

Axiomatic Differential Geometry ?-2

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Axiomatic Differential Geometry III-2
-Its Landscape-
Chapter 2: Model Theory II

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Abstract

Given a complete and (locally) cartesian closed category \mathbf{U} , it is shown that the category of functors from the category of Weil algebras to the category \mathbf{U} is (locally, resp.) cartesian closed. The corresponding axiomatization for differential geometry based upon Weil functors is then given.

1 Introduction

Cartesian closedness is one of the desirable properties that every good category is expected to possess. Indeed, it is surely behind Steenrod's epoch-making notion of a *convenient category* of topological spaces, for which the reader is referred to [30]. Unlike many other desirable properties (e.g., completeness), cartesian closedness is not stable under slicing, and *slicing* within the realm of category theory corresponds to *fibered manifolds* within the realm of differential geometry. Therefore the importance of *locally cartesian closedness* in the arena of differential geometry could not be exaggerated. A few convenient categories of smooth spaces have been proposed (cf. [1] and [29] for their panoramic expositions), but not all of them are locally cartesian closed. By way of example, the category of Chen spaces (cf. [4]) and that of Souriau spaces (cf. [28]) are locally cartesian closed, while that of Frölicher spaces (cf. [6] and [7]) is not.

The principal objective in this paper, as a sequel to [21], is to show that, given a category \mathbf{U} which is complete and (locally, resp.) cartesian closed, the category $\mathcal{K}_{\mathbf{U}}$ of functors on the category of Weil algebras to \mathbf{U} is not only complete but also (locally, resp.) cartesian closed, which will be explained in §4 and §5. A corresponding axiomatization is given in §6.

2 Preliminaries

2.1 Category Theory

Given a category \mathbf{C} and a morphism

$$f : A \rightarrow B$$

in \mathbf{C} , we write

$$A = \text{dom } f$$

$$B = \text{cod } f$$

2.2 Weil Algebras

Let k be a commutative ring. The category of Weil algebras over k (also called Weil k -algebras) is denoted by \mathbf{Weil}_k . It is well known that the category \mathbf{Weil}_k is left exact. The initial and terminal object in \mathbf{Weil}_k is k itself. Given two objects W_1 and W_2 in the category \mathbf{Weil}_k , we denote their tensor algebra by $W_1 \otimes_k W_2$. For a good treatise on Weil algebras, the reader is referred to § 1.16 of [10]. Given a left exact category \mathcal{K} and a k -algebra object \mathbb{R} in \mathcal{K} , there is a canonical functor $\mathbb{R} \otimes_k \cdot$ (denoted by $\mathbb{R} \otimes \cdot$ in [10]) from the category \mathbf{Weil}_k to the category of k -algebra objects and their homomorphisms in \mathcal{K} .

3 The Main Example

Let \mathbf{U} be a complete and cartesian closed category with \mathbb{R} being a k -algebra object in \mathbf{U} . We have in mind a convenient category of smooth spaces as \mathbf{U} .

Notation 1 *We introduce the following notation:*

1. We denote by $\mathcal{K}_{\mathbf{U}}$ the category whose objects are functors from the category \mathbf{Weil}_k to the category \mathbf{U} and whose morphisms are their natural transformations.
2. Given an object W in the category \mathbf{Weil}_k , we denote by

$$\mathbf{T}_{\mathbf{U}}^W : \mathcal{K}_{\mathbf{U}} \rightarrow \mathcal{K}_{\mathbf{U}}$$

the functor obtained as the composition with the functor

$$-\otimes_k W : \mathbf{Weil}_k \rightarrow \mathbf{Weil}_k$$

so that for any object M in the category $\mathcal{K}_{\mathbf{U}}$, we have

$$\mathbf{T}_{\mathbf{U}}^W (M) = M(-\otimes_k W)$$

3. Given a morphism $\varphi : W_1 \rightarrow W_2$ in the category \mathbf{Weil}_k , we denote by

$$\alpha_\varphi^{\mathbf{U}} : \mathbf{T}_{\mathbf{U}}^{W_1} \Rightarrow \mathbf{T}_{\mathbf{U}}^{W_2}$$

the natural transformation such that, given an object W in the category \mathbf{Weil}_k , the morphism

$$\alpha_\varphi^{\mathbf{U}}(M) : \mathbf{T}_{\mathbf{U}}^{W_1}(M) \rightarrow \mathbf{T}_{\mathbf{U}}^{W_2}(M)$$

is

$$M(W \otimes_k \varphi) : M(W \otimes_k W_1) \rightarrow M(W \otimes_k W_2)$$

4. We denote by $\mathbb{R}_{\mathbf{U}}$ the functor

$$\mathbb{R}_{\otimes_k -} : \mathbf{Weil}_k \rightarrow \mathbf{U}$$

4 Cartesian Closedness

Theorem 2 *The category $\mathcal{K}_{\mathbf{U}}$ is cartesian closed.*

Proof. The proof is a modification of Exercise 1.3.7 in [8]. Let M and N be objects in the category $\mathcal{K}_{\mathbf{U}}$. Given an object W in the category \mathbf{Weil}_k , we let $M^N(W)$ denote the intersection of all the equalizers

$$\begin{array}{ccc} \prod_{\text{dom } \varphi = W} M(\text{cod } \varphi)^{N(\text{cod } \varphi)} & \rightarrow & M(\text{cod } \varphi_1)^{N(\text{cod } \varphi_1)} \xrightarrow{M(\varphi_2)^{N(\text{cod } \varphi_1)}} \\ & \rightarrow & M(\text{cod } \varphi_2 \circ \varphi_1)^{N(\text{cod } \varphi_2 \circ \varphi_1)} \xrightarrow{M(\text{cod } \varphi_2 \circ \varphi_1)^{N(\varphi_2)}} \\ M(\text{cod } \varphi_2 \circ \varphi_1)^{N(\text{cod } \varphi_1)} & & \end{array}$$

where φ ranges over all morphisms in the category \mathbf{Weil}_k with $\text{dom } \varphi = W$, φ_1 and φ_2 range over all morphisms in the category \mathbf{Weil}_k with $\text{dom } \varphi_1 = W$ and $\text{cod } \varphi_1 = \text{dom } \varphi_2$, and the morphisms

$$\prod_{\text{dom } \varphi = W} M(\text{cod } \varphi)^{N(\text{cod } \varphi)} \rightarrow M(\text{cod } \varphi_1)^{N(\text{cod } \varphi_1)}$$

and

$$\prod_{\text{dom } \varphi = W} M(\text{cod } \varphi)^{N(\text{cod } \varphi)} \rightarrow M(\text{cod } \varphi_2 \circ \varphi_1)^{N(\text{cod } \varphi_2 \circ \varphi_1)}$$

are the canonical projections. Given a morphism $\psi : W_1 \rightarrow W_2$ in the category \mathbf{Weil}_k , the canonical morphism

$$\begin{array}{ccc} \prod_{\text{dom } \varphi = W_1} M(\text{cod } \varphi)^{N(\text{cod } \varphi)} & \rightarrow & \\ \prod_{\text{dom } \varphi' = W_2} M(\text{cod } \varphi' \circ \psi)^{N(\text{cod } \varphi' \circ \psi)} & = & \prod_{\text{dom } \varphi' = W_2} M(\text{cod } \varphi')^{N(\text{cod } \varphi')} \end{array}$$

naturally gives rise to a morphism $M^N(W_1) \rightarrow M^N(W_2)$ in the category $\mathcal{K}_{\mathbf{U}}$, which we let $M^N(\psi)$. It is easy to see that M^N becomes an object in the category $\mathcal{K}_{\mathbf{U}}$, which works as the exponentiation of M by N within the category $\mathcal{K}_{\mathbf{U}}$. ■

Corollary 3 *Given an object W in the category \mathbf{Weil}_k and objects M and N in the category $\mathcal{K}_{\mathbf{U}}$, we have*

$$\mathbf{T}_{\mathbf{U}}^W(M^N) = \mathbf{T}_{\mathbf{U}}^W(M)^{\mathbf{T}_{\mathbf{U}}^W(N)}$$

Corollary 4 *Given a morphism $\varphi : W_1 \rightarrow W_2$ in the category \mathbf{Weil}_k and objects M and N in the category $\mathcal{K}_{\mathbf{U}}$, the morphism*

$$\alpha_{\varphi}^{\mathbf{U}}(M)^{\mathbf{T}_{\mathbf{U}}^{W_1}(N)} : \mathbf{T}_{\mathbf{U}}^{W_1}(M^N) = \mathbf{T}_{\mathbf{U}}^{W_1}(M)^{\mathbf{T}_{\mathbf{U}}^{W_1}(N)} \rightarrow \mathbf{T}_{\mathbf{U}}^{W_2}(M)^{\mathbf{T}_{\mathbf{U}}^{W_1}(N)}$$

is equal to the morphism

$$\begin{aligned} \mathbf{T}_{\mathbf{U}}^{W_1}(M)^{\mathbf{T}_{\mathbf{U}}^{W_1}(N)} &= \mathbf{T}_{\mathbf{U}}^{W_1}(M^N) \xrightarrow{\alpha_{\varphi}^{\mathbf{U}}(M^N)} \mathbf{T}_{\mathbf{U}}^{W_2}(M^N) = \mathbf{T}_{\mathbf{U}}^{W_2}(M)^{\mathbf{T}_{\mathbf{U}}^{W_2}(N)} \\ &\xrightarrow{\mathbf{T}_{\mathbf{U}}^{W_2}(M)^{\alpha_{\varphi}^{\mathbf{U}}(N)}} \mathbf{T}_{\mathbf{U}}^{W_2}(M)^{\mathbf{T}_{\mathbf{U}}^{W_1}(N)} \end{aligned}$$

5 Locally Cartesian Closedness

In this section we assume that the category \mathbf{U} is locally cartesian closed.

Theorem 5 *The category $\mathcal{K}_{\mathbf{U}}$ is locally cartesian closed.*

Proof. Our present discussion is a localization of the discussion in the proof of Theorem 2 in a sense. Let L be an object in the category $\mathcal{K}_{\mathbf{U}}$. Let $\pi_1 : M \rightarrow L$ and $\pi_2 : N \rightarrow L$ be objects in the slice category $\mathcal{K}_{\mathbf{U}}/L$. Given an object W in the category \mathbf{Weil}_k , we let $(M^N)_L(W)$ denote the intersection of all the equalizers

$$\begin{aligned} &\left(\prod_{\text{dom } \varphi = W} M(\text{cod } \varphi)^{N(\text{cod } \varphi)} \right)_L \\ &\quad \rightarrow \left(M(\text{cod } \varphi_1)^{N(\text{cod } \varphi_1)} \right)_L \xrightarrow{\left(M(\varphi_2)^{N(\text{cod } \varphi_1)} \right)_L} \\ &\quad \rightarrow \left(M(\text{cod } \varphi_2 \circ \varphi_1)^{N(\text{cod } \varphi_2 \circ \varphi_1)} \right)_L \xrightarrow{\left(M(\text{cod } \varphi_2 \circ \varphi_1)^{N(\varphi_2)} \right)_L} \\ &\quad \left(M(\text{cod } \varphi_2 \circ \varphi_1)^{N(\text{cod } \varphi_1)} \right)_L \end{aligned}$$

where φ ranges over all morphisms in the category \mathbf{Weil}_k with $\text{dom } \varphi = W$, φ_1 and φ_2 range over all morphisms in the category \mathbf{Weil}_k with $\text{dom } \varphi_1 = W$ and $\text{cod } \varphi_1 = \text{dom } \varphi_2$, and the morphisms

$$\left(\prod_{\text{dom } \varphi = W} M(\text{cod } \varphi)^{N(\text{cod } \varphi)} \right)_L \rightarrow \left(M(\text{cod } \varphi_1)^{N(\text{cod } \varphi_1)} \right)_L$$

and

$$\left(\prod_{\text{dom } \varphi = W} M(\text{cod } \varphi)^{N(\text{cod } \varphi)} \right)_L \rightarrow \left(M(\text{cod } \varphi_2 \circ \varphi_1)^{N(\text{cod } \varphi_2 \circ \varphi_1)} \right)_L$$

are the canonical projections, and $(-)_L$ denotes the categorical operation within the slice category $\mathcal{K}_{\mathbf{U}}/L$ so that $(M \times N)_L$ denotes the fibered product $M \times_L N$ by way of example. Given a morphism $\psi : W_1 \rightarrow W_2$ in the category \mathbf{Weil}_k , the canonical morphism

$$\begin{aligned} & \left(\prod_{\text{dom } \varphi = W_1} M(\text{cod } \varphi)^{N(\text{cod } \varphi)} \right)_L \rightarrow \\ & \left(\prod_{\text{dom } \varphi' = W_2} M(\text{cod } \varphi' \circ \psi)^{N(\text{cod } \varphi' \circ \psi)} \right)_L = \left(\prod_{\text{dom } \varphi' = W_2} M(\text{cod } \varphi')^{N(\text{cod } \varphi')} \right)_L \end{aligned}$$

naturally gives rise to a morphism $(M^N)_L(W_1) \rightarrow (M^N)_L(W_2)$ in the slice category $\mathcal{K}_{\mathbf{U}}/L$, which we let $(M^N)_L(\psi)$. It is easy to see that $(M^N)_L$ becomes an object in the slice category $\mathcal{K}_{\mathbf{U}}/L$, which works as the exponentiation of M by N within the slice category $\mathcal{K}_{\mathbf{U}}/L$. ■

Corollary 6 *Given an object W in the category \mathbf{Weil}_k , an object L in the category $\mathcal{K}_{\mathbf{U}}$, and objects $\pi_1 : M \rightarrow L$ and $\pi_2 : N \rightarrow L$ in the slice category $\mathcal{K}_{\mathbf{U}}/L$, we have*

$$(\mathbf{T}_{\mathbf{U}})_L^W((M^N)_L) = \left((\mathbf{T}_{\mathbf{U}})_L^W(M)^{(\mathbf{T}_{\mathbf{U}})_L^W(N)} \right)_L$$

where $(\mathbf{T}_{\mathbf{U}})_L^W(M)$ denotes the equalizer of

$$\mathbf{T}_{\mathbf{U}}^W(M) \begin{array}{c} \xrightarrow{\mathbf{T}_{\mathbf{U}}^W(\pi_1)} \\ \xrightarrow{\mathbf{T}_{\mathbf{U}}^W(\pi_1)} \mathbf{T}_{\mathbf{U}}^W(L) \xrightarrow{\alpha_{W \rightarrow k}^{\mathbf{U}}(L)} \mathbf{T}_{\mathbf{U}}^k(L) \xrightarrow{\alpha_{k \rightarrow W}^{\mathbf{U}}(L)} \mathbf{T}_{\mathbf{U}}^W(L) \end{array}$$

with $W \rightarrow k$ and $k \rightarrow W$ being the canonical morphisms in the category \mathbf{Weil}_k . We can naturally extend $(\mathbf{T}_{\mathbf{U}})_L^W$ to a functor

$$\mathcal{K}_{\mathbf{U}}/L \rightarrow \mathcal{K}_{\mathbf{U}}/L$$

in the sense that, given any commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \pi_1 \searrow & & \swarrow \pi_2 \\ & L & \end{array}$$

within the category \mathcal{K} , there exists a unique morphism

$$(\mathbf{T}_{\mathbf{U}})_L^W(f) : (\mathbf{T}_{\mathbf{U}})_L^W(M) \rightarrow (\mathbf{T}_{\mathbf{U}})_L^W(N)$$

making the diagram

$$\begin{array}{ccccc} (\mathbf{T}_{\mathbf{U}})_L^W(M) & \xrightarrow{(\mathbf{T}_{\mathbf{U}})_L^W(f)} & (\mathbf{T}_{\mathbf{U}})_L^W(N) & & \\ \downarrow & & \downarrow & & \\ \mathbf{T}_{\mathbf{U}}^W(M) & \xrightarrow{\overline{\mathbf{T}_{\mathbf{U}}^W(f)}} & \mathbf{T}_{\mathbf{U}}^W(N) & & \end{array}$$

commutative, where the two vertical arrows are the canonical injections.

Corollary 7 Given a morphism $\varphi : W_1 \rightarrow W_2$ in the category \mathbf{Weil}_k , an object L in the category $\mathcal{K}_{\mathbf{U}}$, and objects $\pi_1 : M \rightarrow L$ and $\pi_2 : N \rightarrow L$ in the slice category $\mathcal{K}_{\mathbf{U}}/L$, the morphism

$$\begin{aligned} (\alpha^{\mathbf{U}})_\varphi^L(M)^{(\mathbf{T}_{\mathbf{U}})_L^{W_1(N)}} : (\mathbf{T}_{\mathbf{U}})_L^{W_1}((M^N)_L) &= \left((\mathbf{T}_{\mathbf{U}})_L^{W_1}(M)^{(\mathbf{T}_{\mathbf{U}})_L^{W_1(N)}} \right)_L \\ &\rightarrow \left((\mathbf{T}_{\mathbf{U}})_L^{W_2}(M)^{(\mathbf{T}_{\mathbf{U}})_L^{W_1(N)}} \right)_L \end{aligned}$$

is equal to the morphism

$$\begin{aligned} \left((\mathbf{T}_{\mathbf{U}})_L^{W_1}(M)^{(\mathbf{T}_{\mathbf{U}})_L^{W_1(N)}} \right)_L &= (\mathbf{T}_{\mathbf{U}})_L^{W_1}((M^N)_L) \xrightarrow{(\alpha^{\mathbf{U}})_\varphi^L((M^N)_L)} \\ (\mathbf{T}_{\mathbf{U}})_L^{W_2}((M^N)_L) &= \left((\mathbf{T}_{\mathbf{U}})_L^{W_2}(M)^{(\mathbf{T}_{\mathbf{U}})_L^{W_2(N)}} \right)_L \xrightarrow{(\mathbf{T}_{\mathbf{U}})_L^{W_2}(M)^{(\alpha^{\mathbf{U}})_\varphi^L(N)}} \\ &\left((\mathbf{T}_{\mathbf{U}})_L^{W_2}(M)^{(\mathbf{T}_{\mathbf{U}})_L^{W_1(N)}} \right)_L \end{aligned}$$

where the natural transformation

$$(\alpha^{\mathbf{U}})_\varphi^L : (\mathbf{T}_{\mathbf{U}})_L^{W_1} \Rightarrow (\mathbf{T}_{\mathbf{U}})_L^{W_2}$$

is induced by the natural transformation

$$\alpha_\varphi^{\mathbf{U}} : \mathbf{T}_{\mathbf{U}}^{W_1} \Rightarrow \mathbf{T}_{\mathbf{U}}^{W_2}$$

in the sense of making the diagram

$$\begin{array}{ccccc} (\mathbf{T}_{\mathbf{U}})_L^{W_1}(M) & \xrightarrow{(\alpha^{\mathbf{U}})_\varphi^L(\pi_1)} & \mathbf{T}_{\mathbf{U}}^{W_2}(M) & & \\ \downarrow & & \downarrow & & \\ \mathbf{T}_{\mathbf{U}}^{W_1}(M) & \xrightarrow{\overline{\alpha_\varphi^{\mathbf{U}}(M)}} & \mathbf{T}_{\mathbf{U}}^{W_2}(M) & & \end{array}$$

commutative.

6 The Axiomatics

Definition 8 A *DG-category* (DG stands for Differential Geometry) is a quadruple $(\mathcal{K}, \mathbf{T}, \alpha, \mathbb{R})$, where

1. \mathcal{K} is a category which is complete and cartesian closed.
2. Given an object W in the category \mathbf{Weil}_k , $\mathbf{T}^W : \mathcal{K} \rightarrow \mathcal{K}$ is a functor subject to the conditions:

- \mathbf{T}^W preserves limits.
- $\mathbf{T}^k : \mathcal{K} \rightarrow \mathcal{K}$ is the identity functor.
- We have

$$\mathbf{T}^{W_2} \circ \mathbf{T}^{W_1} = \mathbf{T}^{W_1 \otimes_k W_2}$$

for any objects W_1 and W_2 in the category \mathbf{Weil}_k .

- We have

$$\mathbf{T}^W (M^N) = \mathbf{T}^W (M)^{\mathbf{T}^W(N)}$$

for any objects M and N in the category \mathcal{K} .

3. Given a morphism $\varphi : W_1 \rightarrow W_2$ in the category \mathbf{Weil}_k , $\alpha_\varphi : \mathbf{T}^{W_1} \Rightarrow \mathbf{T}^{W_2}$ is a natural transformation subject to the conditions:

- We have

$$\alpha_{\text{id}_W} = \text{id}_{\mathbf{T}^W}$$

for any identity morphism $\text{id}_W : W \rightarrow W$ in the category \mathbf{Weil}_k .

- We have

$$\alpha_\psi \cdot \alpha_\varphi = \alpha_{\psi \circ \varphi}$$

for any morphisms $\varphi : W_1 \rightarrow W_2$ and $\psi : W_2 \rightarrow W_3$ in the category \mathbf{Weil}_k .

- Given objects M and N in the category \mathcal{K} , the morphism

$$\alpha_\varphi (M)^{\mathbf{T}^{W_1(N)}} : \mathbf{T}^{W_1} (M^N) = \mathbf{T}^{W_1} (M)^{\mathbf{T}^{W_1(N)}} \rightarrow \mathbf{T}^{W_2} (M)^{\mathbf{T}^{W_1(N)}}$$

is equal to the morphism

$$\begin{aligned} \mathbf{T}^{W_1} (M)^{\mathbf{T}^{W_1(N)}} &= \mathbf{T}^{W_1} (M^N) \xrightarrow{\alpha_\varphi (M^N)} \mathbf{T}^{W_2} (M^N) = \mathbf{T}^{W_2} (M)^{\mathbf{T}^{W_2(N)}} \\ &\xrightarrow{\mathbf{T}^{W_2} (M)^{\alpha_\varphi(N)}} \mathbf{T}^{W_2} (M)^{\mathbf{T}^{W_1(N)}} \end{aligned}$$

within the category \mathcal{K} .

4. Given an object W in the category \mathbf{Weil}_k , we have

$$\mathbf{T}^W (\mathbb{R}) = \mathbb{R}_{\otimes_k} W$$

5. Given a morphism $\varphi : W_1 \rightarrow W_2$ in the category \mathbf{Weil}_k , we have

$$\alpha_\varphi (\mathbb{R}) = \mathbb{R}_{\otimes_k} \varphi$$

Notation 9 Given an object W in the category \mathbf{Weil}_k , an object L in the category \mathcal{K} , and an object $\pi : M \rightarrow L$ in the slice category \mathcal{K}/L , we denote by $\mathbf{T}_L^W(M)$ the equalizer of

$$\mathbf{T}^W(M) \begin{array}{c} \xrightarrow{\mathbf{T}^W(\pi)} \\ \xrightarrow{\mathbf{T}^W(\pi)} \mathbf{T}^W(L) \xrightarrow{\alpha_{W \rightarrow k}(L)} \mathbf{T}^k(L) \xrightarrow{\alpha_{k \rightarrow W}(L)} \mathbf{T}^W(L) \end{array}$$

with $W \rightarrow k$ and $k \rightarrow W$ being the canonical morphisms within the category \mathbf{Weil}_k . We can naturally extend \mathbf{T}_L^W to a functor

$$\mathcal{K}/L \rightarrow \mathcal{K}/L$$

in the sense that, given any commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \pi_1 \searrow & & \swarrow \pi_2 \\ & L & \end{array}$$

within the category \mathcal{K} , there exists a unique morphism

$$\mathbf{T}_L^W(f) : \mathbf{T}_L^W(M) \rightarrow \mathbf{T}_L^W(N)$$

making the diagram

$$\begin{array}{ccc} \mathbf{T}_L^W(M) & \xrightarrow{\mathbf{T}_L^W(f)} & \mathbf{T}_L^W(N) \\ \downarrow & & \downarrow \\ \mathbf{T}^W(M) & \xrightarrow{\mathbf{T}^W(f)} & \mathbf{T}^W(N) \end{array}$$

commutative, where the two vertical arrows are the canonical injections.

Notation 10 Given a morphism $\varphi : W_1 \rightarrow W_2$ in the category \mathbf{Weil}_k , an object L in the category \mathcal{K} , and an object $\pi : M \rightarrow L$ within the slice category \mathcal{K}/L , we denote by α_φ^L the natural transformation

$$\mathbf{T}_L^{W_1} \Rightarrow \mathbf{T}_L^{W_2}$$

making the diagram

$$\begin{array}{ccc} \mathbf{T}_L^{W_1}(M) & \xrightarrow{\alpha_\varphi^L(M)} & \mathbf{T}_L^{W_2}(M) \\ \downarrow & & \downarrow \\ \mathbf{T}^{W_1}(M) & \xrightarrow{\alpha_\varphi(M)} & \mathbf{T}^{W_2}(M) \end{array}$$

commutative for any object W in the category \mathbf{Weil}_k , where $\mathbf{T}_L^{W_1}(M) \downarrow \mathbf{T}^{W_1}(M)$ and

$\mathbf{T}_L^{W_2}(M) \downarrow \mathbf{T}^{W_2}(M)$ are the canonical injections.

Definition 11 A local DG-category is a DG-category $(\mathcal{K}, \mathbf{T}, \alpha, \mathbb{R})$ subject to the conditions:

1. The category \mathcal{K} is not only cartesian closed but, what is even more, locally cartesian closed.
2. Given an object W in the category \mathbf{Weil}_k , an object L in the category \mathcal{K} , and objects $\pi_1 : M \rightarrow L$ and $\pi_2 : N \rightarrow L$ in the slice category \mathcal{K}/L , we have

$$\mathbf{T}_L^W ((M^N)_L) = \left(\mathbf{T}_L^W (M)^{\mathbf{T}_L^W (N)} \right)_L$$

within the category \mathcal{K}/L .

3. Given an object W in the category \mathbf{Weil}_k , an object L in the category \mathcal{K} , and objects $\pi_1 : M \rightarrow L$ and $\pi_2 : N \rightarrow L$ in the slice category \mathcal{K}/L , the morphism

$$\alpha_\varphi^L (M)^{\mathbf{T}_L^{W_1} (N)} : \mathbf{T}_L^{W_1} ((M^N)_L) = \left(\mathbf{T}_L^{W_1} (M)^{\mathbf{T}_L^{W_1} (N)} \right)_L \rightarrow \left(\mathbf{T}_L^{W_2} (M)^{\mathbf{T}_L^{W_1} (N)} \right)_L$$

is equal to the morphism

$$\begin{aligned} \left(\mathbf{T}_L^{W_1} (M)^{\mathbf{T}_L^{W_1} (N)} \right)_L &= \mathbf{T}_L^{W_1} ((M^N)_L) \xrightarrow{\alpha_\varphi^L ((M^N)_L)} \\ \mathbf{T}_L^{W_2} ((M^N)_L) &= \left(\mathbf{T}_L^{W_2} (M)^{\mathbf{T}_L^{W_2} (N)} \right)_L \xrightarrow{\mathbf{T}_L^{W_2} (M)^{\alpha_\varphi^L (N)}} \left(\mathbf{T}_L^{W_2} (M)^{\mathbf{T}_L^{W_1} (N)} \right)_L \end{aligned}$$

within the category \mathcal{K}/L .

Proposition 12 Given a local DG-category $(\mathcal{K}, \mathbf{T}, \alpha, \mathbb{R})$ and an object L in the category \mathcal{K} , the quadruple

$$\left(\mathcal{K}/L, \mathbf{T}_L, \alpha^L, \begin{array}{c} L \times \mathbb{R} \\ \downarrow \\ L \end{array} \right),$$

which may be considered to be the localization of the DG-category $(\mathcal{K}, \mathbf{T}, \alpha, \mathbb{R})$

with respect to L in a sense, is a local DG-category, where $\begin{array}{c} L \times \mathbb{R} \\ \downarrow \\ L \end{array}$ is the canonical projection.

Proof. Given an object $\begin{array}{c} M \\ \downarrow \\ L \end{array}$ in the slice category \mathcal{K}/L , we note the following:

1. We can naturally identify the slice category

$$(\mathcal{K}/L) / \left(\begin{array}{c} M \\ \downarrow \\ L \end{array} \right)$$

with the slice category

$$\mathcal{K}/M$$

for which the reader is referred, say, to Page 8 of [9].

2. We have

$$(\mathbf{T}_L)_{M \rightarrow L} = \mathbf{T}_M,$$

since the diagram

$$\begin{array}{ccccc} \mathbf{T}^W(M) & \xrightarrow{\alpha_{W \rightarrow k}(M)} & \mathbf{T}^k(M) & \xrightarrow{\alpha_{k \rightarrow W}(M)} & \mathbf{T}^W(M) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{T}^W(L) & \xrightarrow{\alpha_{W \rightarrow k}(L)} & \mathbf{T}^k(L) & \xrightarrow{\alpha_{k \rightarrow W}(L)} & \mathbf{T}^W(L) \end{array}$$

is commutative.

3. It is easy to see that

$$(\alpha^L)^{M \rightarrow L} = \alpha^M$$

4. We have

$$\begin{aligned} M \times_L (L \times \mathbb{R}) \\ = M \times \mathbb{R} \end{aligned}$$

Therefore the localization

$$\left((\mathcal{K}/L) / \left(\begin{array}{c} M \\ \downarrow \\ L \end{array} \right), (\mathbf{T}_L)_{M \rightarrow L}, (\alpha^L)^{M \rightarrow L}, \begin{array}{ccc} M \times_L (L \times \mathbb{R}) & \rightarrow & L \times \mathbb{R} \\ \downarrow & & \downarrow \\ M & \rightarrow & L \end{array} \right)$$

of the DG-category

$$\left(\mathcal{K}/L, \mathbf{T}_L, \alpha^L, \begin{array}{c} L \times \mathbb{R} \\ \downarrow \\ L \end{array} \right)$$

with respect to

$$\begin{array}{c} M \\ \downarrow \\ L \end{array}$$

is no other than the localization

$$\left(\mathcal{K}/M, \mathbf{T}_M, \alpha^M, \begin{array}{c} M \times \mathbb{R} \\ \downarrow \\ M \end{array} \right)$$

of the DG-category

$$(\mathcal{K}, \mathbf{T}, \alpha, \mathbb{R})$$

with respect to M , so that the desired conclusion follows readily. ■

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