A $tt^*$-bundle associated with a harmonic map from a Riemann surface into a sphere

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Abstract

A $tt^*$-bundle is constructed by a harmonic map from a Riemann surface into an $n$-dimensional sphere. This $tt^*$-bundle is a high-dimensional analogue of a quaternionic line bundle with a Willmore connection. For the construction, a flat connection is decomposed into four parts by a fiberwise complex structure.

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1. Introduction

A $tt^*$-bundle is a real vector bundle equipped with a family of flat connections, parametrized by a circle. The present paper delivers a $tt^*$-bundle derived from a harmonic map from a Riemann surface to an $n$-dimensional sphere.

The notion of $tt^*$-bundles is introduced by Schäfer [10] as a simple solution to a generalized version of the equation of topological-antitopological fusion, introduced by Cecotti and Vafa [2], in terms of real differential geometry. A topological-antitopological fusion of a topological field theory model is a special geometry structure on a Frobenius manifold. As a geometric interpretation of a special geometry structure on a quasi-Frobenius manifold, Dubrovin [6] showed that a solution to the equation is locally a pluriharmonic map from an $n$-dimensional quasi-Frobenius manifold to the symmetric space $\text{GL}(n, \mathbb{R})/\text{O}(n)$.
Schäfer [10] showed that an admissible pluriharmonic map from a simply connected complex manifold \( M \) to a symmetric space \( \text{GL}(r, \mathbb{R})/\text{O}(p, q) \), and that to \( \text{SL}(r, \mathbb{R})/\text{SO}(p, q) \) with \( p + q = r \), gives rise from a metric \( tt^* \)-bundle. A harmonic map from a Riemann surface to \( \text{SU}(1, 1)/\text{SU}(1) \times \text{U}(1) \cong \text{SL}(2, \mathbb{R})/\text{SO}(2) \) is obtained by the generalized Weierstrass representation formula by Dorfmeister, Pedit, and Wu [5]. The Gauss map of a spacelike surface of constant mean curvature in the Minkowski space \( \mathbb{R}^{2,1} \) is a harmonic map from a Riemann surface to \( \text{SL}(2, \mathbb{R})/\text{SO}(2) \). The Sym-Bobenko formula (Bobenko [1], Dorfmeister and Haak [4]) connects a surface and its Gauss map. Applying these formulae, Dorfmeister, Guest, and Rossman [3] gave the description of the quantum cohomology of \( \mathbb{C}P^1 \). The quantum cohomology of \( \mathbb{C}P^1 \) provides a solution to the third Painlevé equation.

A surface of constant mean curvature in \( \mathbb{R}^3 \) is an interesting research subject in the theory of surfaces. Its Gauss map is a harmonic map from a Riemann surface to the two-dimensional sphere \( S^2 \). It is impossible to write \( S^2 \) as a symmetric space \( \text{GL}(r, \mathbb{R})/\text{O}(p, q) \) or \( \text{SL}(r, \mathbb{R})/\text{SO}(p, q) \). This led the authors to find a \( tt^* \)-bundle for a harmonic map into \( S^2 \). The theory of a quaternionic line bundle with a Willmore connection by Ferus, Leschke, Pedit, and Pinkall [8] provides a way to construct a \( tt^* \)-bundle for a harmonic map from a Riemann surface into \( S^2 \). This method is extended and a \( tt^* \)-bundle associated with a harmonic map from a Riemann surface into \( S^n \) \((n \geq 2)\) is obtained (Theorem 4.1).

2. \( tt^* \)-bundles

We recall a \( tt^* \)-bundle (Schäfer [10]).

Let \( M \) be a complex manifold with complex structure \( J^M \). For a one-form \( \omega \) on \( M \), we define a one-form \( \ast \omega \) on \( M \) by \( \ast \omega := \omega \circ J^M \). Let \( E \) be a trivial real vector bundle of rank \( n \) over \( M \), \( \nabla \) a connection on \( E \), and \( S \) a one-form with values in the real endomorphisms of \( E \). A one-form \( S \) is considered as a one-form with values in \( n \)-by-\( n \) real matrices. Define a family of connections \( \{\nabla^\theta\}_{\theta \in \mathbb{R}} \) on \( E \) by

\[
\nabla^\theta := \nabla + \cos(\theta) S + \sin(\theta) \ast S.
\]
The curvature of $\nabla^\theta$ is
\[
d^\nabla^\theta \circ \nabla^\theta = d^\nabla \circ \nabla + (\cos \theta) d^\nabla S + (\sin \theta) d^\nabla *S \\
+ ((\cos \theta) S + (\sin \theta) *S) \wedge ((\cos \theta) S + (\sin \theta) *S) \\
= d^\nabla \circ \nabla + (\cos \theta) d^\nabla S + (\sin \theta) d^\nabla *S \\
+ (\cos \theta)^2 S \wedge S + \cos \theta \sin \theta (S \wedge *S + *S \wedge S) + (\sin \theta)^2 *S \wedge *S \\
= d^\nabla \circ \nabla + (\cos \theta) d^\nabla S + (\sin \theta) d^\nabla *S \\
+ \frac{1 + \cos 2\theta}{2} S \wedge S + \frac{\sin 2\theta}{2} (S \wedge *S + *S \wedge S) + \frac{1 - \cos 2\theta}{2} *S \wedge *S \\
= d^\nabla \circ \nabla + \frac{1}{2} S \wedge S + \frac{1}{2} *S \wedge *S \\
+ (\cos \theta) d^\nabla S + (\sin \theta) d^\nabla *S \\
+ \frac{\cos 2\theta}{2} (S \wedge *S + *S \wedge S) + \frac{\sin 2\theta}{2} (S \wedge *S + *S \wedge S).
\]

A vector bundle $E$ with $\nabla$ and $S$ is called a $tt^*$-bundle if $\nabla^\theta$ is flat for all $\theta \in \mathbb{R}$. By the preceding calculation, a vector bundle $E$ with $\nabla$ and $S$ is a $tt^*$-bundle, if and only if
\[
d^\nabla \circ \nabla + S \wedge S = 0, \quad d^\nabla S = 0, \quad d^\nabla *S = 0, \\
S \wedge S = *S \wedge *S, \quad S \wedge *S = - *S \wedge S.
\]

Indeed,
\[
(S \wedge *S + *S \wedge S)(X, Y) \\
= S(X)S(Y) - S(Y)S(X) - S(J^MX)S(J^MY) + S(J^MY)S(J^MX) \\
= -S(X)S(J^M,J^MY) + S(J^M,J^MY)S(X) \\
- S(J^MX)S(J^MY) + S(J^MY)S(J^MX) \\
= -S(X)S(J^M,J^MY) + S(J^MY)S(J^MX) \\
+ S(J^M,J^MY)S(X) - S(J^MX)S(J^MY) \\
= -(S \wedge *S + *S \wedge S)(X, J^MY)
\]

for any tangent vectors $X, Y$ of $M$. Hence, $S \wedge S = *S \wedge *S$ is equivalent to $S \wedge *S = - *S \wedge S$. Then, a vector bundle $E$ with $\nabla$ and $S$ is a $tt^*$-bundle, if and only if
\[
d^\nabla \circ \nabla + S \wedge S = 0, \quad d^\nabla S = 0, \quad d^\nabla *S = 0, \quad S \wedge S = *S \wedge *S.
(see Schäfer [10], Proposition 1).

Assume that $E$ with $\nabla$ and $S$ forms a $tt^*$-bundle. Define $F$ as the complexification of $E$, that is, $F := \mathbb{C} \otimes E$. Denote the complex-linear extensions of $\nabla$ and $S$ by the same notations respectively. Define a family of connections \[\{\nabla^\mu\}_{\mu \in \mathbb{C} \setminus \{0\}}\] of $F$ by

\[
\nabla^\mu = \nabla + \frac{1}{\mu} C + \mu \bar{C}, \quad C = \frac{1}{2} (S - i * S). \tag{1}
\]

Then $C$ is a $(1,0)$-form on $M$ with values in complex linear endmorphisms of $F$. The $tt^*$-bundle $E$ with $\nabla$ and $S$ is the real part of $F$ with $\nabla^\mu$ if and only if $|\mu| = 1$.

**Proposition 2.1.** For each $\mu \in \mathbb{C} \setminus \{0\}$, the connection $\nabla^\mu$ is flat.

**Proof.** As $E$ with $\nabla$ and $S$ is a $tt^*$-bundle, it follows that

\[
d^C = 0, \quad d\bar{C} = 0,
\]

\[
C \wedge C = \frac{1}{4} (S \wedge S - iS \wedge *S - i * S \wedge S - *S \wedge *S) = 0,
\]

\[
C \wedge \bar{C} = \frac{1}{4} (S \wedge S + iS \wedge *S - i * S \wedge S + *S \wedge *S) = \frac{1}{2} (S \wedge S + iS \wedge *S).
\]

Then

\[
d^{\nabla^\mu} \circ \nabla^\mu = d^C \circ \nabla + \left( \frac{1}{\mu} C + \mu \bar{C} \right) \wedge \left( \frac{1}{\mu} C + \mu \bar{C} \right)
\]

\[= d^C \circ \nabla + C \wedge \bar{C} + \bar{C} \wedge C
\]

\[= d^C \circ \nabla + S \wedge S = 0.
\]

Hence $\nabla^\mu$ is flat. \hfill $\Box$

Adding the assumption in Proposition 2.1, we assume that there exists a hermitian pseudo-metric $h$ on $F$, and a metric connection $\nabla$ with respect to $h$, such that

\[h(C(X)a,b) = h(a, \bar{C}(X)b),\]

where $a, b \in \Gamma(F)$, and $X$ is a vector field of type $(1,0)$ on $M$. Then $(F, \nabla, C, \bar{C}, h)$ becomes a harmonic bundle defined in Schäfer [11].
3. Decomposition of a connection

We obtain a condition for a map from a Riemann surface into a sphere, to become a harmonic map, by decomposing a flat connection into four parts.

Let $\text{Cl}_n$ be the Clifford algebra associated with $\mathbb{R}^n$ and the quadratic form $x_1^2 + x_2^2 + \cdots + x_n^2$ (see Lawson and Michelsohn [9]). The Clifford algebra $\text{Cl}_n$ is the algebra generated by an orthonormal basis $e_1, \ldots, e_n$ subject to the relation

$$e_i e_j + e_j e_i = -2\delta_{ij}.$$ 

Then $\text{Cl}_n$ is identified with $\mathbb{R}^{2^n}$. The set

$$\{a \in \mathbb{R}^n \subset \text{Cl}_n \mid a^2 = -1\}$$

is an $(n - 1)$-dimensional unit sphere $S^{n-1} \subset \mathbb{R}^n \subset \text{Cl}_n \cong \mathbb{R}^{2^n}$.

Let $M$ be a Riemann surface with complex structure $J^M$ and $V$ be the trivial associate bundle of a principal $\text{Cl}_n$-bundle, with right $\text{Cl}_n$ action, over $M$. We denote the set of smooth sections of $V$ by $\Gamma(V)$ and the fiber of $V$ at $p$ by $V_p$. Let $\Omega^m(V)$ be the set of $V$-valued $m$-forms on $M$ for every non-negative integer $m$. Then $\Omega^0(V) = \Gamma(V)$. Let $W$ be another trivial associate bundle of a principal $\text{Cl}_n$-bundle, with right $\text{Cl}_n$ action, over $M$. We denote by $\text{Hom}(V,W)$ the $\text{Cl}_n$-homomorphism bundle from $V$ to $W$. Let $N$ be a smooth section of the Clifford endomorphism bundle $\text{End}(V)$ of $V$ such that $-N_p \circ N_p$ is the identity map $\text{Id}_p$ on $V_p$ for every $p \in M$. The section $N$ is a complex structure at each fiber of $V$. We have a splitting $\text{End}(V) = \text{End}(V)_+ \oplus \text{End}(V)_-$, where

$$\text{End}(V)_+ = \{\xi \in \text{End}(V) : N\xi = \xi N\},$$

$$\text{End}(V)_- = \{\xi \in \text{End}(V) : N\xi = -\xi N\}.$$

This splitting induces a decomposition of $\xi \in \text{End}(V)$ into $\xi = \xi_+ + \xi_-$, where $\xi_+ = (\xi - N\xi N)/2 \in \text{End}(V)_+$ and $\xi_- = (\xi + N\xi N)/2 \in \text{End}(V)_-.$

Let $T^*M \otimes_{\mathbb{R}} V$ be the tensor bundle of the cotangent bundle $T^*M$ of $M$ and $V$ over real numbers. We set $\ast \omega = \omega \circ J^{T^*M}$ for every $\omega \in \Omega^1(V)$. We have a splitting $T^*M \otimes_{\mathbb{R}} V = KV \oplus \bar{KV}$, where

$$KV = \{\eta \in T^*M \otimes_{\mathbb{R}} V : \ast \eta = N\eta\}, \quad \bar{KV} = \{\eta \in T^*M \otimes_{\mathbb{R}} V : \ast \eta = -N\eta\}.$$

This splitting induces the type decomposition of $\eta \in T^*M \otimes_{\mathbb{R}} V$ into $\eta = \eta' + \eta''$, where $\eta' = (\eta - N \ast \eta)/2 \in KV$ and $\eta'' = (\eta + N \ast \eta)/2 \in \bar{KV}$.
Let $C$ be the right trivial Clifford bundle over $M$ with fiber $C\ell_n$. We identify a smooth map $\phi: M \to C\ell_n$ with a smooth section $p \mapsto (p, \phi(p))$ of $C$. The bundle $\text{End}(C)$ is identified with $C$, by the identification of $\xi_p \in \text{End}(C)_p$ with $P_p \in C_p$ such that $\xi_p(1) = P_p$ for every $p \in M$. We assume that $N$ takes values in $\mathbb{R}^n \subset C\ell_n$. Then $N$ is considered as a map from $M$ to $S^{n-1} \subset \mathbb{R}^n$. Then $T^*M \otimes \mathbb{R} C$ decomposes as

$$T^*M \otimes \mathbb{R} C = (KC)_+ \oplus (KC)_- \oplus (\bar{K}C)_+ \oplus (\bar{K}C)_-.$$ 

According to this decomposition, a connection $\nabla: \Gamma(C) \to \Omega^1(C)$ of the Clifford bundle $C$ decomposes as

$$\nabla = \partial^\nabla + A^\nabla + \bar{\partial}^\nabla + Q^\nabla,$$

$$\nabla': \Gamma(C) \to \Gamma(KC), \quad \nabla'\phi = (\nabla\phi)',$$

$$\nabla'': \Gamma(C) \to \Gamma(\bar{K}C), \quad \nabla''\phi = (\nabla\phi)'',$$

$$\partial^\nabla: \Gamma(C) \to \Gamma((KC)_+), \quad \partial^\nabla\phi = (\nabla\phi)_+,$$

$$A^\nabla \in \Gamma(\text{Hom}(C, (KC)_-)), \quad A^\nabla\phi = (\nabla'\phi)_-,$$

$$\bar{\partial}^\nabla: \Gamma(C) \to \Gamma((\bar{K}C)_+), \quad \bar{\partial}^\nabla\phi = (\nabla''\phi)_+,$$

$$Q^\nabla \in \Gamma(\text{Hom}(C, (\bar{K}C)_-)), \quad Q^\nabla\phi = (\nabla''\phi)_-,$$

where $\phi$ is any smooth section of $C$. We see that $A^\nabla$ and $Q^\nabla$ are tensorial, that is, $A^\nabla \in \Gamma(\text{Hom}(C, (KC)_-))$ and $Q^\nabla \in \Gamma(\text{Hom}(C, (\bar{K}C)_-))$. The sections $A^\nabla$ and $Q^\nabla$ are called the Hopf fields of $\nabla'$ and $\nabla''$ respectively.

We denote by $d$ the trivial connection on $C$.

**Lemma 3.1.** A map $N: M \to S^{n-1} \subset \mathbb{R}^n \subset C\ell_n$ is a harmonic map, if and only if $d * A^d = 0$.

**Proof.** The Hopf field $A^d$ satisfies the equation

$$A^d\phi = \frac{1}{2} \left[ (d'+ Jd'J) \right] \phi$$

$$= \frac{1}{4} \left[ (d - J * d + J(d - J * d)J) \right] \phi$$

$$= \frac{1}{4} \left[ (d\phi) - N * (d\phi) + [N(dN)\phi - d\phi] + [*(dN)\phi + N * d\phi] \right]$$

$$= \frac{1}{4} \left[ N(dN) + *(dN) \right] \phi$$

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for every \( \phi \in \Gamma(C) \). Hence

\[
d * A^d = \frac{1}{4} (dN \wedge * dN + Nd * dN).
\]

Hence \( d * A^d = 0 \) if and only if

\[
dN \wedge * dN + Nd * dN = 0.
\]

For an isothermal coordinate \((x, y)\) such that \(x + yi\) is a holomorphic coordinate, a map \( N: M \to S^{n-1} \subset \mathbb{R}^n \subset C\ell_n \) is a harmonic map if and only if

\[
\Delta N = -(N_{xx} + N_{yy})dx \wedge dy = |dN|^2 N
\]

(see Eells and Lemaire [7]). We have

\[
d * dN = d * (N_x dx + N_y dy) = d(-N_x dy + N_y dx)
\]

\[
= -(N_{xx} + N_{yy})dx \wedge dy = \Delta N,
\]

\[
dN \wedge * dN = (N_x dx + N_y dy) \wedge (-N_x dy + N_y dx) = (-N_x^2 - N_y^2)dx \wedge dy
\]

\[
= (|N_x|^2 + |N_y|^2)dx \wedge dy = |dN|^2,
\]

where the Clifford multiplication is used. Hence, \( N \) is a harmonic map if and only if \( d * A^d = 0 \).

4. Harmonic maps into a sphere

We construct a \( tt^* \)-bundle for a harmonic map from a Riemann surface to an \( n \)-dimensional sphere.

Let \( M \) be a Riemann surface with complex structure \( J_M \). For a one-form \( \omega \) on \( M \), define a one-form \( * \omega \) on \( M \) by

\[
* \omega = \omega \circ J_M.
\]

For one-forms \( \omega \) and \( \eta \) on \( M \) with values in \( C\ell_n \), we have the relation

\[
* \omega \wedge * \eta = \omega \wedge \eta.
\]

Indeed, for a basis \( E_1, E_2 \) of a tangent space of \( M \) with \( J^M E_1 = E_2 \), we have

\[
(\omega \wedge \eta)(qE_1 + rE_2, sE_1 + tE_2) = (qt - rs)(\omega(E_1)\eta(E_2) - \omega(E_2)\omega(E_1)),
\]

\[
(* \omega \wedge * \eta)(qE_1 + rE_2, sE_1 + tE_2) = (\omega \wedge \eta)(qE_2 - rE_1, sE_2 - tE_1)
\]

\[
= (qt - rs)(\omega(E_1)\eta(E_2) - \omega(E_2)\omega(E_1)).
\]
where \( q, r, s, t \in \mathbb{R} \).

Let \( F := M \times \mathbb{R}^2 \cong M \times C\ell_n \). For a map \( N: M \to S^{n-1} \subset \mathbb{R}^n \subset C\ell_n \), define a one-form \( S \) on \( M \) with values in \( C\ell_n \) by

\[
S := \frac{1}{4}(*dN + N dN).
\]

**Lemma 4.1.** \( N \) is a harmonic map if and only if the one-form \( S \) satisfies

\[
d^*S = 0.
\]

**Proof.** Since we have

\[
4d \ast S = d(-dN + N \ast dN) = dN \wedge \ast dN + N d \ast dN = 4d \ast A^d,
\]

this lemma follows from Lemma 3.1. \( \square \)

**Theorem 4.1.** A vector bundle \( F \) with \( \nabla := d - S \) and \( S \) is a \( tt^* \)-bundle.

**Proof.** We see that

\[
4dS = d \ast dN + dN \wedge dN = dN \wedge dN + N dN \wedge \ast dN,
\]

\[
16S \wedge S = (*dN + N dN) \wedge (*dN + N dN)
= *dN \wedge *dN + *dN \wedge N dN + N dN \wedge *dN + N dN \wedge N dN
= dN \wedge dN + N dN \wedge *dN + N dN \wedge *dN + dN \wedge dN
= 2(dN \wedge dN + N dN \wedge *dN).
\]

Hence \( dS = 2S \wedge S \) holds.

Lemma 4.1 and a direct calculation yield

\[
\nabla^\theta = d + (\cos \theta - 1)S + (\sin \theta) \ast S,
\]

\[
d^{\nabla^\theta} \circ \nabla^\theta
= (\cos \theta - 1)dS + ((\cos \theta - 1)S + (\sin \theta) \ast S) \wedge ((\cos \theta - 1)S + (\sin \theta) \ast S)
= (\cos \theta - 1)dS + (\cos \theta - 1)^2 S \wedge S + (\cos \theta - 1)(\sin \theta) S \wedge \ast S
+ (\sin \theta)(\cos \theta - 1) \ast S \wedge S + (\sin \theta)^2 \ast S \wedge \ast S
= (\cos \theta - 1)dS - 2(\cos \theta - 1)S \wedge S = 0.
\]

Hence \( F \) with \( \nabla \) and \( S \) is a \( tt^* \)-bundle. \( \square \)

For a harmonic map from a Riemann surface to \( S^2 \), we have two \( tt^* \)-bundles. One is the \( tt^* \)-bundle in Theorem 4.1. The other is that in the theory of quaternionic holomorphic line bundles (see [8]). These do not coincide directly as the fiber of the former is \( C\ell_3 \) and that of the latter is \( C\ell_2 \).
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References


