Axiomatic Differential Geometry ?-1

Axiomatic Differential Geometry III-1 -Its Landscape- Chapter 1: Model Theory I

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Abstract

In this paper is proposed a kind of model theory for our axiomatic differential geometry. It is claimed that smooth manifolds, which have occupied the center stage in differential geometry, should be replaced by functors on the category of Weil algebras. Our model theory is geometrically natural and conceptually motivated, while the model theory of synthetic differential geometry is highly artificial and exquisitely technical.

1 Introduction

Consider two smooth curves sharing a unique point, say the origin, in the 2-dimensional Euclidean space. This is a familiar situation in high school mathematics. From the authentic viewpoint of contemporary mathematics, whether the two curves are transversal at the common point or they are tangential therein, their intersection is the shared point, the same figure consisting of a unique point. Now the story is over to most of the schoolboys and schoolgirls. However, a good mathematician endowed with a cornucopia of geometric acumen and academically sophisticated instinct feels a flavor of impropriety here. What is wrong ?

To resolve the above paradoxical situation, *synthetic differential geometers* resort to the resurrection of *nilpotent infinitesimals*, which were abandoned as anathema and replaced by so-called $\varepsilon - \delta$ arguments in the 19th century. They insist that, were one able to recognize nilpotent infinitesimals by looking closer and closer and using highly sensitive microscopes if necessay, he or she would find distinct intersections at the above two contradistinctive cases. The intersection of the transversal curves is really one point, while that of the tangential curves would be that point accompanied by its microcosm of numerous lurking points infinitesimally close to it. In any case, synthetic differential geometers were forced to invent an artifact, called *well-adapted models*, in which they are generously able to indulge in their favorite nilpotent infinitesimals. For synthetic differential geometry, the reader is referred to [8] and [11].

Our solution to the above paradox is more realistic and highly geometrical. We plumb impropriety in the definition of a *figure*. It is nothing but platitudinous to say that *geometry is the study of figures*, but the notion of a figure in mathematics has changed dramatically several times since the days of ancient Greeks such as Euclid and Pythagoras. We ask again what a figure is, as Riemann did at his trial lecture for his habilitation (entitled "The hypotheses on which geometry is based"). We insist that the definition of a figure should be hierarchical. The figure is to be depicted not only at its 0-th order level corresponding to the Weil algebra $\mathbf{R}$ (the degenerate Weil algebra of real numbers) but at various levels corresponding to various Weil algebras. Consider the two curves tangential at the unique common point as above, whose intersection at the 0-th order level is surely one point. The depiction of each curve at the 1-st order level corresponding to the Weil algebra $\mathbf{R}[x]/\left(x^2\right)$ is its tangent bundle, and the intersection of the two curves at the 1-st order level is the same tangent space of both curves to the tangential point. We note in passing that if the intersection of two figures should always be a figure, this example forces us to admit a figure whose 0-th order description is one point but whose 1-st order level description is a one-dimensional linear space over $\mathbf{R}$. We note also that the depiction of a figure at the 0-th order level does not determine its depiction at the 1-st order level uniquely. Formally speaking from a coign of vantage of category theory, we propose that a *figure* is a functor on the category of Weil algebras.

The principal objective in this paper is to give a model theory to the axiomatics in [15] by exploiting the above notion of a figure. The model theory is explained in §4. We review our axiomatics of [15] in §3. We give some preliminaries on Weil algebras and convenient categories in §2.

2 Preliminaries

2.1 Weil Algebras

Let k be a commutative ring. The category of Weil algebras over k (also called Weil k-algebras) is denoted by $\mathbf{Weil}_k$. It is well known that the category $\mathbf{Weil}_k$ is left exact. The initial and terminal object in $\mathbf{Weil}_k$ is k itself. Given two objects W_1 and W_2 in the category $\mathbf{Weil}_k$, we denote their tensor algebra by $W_1 \otimes_k W_2$. For a good treatise on Weil algebras, the reader is referred to § 1.16 of [8]. Given a left exact category $\mathcal{K}$ and a k-algebra object $\mathbb{R}$ in $\mathcal{K}$, there is a canonical functor $\mathbb{R}\underline{\otimes}_k\cdot$ (denoted by $\mathbb{R}\otimes\cdot$ in [8]) from the category $\mathbf{Weil}_k$ to the category of k-algebra objects and their homomorphisms in $\mathcal{K}$.

2.2 Convenient Categories

The category of topological spaces and continuous mappins is by no means cartesian closed. In 1967 Steenrod [28] popularized the idea of *convenient category* by announcing that the category of compactly generated spaces and continuous mappings renders a good setting for algebraic topology. The proposed category is cartesian closed, complete and cocomplete, and contains all CW complexes.

About the same time, an attempt to give a convenient category to smooth spaces began, and we have a few candidates. For a thorough study on the relationship among these proposed candidates, the reader is referred to [27], in which the reader finds by way of example that the category of Frölicher spaces is a full subcategory of that of Souriau spaces, and the category of Souriau spaces is in turn a full subcategory of Chen spaces. We have no intention to discuss which is the best convenient category of smooth spaces here. We content ourselves with denoting some of such categories by **Smooth**, which is required to be complete and cartesian closed at least containing the category **Mf** of smooth manifolds as a full subcategory.

3 The Axiomatics

We review the axiomatics in [15].

Definition 1 *A* <u>*DG-category*</u> *(DG stands for Differential Geometry) is a quadruple* $(\mathcal{K}, \mathbb{R}, \mathbf{T}, \alpha)$*, where*

1. $\mathcal{K}$ *is a category which is left exact and cartesian closed.*
2. $\mathbb{R}$ *is a commutative* k*-algebra object in* $\mathcal{K}$*.*
3. *Given a Weil* k*-algebra* W*,* $\mathbf{T}^W : \mathcal{K} \to \mathcal{K}$ *is a left exact functor for any Weil* k*-algebra* W *subject to the condition that* $\mathbf{T}^k : \mathcal{K} \to \mathcal{K}$ *is the identity functor, while we have*
$$\mathbf{T}^{W_2} \circ \mathbf{T}^{W_1} = \mathbf{T}^{W_1 \otimes_k W_2}$$
for any Weil k*-algebras* W_1 *and* W_2*.*
4. *Given a Weil* k*-algebra* W*, we have*
$$\mathbf{T}^W \mathbb{R} = \mathbb{R} \underline{\otimes}_k W$$
5. $\alpha_\varphi : \mathbf{T}^{W_1} \dot{\to} \mathbf{T}^{W_2}$ *is a natural transformation for any morphism* $\varphi : W_1 \to W_2$ *in the category* $\mathbf{Weil}_k$ *such that we have*
$$\alpha_\psi \cdot \alpha_\varphi = \alpha_{\psi \circ \varphi}$$
for any morphisms $\varphi : W_1 \to W_2$ *and* $\psi : W_2 \to W_3$ *in the category* $\mathbf{Weil}_k$*, while we have*
$$\alpha_{\mathrm{id}_W} = \mathrm{id}_{\mathbf{T}^W}$$
for any identity morphism $\mathrm{id}_W : W \to W$ *in the category* $\mathbf{Weil}_k$*.*

6. *Given a morphism* $\varphi : W_1 \to W_2$ *in the category* $\mathbf{Weil}_k$*, we have*

$$\alpha_\varphi (\mathbb{R}) = \mathbb{R} \underline{\otimes}_k \varphi$$

4 Model Theory

Let $\mathbf{U}$ be a complete and cartesian closed category with $\mathbb{R}$ being a k-algebra object in $\mathbf{U}$.

Notation 2 *We denote by* $\mathcal{K}_{\mathbf{U}}$ *the category whose objects are functors from the category* $\mathbf{Weil}_k$ *to the category* $\mathbf{U}$ *and whose morphisms are their natural transformations.*

It is easy to see that

Proposition 3 *The category* $\mathcal{K}_{\mathbf{U}}$ *is complete and cartesian closed.*

Proof. The proof is tremendously similar to that of the familiar fact that the arrow category of a complete and cartesian closed category is complete and cartesian closed. That the category $\mathcal{K}_{\mathbf{U}}$ is complete follows from Theorem 7.5.2 in [25]. The cartesian closedness of $\mathcal{K}_{\mathbf{U}}$ is discussed in a subsequent paper, but the reader is referred to Exercise 1.3.7 in [7] for the cartesian closedness of the arrow category of a complete and cartesian closed category. ■

Notation 4 *Given an object* W *in the category* $\mathbf{Weil}_k$*, we denote by*

$$\mathbf{T}_{\mathbf{U}}^{W} : \mathcal{K}_{\mathbf{U}} \to \mathcal{K}_{\mathbf{U}}$$

the functor obtained as the composition with the functor

$$\_ \otimes_k W : \mathbf{Weil}_k \to \mathbf{Weil}_k$$

so that for any object M *in the category* $\mathcal{K}_{\mathbf{U}}$*, we have*

$$\mathbf{T}_{\mathbf{U}}^{W} (M) = M (\_ \otimes_k W)$$

It is easy to see that

Proposition 5 *We have*

$$\mathbf{T}_{\mathbf{U}}^{W_2} \circ \mathbf{T}_{\mathbf{U}}^{W_1} = \mathbf{T}_{\mathbf{U}}^{W_1 \otimes W_2}$$

for any objects W_1, W_2 *in the category* $\mathbf{Weil}_k$*.*

Proof. We have

$$\begin{aligned} \mathbf{T}_{\mathbf{U}}^{W_2} \circ \mathbf{T}_{\mathbf{U}}^{W_1} &= (\_ \otimes_k W_1) \otimes_k W_2 \\ &= \_ \otimes_k (W_1 \otimes W_2) \\ &= \mathbf{T}_{\mathbf{U}}^{W_1 \otimes W_2} \end{aligned}$$

■

It is also easy to see that

Proposition 6 *The functor*

$$\mathbf{T}_{\mathbf{U}}^{W}:\mathcal{K}_{\mathbf{U}}\rightarrow\mathcal{K}_{\mathbf{U}}$$

preserves limits for each object W in the category $\mathbf{Weil}_k$.

Proof. This follows easily from 7.5.2 and 7.5.3 in [25]. ■

Proposition 7 *Given a morphism $\varphi:W_1\rightarrow W_2$ in the category $\mathbf{Weil}_k$ and an object M in the category $\mathcal{K}_{\mathbf{U}}$, the assignment of the morphism*

$$M\left(W\otimes_k\varphi\right):M\left(W\otimes_k W_1\right)\rightarrow M\left(W\otimes_k W_2\right)$$

in the category $\mathbf{U}$ to each object W in the category $\mathbf{Weil}_k$ is a morphism

$$\mathbf{T}_{\mathbf{U}}^{W_1}\left(M\right)\rightarrow\mathbf{T}_{\mathbf{U}}^{W_2}\left(M\right)$$

in the category $\mathcal{K}_{\mathbf{U}}$, which we denote by $\alpha_{\varphi}^{\mathbf{U}}\left(M\right)$.

Proof. Given a morphism $\psi:W\rightarrow W'$ in the category $\mathbf{Weil}_k$, the diagram

$$\begin{array}{ccc}
M\left(W\otimes_k W_1\right) & \underrightarrow{M\left(W\otimes_k\varphi\right)} & M\left(W\otimes_k W_2\right)\\
M\left(\psi\otimes_k W_1\right)\downarrow & & \downarrow M\left(\psi\otimes_k W_2\right)\\
M\left(W'\otimes_k W_1\right) & \overrightarrow{M\left(W'\otimes_k\varphi\right)} & M\left(W'\otimes_k W_2\right)
\end{array}$$

is commutative, so that the desired conclusion follows. ■

Proposition 8 *Given a morphism $\varphi:W_1\rightarrow W_2$ in the category $\mathbf{Weil}_k$, the assignment of the morphism*

$$\alpha_{\varphi}^{\mathbf{U}}\left(M\right):\mathbf{T}_{\mathbf{U}}^{W_1}\left(M\right)\rightarrow\mathbf{T}_{\mathbf{U}}^{W_2}\left(M\right)$$

in the category $\mathcal{K}_{\mathbf{U}}$ to each object W in the category $\mathbf{Weil}_k$ is a natural transformation

$$\mathbf{T}_{\mathbf{U}}^{W_1}\Rightarrow\mathbf{T}_{\mathbf{U}}^{W_2}$$

which we denote by $\alpha_{\varphi}^{\mathbf{U}}$.

Proof. Given a morphism $f:M_1\rightarrow M_2$ in the category $\mathcal{K}_{\mathbf{U}}$, the diagram

$$\begin{array}{ccc}
M_1\left(W\otimes_k W_1\right) & \underrightarrow{M_1\left(W\otimes_k\varphi\right)} & M_1\left(W\otimes_k W_2\right)\\
f_{W\otimes_k W_1}\downarrow & & \downarrow f_{W\otimes_k W_2}\\
M_2\left(W\otimes_k W_1\right) & \overrightarrow{M_2\left(W\otimes_k\varphi\right)} & M_2\left(W\otimes_k W_2\right)
\end{array}$$

is commutative, so that the desired conclusion follows. ■

It is easy to see that

Proposition 9 *We have*

$$\alpha_{\psi}^{\mathbf{U}}\circ\alpha_{\varphi}^{\mathbf{U}}=\alpha_{\psi\circ\varphi}^{\mathbf{U}}$$

for any morhisms $\varphi:W_1\rightarrow W_2$ and $\psi:W_2\rightarrow W_3$ in the category $\mathbf{Weil}_k$.

Proof. Given an object M in the category $\mathcal{K}_{\mathbf{U}}$, we have

$$M\left(\_\otimes_k \psi\right) \circ M\left(\_\otimes_k \varphi\right) = M\left(\_\otimes_k \left(\psi \circ \varphi\right)\right)$$

so that the desired conclusion follows. ■

Notation 10 *We denote by* $\mathbb{R}_{\mathbf{U}}$ *the functor*

$$\mathbb{R}\underline{\otimes}_k\_ : \mathbf{Weil}_k \to \mathbf{U}$$

It is easy to see that

Proposition 11 *We have*

$$\mathbf{T}_{\mathbf{U}}^{W}\left(\mathbb{R}_{\mathbf{U}}\right) = \mathbb{R}_{\mathbf{U}}\underline{\otimes}_k W$$

for any object W *in the category* $\mathbf{Weil}_k$*.*

It is also easy to see that

Proposition 12 *We have*

$$\alpha_{\varphi}^{\mathbf{U}}\left(\mathbb{R}_{\mathbf{U}}\right) = \mathbb{R}_{\mathbf{U}}\underline{\otimes}_k \varphi$$

for any morphism $\varphi : W_1 \to W_2$ *in the category* $\mathbf{Weil}_k$*.*

Now we recapitulate as follows.

Theorem 13 *The quadruple*

$$\left(\mathcal{K}_{\mathbf{U}}, \mathbb{R}_{\mathbf{U}}, \mathbf{T}_{\mathbf{U}}, \alpha^{\mathbf{U}}\right)$$

is a DG-category.

Example 14 *Let* $\mathbf{U} = \mathbf{Smooth}$ *with* $k = \mathbf{R}$ *and* $\mathbb{R} = \mathbf{R}$*. We denote by* $\mathbf{Mf}$ *the category of smooth manifolds, which can be regarded as a subcategory of* $\mathbf{Smooth}$*. It is well known (cf. Theorem 31.7 in [10]) that there is a bifunctor*

$$T : \mathbf{Weil}_{\mathbf{R}} \times \mathbf{Mf} \to \mathbf{Mf}$$

Therefore each smooth manifold M *can be regarded as the functor*

$$T(\_, M) : \mathbf{Weil}_{\mathbf{R}} \to \mathbf{Smooth}$$

which is an object in $\mathcal{K}_{\mathbf{Smooth}}$*. This gives rise to a functor from the category* $\mathbf{Mf}$ *to the category* $\mathcal{K}_{\mathbf{Smooth}}$*.*

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