{c}
ABCDEF \\
\downarrow \otimes \\
ABDCEF \\
\downarrow \otimes \\
ABCDFE \\
\downarrow \otimes \\
ABDCF \end{array}
\]

\[
\begin{array}{c}
BACDEF \\
\downarrow \otimes \\
BADCEF \\
\downarrow \otimes \\
BACDFE \\
\downarrow \otimes \\
BADCFE
\end{array}
\]
For prisms, we can use the following one:

\[ A B C E D \rightarrow A B E C D \]

\[ A B C D E \rightarrow \downarrow S \rightarrow A B E D C \]

\[ \downarrow \otimes A B D C E \rightarrow A B D E C \rightarrow \downarrow \otimes B A E C D \]

\[ B A C D E \rightarrow \downarrow \otimes \rightarrow \downarrow S \rightarrow B A E D C \]

\[ B A D C E \rightarrow \downarrow \otimes \rightarrow B A D E C \]

The commutativity of the above diagram is an easy consequence of the definition of a braiding on a Gray monoid. Finally, for permutohedra, we can use the result of Kapranov and Voevodsky [15], which states that the following two pastings coincide:

\[ A B C D \]

\[ B A C D \rightarrow \downarrow \otimes \rightarrow B A D C \]

\[ B C A D \rightarrow \downarrow S \rightarrow B D A C \rightarrow A D B C \]

\[ C B A D \rightarrow \downarrow \otimes \rightarrow C D B A \rightarrow D A B C \]

\[ C B D A \rightarrow \downarrow S \rightarrow C D B A \rightarrow D A C B \]

\[ C D B A \rightarrow \downarrow \otimes \rightarrow D C B A \rightarrow D A C B \]

\[ D C B A \rightarrow \downarrow S \rightarrow D C A B \]
Hence, modulo the directions of arrows, Step 3 reduces to Theorem 3.2.1. Further, once the direction of an arrow is specified, the content of the arrow is uniquely determined; for two 1-morphisms \( f : A \to A' \) and \( g : B \to B' \) in \( B \), we have a 2-isomorphism

\[
AB \quad \xrightarrow{\downarrow \otimes f,g} \quad A'B
\]

in \( B \). If an arrow has the direction \((f \otimes 1)(1 \otimes g) \Rightarrow (1 \otimes g)(f \otimes 1)\), then the content of the arrow becomes \( \otimes_{f,g} \). On the other hand, if the direction is \((1 \otimes g)(f \otimes 1) \Rightarrow (f \otimes 1)(1 \otimes g)\), then the content of the arrow becomes \( \otimes_{f,g}^{-1} \). The same is true for 2-morphisms of the form \( S \). In other words, a 2-morphism with the opposite direction is given by the inverse. This completes the proof of Proposition 8.4.1. \( \square \)

Putting together the proofs of Step 1, Step 2, and Step 3, we obtain Theorem 8.1.1.
Chapter 9

Braid monoid actions on a category

9.1 Deligne’s work

In the paper [8], Deligne studied actions of a braid monoid on a category. In general, an action of an monoid $M$ on a category $C$ is defined by giving functors $T(f)$ ($f \in M$) and isomorphisms of functors $c_{f,g} : T(f) \circ T(g) \to T(fg)$ making the following diagram commutative:

$$
\begin{array}{ccc}
T(f) \circ T(g) \circ T(h) & \longrightarrow & T(fg) \circ T(h) \\
\downarrow & & \downarrow \\
T(f) \circ T(gh) & \longrightarrow & T(fgh)
\end{array}
$$

As Deligne explained in the paper, in order to define an action of $B^+_n$ on a category $C$, it is not enough to give isomorphisms of functors

\[ (*) = \begin{cases} 
T(s_i) \circ T(s_{i+1}) \circ T(s_i) \to T(s_{i+1}) \circ T(s_i) \circ T(s_{i+1}), \\
T(s_i) \circ T(s_j) \to T(s_j) \circ T(s_i) 
\end{cases} \quad \text{for } j \geq i + 2, \]

but requires extra conditions. In particular, he mentioned that for the case of $B^+_3$ the functors $T(s_1), T(s_2), T(s_3) must further satisfy the following condition. Write $a, b, c$ for $T(s_1), T(s_2), T(s_3)$, and consider the composite of isomorphisms of functors
Then the composite must be the identity. As one can see, this condition is nothing but the condition on our permutohedron. Hence it is quite natural to expect that the following conjecture is true:

**Conjecture 9.1.1** Giving an action of the braid monoid $B_n^+$ on $C$ is equivalent to giving functors $T(s_i) : C \to C$ and the isomorphisms $(\ast)$ such that all the composites of such isomorphisms on inessential self-homotopies, cubes, prisms, and permutohedra are the identity.

### 9.2 Two presentations

Using another presentation of the braid monoid, which is given by generators $\tau(w)$ ($w \in S_n$) and the relations

$$
\begin{align*}
\tau(e) &= e, \\
\tau(w'w'') &= \tau(w')\tau(w'') & \text{for } l(w'w'') = l(w') + l(w''),
\end{align*}
$$

Deligne proved the following theorem.

**Theorem 9.2.1** (Deligne) Giving an action of the braid monoid $B_n^+$ on $C$ is equivalent to giving the following:

For $w \in S_n$, a functor $T(w) : C \to C$.

For $w', w'' \in S_n$ satisfying $l(w'w'') = l(w') + l(w'')$, an isomorphism of functors

$$
T(w')T(w'') \to T(w'w'')
$$

making the following diagram
We now explain how Deligne’s theorem implies a positive answer to our conjecture. Recall that the two presentations are related by the following rules:

\[
T(s_i)T(s_{i+1})T(s_i) \rightarrow T(s_is_{i+1}s_i) = T(s_{i+1}s_is_{i+1}) \leftarrow T(s_{i+1})T(s_i)T(s_{i+1}),
\]

\[
T(s_i)T(s_j) \rightarrow T(s_is_j) = T(s_js_i) \leftarrow T(s_j)T(s_i).
\]

For example, the next diagram shows that the composite of the isomorphisms on a cube becomes the identity.

\[
\begin{array}{ccc}
T(w')T(w'')T(w''') & \longrightarrow & T(w'w''w''') \\
\downarrow & & \downarrow \\
T(w')T(w''w''') & \longrightarrow & T(w'w''w''')
\end{array}
\]

commutative for \( l(w'w''w''') = l(w') + l(w'') + l(w'''). \)
\[
T(1)T(3)T(5) \rightarrow T(13)T(5) = T(31)T(5) \rightarrow T(3)T(1)T(5)
\]
\[
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow
\]
\[
T(1)T(35) \rightarrow T(135) = T(315) \rightarrow T(3)T(15)
\]
\[
\| \quad \| \quad \| \quad \|
\]
\[
T(1)T(53) \rightarrow T(153) \quad T(351) \rightarrow T(3)T(51)
\]
\[
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow
\]
\[
T(1)T(5)T(3) \rightarrow T(15)T(3) \quad T(35)T(1) \rightarrow T(3)T(5)T(1)
\]
\[
\downarrow \quad \| \quad \downarrow \quad \|
\]
\[
T(15)T(3) \equiv T(51)T(3) \quad T(53)T(1) \equiv T(35)T(1)
\]
\[
\| \quad \downarrow \quad \| \quad \|
\]
\[
T(51)T(3) \rightarrow T(513) = T(531) \rightarrow T(53)T(1)
\]
\[
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow
\]
\[
T(5)T(1)T(3) \rightarrow T(5)T(13) = T(5)T(31) \rightarrow T(5)T(3)T(1)
\]
Bibliography


Part II

Tortile Yang-Baxter operators for crossed group-categories
Chapter 1

Introduction

The category of tangles in 3 dimension has a beautiful algebraic characterization in terms of a universal property. This was initially developed by Yetter [11], Turaev [9], Freyd-Yetter [1] and Joyal-Street [4], and has culminated in the work of Shum [8] asserting that the category of framed tangles $\mathcal{FT}$ is monoidally equivalent to the tortile category freely generated by a single object. Joyal and Street [3] gave another purely algebraic interpretation of this category as the free tensor category containing an object equipped with a tortile Yang-Baxter operator.

Recently, Turaev [10] introduced the notion of a modular crossed group-category, and used it to develop 3-dimensional homotopy quantum field theory (HQFT). He started with defining the notion of a tortile (ribbon) crossed $\pi$-category for a group $\pi$, and showed that modular crossed $\pi$-categories induce invariants of 3-dimensional $\pi$-manifolds.

The aim of the second part of this thesis is to give the Joyal and Street’s interpretation for a crossed group-category. To do this, we define a balanced Yang-Baxter operator and a tortile Yang-Baxter operator in a crossed group-category. Then we prove that the free crossed group-category $\mathcal{F}$ generated by a single object equipped with a tortile Yang-Baxter operator admits a unique braiding and a twist. Although our construction owes much to the paper [3], several new aspects appear. First, it turns out that one should define a twist before a Yang-Baxter operator. This statement means that in a general crossed group-category, it is not possible to define a Yang-Baxter operator without a twist. Thus one can define only balanced Yang-Baxter

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operators in a crossed group-category. Second, we use the fact that the category $\mathcal{F}$ admits a connectivity structure, which we feel non-trivial. In general, for an object $U$ in a crossed $\pi$-category $\mathcal{C}$, the centralizer $\mathcal{L}_C(U)$ does not admit a crossed $\pi$-category structure. However, if a crossed $\pi$-category $\mathcal{C}$ is connected, then the category $\mathcal{L}_C(U)$ admits a crossed $\pi$-category structure, so that we can apply this procedure to $\mathcal{F}$. Third, since we have to consider a Yang-Baxter operator with a group action, various identities which were simple in [3] become much more complicated. To overcome this difficulty, we use a diagrammatic notion. Then, we can check that each equality between diagrams corresponds to a certain equality between morphisms in $\mathcal{F}$. As a result, we see that the constructions above are all well done and the theorem holds.
Chapter 2

Tortile crossed group-categories

2.1 Crossed group-categories

Definition 2.1.1 Let $\pi$ be a group and let $\mathcal{C}$ be a strict monoidal category with a unit object $I$. Then the category $\mathcal{C}$ is called a $\pi$-category if it satisfies the following conditions:

(a) there are full subcategories $\mathcal{C}_\alpha (\alpha \in \pi)$ of $\mathcal{C}$ such that each object of $\mathcal{C}$ belongs to $\mathcal{C}_\alpha$ for a unique $\alpha \in \pi$;

(b) if $U \in \mathcal{C}_\alpha$ and $V \in \mathcal{C}_\beta$ with $\alpha \neq \beta$ then there is not any morphism from $U$ to $V$;

(c) $I \in \mathcal{C}_1$, and if $U \in \mathcal{C}_\alpha$ and $V \in \mathcal{C}_\beta$ then $U \otimes V \in \mathcal{C}_{\alpha\beta}$.

In [10] a $K$-additivity and a left duality were assumed in the monoidal category $\mathcal{C}$. In this paper, we do not assume those structures in $\mathcal{C}$.

Definition 2.1.2 In the setting above, an automorphism of $\mathcal{C}$ is defined as a functor $\varphi : \mathcal{C} \to \mathcal{C}$ which preserves the tensor product and the unit object. Thus,

$$\varphi(I) = I, \quad \varphi(U \otimes V) = \varphi(U) \otimes \varphi(V), \quad \varphi(f \otimes g) = \varphi(f) \otimes \varphi(g),$$

for any objects $U, V$ and any morphisms $f, g$ in $\mathcal{C}$. We denote by $\text{Aut}(\mathcal{C})$ the group of automorphisms of $\mathcal{C}$. A crossed $\pi$-category is a $\pi$-category $\mathcal{C}$ endowed with a group homomorphism $\varphi : \pi \to \text{Aut}(\mathcal{C})$ such that for all $\alpha, \beta \in \pi$ the functor $\varphi_\alpha = \varphi(\alpha) : \mathcal{C} \to \mathcal{C}$ maps $\mathcal{C}_\beta$ to $\mathcal{C}_{\alpha\beta^{-1}}$. For objects $U \in \mathcal{C}_\alpha, V \in \mathcal{C}_\beta$, set $\uparrow V = \varphi_\alpha(V)$.

For crossed $\pi$-categories $\mathcal{C}, \mathcal{C}'$, a tensor functor $\mathcal{C} \to \mathcal{C}'$ is called a crossed $\pi$-functor if it preserves the action of $\pi$. 
2.2 Balanced crossed group-category

**Definition 2.2.1** Let $\mathcal{C}$ be a crossed $\pi$-category. A braiding in $\mathcal{C}$ is a system of invertible morphisms $c_{U,V} : U \otimes V \rightarrow ^U V \otimes U$ satisfying the following conditions:

(a) for any morphisms $f : U \rightarrow U'$ and $g : V \rightarrow V'$ such that $U, U'$ lie in the same component of $\mathcal{C}$, we have

$$c_{U',U}(f \otimes g) = (^U g \otimes f)c_{U,V};$$

(b) for any objects $U, V, W$ in $\mathcal{C}$ we have

$$c_{U \otimes V, W} = (c_{U,V} \otimes 1)(1 \otimes c_{V,W});$$

(c) for any objects $U, V, W$ in $\mathcal{C}$ we have

$$c_{U,V \otimes W} = (1 \otimes c_{U,W})(c_{U,V} \otimes 1);$$

(d) the action of $\pi$ on $\mathcal{C}$ preserves the braiding, i.e., for any $\alpha \in \pi$ and any $V, W \in \mathcal{C}$ we have

$$\varphi_{\alpha}(c_{V,W}) = c_{\varphi_{\alpha}(V),\varphi_{\alpha}(W)}.$$

A crossed $\pi$-category equipped with a braiding is called a braided crossed $\pi$-category. A braided crossed $\pi$-category $\mathcal{C}$ is called balanced if it is equipped with a natural family of invertible morphisms $\theta_U : U \rightarrow ^U U$ (called twist) satisfying the following conditions:

1. $\theta_I = id_I : I \rightarrow I$;
2. for any object $U, V$ in $\mathcal{C}$ we have

$$\theta_{U \otimes V} = c_{(UV),U}(c_{U,V}(\theta_U \otimes \theta_V));$$

3. the action of $\pi$ on $\mathcal{C}$ preserves the twist, i.e., for any $\alpha \in \pi$ and any $U \in \mathcal{C}$ we have $\varphi_\alpha(\theta_U) = \theta_{\varphi_\alpha(U)}$.

A braided crossed $\pi$-category $\mathcal{C}$ is called tortile if it is balanced and each object $U$ has a dual $U^*$ such that $\theta_{U^*} = (\theta_U)^*$.
Chapter 3

Tortile Yang-Baxter operators

3.1 Yang-Baxter operators and twists

In this chapter we consider Yang-Baxter operators and twists in a crossed \(\pi\)-category \(\mathcal{C}\). When \(\pi = 1\), one can define a Yang-Baxter operator on each object \(U\) in \(\mathcal{C}\) without a twist. However, for a general crossed \(\pi\)-category \(\mathcal{C}\), one must define a twist first, then proceed to define a balanced Yang-Baxter operator by using the twist.

**Definition 3.1.1** A twist on an object \(U\) of a crossed \(\pi\)-category \(\mathcal{C}\) is an invertible arrow \(z : U \to {}^U U\). A balanced Yang-Baxter operator on an object \(U\) is an invertible arrow \(y : U \otimes U \to {}^U U \otimes U\) satisfying the hexagonal condition

\[
({}^U y \otimes 1)(1 \otimes y)(y \otimes 1) = (1 \otimes y)(y' \otimes 1)(1 \otimes y)
\]

where \(y' = ({}^U z \otimes 1)y(1 \otimes z^{-1})\).

A left dual for an object \(U\) of \(\mathcal{C}_\alpha\) is an object \(U^*\) in \(\mathcal{C}_{\alpha^{-1}}\) together with arrows

\[
b_U : I \to U \otimes U^*\quad\text{and}\quad d_U : U^* \otimes U \to I
\]

such that

\[
(d_U \otimes 1)(1 \otimes b_U) = 1\quad\text{and}\quad(1 \otimes d_U)(b_U \otimes 1) = 1.
\]

If both \(U, V\) have duals, then each arrow \(f : U \to V\) gives rise to an arrow

\[
f^* : (d_V \otimes 1)(1 \otimes f \otimes 1)(1 \otimes b_U) : V^* \to U^*.
\]
A balanced Yang-Baxter operator on an object $U \in \mathcal{C}$ is called \textit{dualizable} if $U$ has a dual and, both the arrows $u : UU^* \otimes U \rightarrow U \otimes U^*$ and $v : U^* \otimes U \rightarrow U^* U \otimes U^*$, given by the equations
\[ u = (d_{U^*} \otimes 1 \otimes 1)(1 \otimes y \otimes 1)(1 \otimes 1 \otimes b_U) \]
and
\[ v = (d_U \otimes 1 \otimes 1)(1 \otimes 1 \otimes U^* z^{-1} \otimes 1)(1 \otimes y^{-1} \otimes 1) \]

\[ (1 \otimes z \otimes 1 \otimes 1)(1 \otimes 1 \otimes b_U) \]
are invertible. A balanced Yang-Baxter operator on an object $U$ is called \textit{tortile} if it is dualizable and the following identity holds.
\[ U \circ z = (1 \otimes d_{U^*})(1 \otimes U^* v^{-1})(y' \otimes 1)(1 \otimes b_{U^*}) : U \rightarrow U(U^*). \]

In a balanced crossed $\pi$-category $\mathcal{C}$, we have a balanced YB-operator ($y = c_{U,U}, z = \theta_U$) on each object $U$. If $U$ has a dual, then we have the identities $u = c_{U,U}^{-1}$ and $v = c_{U^*,U}$ in $\mathcal{C}$. Hence ($y = c_{U,U}, z = \theta_U$) is dualizable. The next proposition shows that a balanced crossed $\pi$-category $\mathcal{C}$ becomes a tortile crossed $\pi$-category iff the above balanced YB-operators ($y = c_{U,U}, z = \theta_U$) become tortile for all objects $U$ in $\mathcal{C}$.

\textbf{Proposition 3.1.1} \textit{In a balanced crossed $\pi$-category $\mathcal{C}$, if $U$ is an object with a dual $U^*$, then the pair $(c_{U,U}, \theta_U)$ is a tortile YB-operator iff $\theta_{U^*} = (\theta_U)^*$.}

\textit{Proof.} We first observe that if $(U^*, d_U, b_U)$ is a dual for $U$, then $(U^*, d_{U^*} c_{v_{U,U^*}}, c_{U^*,v_{U^*}} b_{U^*})$ is a dual for $U^*$. Then for an arrow $f : U^* \rightarrow V^*$ in $\mathcal{C}$, we obtain an arrow
\[ f^* = (1 \otimes d_{v_{V}})(1 \otimes f \otimes 1)(b_{v_{U}} \otimes 1) : V^* \rightarrow U^*. \]
Applying this construction to the arrow $\theta_{v_{v_{U^*}}} : UU^* \rightarrow U^*$, we see that
\[ (\theta_{v_{v_{U^*}}})^* \theta_U = (d_{v_{U^*}} \otimes 1)(c_{v_{v_{U^*}},v_{U^*}} \otimes 1)(1 \otimes \theta_{v_{v_{U^*}}} \otimes 1)(1 \otimes c_{v_{v_{U^*}},v_{v_{U^*}}}^{-1})(1 \otimes b_{v_{U^*}}) \theta_U \]
\[ = (d_{v_{U^*}} \otimes 1)(c_{v_{v_{U^*}},v_{U^*}} \otimes 1)(1 \otimes \theta_{v_{v_{U^*}}} \otimes 1)(\theta_U \otimes 1 \otimes 1)(1 \otimes c_{v_{v_{U^*}},v_{v_{U^*}}}^{-1})(1 \otimes b_{v_{U^*}}) \]
\[ = (d_{v_{U^*}} \otimes 1)(c_{v_{v_{U^*}},v_{U^*}} \otimes 1)(\theta_U \otimes \theta_{v_{v_{U^*}}} \otimes 1)(1 \otimes c_{v_{v_{U^*}},v_{v_{U^*}}}^{-1})(1 \otimes b_{v_{U^*}}) \]
\[ = (d_{v_{U^*}} \otimes 1)(\theta_{v_{v_{U^*}}} \otimes \theta_{v_{v_{U^*}}} \otimes 1)(c_{v_{v_{U^*}},v_{v_{U^*}}}^{-1})(1 \otimes c_{v_{v_{U^*}},v_{v_{U^*}}}^{-1})(1 \otimes b_{v_{U^*}}) \]
\[ = (d_{v_{U^*}} \otimes 1)(c_{v_{v_{U^*}},v_{v_{U^*}}}^{-1})(1 \otimes c_{v_{v_{U^*}},v_{v_{U^*}}}^{-1})(1 \otimes b_{v_{U^*}}) \]
\[ = (d_{v_{U^*}} \otimes 1)c_{v_{v_{U^*}},v_{v_{U^*}}}^{-1}(1 \otimes b_{v_{U^*}}) \]

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that \(\beta\) category whose objects are pairs \((f, g)\) of objects in \(U\) and \(V\), respectively, and whose arrows \((X, Y)\) are triples \((X, Y, \alpha)\) where \(\alpha : X \otimes X \to V_X \otimes U_Y\) are isomorphisms in \(C\) such that \(\beta(1 \otimes f) = (U f \otimes 1) \alpha\).

Then \(L_C(U)\) becomes a \(\pi\)-category by the rule \((X, \alpha) \in C_x \iff X \in C_x\) for \(x \in \pi\) and the tensor product \((X, \alpha) \otimes (Y, \beta) = (X \otimes Y, (\alpha \otimes 1)(1 \otimes \beta))\).

Similarly, one can define the centralizer \(L_C(h)\) of an arrow \(h : U \to V\) in \(C\) as follows. The objects of \(L_C(h)\) are triples \((X, \alpha, \beta)\) where \(X\) is an object in \(C\) and \(\alpha : U \otimes X \to U X \otimes U\) and \(\beta : V \otimes X \to V X \otimes V\) are isomorphisms in \(C\) such that \(\beta(1 \otimes 1) = (1 \otimes 1) \alpha\). The arrows \((X, \alpha, \beta) \to (Y, \gamma, \delta)\) in \(L_C(h)\) are arrows \(f : X \to Y\) in \(C\) such that \((U f \otimes 1) \alpha = \gamma(1 \otimes f)\) and \((V f \otimes 1) \beta = \delta(1 \otimes f)\). This category \(L_C(h)\) also admits a \(\pi\)-category structure.

**Definition 3.1.3** A crossed \(\pi\)-category \(C\) is called connected if for each pair \((X, Y)\) of objects, there exists an invertible arrow \(g(X, Y) : X Y \to Y\) such that \(g(I, Y) = id_Y, g(X, I) = id_I, g(X' \otimes X, Y) = g(X', Y) \cdot g(X, Y)\) and \(g(X, Y \otimes Y') = g(X, Y) \cdot g(X, Y')\).

When a crossed \(\pi\)-category \(C\) is connected, the categories \(L_C(U)\) and \(L_C(h)\) become crossed \(\pi\)-categories via \((X, \alpha)(Y, \beta) = (X Y, (U g^{-1}(X, Y) \otimes 1) \beta(1 \otimes g(X, Y)))\) and \((X, \alpha, \beta)(Y, \gamma, \delta) = (X Y, (U g^{-1}(X, Y) \otimes 1) \gamma(1 \otimes g(X, Y)), (V g^{-1}(X, Y) \otimes 1) \delta(1 \otimes g(X, Y)))\).

**Definition 3.1.4** For a crossed \(\pi\)-category \(C\), the center \(L_C\) of \(C\) is the category whose objects are pairs \((U, \alpha)\) where \(U \in C\) and \(\alpha : U \otimes - \to U - \otimes U\) is a natural isomorphism obeying the following two conditions:

\[
\begin{align*}
&= (1 \otimes dv_U)(cv_U \otimes v_U v(u v_U c v^{-1}_U \cdot v_U v_U)(1 \otimes bv_U) \\
&= (1 \otimes dv_U)(cv_U \otimes v_U v(u v_U c v^{-1}_U \cdot v_U v_U)(1 \otimes bv_U) \\
&= (1 \otimes dv_U)(cv_U \otimes v_U v(u v_U c v^{-1}_U \cdot v_U v_U)(1 \otimes bv_U) \\
&= (1 \otimes dv_U)(cv_U \otimes v_U v(u v_U c v^{-1}_U \cdot v_U v_U)(1 \otimes bv_U) \\
&= (1 \otimes dv_U)(cv_U \otimes v_U v(u v_U c v^{-1}_U \cdot v_U v_U)(1 \otimes bv_U) \\
&= (1 \otimes dv_U)(cv_U \otimes v_U v(u v_U c v^{-1}_U \cdot v_U v_U)(1 \otimes bv_U).
\end{align*}
\]
(1) $\alpha_I = 1$;
(2) $\alpha_{X \otimes Y} = (1 \otimes \alpha_Y)(\alpha_X \otimes 1)$ for all $X, Y \in \mathcal{C}$.

An arrow $f : (U, \alpha) \to (V, \beta)$ in $\mathcal{L}_C$ is an arrow $f : U \to V$ in $\mathcal{C}$ such that $\beta_X(f \otimes 1) = (1 \otimes f)\alpha_X$ for all $X \in \mathcal{C}$.

Then $\mathcal{L}_C$ becomes a crossed $\pi$-category with $(U, \alpha) \otimes (V, \beta) = (U \otimes V, (\alpha \otimes 1)(1 \otimes \beta))$ and $(U, \alpha)(V, \beta) = (U V, U v \beta^* X)$.

**Proposition 3.1.2** (a) For a crossed $\pi$-category $\mathcal{C}$, the crossed $\pi$-category $\mathcal{L}_C$ is braided via $\alpha_U : (U, \alpha) \otimes (V, \beta) \to (U, \alpha)(V, \beta)$. (b) Let $\mathcal{C}$ be a crossed $\pi$-category. Then for each object $U \in \mathcal{C}$, the equation $F(X) = (X, \alpha_X)$ determines a bijection between objects $(U, \alpha) \in \mathcal{L}_C$ and tensor functors $F : \mathcal{C} \to \mathcal{L}_C(U)$. Similarly, for each arrow $h : U \to V$ in $\mathcal{C}$, the equation $F'(X) = (X, \alpha_X, \beta_X)$ determines a bijection between arrows $h : (U, \alpha) \to (V, \beta) \in \mathcal{L}_C$ and tensor functors $F' : \mathcal{C} \to \mathcal{L}_C(h)$. (c) For a crossed $\pi$-category $\mathcal{C}$, the equation $G(U) = (U, c_{U, -})$ determines a bijection between braiding $c$ on $\mathcal{C}$ and crossed $\pi$-functors $G : \mathcal{C} \to \mathcal{L}_C$.

**Proof.** Straightforward. \qed

For a connected crossed $\pi$-category $\mathcal{C}$, let $(y, z)$ be a balanced YB-operator on an object $U$ such that $g(U, U) = z^{-1}$. Then we have the following lemma:

**Lemma 3.1.1** (a) The balanced YB-operator $(y, z)$ defines a balanced YB-operator on the object $(U, y) \in \mathcal{L}_C(U)$. If $(y, z)$ is dualizable, then $(U^*, u^{-1}) \in \mathcal{L}_C(U)$ is a left dual for the object $(U, y) \in \mathcal{L}_C(U)$. Moreover, $(y, z)$ defines a dualizable balanced YB-operator on $(U, y)$.

(b) The centralizer $\mathcal{L}_C(U^*)$ contains $(U, v)$ and $(U^*, w)$ where $w = (d_U \otimes 1 \otimes 1 \otimes 1 \otimes z^* \otimes 1)(1 \otimes 1 \otimes 1)(1 \otimes z^* \otimes 1 \otimes 1 \otimes 1)(1 \otimes 1 \otimes b_U)$. The object $(U^*, w)$ is dual to $(U, v)$ and $(y, z)$ defines a balanced YB-operator on $(U, v)$.

(c) If $(y, z)$ is a tortile Yang-Baxter operator, then it is also a tortile Yang-Baxter operator on $(U, y) \in \mathcal{L}_C(U)$ and $(U, v) \in \mathcal{L}_C(U^*)$.

**Proof.** (a) For arrows $(U, y) \to (U, y)(U, y)$ and $(U, y) \otimes (U, y) \to (U, y)(U, y)$ in $\mathcal{L}_C(U)$, we take the arrows $z : U \to U$ and $y : U \otimes U \to U \otimes U$ in $\mathcal{C}$. Then by the hexagonal condition on $(y, z)$ and the assumption $g(U, U) = z^{-1}$, we see that these arrows are indeed arrows in $\mathcal{L}_C(U)$.  

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(b) We have to show that the object \((U^*, w)\) is dual to \((U, v)\). For this it is convenient to use a diagrammatic notion as used in [3], [5]. For example, the following equalities show that the arrow \(d_U : U^* \otimes U \to I\) becomes an arrow \((U^*, w) \otimes (U, v) \to (I, id_{U^*})\) in \(\mathcal{L}_C(U^*)\).
\[ U^* z = z^{-1} U^* z - 1 = U^* z \]
\[ z^{-1} U^* \quad U \quad U^* \quad z = z \quad z^{-1} U^* \quad U \quad U^* \quad z^{-1} U^* \quad U \quad U^* \quad z \]

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\[ U^* U^* U U^* U^* U^* \]

\[ U^* U^* U^* U^* U^* U \]

\[ U^* U^* U U^* U^* U^* \]

\[ U^* U^* U^* U^* U^* U \]
The next equalities show that the arrow \( y : U \otimes U \to U \otimes U^* \) defines an arrow \( (U, v) \otimes (U, v) \to (U,v)(U, v) \otimes (U, v) \) in \( \mathcal{L}_C(U^*) \).
\[(e) \text{ Straightforward.} \]
Chapter 4

Free crossed group-category

4.1 Free crossed $\mathbb{Z}$-category

Let $\mathcal{F}$ be the crossed $\mathbb{Z}$-category freely generated by objects $X$ and $X^*$ and arrows $d_X : X^* \otimes X \to I$, $b_X : I \to X \otimes X^*$, $y : X \otimes X \to XX \otimes X$, $z : X \to XX$, subject to the conditions that $X \in \mathcal{F}_1$, $X^* \in \mathcal{F}_{-1}$, $X^*$ is left dual to $X$ via $d_X$ and $b_X$, and $(y, z)$ is a tortile $YB$-operators on $X$.

We now state our main theorem in Part II.

Theorem 4.1.1 The category $\mathcal{F}$ admits a unique braiding $c$ and a twist $\theta$ such that $c_{X,X} = y$, $\theta_X = z$ and $\theta_{X^*} = X^*(\theta_X)^*$. 

To prove the existence of such a braiding, we begin with the following proposition.

Proposition 4.1.1 There is a natural isomorphisms

$$c_{X,-} : X \otimes - \to X - \otimes X \quad \text{and} \quad c_{X^*,-} : X^* \otimes - \to X^* - \otimes X^*$$

such that $(X, c_{X,-})$ and $(X^*, c_{X^*,-}) \in \mathcal{L}_\mathcal{F}$ where $c_{X,X} = y$ and $c_{X^*,X} = v$.

Proof. We first show that the category $\mathcal{F}$ is a connected crossed $\mathbb{Z}$-category. For this, observe that the category $\mathcal{F}$ is generated as a monoidal category by the objects $X^nX$, $X^nX^*$, $X^*nX^*$, $X^*nX$, where $X^n$ and $X^*n$ mean $X \otimes X \otimes ... \otimes X (n \text{ times})$ and $X^* \otimes X^* \otimes ... \otimes X^* (n \text{ times})$ respectively, and $n$ runs through the set of non-negative integers. Besides, the objects $X^nX$ and $X^*nX$ belong to the same class of $X$, and the objects $X^nX^*$ and $X^*nX^*$ belong to
the same class of $X^*$. Using this property we can prove that for any object $Y$ there exists an isomorphism $YX \to X$. Indeed, for any objects $Y$ and $Z$ we can define an isomorphism $Y \otimes Z \to X$ once we obtain isomorphisms $f: YX \to X$ and $g: ZX \to X$ since we can use the composition $f^Y g$. Thus it is enough to show that there are isomorphisms $X X \to X$ and $X^* X \to X$, and we can take the arrows $z^{-1}$ and $X^* z$. The same argument apply to show that for any object $Y$ there exists an isomorphism $Y^X X \to X^X X$. Actually, we only need isomorphisms $X(X^n X) \to X^n X$ and $X^*(X^n X) \to X^n X$, and we can take the arrows $X^n z^{-1}$ and $X^n(X^* z)$. Similarly, one can prove that for any object $Y$ there are isomorphisms $Y(X^n X) \to X^n X$, $Y(X^n X^*) \to X^n X^*$ and $Y(X^n X^*) \to X^n X^*$. Finally, we observe that for any object $Y$, $Z$ and $W$, an isomorphism $Y(Z \otimes W) = YZ \otimesYW \to Z \otimes W$ is obtained from $YZ \to Z$ and $YW \to W$. Hence the category $F$ is connected, and the categories $L_F(X)$ and $L_F(X^*)$ become crossed $Z$-categories. Moreover, the condition $g(X, X) = z^{-1}$ is satisfied. Thus we can use the universality of the category $F$ and Lemma 1 to get crossed $Z$-functors $F \to L_F(X)$ and $F \to L_F(X^*)$. In particular, these functors are tensor functors, hence by Proposition 3.1.2 (b) we obtain the natural isomorphisms $c_{X, -} : X \otimes - \to X^* \otimes X$ and $c_{X^*, -} : X^* \otimes - \to X^* \otimes X^*$.

**Lemma 4.1.1** The pair $(y, z)$ becomes a tortile $YB$-operator on the object $(X, c_{X, -})$ in $L_F$. Also, the object $(X^*, c_{X^*, -})$ is left dual to the object $(X, c_{X, -})$ in $L_F$.

**Proof.** To obtain a tortile $YB$-operator on the object $(X, c_{X, -})$ in $L_F$, we first consider the centralizer $L_F(y)$ of $y : X \otimes X \to X \otimes X$. Then we see that the category $L_F(y)$ contains the objects $(X, \alpha, \beta)$ and $(X^*, \gamma, \delta)$ where $\alpha = (y' \otimes 1)(1 \otimes y)$, $\beta = (X y \otimes 1)(1 \otimes y)$, $\gamma = (X z^{-1} \otimes 1)(1 \otimes u^{-1})(1 \otimes z \otimes 1)(1 \otimes u^{-1})$, $\delta = (X u^{-1} \otimes 1)(1 \otimes u^{-1})$. Moreover, the object $(X^*, \gamma, \delta)$ is dual to $(X, \alpha, \beta)$, and we obtain a tortile $YB$-operator $(y, z)$ on $(X, \alpha, \beta)$ in $L_F(y)$. Since $L_F(y)$ is a crossed $Z$-category, the universal property of $F$ induces a crossed tensor functor $F \to L_F(y)$, which is a section of the projection $L_F(y) \to F$. In particular, this functor is a tensor functor, hence by Proposition 3.1.2 (b), we obtain an arrow $y : (X, c_{X, -}) \otimes (X, c_{X, -}) \to (X, c_{X, -})(X, c_{X, -}) \otimes (X, c_{X, -})$ in $L_F$.

Next consider the category $L_F(z)$. Then we see that the category $L_F(z)$ contains the objects $(X, y, \eta)$ and $(X^*, u^{-1}, \zeta)$ where $\eta = (1 \otimes z)y(z^{-1} \otimes 1)$ and $\zeta = (1 \otimes z)u^{-1}(z^{-1} \otimes 1)$. In addition, $(X^*, u^{-1}, \zeta)$ is dual to $(X, y, \eta)$,
and we obtain a tortile $YB$-operator $(y, z)$ on the objects $(X, y, \eta)$ in $\mathcal{L}_F(z)$. Thus we have a crossed tensor functor $\mathcal{F} \to \mathcal{L}_F(z)$, which corresponds to an arrow $z : (X, c_{X,-}) \to (X, c_{X,-})_{\mathcal{L}_F}$ in $\mathcal{L}_F$.

Then it is easy to see that the arrows $y : (X, c_{X,-}) \otimes (X, c_{X,-}) \to (X, c_{X,-})_{\mathcal{L}_F}$ and $z : (X, c_{X,-}) \to (X, c_{X,-})_{\mathcal{L}_F}$ define a tortile $YB$-operator on the object $(X, c_{X,-})$ in $\mathcal{L}_F$.

Finally, we check that the object $(X^*, c_{X^*,-})$ is left dual to the object $(X, c_{X,-})$ in $\mathcal{L}_F$. For this we consider the centralizers $\mathcal{L}_F(d_X)$ and $\mathcal{L}_F(b_X)$. We see that the category $\mathcal{L}_F(d_X)$ contains the objects $(X, \tau, id_X)$ and $(X^*, \sigma, id_{X^*})$ where $\tau = (X^* z \otimes 1 \otimes 1)(v \otimes 1)(1 \otimes z^{-1} \otimes 1)(1 \otimes y)$ and $\sigma = (X^* z^{-1} \otimes 1 \otimes 1)(w \otimes 1)(1 \otimes z^* \otimes 1)(1 \otimes u^{-1})$. Moreover, the object $(X^*, \sigma, id_{X^*})$ is dual to $(X, \tau, id_X)$, and $(y, z)$ defines a tortile $YB$-operator on the object $(X, \tau, id_X)$ in $\mathcal{L}_F(d_X)$. Thus we obtain a crossed tensor functor $\mathcal{F} \to \mathcal{L}_F(d_X)$, which corresponds to an arrow $(X^*, c_{X^*,-})_{\mathcal{L}_F}$. We can define a tensor product and a cross action on $\mathcal{F}$ by putting

$$(U, \xi) \otimes (V, \zeta) = (U \otimes V, \chi) \quad \text{and} \quad (U, \xi)(V, \zeta) = (U \otimes U, \chi)(V, \zeta)$$

where $\chi = c_{U, V} c_{U, V} c_{U, V} (\xi \otimes \zeta)$. This tensor product makes $\mathcal{C}'$ into a crossed $\pi$-category. Applying this procedure to the braided crossed $\mathcal{Z}$-category $\mathcal{F}$, we obtain a crossed $\mathcal{Z}$-category $\mathcal{F}'$. To get a left dual for the object $(X, z)$ in $\mathcal{F}'$, we use the following lemma:

**Lemma 4.1.2** The following identity holds in $\mathcal{F}$.

$$z X^* z (d_X \otimes 1)(1 \otimes c_{X, X^*}^{-1})(c_{X, X^*} \otimes 1)(b_X \otimes 1) = 1 : X^* X \to X^* X.$$
Using the lemma above, we obtain a crossed tensor functor \( \mathcal{F} \to \mathcal{F}' \) which takes \( X \to (X, z) \) and takes \( X^* \) to \( (X^*, X^* z^*) \). Then for any object \( U \) in \( \mathcal{F} \), the value \( (U, \theta_U) \) of this tensor functor at \( U \) gives the twist \( \theta : U \to U^U \). The
uniqueness of the braiding and the twist follows from the same argument in [3]. Let $c, c'$ be braidings on $\mathcal{F}$ such that $c_{X,X} = y = c'_{X,X}$. For any object $U$ of $\mathcal{F}$, let $\mathcal{E}(U)$ be the set of objects $Z$ for which $c_{U,Z} = c'_{U,Z}$, and let $\mathcal{E}$ be the set of objects $U$ for which $\mathcal{E}(U) = \text{obj} \mathcal{F}$. Then using the connectivity structure on $\mathcal{F}$, we see that both $\mathcal{E}(U)$ and $\mathcal{E}$ are closed under tensor product and crossed action. Moreover, using the fact that $u = c^{-1}_{X,X^*}, v = c_{X^*,X}$ and $w = c_{X^*,X^*}$, we see that $X, X^* \in \mathcal{E}(X)$ and $X, X^* \in \mathcal{E}(X^*)$. Thus $X, X^* \in \mathcal{E}$, and since $X, X^*$ generate $\text{obj} \mathcal{F}$, we have $\mathcal{E} = \text{obj} \mathcal{F}$. Similarly, one can prove the uniqueness of twist. This completes our proof of Theorem 4.1.1.
Bibliography


Part III

Perfect braided crossed modules and their mod-$q$ analogues
Chapter 1

Introduction

In the last part of this paper, we consider the extension theory of braided crossed modules and its mod-$q$ analogues.

Crossed modules arise in several contexts. For example, in non-Abelian homological algebra, crossed modules play the role of coefficients for degree two cohomology groups (see [1]). Alternatively, Brown and Spencer [8] obtained certain crossed modules as the fundamental groupoids of topological groups. This point of view can be extended to higher dimensional groupoids as well. For example, Brown and Higgins [5] defined the fundamental double groupoid of a pair of spaces, and Loday [16] developed this to the fundamental $cat^n$-group $\prod X$ of a $n$-cube of spaces $X$. Among other results, he proved the equivalence between $cat^2$-groups and crossed squares, and braided crossed modules appeared as a special case of crossed squares. In the work of Bullejos and Cegarra [9], braided crossed modules were used as coefficients for certain degree three non-Abelian cohomology groups. More generally, Breen [1] considered, as the objects of degree three non-Abelian cohomology groups, the extensions of the form

$$1 \to \mathcal{G} \to \mathcal{H} \to k$$

where $\mathcal{G}$, $\mathcal{H}$ are crossed modules and $k$ is a group. Thus it is quite natural to consider the case where $k$ is also a crossed module, braided crossed module and so on.

By use of the Brown-Loday non-Abelian tensor product of groups, Norrie [18] determined the universal central extensions of perfect crossed modules.

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The Brown-Loday non-Abelian tensor product of groups was extended to mod-$q$ tensor product by D. Conduché and C. Rodriguez-Fernández, and Doncel-Juárez and Grandjean L.-Valcárcel used this to obtain the mod-$q$ analogue of Norrie’s theorem.

The goal of Part III is to prove the braided version of Norrie’s theorem and its mod-$q$ analogues. The construction is simple, but we have to use very complicated arguments to prove the main theorem. Besides, it depends on an unexpected relationship between non-Abelian tensor products and braidings on crossed modules, which makes the whole story rather interesting!
Chapter 2

Braided crossed modules

2.1 Preliminaries

In this section we recall some definitions and properties of crossed modules and braidings on them.

**Definition 2.1.1** Let $N$ and $G$ be groups together with a homomorphism $N \to G$. This $N \to G$ is called a crossed module if $G$ acts on $N$ and satisfies the following conditions:

1. $\partial(gn) = g\partial(n)g^{-1}$, $g \in G, n \in N$,
2. $\partial nn' = nn'n^{-1}$, $n, n' \in N$.

**Example 2.1.1** For a group $G$, the identity map $G \to G$ together with the action $gg' = gg'g^{-1}$ defines a crossed module.

**Definition 2.1.2** Let $(M, P, \partial)$ and $(N, G, \partial')$ be crossed modules. A crossed module morphism $(\varphi, \psi) : (M, P) \to (N, G)$ is a pair of group homomorphisms $\varphi : M \to N$ and $\psi : P \to G$ such that

1. $\psi \partial = \partial' \varphi$,
2. $\varphi(gn) = \psi(g)\varphi(n)$, $g \in P, n \in M$.

When $\varphi$ and $\psi$ are surjective, the morphism is called an extension.
Definition 2.1.3 For a non-negative integer \(q\), the \(q\)-center of a crossed module \(N \rightarrow G\) is the crossed module \((N^G)^q \rightarrow Z(G)^q\), where \((N^G)^q = \{n \in N; n^q = 1, g^n = n, g \in G\}\) and \(Z(G)^q = \{g \in Z(G); g^q = 1\}\). In particular, we call the 0-center the center of \(N \rightarrow G\).

Definition 2.1.4 An extension \((\varphi, \psi) : (M, P) \rightarrow (N, G)\) of a crossed module \((N, G)\) is called \(q\)-central if the crossed module \(\ker \varphi \rightarrow \ker \psi\) is contained in the \(q\)-center of the crossed module \(M \rightarrow P\). In particular, we call the 0-central extension the central extension.

Definition 2.1.5 When \(N \rightarrow G\) is a crossed module, the \(q\)-commutator crossed module is defined as a crossed module \(D^q_G(N) \rightarrow [G, G]^q\) where \(D^q_G(N)\) is the subgroup of \(N\) generated by \(\{g n r^{-1}; g \in G, n, r \in N\}\) and \([G, G]^q\) is the subgroup of \(G\) generated by \(\{[g, h]^k; g, h, k \in G\}\). In particular, we call the 0-commutator crossed module the commutator crossed module.

Definition 2.1.6 A crossed module \(N \rightarrow G\) is called \(q\)-perfect if it coincides with the \(q\)-commutator crossed module. In particular, we call the 0-perfect crossed module the perfect crossed module.

Based on the earlier works of Dennis [12] and Miller [17], Brown and Loday [6] defined the notion of non-Abelian tensor product \(M \otimes N\) of two crossed modules. Later, the notion of mod-\(q\) exterior product of groups, for a non-negative integer \(q\), was introduced by Ellis [14], and Brown [3] defined the mod-\(q\) non-Abelian tensor product \(G \otimes^q G\) of a group \(G\).

The following definition of the mod-\(q\) non-Abelian tensor product of crossed modules is due to Conduché and Rodriguez-Fernández [11].

Definition 2.1.7 Let \((M, G, \partial)\) and \((N, G, \partial')\) be crossed modules and let \(q\) be a non-negative integer. Then the tensor product \(M \otimes^q N\) is defined as a group generated by the symbols

\[ a \otimes^q b \ (a \in M, b \in N) \text{ and } \{k\} \ (k \in M \times_G N) \]

with the following relations:

1. \(a \otimes^q bc = (a \otimes^q b)(^a a \otimes^q b)c\),
2. \(ab \otimes^q c = (^ac a \otimes^q a)c\).
\( k \{ a \otimes^q b \} \{ k \}^{-1} = \alpha(k)^a a \otimes^q \alpha(k)^a b, \)

(4) \( \{ k \} \{ h \} = \pi_1(k)^q \otimes^q \pi_2(h)^q, \)

(5) \( \{ k h \} = \{ k \} \{ \Pi(\pi_1(k)^{-1} \otimes^q (\alpha(k)^1-q^{-1} \pi_2(h)^i)) \} \{ h \}, \)

(6) \( \{ a^b a^{-1}, a b b^{-1} \} = (a \otimes^q b)^q, \)

where \( \alpha = \partial \circ \pi_1. \)

Note that the Brown-Loday non-Abelian tensor product \( M \otimes N \) can be regarded as the special case where the generators are just \( a \otimes^0 b \) and the relations are just (1) and (2). Besides, it was shown in [6] that, for a group \( G \), the following identities hold in \( G \otimes G \):

(a) \( (a \otimes b)(c \otimes d)(a \otimes b)^{-1} = [a,b]c \otimes [a,b]d, \)

(b) \( [a,b] \otimes c = (a \otimes b)(c \otimes a b), \)

(c) \( a \otimes [b,c] = (a b \otimes a c)(b \otimes c)^{-1}, \)

where \( a, b, c \in G \) and \( [a,b] = aba^{-1}b^{-1}. \)

We next consider braidings on crossed modules.

**Definition 2.1.8** A braiding on a crossed module \( \partial : N \to G \) is a map \( \{ , \} : G \times G \to N \) (bracket operation) satisfying the following conditions:

1. \( \partial\{a, b\} = aba^{-1}b^{-1}, \)
2. \( \{\partial(n), b\} = n^b n^{-1}, \)
3. \( \{a, \partial(n)\} = a n n^{-1}, \)
4. \( \{a, b c\} = \{a, b\} b \{a, c\}, \)
5. \( \{a b, c\} = a \{b, c\} \{a, c\}, \)

for \( a, b, c \in G \) and \( n \in N. \)

**Example 2.1.2** There are canonical braidings on the crossed modules \( id : G \to G \) and \( G \otimes G \to G \), \( a \otimes b \mapsto [a, b] \) by the following maps:

\( G \times G \to G, \ (a, b) \mapsto [a, b] = aba^{-1}b^{-1}, \)

\( G \times G \to G \otimes G, \ (a, b) \mapsto a \otimes b. \)

**Definition 2.1.9** A morphism between two braided crossed modules is defined as a crossed module morphism which preserves the braiding structures. In particular, a \( q \)-central extension of a braided crossed module is a \( q \)-central extension of the underlying crossed module which preserves the braiding structures.
2.2 Canonical braidings

To construct new braidings, we start from the following observation:

**Proposition 2.2.1** If a crossed module \( \partial : N \to G \) has a braiding \( \{ \ , \ \} \), then there is a group homomorphism \( f : G \otimes G \to N, \ a \otimes b \mapsto \{a, b\} \).

**Proof.** Let us check that \( f \) preserves the defining relations in \( G \otimes G \). By the definitions, we have

\[
 f(a \otimes bc) = \{a, bc\} = \{a, b\}^b\{a, c\}, \\
 f(a \otimes b)f(b \otimes c) = \{a, b\}^b\{a, bc\}.
\]

Then we can use the result of Conduché [10] which says that any braiding is equivariant (i.e., \( a\{b, c\} = \{ab, c\} \)), so that \( f(a \otimes bc) = f(a \otimes b)f(b \otimes c) \). The other relation can be proved by the same computation. \( \square \)

We next consider the \( q \)-tensor analogues. The main difference is the existence of the elements \( \{k\} \), and to construct a well behaved map on \( G \otimes^q G \), we assume that the crossed modules \( N \to G \) are \( q \)-central extensions of \( G \).

**Proposition 2.2.2** When a crossed module \( \partial : N \to G \) is a \( q \)-central extension and has a braiding \( \{ \ , \ \} \), there is a group homomorphism \( f : G \otimes^q G \to N, \ a \otimes b \mapsto \{a, b\}, \ {k} \mapsto s(k)^q \) (\( s \) is a section of \( \partial \)).

**Proof.** We have to check that \( f \) preserves the relations (3)-(6) in \( \mod-q \) tensor product. We first consider the relation (3). The we have \( f(\{k\}(a \otimes^q b)\{k\}^{-1}) = s(k)^q\{a, b\} s(k)^{-q} = k^q\{a, b\} = \{k^q a, k^q b\} = f(k^q a \otimes^q k^q b) \). We next consider the relation (4). The we have \( f(\{(k_1 h_1)\}) = [s(k)^q, s(h)^q] = s(k)^q s(h)^q (s(h)^q)^{-1} = k^q (s(h)^q)^{-1} = \{k^q, h^q\} \). For the relation (5), we have \( f(\{kh\}) = s(kh)^q = (s(k)s(h))^q = s(k)^q (\prod[(s(k)^{-1}, (k_1^{-q} h_1)^i)s(h)^q = s(k)^q (\prod[k_1^{-q}, (k_1^{-q} h_1)^i])s(h)^q \). Finally, we consider the relation (6). Then we have \( f(\{(ab^{-1}, ab^{-1})\}) = s([a, b])^q \), and because \( s([a, b]) \) and \( \{a, b\} \) have the same image under \( \partial \), \( s([a, b])^q \) coincides with \( \{a, b\}^q \). \( \square \)

We proceed to construct a canonical braiding on \( \rho : N \otimes G \to G \otimes G \) when \( N \to G \) is braided with a braiding \( \{ \ , \ \} \). Define \( \{ \ , \ \} : G \otimes G \times G \otimes G \to N \otimes G \) by

\[
 \{ \ , \ \} : (a \otimes b, c \otimes d) \mapsto \{a, b\} \otimes [c, d].
\]

Then we have the following proposition:
Proposition 2.2.3 \( \{ \_ , \_ \} \) satisfies the braiding conditions.

Proof. We first consider the identity (1). If we take \( a = a \otimes b \) and \( b = c \otimes d \), we have \( \{ a \otimes b, c \otimes d \} = \partial(\{ a, b \} \otimes [c, d]) = \vartheta(\{ a, b \} \otimes [c, d]) = [a, b] \otimes [c, d] \), so we need the following identity:

\[
(a \otimes b)(c \otimes d)(a \otimes b)^{-1}(c \otimes d)^{-1} = [a, b] \otimes [c, d],
\]

but this is the product of \( (a) \) and \( (b) \) in page 92.

The identities (2) and (3) are proved by a result in Brown and Loday [6]. Alternatively, one can prove them using a technique which will be described in Lemma 1 below.

We next consider the identity (4). If we take \( a = a \otimes b \) and \( bc = (c \otimes d)(c' \otimes d') \), we have \( \{ a \otimes b, (c \otimes d)(c' \otimes d') \} = \{ a, b \} \otimes [c, d][c', d'] \). On the other hand, we have \( \{ a \otimes b, c \otimes d \} = (\{ a, b \} \otimes [c, d]) \otimes (\{ a, b \} \otimes [c', d']) = (\{ a, b \} \otimes [c, d]) \otimes (\{ a, b \} \otimes [c', d']) \).

Finally, we consider the identity (5). If we take \( ab = (a \otimes b)(a' \otimes b') \) and \( c = c \otimes d \), we have \( \{ (a \otimes b)(a' \otimes b'), c \otimes d \} = \{ a, b \} \{ a', b' \} \otimes [c, d] \). On the other hand, \( a \otimes b \{ a' \otimes b', c \otimes d \} = \{ a, b \} \otimes [c, d] ) \{ a, b \} \otimes [c, d] \).

Remark. In (4) and (5), the properties \( \partial(\{ a, b \}) = [a, b] \) and \( \vartheta(\{ a, b \}) = n \) were used.

When a crossed module \( N \rightarrow G \) is a \( q \)-central extension of \( G \) and equipped with a braiding \( \{ , \} \), one can use Proposition 2.2.2 to define a canonical braiding \( \{ , \}^q \) on the crossed module \( N \otimes^q G \rightarrow G \otimes^q G \).

Before checking the braiding conditions, we prove the next lemma.

Lemma 2.2.1 In \( N \otimes^q G \), the next identities hold:

\[
\begin{align*}
(a) \quad a^b a^{-1} \otimes^q h^q & = (a \otimes^q b)(h^q a \otimes^q h^q b)^{-1}, \\
(b) \quad \{ n \}^q \otimes^q [a, b] & = \{ n \} \{ [a, b] n \}^{-1}, \\
(c) \quad n^q \otimes^q h^q & = \{ n \} \{ h^q n \}^{-1}.
\end{align*}
\]

Proof. Recall that for two crossed modules \( (M, G, \partial) \) and \( (N, G, \partial') \), Doncel-Juárez and Grandjean L.-Valcárcel constructed the following crossed module \( \rho : M \otimes^q N \rightarrow G \otimes^q G \):
\[ \rho(m \otimes^q n) = \partial(m) \otimes^q \partial'(n), \quad \rho(k) = \{ \partial(\pi_1(k)) \}, \]
\[ a \otimes^b (m \otimes^q n) = [a, b]_m \otimes^q [a, b]_n, \quad a \otimes^b \{ k \} = \{ [a, b]k \}, \]
\[ h(m \otimes^q n) = h^m \otimes^q h^q n, \quad h \{ k \} = \{ h^q k \}, \]
and proved that when \( N \rightarrow G \) is a \( q \)-perfect crossed module the crossed module \( \varphi \otimes^q G \rightarrow G \otimes^q G \) becomes the universal \( q \)-central extension of it.

To prove the identities \((a) \sim (c)\), we use the universality of \( N \otimes^q G \), and show that, for any \( q \)-central extension \((X_1, X_2, \partial')\) of \((N, G, \partial)\), the unique map \( \varphi_1 : N \otimes^q G \rightarrow X_1 \) defined by \( \varphi_1(n \otimes^q g) = s_1(n) \otimes^q s_1(n)^{-1} \), \( \varphi_1 \{ h \} = s_1(h)^q \), where \( s_1 \) and \( s_2 \) are sections of \( \psi_1 : X_1 \rightarrow N \) and \( \psi_2 : X_2 \rightarrow G \) respectively, preserves the relations.

We first check the identity \((a)\). By the definitions, we have
\[ \varphi_1(a^b a^{-1} \otimes^q h^q) = s_1(a^b a^{-1}) \otimes^q s_1(a^b a^{-1})^{-1}. \]

Since \( s_1(a^b a^{-1}) \otimes^q s_1(a^b a^{-1})^{-1} \) has the form \( x^q x^{-1} \) in \( X_1 \), we can change \( s_1(a^b a^{-1}) \) to \( s_1(a) \otimes^q s_1(a)^{-1} \). Then we have
\[ s_1(a^b a^{-1}) \otimes^q s_1(a^b a^{-1})^{-1} = (s_1(a) \otimes^q s_1(a)^{-1}) \otimes^q (s_1(a) \otimes^q s_1(a)^{-1})^{-1}. \]

On the other hand, we have
\[ \varphi((a \otimes^q b)(h^q a \otimes^q h^q b)^{-1}) = (s_1(a) \otimes^q s_1(a)^{-1})(s_1(h^q a) \otimes^q s_1(h^q b))^{-1}. \]

Hence we should prove the formula:
\[ s_1(h^q a) \otimes^q s_1(a)^{-1} = (s_1(h^q a)^{-1}) \otimes^q s_1(h^q a)^{-1}, \]
but notice that the latter has the form \( (x^q x^{-1})^{-1} \). Thus we can replace \( s_1(h^q a) \) by \( s_2(h^q) s_1(a) \) and \( s_2(h^q b) \) by \( s_2(h^q) s_2(b) s_2(h^q)^{-1} \).

We next check the identity \((b)\). By the definition, we have
\[ \varphi_1(\{ n \} \otimes^q [a, b]) = s_1(n) \otimes^q s_1(n)^{-1} = (s_1(n) \otimes^q s_1(n)^{-1}) \otimes^q (s_1(n) \otimes^q s_1(n)^{-1})^{-1}. \]

On the other hand, we have
\[ \varphi_1(\{ n \} [a, b]^{-1}) = s_1(n)^{-1}, \]
Since the elements \( s_2([a, b]) \) and \( s_1([a, b] n) \) have the same image under \( \psi_1 : X_1 \rightarrow N \), one sees that, by the property of \( q \)-central extensions of a crossed module, the element \( s_2([a, b]) (s_1(n)^{-1}) \) coincides with \( (s_1([a, b] n)^{-1}) \).

Finally, we check the identity \((c)\). By the definition, we have
\[ \varphi_1(n^q \otimes^q h^q) = s_1(n^q) s_2(h^q) s_1(n^q)^{-1} = s_1(n)^q(s_2(h^q) s_1(n))^{-q}. \]

On the other hand, we have
\[ \varphi_1(n\{n\}^{h^q n\}^{-1}) = s_1(n)^q s_1(h^q n)^{-q}. \]

Then one can easily see that the elements \( s_2(h^q) s_1(n) \) and \( s_1(h^q n) \) have the same image under \( \psi_1 \). Thus the result follows. \( \square \)

**Proposition 2.2.4** \( \{ , \}^q \) becomes a braiding on \( N \otimes^q G \rightarrow G \otimes^q G \).

**Proof.** When the elements \( \{ k \} \) do not appear in the relations, they are derived from the results for \( \{ , \} \). So we consider the case where the elements \( \{ k \} \) are appearing in the relations. Below, we denote \( \{ , \}^q \) and \( \otimes^q \) by \( \{ , \} \) and \( \otimes \) respectively. We first consider the relation (1). If we take \( a = \{ k \} \) and \( b = c \otimes d \), we have \( \rho\{k\}, c \otimes d) = \rho(s(k)^q \otimes \{ c, d \}) = k^q \otimes [c, d] \). On the other hand, we have \( \{k\}(c \otimes d) \{k\}^{-1}(c \otimes d)^{-1} = (k^q c \otimes k^q d)(c \otimes d)^{-1} \). Hence we need the identity:
\[ k^q \otimes [c, d] = (k^q c \otimes k^q d)(c \otimes d)^{-1}, \]
but this is the formula (c) applied to mod-\( q \) tensor product with \( a = k^q \), \( b = c, d = c \).

We next consider the relation (2). If we take \( n = a \otimes b \) and \( b = \{ h \} \), then by the definition we have \( \{ \partial(a) \otimes b, \{ h \} \} = \{ \partial(a), b \} \otimes h^q = a^q a^{-1} \otimes h^q \). On the other hand, we have \( (a \otimes b)^{h^q}(a \otimes b)^{-1} = (a \otimes b)(h^q a \otimes h^q b)^{-1} \). Thus by Lemma 2.2.1 (a), they coincide. If we take \( n = \{ n \} \) and \( b = a \otimes b \), then we have \( \{ \rho(n), a \otimes b \} = n^q \otimes [a, b] \). On the other hand, we have \( \{ n \}^{a \otimes b} \{ n \}^{-1} = \{ n \}^{[a, b] n^{-1}} \). Thus by Lemma 2.2.1 (b), they coincide. If we take \( n = \{ n \} \) and \( b = \{ h \} \), we have \( \{ \rho(n), \{ h \} \} = n^q \otimes h^q \). On the other hand, we have \( \{ n \}^{h^q} \{ n \}^{-1} = \{ n \}^{h^q n^{-1}} \). Thus by Lemma 2.2.1 (c), they coincide.

The relation (3) follows by the same computations. We next consider the relation (4). If we take \( a = \{ k \} \) and \( bc = (a \otimes b)(c \otimes d) \), we have \( \{k\}, (a \otimes b)(c \otimes d) = (s(k)^q \otimes [a, b][c, d]) \). On the other hand, we have \( \{k\}, a \otimes b \otimes [a, b] = (s(k)^q \otimes [a, b]) = (s(k)^q \otimes [a, b]) \). When \( a = \{ k \} \) and \( bc = \{ h \}(c \otimes d) \), we have \( \{k\}, \{ h \}(c \otimes d) = s(k)^q \otimes s(h)^q[c, d] \). On the other hand, we have \( (\{k\}, \{ h \})(c \otimes d) = s(k)^q \otimes s(h)^q[c, d]) = (s(k)^q \otimes s(h)^q)[c, d] \). If we take \( a = \{ k \} \)
and $bc = (c \otimes d)\{h\}$, we have $\{\{k\}, (c \otimes d)\{h\}\} = s(k)^q \otimes [c, d]s(h)^q$. On the other hand, we have $\{\{k\}, (c \otimes d)^{\otimes d}\{\{k\}, \{h\}\}\} = (s(k)^q \otimes [c, d])^{\otimes d}(s(k)^q \otimes s(h)^q) = (s(k)^q \otimes [c, d])([c, d]s(k)^q \otimes [c, d]s(h)^q) = s(k)^q \otimes [c, d]s(h)^q$.

(5) Omitted.

2.3 Universality

In the previous section, we have constructed canonical braidings on the crossed modules $N \otimes G \to G \otimes G$ and $N \otimes^q G \to G \otimes^q G$. Since it is known that they are the universal central extensions of a perfect ($q$-perfect) crossed module $N \to G$, it is quite natural to consider their braided version.

The next proposition shows that the canonical braiding $\{\ldots\}$ on the crossed module $N \otimes G \to G \otimes G$ is compatible with $\{\ldots\}$.

**Proposition 2.3.1** The next diagram becomes commutative.

\[
\begin{array}{ccc}
(G \otimes G) \times (G \otimes G) & \longrightarrow & N \otimes G \\
\downarrow & & \downarrow \\
G \otimes G & \longrightarrow & N
\end{array}
\]

**Proof.** It is enough to show that the next diagrams commute:

(1) $(G \otimes G) \times (G \otimes G) \longrightarrow N \otimes G$

\[
\begin{array}{c}
G \otimes G \hspace{1cm} G \otimes G
\end{array}
\]

The diagram (1) becomes commutative because of the braiding condition (1). The triangle (2) also becomes commutative by the braiding condition (2) for $\{\ldots\}$. □

Thus we know that the braided crossed module $(N \otimes G \to G \otimes G, \{\ldots\})$ is an extension of $(N \to G, \{\ldots\})$. Furthermore, this braiding has a universal property.
Theorem 2.3.1 If \((N \to G, \{\ , \} )\) is a perfect braided crossed module, and \((X_1 \cong X_2, \{\ , \}' )\) is a central extension of it with a compatible braiding, then the next diagram becomes commutative.

\[
\begin{array}{cccc}
(G \otimes G) \times (G \otimes G) & \to & N \otimes G \\
\uparrow & & \downarrow \\
X_2 \times X_2 & \to & X_1
\end{array}
\]

Proof. Let \(s_2\) be a section of the map \(X_2 \to G\), and let \(u : G \otimes G \to X_2 \times X_2\) be the map \(a \otimes b \to (s_2(a), s_2(b))\). Define \(r : G \otimes G \to X_1\) to be \(\{\ , \}' \circ u\) and let \(t : G \otimes G \to X_2\) be the map \(a \otimes b \mapsto [s_2(a), s_2(b)]\). Set \(p = r \times t\) and \(q = \omega \times \text{id}\).

We now consider the following decomposition of the diagram in the theorem and show that each diagram inside commutes.

\[
\begin{array}{cccc}
(G \otimes G) \times (G \otimes G) & \to & N \otimes G \\
\uparrow & & \downarrow \\
X_2 \times X_2 & \to & X_1
\end{array}
\]

The diagram \((A)\) becomes commutative because the braiding \(\{\ , \}'\) is compatible with \(\{\ , \}\). The diagram \((B)\) also becomes commutative because of the braiding condition (2) and the choice of \(r\). Finally the commutativity of the diagram \((C)\) follows from the braiding condition (2) and the constructions.

\[\square\]

Corollary 2.3.1 If \((N \to G, \{\ , \} )\) is a \(q\)-perfect braided crossed module with \(N\) being a \(q\)-central extension of \(G\), then \((N \otimes^q G \to G \otimes^q G, \{\ , \}^q )\) becomes the universal central extension of it.

Proof. This follows since we can construct the similar maps by \(r(\{k\}) = (s_1 \circ s(k))^q\) and \(t(\{k\}) = (\omega \circ s_1 \circ s(k))^q\). \[\square\]
Bibliography


