SUMMARY This paper is concerned with coding theorems in the optimistic sense for separate coding of two correlated general sources $X_1$ and $X_2$. We investigate the achievable rate region $\mathcal{R}_{\text{opt}}(X_1, X_2)$ such that the decoding error probability caused by two encoders and one decoder can be arbitrarily small infinitely often under a certain rate constraint. We give an inner and an outer bounds of $\mathcal{R}_{\text{opt}}(X_1, X_2)$, where the outer bound is described by using new information-theoretic quantities. We also give two simple sufficient conditions under which the inner bound coincides with the outer bound.

**key words:** correlated sources, general source, optimistic coding, achievable rate region, information-spectrum methods

1. Introduction

In information-spectrum methods that originate from [3] and are described in detail in [4], fixed-length coding of a general source $X = \{X_n\}_{n=1}^\infty$ is one of fundamental problems, where, letting $\mathcal{X}$ be a finite or countably infinite alphabet, $X^n$ is defined as a random variable taking values in $\mathcal{X}^n$. The class of general sources includes vast classes of sources such as memoryless sources, Markov sources, stationary ergodic sources and stationary sources. Therefore, coding theorems for the class of general sources are valid to such classes of sources.

In coding of a general source we can formulate a new kind of coding problem in the optimistic sense. We are interested in the infimum of the rate $R$ such that there exists a sequence $\{(\varphi_n, \psi_n)\}_{n=1}^{\infty}$ of pairs of an encoder $\varphi_n$ and a decoder $\psi_n$ satisfying for any $\gamma > 0$ both $\frac{1}{n} \log M_n \leq R + \gamma$ and $\varepsilon_n \leq \gamma$ with a subsequence $n = n_i$, $i \geq 1$, where $M_n$ and $\varepsilon_n$ denote the number of codewords and the decoding error probability, respectively. This problem was first formulated by Vembu, Verdú and Steinberg [11] and was discussed by Chen and Alajaji [1]. Results related to the optimistic coding problem can be found in [5], [9]. Quite recently, the author defined new information-theoretic quantities and clarified their operational meanings in the optimistic sense [6].

In this paper we discuss the achievable rate region in the optimistic sense for separate coding of a pair $(X_1, X_2) = \{(X_1^n, X_2^n)\}_{n=1}^{\infty}$ of two correlated general sources, where for each $k = 1, 2$, $X_k^n \in \mathcal{X}_k^n$ and $\mathcal{X}_k$ denotes a finite or countably infinite alphabet. We consider two encoders $\varphi_n^{(k)}$ : $X_k^n \rightarrow \{1, 2, \ldots, M_{n_k}^{(k)}\}$, $k = 1, 2$, for coding of $X_1^n$ and $X_2^n$, respectively, and a decoder $\psi_n$ that outputs an estimate $(\hat{X}_1^n, \hat{X}_2^n) \in \mathcal{X}_1^n \times \mathcal{X}_2^n$ of $(X_1^n, X_2^n)$ from two transmitted codewords $I_n^{(1)}$ and $I_n^{(2)}$ (see Fig. 1). We are interested in the achievable rate region $\mathcal{R}_{\text{opt}}(X_1, X_2)$ in the optimistic sense, which is defined as the collection of all the rate pairs $(R_1, R_2)$ such that there exists a sequence $\{(\varphi_n^{(1)}, \varphi_n^{(2)}, \psi_n)\}_{n=1}^{\infty}$ satisfying $\frac{1}{n} \log M_{n_k}^{(k)} \leq R_k + \gamma (k = 1, 2)$ and $\varepsilon_n \leq \gamma$ for a subsequence $n = n_i$, $i \geq 1$, where $\gamma > 0$ is an arbitrarily small constant and $\varepsilon_n$ denotes the decoding error probability.

We give an inner and an outer bounds of $\mathcal{R}_{\text{opt}}(X_1, X_2)$, where the outer bound can be expressed by using new quantities introduced in [6]. In particular, it is shown that the outer bound is easily treated by characterizing the unachievable region in the strong sense [6]. We also give two simple sufficient conditions under which the outer bound coincides with the inner bound.

The problem of separate coding of correlated sources was first formulated by Slepian and Wolf [10] for the memoryless case. A simple proof using the bin coding is given in Cover [2]. The achievable rate region for general sources $X_1$ and $X_2$ in the ordinary sense is given in Miyake and Kanaya [8]. The achievable rate region in [8] is extended to the case of $\varepsilon$-error by Han [4]. However, to the author’s knowledge, no coding theorem in the optimistic sense is obtained so far. Our approach using the inner and the outer bounds clarifies a difference between coding in the ordinary sense and in the optimistic sense.

2. Problem Setup

In this section we consider separate coding of two correlated sources [10]. The block diagram is given in Fig. 1. Let $(X_1^n, X_2^n) = \{(X_1^n, X_2^n)\}_{n=1}^{\infty}$ be two correlated general sources. For each $n \geq 1$, $X_1^n$ and $X_2^n$ take values in $\mathcal{X}_1^n$ and $\mathcal{X}_2^n$, respectively, where $X_1$ and $X_2$ denote finite or countably infinite alphabets. Denote by $P_{X_1^n|X_2^n}$ and $P_{X_1^n|x_2^n}$ the joint probabil-

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**Fig. 1** Separate coding of two correlated general sources.
Theorem 1: A rate pair \((R_1, R_2)\) is called achievable in the ordinary sense if there exists a code \(\{(\varphi_n^{(1)}, \varphi_n^{(2)}, \psi_n)\}_{n=1}^{\infty}\) satisfying
\[
\limsup_{n \to \infty} \frac{1}{n} \log M_n^{(k)} \leq R_k, \quad k = 1, 2
\]
and
\[
\lim_{n \to \infty} \epsilon_n = 0,
\]
where \(\log(\cdot) = \log_2(\cdot)\) throughout this paper. Define the achievable rate region in the ordinary sense by
\[
\mathcal{R}(X_1, X_2) = \text{Cl}(\{(R_1, R_2) : (R_1, R_2) \text{ is achievable in the ordinary sense}\}),
\]
where \(\text{Cl}(\cdot)\) denotes the closure of the set.

Theorem 1 ([8]): It holds that
\[
\mathcal{R}(X_1, X_2) = \mathcal{S}(X_1, X_2),
\]
where
\[
\mathcal{S}(X_1, X_2) = \{(R_1, R_2) : R_1 \geq \overline{H}(X_1|X_2), \quad R_2 \geq \overline{H}(X_2|X_1) \text{ and } R_1 + R_2 \geq \overline{H}(X_1, X_2)\}
\]
and
\[
\overline{H}(X_1, X_2) = \inf \left\{ \alpha : \liminf_{n \to \infty} \Pr \left( \frac{1}{n} \log \frac{1}{P_{X_1^{n}X_2^{n}}(X_1^n, X_2^n)} \leq \alpha \right) = 1 \right\},
\]
\[
\overline{H}(X_1|X_2) = \inf \alpha : \liminf_{n \to \infty} \Pr \left( \frac{1}{n} \log \frac{1}{P_{X_1^{n}X_2^{n}}(X_1^n)} \leq \alpha \right) = 1
\]
for \(k, l = (1, 2)\) and \((2, 1)\).

It is important to notice that (1) and (2) require that for any \(\gamma > 0\)
\[
\frac{1}{n} \log M_n^{(k)} \leq R_k + \gamma, \quad k = 1, 2
\]
and sufficiently large \(n\).

In this paper, we are interested in the achievable rate region in the optimistic sense defined as follows:

Definition 2: A rate pair \((R_1, R_2)\) is called achievable in the optimistic sense if there exists a code \(\{(\varphi_n^{(1)}, \varphi_n^{(2)}, \psi_n)\}_{n=1}^{\infty}\) such that for any \(\gamma > 0\) there exists a subsequence \(\{n_k\}_{k=1}^{\infty}\) satisfying
\[
\frac{1}{n_k} \log M_n^{(k)} \leq R_k + \gamma, \quad k = 1, 2
\]
and
\[
\epsilon_n = \gamma
\]
for all \(n = n_k, i \geq 1\). Define the achievable rate region in the optimistic sense by
\[
\mathcal{R}_{\text{opt}}(X_1, X_2) = \text{Cl}(\{(R_1, R_2) : (R_1, R_2) \text{ is achievable in the optimistic sense}\}).
\]

Note that in Definition 2 (3) and (4) require that for any \(\gamma > 0\)
\[
\frac{1}{n} \log M_n^{(k)} \leq R_k + \gamma, \quad k = 1, 2
\]
and \(\epsilon_n \leq \gamma\) simultaneously for infinitely many \(n\). Hence, it is obvious that \(\mathcal{R}(X_1, X_2) \subset \mathcal{R}_{\text{opt}}(X_1, X_2)\). While the formula of the infimum achievable rate in the optimistic sense for fixed-to-fixed length coding of a general source \(X\) is given in [1], no result is known on the achievable rate region \(\mathcal{R}_{\text{opt}}(X_1, X_2)\) in the optimistic sense.

We also consider the following region.

Definition 3: A rate pair \((R_1, R_2)\) is called unachievable in the strong sense if for any code \(\{(\varphi_n^{(1)}, \varphi_n^{(2)}, \psi_n)\}_{n=1}^{\infty}\) satisfying (1) it holds that
\[
\liminf_{n \to \infty} \epsilon_n > 0.
\]

Define the unachievable rate region in the strong sense by
\[
\mathcal{U}(X_1, X_2) = \{(R_1, R_2) : (R_1, R_2) \text{ is unachievable in the strong sense}\}.
\]

Notice that (5) means that \(\epsilon_n\) is positive for all sufficiently large \(n\). This \(\mathcal{U}(X_1, X_2)\) is closely related to \(\mathcal{R}_{\text{opt}}(X_1, X_2)\) and facilitates characterization of \(\mathcal{R}_{\text{opt}}(X_1, X_2)\).

The following proposition is easily obtained from the definitions of \(\mathcal{R}_{\text{opt}}(X_1, X_2)\) and \(\mathcal{U}(X_1, X_2)\).

Proposition 1: Define \(\mathcal{U}^c(X_1, X_2) = \text{Cl}(\mathcal{U}(X_1, X_2))\), where the superscript \(c\) denotes the complement. Then, it holds that
\[
\mathcal{U}^c(X_1, X_2) \subset \mathcal{R}_{\text{opt}}(X_1, X_2).
\]

**Proof:** If \((R_1, R_2) \in \mathcal{U}(X_1, X_2)\), there exists a code \(\{(\varphi_n^{(1)}, \varphi_n^{(2)}, \psi_n)\}_{n=1}^{\infty}\) satisfying (1) and \(\liminf_{n \to \infty} \epsilon_n = 0\). Such a code \(\{(\varphi_n^{(1)}, \varphi_n^{(2)}, \psi_n)\}_{n=1}^{\infty}\) satisfies, letting \(\gamma > 0\) be an arbitrary constant, \(\frac{1}{n} \log M_n^{(k)} \leq R_k + \gamma, k = 1, 2\), for all sufficiently large \(n\) and \(\epsilon_n \leq \gamma\) infinitely often. This means \(\mathcal{U}^c(X_1, X_2) \subset \mathcal{R}_{\text{opt}}(X_1, X_2)\) because \((R_1, R_2) \in \mathcal{U}^c(X_1, X_2)\) is arbitrary. Since \(\mathcal{R}_{\text{opt}}(X_1, X_2)\) is defined as the closure, the claim of this proposition follows.  \(\Box\)
Remark: In [4], [8] the achievable rate region $\mathcal{R}(X_1, X_2)$ is defined without taking the closure. In this paper, however, $\mathcal{R}(X_1, X_2)$ and $\mathcal{R}_{\text{opt}}(X_1, X_2)$ are defined as the closures so that we can ensure technical soundness especially of Proposition 1 and discussions in Sect. 4.1.

3. Main Results

Before giving inner and outer bounds of $\mathcal{R}_{\text{opt}}(X_1, X_2)$ and $\mathcal{U}(X_1, X_2)$, we define the three events for $R \geq 0$ as follows:

$$
E_n^{(1)}(R) = \left\{ (x_1^n, x_2^n) \in X_1^n \times X_2^n : \frac{1}{n} \log \frac{1}{P_{X_1^n|X_2^n}(x_1^n|X_2^n)} \geq R \right\},
$$

$$
E_n^{(2)}(R) = \left\{ (x_1^n, x_2^n) \in X_1^n \times X_2^n : \frac{1}{n} \log \frac{1}{P_{X_1^n|X_2^n}(x_2^n|X_1^n)} \geq R \right\},
$$

$$
E_n^{(3)}(R) = \left\{ (x_1^n, x_2^n) \in X_1^n \times X_2^n : \frac{1}{n} \log \frac{1}{P_{X_1^n|X_2^n}(x_1^n|X_2^n)} \geq R \right\}.
$$

Furthermore, define

$$
\mathcal{A}(X_1, X_2) = \{ (R_1, R_2) : \text{for any } \gamma > 0 \}
$$

$$
\liminf_{n \to \infty} \Pr\{E_n^{(1)}(R_1 + \gamma) \cup E_n^{(2)}(R_2 + \gamma) \cup E_n^{(3)}(R_1 + R_2 + \gamma) \} = 0 \} \quad (6)
$$

and

$$
\mathcal{B}(X_1, X_2) = \{ (R_1, R_2) : \text{for any } \gamma > 0 \}
$$

$$
\liminf_{n \to \infty} \Pr\{E_n^{(1)}(R_1 + \gamma) = 0,
$$

$$
\liminf_{n \to \infty} \Pr\{E_n^{(2)}(R_2 + \gamma) = 0,
$$

$$
\liminf_{n \to \infty} \Pr\{E_n^{(3)}(R_1 + R_2 + \gamma) = 0 \} \} \quad (7)
$$

In fact, $\mathcal{B}(X_1, X_2)$ can be written in the following form.

**Proposition 2:** $\mathcal{B}(X_1, X_2) = S^*(X_1, X_2)$, where

$$
S^*(X_1, X_2) = \{ (R_1, R_2) : R_1 \geq \overline{H}(X_1|X_2),
$$

$$
R_2 \geq \overline{H}(X_2|X_1) \text{ and } R_1 + R_2 \geq \overline{H}(X_1, X_2) \}.
$$

and

$$
\overline{H}(X_1, X_2) = \inf \left\{ \alpha : \limsup_{n \to \infty} \Pr\left\{ \frac{1}{n} \log \frac{1}{P_{X_1^n|X_2^n}(X_1^n|X_2^n)} \leq \alpha \right\} = 1 \right\}.
$$

$$
\overline{H}(X_k|X_l) = \inf \left\{ \alpha : \limsup_{n \to \infty} \Pr\left\{ \frac{1}{n} \log \frac{1}{P_{X_k^n|X_l^n}(X_k^n|X_l^n)} \leq \alpha \right\} = 1 \right\}
$$

for $(k, l) = (1, 2)$ and $(2, 1)$.

**Proof:** We first prove $\mathcal{B}(X_1, X_2) \subset S^*(X_1, X_2)$. Let $(R_1, R_2)$ be an arbitrary element of $\mathcal{B}(X_1, X_2)$. Then, letting $\gamma > 0$ be an arbitrary constant, we have

$$
\liminf_{n \to \infty} \Pr\{E_n^{(1)}(R_1 + \gamma) \}
$$

$$
= \liminf_{n \to \infty} \Pr\left\{ \frac{1}{n} \log \frac{1}{P_{X_1^n|X_2^n}(X_1^n|X_2^n)} \geq R_1 + \gamma \right\} = 0,
$$

which implies that

$$
\limsup_{n \to \infty} \Pr\left\{ \frac{1}{n} \log \frac{1}{P_{X_1^n|X_2^n}(X_1^n|X_2^n)} \leq R_1 + \gamma \right\} = 1 \quad (8)
$$

for any $\gamma > 0$. Note that we can use $\leq$ instead of $<$ in (8). Hence, it holds that $\overline{H}(X_1|X_2) \leq R_1$ owing to the definition of $\overline{H}(X_1|X_2)$. Similarly, we can obtain $\overline{H}(X_2|X_1) \leq R_2$ and $\overline{H}(X_1, X_2) \leq R_1 + R_2$. Thus, we obtain $(R_1, R_2) \in S^*(X_1, X_2)$. Since $(R_1, R_2) \in \mathcal{B}(X_1, X_2)$ is arbitrary, $\mathcal{B}(X_1, X_2) \subset S^*(X_1, X_2)$ is established.

Next, we prove $S^*(X_1, X_2) \subset \mathcal{B}(X_1, X_2)$. Fix $(R_1, R_2) \in S^*(X_1, X_2)$ arbitrarily. Since $R_1 \geq \overline{H}(X_1|X_2)$ is guaranteed, for any $\gamma > 0$ it follows that

$$
\Pr\{E_n^{(1)}(R_1 + \gamma)\} = \Pr\left\{ \frac{1}{n} \log \frac{1}{P_{X_1^n|X_2^n}(X_1^n|X_2^n)} \geq R_1 + \gamma \right\}
$$

$$
\leq \Pr\left\{ \frac{1}{n} \log \frac{1}{P_{X_1^n|X_2^n}(X_1^n|X_2^n)} > \overline{H}(X_1|X_2) + \gamma \right\}
$$

$$
= 1 - \Pr\left\{ \frac{1}{n} \log \frac{1}{P_{X_1^n|X_2^n}(X_1^n|X_2^n)} \leq \overline{H}(X_1|X_2) + \gamma \right\}.
$$

By taking the limit inferior of both sides, we have

$$
\liminf_{n \to \infty} \Pr\{E_n^{(1)}(R_1 + \gamma)\} = 0
$$

$$
\leq 1 - \limsup_{n \to \infty} \Pr\left\{ \frac{1}{n} \log \frac{1}{P_{X_1^n|X_2^n}(X_1^n|X_2^n)} \leq \overline{H}(X_1|X_2) + \gamma \right\} = 0
$$

due to the definition of $\overline{H}(X_1|X_2)$. Since $\Pr\{E_n^{(1)}(R_1 + \gamma)\} \geq 0$ for all $n \geq 1$, this establishes $\liminf_{n \to \infty} \Pr\{E_n^{(1)}(R_1 + \gamma)\} = 0$. Similarly, we can prove $\liminf_{n \to \infty} \Pr\{E_n^{(2)}(R_2 + \gamma)\} = 0$ and $\liminf_{n \to \infty} \Pr\{E_n^{(3)}(R_1 + R_2 + \gamma)\} = 0$. Thus, we obtain $(R_1, R_2) \in \mathcal{B}(X_1, X_2)$. Since $(R_1, R_2) \in S^*(X_1, X_2)$ is arbitrary, $S^*(X_1, X_2) \subset \mathcal{B}(X_1, X_2)$ follows.

**Proposition 3:** $\mathcal{A}(X_1, X_2) \subset \mathcal{B}(X_1, X_2)$.

**Proof:** For any $(R_1, R_2) \in \mathcal{A}(X_1, X_2)$ it holds that

$$
\liminf_{n \to \infty} \Pr\{E_n^{(1)}(R_1 + \gamma)\} = \liminf_{n \to \infty} \Pr\{E_n^{(2)}(R_2 + \gamma)\}
$$

$$
= \liminf_{n \to \infty} \Pr\{E_n^{(3)}(R_1 + R_2 + \gamma)\} = 0
$$

because all of $E_n^{(1)}(R_1 + \gamma)$, $E_n^{(2)}(R_2 + \gamma)$ and $E_n^{(3)}(R_1 + R_2 + \gamma)$ are subsets of $E_n^{(1)}(R_1 + \gamma) \cup E_n^{(2)}(R_2 + \gamma) \cup E_n^{(3)}(R_1 + R_2 + \gamma)$. Thus, the claim of this proposition follows immediately.

Now, we are ready to state the main results.

**Theorem 2:** For any two correlated general sources $X_1$ and $X_2$ we have

$$
\mathcal{A}(X_1, X_2) \subset \mathcal{U}^i(X_1, X_2) \subset \mathcal{R}_{\text{opt}}(X_1, X_2) \subset \mathcal{B}(X_1, X_2).
$$

We use the following two lemmas in the proof of Theorem 2.
Lemma 1 ([4]): Let $M_n^{(1)}$ and $M_n^{(2)}$ be positive integers arbitrarily given. Then, there exists a code $\{(\varphi_n^{(1)}, \varphi_n^{(2)}, \psi_n)\}_{n=1}^\infty$ satisfying

$$
\varepsilon_n \leq \Pr\left( \frac{1}{n} \log \frac{1}{P_{X_n,Y_n}(X_n,Y_n)} \geq \frac{1}{n} \log M_n^{(1)} - \gamma \right) \\
+ \frac{1}{n} \log \frac{1}{P_{X_n,Y_n}(X_n,Y_n)} \geq \frac{1}{n} \log M_n^{(2)} - \gamma \\
+ \frac{1}{n} \log \frac{1}{P_{X_n,Y_n}(X_n,Y_n)} \geq \frac{1}{n} \log M_n^{(1)} M_n^{(2)} - \gamma \right)
$$

(9)

for any $\gamma > 0$ and $n \geq 1$.

Lemma 2 ([4]): Any code $\{(\varphi_n^{(1)}, \varphi_n^{(2)}, \psi_n)\}_{n=1}^\infty$ satisfies

$$
\varepsilon_n \geq \Pr\left( \frac{1}{n} \log \frac{1}{P_{X_n,Y_n}(X_n,Y_n)} \geq \frac{1}{n} \log M_n^{(1)} + \gamma \\
+ \frac{1}{n} \log \frac{1}{P_{X_n,Y_n}(X_n,Y_n)} \geq \frac{1}{n} \log M_n^{(2)} + \gamma \\
+ \frac{1}{n} \log \frac{1}{P_{X_n,Y_n}(X_n,Y_n)} \geq \frac{1}{n} \log M_n^{(1)} M_n^{(2)} + \gamma \right)
$$

(10)

for all $n \geq 1$ and $\gamma > 0$.

Proof of Theorem 2: Since Proposition 1 guarantees $\mathcal{U}(X_1, X_2) \subseteq \mathcal{R}_{opt}(X_1, X_2)$, we have only to prove $\mathcal{R}(X_1, X_2) \subseteq \mathcal{U}(X_1, X_2)$ and $\mathcal{R}_{opt}(X_1, X_2) \subseteq \mathcal{B}(X_1, X_2)$.

First, we prove $\mathcal{R}(X_1, X_2) \subseteq \mathcal{U}(X_1, X_2)$ and $\mathcal{R}_{opt}(X_1, X_2)$ and $\mathcal{B}(X_1, X_2)$. We use the idea in [4, Theorem 7.4.1]. Fix $\gamma > 0$ arbitrarily. We first show that $(R_1, R_2) \in \mathcal{R}(X_1, X_2)$ implies $(R_1 + 2\gamma, R_2 + 2\gamma) \in \mathcal{U}(X_1, X_2)$. Set $M_n^{(1)} = 2^{\gamma n} + R_1$ and $M_n^{(2)} = 2^{\gamma n} + R_2$. Clearly, these $M_n^{(1)}$ and $M_n^{(2)}$ satisfy

$$
\limsup_{n \to \infty} \frac{1}{n} \log M_n^{(k)} \leq R_k + 2\gamma, \quad k = 1, 2.
$$

Since Lemma 1 guarantees the existence of a code $\{(\varphi_n^{(1)}, \varphi_n^{(2)}, \psi_n)\}_{n=1}^\infty$ satisfying (9), we have

$$
\varepsilon_n \leq \Pr\left( \frac{1}{n} \log \frac{1}{P_{X_n,Y_n}(X_n,Y_n)} \geq R_1 + \gamma \\
+ \frac{1}{n} \log \frac{1}{P_{X_n,Y_n}(X_n,Y_n)} \geq R_2 + \gamma \\
+ \frac{1}{n} \log \frac{1}{P_{X_n,Y_n}(X_n,Y_n)} \geq R_1 + R_2 + 3\gamma \right)
$$

$$
\leq \Pr\left( E_n^{(1)}(R_1 + \gamma) \cap E_n^{(2)}(R_2 + \gamma) \cap E_n^{(3)}(R_1 + R_2 + \gamma) \right) + 3 \cdot 2^{-n\gamma}
$$

(11)

where the left side is less than $\gamma$ for all $n = n_i, i \geq 1$, and the right side is greater than $\delta_0 (> \gamma)$ for all sufficiently large $n = n_i$ due to (12). This is a contradiction.

Since this argument can be applied to establishing (b) $R_2 \geq \overline{H}(X_2|X_1)$ and (c) $R_1 + R_2 \geq \overline{H}(X_1, X_2)$ as well, we can conclude that $(R_1, R_2) \in S' \subseteq S$ is achievable in the optimistic sense. This establishes $\mathcal{R}_{opt}(X_1, X_2) \subseteq S' \subseteq S$ because $S' \subseteq S$ includes the boundaries.

4. Discussions

4.1 Characterization of $\mathcal{R}(X_1, X_2)$

So far, we have established an inner and an outer bounds
of $\mathcal{R}_{\text{opt}}(X_1, X_2)$. In this subsection we revisit $\mathcal{R}(X_1, X_2)$ in Definition 1 and characterize $\mathcal{R}(X_1, X_2)$ in the same manner as Theorem 2.

To this end, define $\tilde{\mathcal{A}}(X_1, X_2)$ and $\tilde{\mathcal{B}}(X_1, X_2)$ by replacing the limit inferiors in (6) and (7) with the limit superiors, respectively. Define $\mathcal{W}'(X_1, X_2)$ by replacing the limit inferior in (5) with the limit superior. Then, it is easily verified that $\mathcal{R}(X_1, X_2)$ satisfies

$$\tilde{\mathcal{A}}(X_1, X_2) \subset \mathcal{W}'(X_1, X_2) \subset \mathcal{R}(X_1, X_2) \subset \tilde{\mathcal{B}}(X_1, X_2)$$

similarly to the proof of Theorem 2, where $\mathcal{W}'(X_1, X_2)$ denotes the closure of the complement of $\mathcal{W}(X_1, X_2)$. We can also verify that $\tilde{\mathcal{B}}(X_1, X_2)$ is expressed as $S(X_1, X_2)$ in Theorem 1 similarly to Proposition 2.

In the ordinary case, however, we can prove $\tilde{\mathcal{B}}(X_1, X_2) \subset \tilde{\mathcal{A}}(X_1, X_2)$ as well and therefore $\mathcal{W}'(X_1, X_2) = \mathcal{R}(X_1, X_2) = S(X_1, X_2)$ without any assumption on $(X_1, X_2)$. This explains a reason why $\mathcal{R}(X_1, X_2)$ is expressed in a closed form and is coincident with $S(X_1, X_2)$.

In order to verify $\tilde{\mathcal{B}}(X_1, X_2) \subset \tilde{\mathcal{A}}(X_1, X_2)$, let $(R_1, R_2)$ be an arbitrary element of $\tilde{\mathcal{B}}(X_1, X_2)$. By the definition of $\tilde{\mathcal{B}}(X_1, X_2)$, it holds that $\limsup_{n \to \infty} \text{Pr}(E_n^{(1)}(R_1 + \gamma)) = 0$ for $k = 1, 2$ and $\limsup_{n \to \infty} \text{Pr}(E_n^{(2)}(R_1 + R_2 + \gamma)) = 0$. Then, we have

$$\limsup_{n \to \infty} \text{Pr}(E_n^{(1)}(R_1 + \gamma) \cup E_n^{(2)}(R_2 + \gamma) \cup E_n^{(3)}(R_1 + R_2 + \gamma))$$

$$\leq \limsup_{n \to \infty} \text{Pr}(E_n^{(1)}(R_1 + \gamma)) + \limsup_{n \to \infty} \text{Pr}(E_n^{(2)}(R_2 + \gamma))$$

$$+ \limsup_{n \to \infty} \text{Pr}(E_n^{(3)}(R_1 + R_2 + \gamma)) = 0,$$

where the inequality follows from the union bound and

$$\limsup_{n \to \infty} \text{Pr}(E_n^{(1)}(a_n + b_n)) \leq \limsup_{n \to \infty} \text{Pr}(a_n) + \limsup_{n \to \infty} \text{Pr}(b_n)$$

for any real-valued sequences $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$. Since the left side is nonnegative, (15) implies that $(R_1, R_2) \in \tilde{\mathcal{A}}(X_1, X_2)$. This establishes $\tilde{\mathcal{B}}(X_1, X_2) \subset \tilde{\mathcal{A}}(X_1, X_2)$.

Notice that we cannot use the same argument to establish $\tilde{\mathcal{B}}(X_1, X_2) \subset \tilde{\mathcal{A}}(X_1, X_2)$ because only the opposite side the inequality holds with respect to the limit inferior. In Example 3 in Sect. 4.2 we will see an example of $(X_1, X_2)$ satisfying $\mathcal{A}(X_1, X_2) \subset \mathcal{B}(X_1, X_2)$.

### 4.2 Sufficient Conditions for $\mathcal{A}(X_1, X_2) = \mathcal{B}(X_1, X_2)$

In this subsection, we investigate sufficient conditions on $(X_1, X_2)$ under which $\mathcal{A}(X_1, X_2) = \mathcal{B}(X_1, X_2)$ holds. Theorem 2 and Proposition 2 guarantee that $\mathcal{R}_{\text{opt}}(X_1, X_2) = \mathcal{U}'(X_1, X_2) = \mathcal{S}'(X_1, X_2)$ under such sufficient conditions.

We begin with the simple sufficient condition given in the following proposition. This sufficient condition is shown to be valid by using the argument given in Sect. 4.1.

**Proposition 4:** If $\overline{H}(X_1|X_2) = \overline{H}'(X_1|X_2)$, $\overline{H}'(X_2|X_1) = \overline{H}(X_2|X_1)$ and $\overline{H}(X_1, X_2) = \overline{H}'(X_1, X_2)$, then it holds that $\mathcal{R}_{\text{opt}}(X_1, X_2) = \mathcal{U}'(X_1, X_2) = \mathcal{S}'(X_1, X_2)$.

**Proof:** Clearly, $S(X_1, X_2) = S'(X_1, X_2)$ holds by the assumption. So far, we have already established that $\mathcal{B}(X_1, X_2) = S'(X_1, X_2)$ in Proposition 2 and $\mathcal{B}(X_1, X_2) = \mathcal{S}(X_1, X_2)$ in Sect. 4.1. Hence, $\mathcal{S}(X_1, X_2) = S'(X_1, X_2)$ implies $\mathcal{B}(X_1, X_2) = \mathcal{B}(X_1, X_2)$. In addition, we should note $\tilde{\mathcal{A}}(X_1, X_2) \subset \mathcal{A}(X_1, X_2)$, which immediately follows from their definitions. Then, in view of Theorem 2 it holds that

$$\tilde{\mathcal{A}}(X_1, X_2) \subset \mathcal{A}(X_1, X_2) \subset \mathcal{U}'(X_1, X_2)$$

$$\subset \mathcal{R}_{\text{opt}}(X_1, X_2) \subset \mathcal{B}(X_1, X_2) = \mathcal{B}(X_1, X_2).$$

Since we have proved that $\tilde{\mathcal{A}}(X_1, X_2) = \mathcal{B}(X_1, X_2)$ in Sect. 4.1, the claim in this proposition follows.

**Example 1:** Let $P_{X_1X_2}$ be a joint probability distribution on $X_1 \times X_2$ satisfying $H(X_1|X_2) < \infty$, where $H(X_1|X_2)$ denotes the joint entropy of $X_1$ and $X_2$. Let $X_1$ and $X_2$ be two correlated stationary memoryless sources induced by $P_{X_1X_2}$. Clearly, by the weak law of large numbers we have $\overline{H}(X_1|X_2) = \overline{H}'(X_1|X_2) = H(X_1|X_2)$. Hence, Proposition 4 guarantees that

$$\mathcal{R}(X_1, X_2) = \mathcal{R}_{\text{opt}}(X_1, X_2)$$

$$= \{ (R_1, R_2) : R_1 \geq H(X_1|X_2), R_2 \geq H(X_2|X_1)$$

$$\text{and } R_1 + R_2 \geq H(X_1,X_2) \}.$$

Next, we introduce a new class of correlated sources.

**Definition 4:** Let $X_1$ and $X_2$ be two correlated general sources. We say that $(X_1, X_2)$ is synchronizing if for any $\theta > 0$ and $\eta \in (0, 1)$ there exists a subsequence $(n_i)_{i=1}^\infty$ such that all of

$$\frac{1}{n} \log \frac{1}{P_{X_1X_2}(X_1^n|X_2^n)} \geq \overline{H}(X_1|X_2) + \gamma \leq \eta,$$  

(17)

$$\frac{1}{n} \log \frac{1}{P_{X_1X_2}(X_1^n|X_2^n)} \geq \overline{H}(X_2|X_1) + \gamma \leq \eta,$$  

(18)

$$\frac{1}{n} \log \frac{1}{P_{X_1X_2}(X_1^n|X_2^n)} \geq \overline{H}(X_1, X_2) + \gamma \leq \eta$$  

(19)

are satisfied for all $n = n_i$, $i \geq 1$.

Note that owing to the definitions of $\overline{H}(X_1|X_2)$, $\overline{H}'(X_2|X_1)$ and $\overline{H}(X_1, X_2)$, each of the above three inequalities in Definition 4 is satisfied infinitely often. The synchronizing property in Definition 4 actually requires that all the three inequalities are satisfied simultaneously with the same subsequence $(n_i)_{i=1}^\infty$.

The following proposition indicates that the synchronizing property of $(X_1, X_2)$ is another sufficient condition for $\mathcal{A}(X_1, X_2) = \mathcal{B}(X_1, X_2)$.

**Proposition 5:** If $(X_1, X_2)$ is synchronizing, $\mathcal{R}_{\text{opt}}(X_1, X_2) = \mathcal{U}'(X_1, X_2) = \mathcal{S}'(X_1, X_2)$.

**Proof:** In view of Theorem 2, it suffices to establish
Similarly, we have $\Pr[E_{n}^{(1)}(R_{1} + \gamma)] = \Pr\left\{ \frac{1}{n} \log \frac{1}{P_{X_{1}|X_{2}}(X_{1}^{n}|X_{2})} \geq R_{1} + \gamma \right\}$ satisfying

$$\Pr[E_{n}^{(1)}(R_{1} + \gamma)] \leq \eta \quad \text{for all } n = n_{i}, i \geq 1.$$ 
Similarly, we have $\Pr[E_{n}^{(2)}(R_{2} + \gamma)] \leq \eta$ and $\Pr[E_{n}^{(3)}(R_{1} + R_{2} + \gamma)] \leq \eta$ for all $n = n_{i}, i \geq 1$. Therefore, it holds that

$$\Pr[E_{n}^{(1)}(R_{1} + \gamma) \cup E_{n}^{(2)}(R_{2} + \gamma) \cup E_{n}^{(3)}(R_{1} + R_{2} + \gamma)] \leq \Pr[E_{n}^{(1)}(R_{1} + \gamma)] + \Pr[E_{n}^{(2)}(R_{2} + \gamma)] + \Pr[E_{n}^{(3)}(R_{1} + R_{2} + \gamma)] \leq 3\eta \quad \text{for all } n = n_{i}, i \geq 1,$$

which means that the limit inferior of the left side is equal to zero. Since $(R_{1}, R_{2}) \in \mathcal{B}(X_{1}, X_{2})$ is arbitrary, $\mathcal{B}(X_{1}, X_{2}) \subset \mathcal{A}(X_{1}, X_{2})$ follows. □

The notion of synchronizing property in Definition 4 leads to an interesting implication on inner bounds of $\mathcal{R}_{\text{opt}}(X_{1}, X_{2})$. Assume that for any $\gamma > 0$ there exists a subsequence $\{n_{i}\}_{i=1}^{\infty}$ such that (17) and (19) hold for all $n = n_{i}, i \geq 1$. Then, we can show that

$$\mathcal{V}_{1}(X_{1}, X_{2}) = \{(R_{1}, R_{2}) : R_{1} \geq H(X_{1}|X_{2}),$$
$$R_{2} \geq \overline{H}(X_{2}|X_{1}) \text{ and } R_{1} + R_{2} \geq \overline{H}(X_{1}, X_{2})\}$$
is a subset of $\mathcal{R}_{\text{opt}}(X_{1}, X_{2})$. This fact is verified by checking $\mathcal{V}_{1}(X_{1}, X_{2}) \subset \mathcal{A}(X_{1}, X_{2})$ as follows. Fix $(R_{1}, R_{2}) \in \mathcal{V}_{1}(X_{1}, X_{2})$ arbitrarily. Then, we have

$$\Pr[E_{n}^{(1)}(R_{1} + \gamma)] + \Pr[E_{n}^{(2)}(R_{2} + \gamma)] \leq 2\eta$$
for all $n = n_{i}$ and $i \geq 1$, and

$$\Pr[E_{n}^{(2)}(R_{2} + \gamma)] \leq \eta \quad \text{for all sufficiently large } n.$$
Thus, there exists an integer $i_{0}$ such that

$$\Pr[E_{n}^{(1)}(R_{1} + \gamma) \cup E_{n}^{(2)}(R_{2} + \gamma) \cup E_{n}^{(3)}(R_{1} + R_{2} + \gamma)] \leq 3\eta$$
for all $n = n_{i}$ and $i \geq i_{0},$

which means the limit inferior of the left side is equal to zero. On the other hand, we can similarly prove that

$$\mathcal{V}_{2}(X_{1}, X_{2}) = \{(R_{1}, R_{2}) : R_{1} \geq \overline{H}(X_{1}|X_{2}),$$
$$R_{2} \geq \overline{H}(X_{2}|X_{1}) \text{ and } R_{1} + R_{2} \geq \overline{H}(X_{1}, X_{2})\}$$
is a subset of $\mathcal{A}(X_{1}, X_{2})$ with no assumption on $(X_{1}, X_{2})$. Example 2: Let $P_{X_{1}X_{2}}^{(1)}$ and $P_{X_{1}X_{2}}^{(2)}$ be the joint probability distributions on $X_{1} \times X_{2}$ satisfying $H(X_{1}|X_{2}) \leq H(X_{1}^{(1)}|X_{2}^{(1)})$, $H(X_{2}|X_{1}) \leq H(X_{2}^{(2)}|X_{1}^{(2)})$ and $H(X_{1}|X_{2}) \leq H(X_{1}^{(3)}|X_{2}^{(3)})$. Define $\mathcal{R}(X_{1}, X_{2}) = \{(X_{1}^{n}, X_{2}^{n})\}_{n=1}^{\infty}$ as the pair of general sources satisfying

$$P_{X_{1}X_{2}}^{(i)}(x_{1}^{n}, x_{2}^{n}) = \begin{cases} \prod_{i=1}^{n} P_{X_{1}X_{2}}^{(1)}(x_{1}, x_{2}), & \text{if } n \text{ is odd}, \\ \prod_{i=1}^{n} P_{X_{1}X_{2}}^{(2)}(x_{1}, x_{2}), & \text{if } n \text{ is even}, \end{cases}$$
where $x_{1}^{n} = (x_{1}, \ldots, x_{n})$ and $x_{2}^{n} = (x_{1}, \ldots, x_{n})$. Since it holds that $\overline{H}(X_{1}|X_{2}) = H(X_{1}^{(2)}|X_{2}^{(2)})$, $\overline{H}(X_{2}|X_{1}) = H(X_{2}^{(2)}|X_{1}^{(2)})$, and $\overline{H}(X_{1}, X_{2}) = H(X_{1}^{(3)}|X_{2}^{(3)})$, we have

$$\mathcal{R}(X_{1}, X_{2}) = \{(R_{1}, R_{2}) : R_{1} \geq H(X_{1}^{(1)}|X_{1}^{(1)}),$$
$$R_{2} \geq H(X_{2}^{(2)}|X_{1}^{(1)}) \text{ and } R_{1} + R_{2} \geq H(X_{1}^{(3)}|X_{2}^{(3)})\}.$$
In addition, since $(X_{1}, X_{2})$ is synchronizing and satisfies $\overline{H}(X_{1}|X_{2}) = H(X_{1}^{(1)}|X_{2}^{(1)})$, $\overline{H}(X_{2}|X_{1}) = H(X_{2}^{(2)}|X_{1}^{(2)})$, and $\overline{H}(X_{1}, X_{2}) = H(X_{1}^{(3)}|X_{2}^{(3)})$, we have

$$\mathcal{R}_{\text{opt}}(X_{1}, X_{2}) = \{(R_{1}, R_{2}) : R_{1} \geq H(X_{1}^{(1)}|X_{2}^{(1)}),$$
$$R_{2} \geq H(X_{2}^{(2)}|X_{1}^{(1)}) \text{ and } R_{1} + R_{2} \geq H(X_{1}^{(3)}|X_{2}^{(3)})\}.$$
Figure 2 shows $\mathcal{R}(X_{1}, X_{2})$ and $\mathcal{R}_{\text{opt}}(X_{1}, X_{2})$ in this example.

Example 3: Suppose that $P_{X_{1}X_{2}}^{(1)}$ and $P_{X_{1}X_{2}}^{(2)}$ satisfy $H(X_{1}^{(1)}|X_{2}^{(1)}) \leq H(X_{1}^{(2)}|X_{2}^{(2)}), H(X_{2}^{(1)}|X_{1}^{(1)}) \leq H(X_{2}^{(2)}|X_{1}^{(2)}) < \infty$, and $H(X_{2}^{(1)}|X_{1}^{(1)}) \leq H(X_{2}^{(2)}|X_{1}^{(2)})$ in Example 2. In this case, we can easily verify from Theorems 1 and 2 that

$$\mathcal{R}(X_{1}, X_{2}) = \{(R_{1}, R_{2}) : R_{1} \geq H(X_{1}^{(2)}|X_{2}^{(2)}),$$
$$R_{2} \geq H(X_{2}^{(1)}|X_{1}^{(1)}) \text{ and } R_{1} + R_{2} \geq H(X_{1}^{(3)}|X_{2}^{(3)})\}$$
and

$$\mathcal{V}(X_{1}, X_{2}) \subset \mathcal{R}_{\text{opt}}(X_{1}, X_{2}) \subset \mathcal{A}(X_{1}, X_{2}),$$
where $\mathcal{V}(X_{1}, X_{2}) = \mathcal{V}_{1}(X_{1}, X_{2}) \cup \mathcal{V}_{2}(X_{1}, X_{2})$.

$$\mathcal{V}_{1}(X_{1}, X_{2}) = \{(R_{1}, R_{2}) : R_{1} \geq H(X_{1}^{(2)}|X_{1}^{(2)}),$$
$$R_{2} \geq H(X_{2}^{(1)}|X_{1}^{(1)}) \text{ and } R_{1} + R_{2} \geq H(X_{1}^{(3)}|X_{2}^{(3)})\}$$
for $i = 1, 2$ and

![Fig. 2](image2.png)
of the obtained results to the case of \( m \geq 2 \) correlated general sources is straightforward.

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**References**


