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doi: 10.1016/j.difgeo.2012.04.003
A $tt^*$-bundle associated with a harmonic map from a Riemann surface into a sphere

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Abstract
A $tt^*$-bundle is constructed by a harmonic map from a Riemann surface into an $n$-dimensional sphere. This $tt^*$-bundle is a high-dimensional analogue of a quaternionic line bundle with a Willmore connection. For the construction, a flat connection is decomposed into four parts by a fiberwise complex structure.

Keywords: $tt^*$-bundle, harmonic map, the Clifford algebra

2010 MSC: 53C43, 81R12

1. Introduction

A $tt^*$-bundle is a real vector bundle equipped with a family of flat connections, parametrized by a circle. The present paper delivers a $tt^*$-bundle derived from a harmonic map from a Riemann surface to an $n$-dimensional sphere.

The notion of $tt^*$-bundles is introduced by Schäfer [10] as a simple solution to a generalized version of the equation of topological-antitopological fusion, introduced by Cecotti and Vafa [2], in terms of real differential geometry. A topological-antitopological fusion of a topological field theory model is a special geometry structure on a Frobenius manifold. As a geometric interpretation of a special geometry structure on a quasi-Frobenius manifold, Dubrovin [6] showed that a solution to the equation is locally a pluriharmonic map from an $n$-dimensional quasi-Frobenius manifold to the symmetric space $GL(n, \mathbb{R})/O(n)$.
Schäfer [10] showed that an admissible pluriharmonic map from a simply connected complex manifold \( M \) to a symmetric space \( \text{GL}(r, \mathbb{R})/\text{O}(p, q) \), and that to \( \text{SL}(r, \mathbb{R})/\text{SO}(p, q) \) with \( p + q = r \), gives rise from a \textit{metric} \( tt^* \)-bundle. A harmonic map from a Riemann surface to \( \text{SU}(1, 1)/\text{SU}(1) \times \text{U}(1) \cong \text{SL}(2, \mathbb{R})/\text{SO}(2) \) is obtained by the generalized Weierstrass representation formula by Dorfmeister, Pedit, and Wu [5]. The Gauss map of a spacelike surface of constant mean curvature in the Minkowski space \( \mathbb{R}^{2, 1} \) is a harmonic map from a Riemann surface to \( \text{SL}(2, \mathbb{R})/\text{SO}(2) \). The Sym-Bobenko formula (Bobenko [1], Dorfmeister and Haak [4]) connects a surface and its Gauss map. Applying these formulae, Dorfmeister, Guest, and Rossman [3] gave the description of the quantum cohomology of \( \mathbb{C}P^1 \). The quantum cohomology of \( \mathbb{C}P^1 \) provides a solution to the third Painlevé equation.

A surface of constant mean curvature in \( \mathbb{R}^3 \) is an interesting research subject in the theory of surfaces. Its Gauss map is a harmonic map from a Riemann surface to the two-dimensional sphere \( S^2 \). It is impossible to write \( S^2 \) as a symmetric space \( \text{GL}(r, \mathbb{R})/\text{O}(p, q) \) or \( \text{SL}(r, \mathbb{R})/\text{SO}(p, q) \). This led the authors to find a \( tt^* \)-bundle for a harmonic map into \( S^2 \). The theory of a quaternionic line bundle with a Willmore connection by Ferus, Leschke, Pedit, and Pinkall [8] provides a way to construct a \( tt^* \)-bundle for a harmonic map from a Riemann surface into \( S^2 \). This method is extended and a \( tt^* \)-bundle associated with a harmonic map from a Riemann surface into \( S^n \) \((n \geq 2)\) is obtained (Theorem 4.1).

2. \( tt^* \)-bundles

We recall a \( tt^* \)-bundle (Schäfer [10]).

Let \( M \) be a complex manifold with complex structure \( J^M \). For a one-form \( \omega \) on \( M \), we define a one-form \(*\omega\) on \( M \) by \(*\omega := \omega \circ J^M\). Let \( E \) be a trivial real vector bundle of rank \( n \) over \( M \), \( \nabla \) a connection on \( E \), and \( S \) a one-form with values in the real endomorphisms of \( E \). A one-form \( S \) is considered as a one-form with values in \( n \)-by-\( n \) real matrices. Define a family of connections \( \{\nabla^\theta\}_{\theta \in \mathbb{R}} \) on \( E \) by

\[
\nabla^\theta := \nabla + (\cos \theta)S + (\sin \theta) * S.
\]
The curvature of $\nabla^\theta$ is

$$d\nabla^\theta \circ \nabla^\theta$$

$$= d\nabla \circ \nabla + (\cos \theta)d\nabla S + (\sin \theta)d\nabla \ast S$$

$$+ ((\cos \theta)S + (\sin \theta) \ast S) \wedge ((\cos \theta)S + (\sin \theta) \ast S)$$

$$= d\nabla \circ \nabla + (\cos \theta)d\nabla S + (\sin \theta)d\nabla \ast S$$

$$+ (\cos \theta)^2 S \wedge S + \cos \theta \sin \theta(S \wedge \ast S + \ast S \wedge S) + (\sin \theta)^2 S \wedge \ast S$$

$$= d\nabla \circ \nabla + (\cos \theta)d\nabla S + (\sin \theta)d\nabla \ast S$$

$$+ \frac{1 + \cos 2\theta}{2} S \wedge S + \frac{\sin 2\theta}{2} (S \wedge \ast S + \ast S \wedge S) + \frac{1 - \cos 2\theta}{2} \ast S \wedge \ast S$$

$$= d\nabla \circ \nabla + \frac{1}{2} S \wedge S + \frac{1}{2} \ast S \wedge \ast S$$

$$+ (\cos \theta)d\nabla S + (\sin \theta)d\nabla \ast S$$

$$+ \frac{\cos 2\theta}{2} (S \wedge S - \ast S \wedge \ast S) + \frac{\sin 2\theta}{2} (S \wedge \ast S + \ast S \wedge S) \ast S \wedge \ast S \ast S \wedge S$$

A vector bundle $E$ with $\nabla$ and $S$ is called a $tt^*$-bundle if $\nabla^\theta$ is flat for all $\theta \in \mathbb{R}$. By the preceding calculation, a vector bundle $E$ with $\nabla$ and $S$ is a $tt^*$-bundle, if and only if

$$d\nabla \circ \nabla + S \wedge S = 0, \quad d\nabla S = 0, \quad d\nabla \ast S = 0,$$

$$S \wedge S = \ast S \wedge \ast S, \quad S \wedge \ast S = - \ast S \wedge S.$$

Indeed,

$$(S \wedge S - \ast S \wedge \ast S)(X, Y)$$

$$= S(X)S(Y) - S(Y)S(X) - S(J^M X)S(J^M Y) + S(J^M Y)S(J^M X)$$

$$= -S(X)S(J^M J^M Y) + S(J^M J^M Y)S(X)$$

$$- S(J^M X)S(J^M Y) + S(J^M Y)S(J^M X)$$

$$= -S(X)S(J^M J^M Y) + S(J^M Y)S(J^M X)$$

$$+ S(J^M J^M Y)S(X) - S(J^M X)S(J^M Y)$$

$$= -(S \wedge S + \ast S \wedge \ast S)(X, J^M Y)$$

for any tangent vectors $X, Y$ of $M$. Hence, $S \wedge S = \ast S \wedge \ast S$ is equivalent to $S \wedge \ast S = - \ast S \wedge S$. Then, a vector bundle $E$ with $\nabla$ and $S$ is a $tt^*$-bundle, if and only if

$$d\nabla \circ \nabla + S \wedge S = 0, \quad d\nabla S = 0, \quad d\nabla \ast S = 0, \quad S \wedge S = \ast S \wedge \ast S.$$
Assume that $E$ with $\nabla$ and $S$ forms a $tt^*$-bundle. Define $F$ as the complexification of $E$, that is, $F := \mathbb{C} \otimes E$. Denote the complex-linear extensions of $\nabla$ and $S$ by the same notations respectively. Define a family of connections $\\{\nabla^\mu\}_{\mu \in \mathbb{C} \setminus \{0\}}$ of $F$ by

$$\nabla^\mu = \nabla + \frac{1}{\mu} C + \mu \bar{C}, \quad C = \frac{1}{2}(S - i \ast S).$$

Then $C$ is a $(1,0)$-form on $M$ with values in complex linear endmorphisms of $F$. The $tt^*$-bundle $E$ with $\nabla$ and $S$ is the real part of $F$ with $\nabla^\mu$ if and only if $|\mu| = 1$.

**Proposition 2.1.** For each $\mu \in \mathbb{C} \setminus \{0\}$, the connection $\nabla^\mu$ is flat.

**Proof.** As $E$ with $\nabla$ and $S$ is a $tt^*$-bundle, it follows that

$$d^\nabla C = 0, \quad d^\nabla \bar{C} = 0,$$

$$C \wedge C = \frac{1}{4}(S \wedge S - iS \wedge \ast S - i \ast S \wedge S - \ast S \wedge \ast S) = 0,$$

$$C \wedge \bar{C} = \frac{1}{4}(S \wedge S + iS \wedge \ast S - i \ast S \wedge S + \ast S \wedge \ast S) = \frac{1}{2}(S \wedge S + iS \wedge \ast S).$$

Then

$$d^\nabla^\mu \circ \nabla^\mu = d^\nabla \circ \nabla + \left(\frac{1}{\mu} C + \mu \bar{C}\right) \wedge \left(\frac{1}{\mu} C + \mu \bar{C}\right)$$

$$= d^\nabla \circ \nabla + C \wedge \bar{C} + \bar{C} \wedge C$$

$$= d^\nabla \circ \nabla + S \wedge S = 0.$$

Hence $\nabla^\mu$ is flat. \qed

Adding the assumption in Proposition 2.1, we assume that there exists a hermitian pseudo-metric $h$ on $F$, and a metric connection $\nabla$ with respect to $h$, such that

$$h(C(X)a, b) = h(a, \bar{C}(X)b),$$

where $a, b \in \Gamma(F)$, and $X$ is a vector field of type $(1,0)$ on $M$. Then $(F, \nabla, C, \bar{C}, h)$ becomes a harmonic bundle defined in Schäfer [11].

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3. Decomposition of a connection

We obtain a condition for a map from a Riemann surface into a sphere, to become a harmonic map, by decomposing a flat connection into four parts.

Let $\mathcal{C} = \mathbb{C}^n$ and the quadratic form $x^2_1 + x^2_2 + \cdots + x^2_n$ (see Lawson and Michelsohn [9]). The Clifford algebra $\mathcal{C} = \mathbb{C}^n$ is the algebra generated by an orthonormal basis $e_1, \ldots, e_n$ subject to the relation

$$e_ie_j + e_je_i = -2\delta_{ij}.$$  

Then $\mathcal{C} = \mathbb{C}^n$ is identified with $\mathbb{R}^{2n}$. The set

$$\{a \in \mathbb{R}^n \subset \mathcal{C} = \mathbb{C}^n | a^2 = -1\}$$

is an $(n-1)$-dimensional unit sphere $S^{n-1} \subset \mathbb{R}^n \subset \mathcal{C} = \mathbb{C}^n \cong \mathbb{R}^{2n}$.

Let $M$ be a Riemann surface with complex structure $J_M$ and $V$ be the trivial associate bundle of a principal $\mathcal{C} = \mathbb{C}^n$-bundle, with right $\mathcal{C} = \mathbb{C}^n$ action, over $M$. We denote the set of smooth sections of $V$ by $\Gamma(V)$ and the fiber of $V$ at $p$ by $V_p$. Let $\Omega^m(V)$ be the set of $V$-valued $m$-forms on $M$ for every non-negative integer $m$. Then $\Omega^0(V) = \Gamma(V)$. Let $W$ be another trivial associate bundle of a principal $\mathcal{C} = \mathbb{C}^n$-bundle, with right $\mathcal{C} = \mathbb{C}^n$ action, over $M$. We denote by $\text{Hom}(V, W)$ the $\mathcal{C} = \mathbb{C}^n$-homomorphism bundle from $V$ to $W$. Let $N$ be a smooth section of the Clifford endomorphism bundle $\text{End}(V)$ of $V$ such that $-N_p \circ N_p$ is the identity map $\text{Id}_p$ on $V_p$ for every $p \in M$. The section $N$ is a complex structure at each fiber of $V$. We have a splitting

$$\text{End}(V) = \text{End}(V)_+ \oplus \text{End}(V)_-,$$

where

$$\text{End}(V)_+ = \{\xi \in \text{End}(V) : N\xi = \xi N\},$$

$$\text{End}(V)_- = \{\xi \in \text{End}(V) : N\xi = -\xi N\}.$$  

This splitting induces a decomposition of $\xi \in \text{End}(V)$ into $\xi = \xi_+ + \xi_-$, where $\xi_+ = (\xi - N\xi N)/2 \in \text{End}(V)_+$ and $\xi_- = (\xi + N\xi N)/2 \in \text{End}(V)_-$. 

Let $T^*M \otimes_{\mathbb{R}} V$ be the tensor bundle of the cotangent bundle $T^*M$ of $M$ and $V$ over real numbers. We set $*\omega = \omega \circ J^{TM}$ for every $\omega \in \Omega^1(V)$. We have a splitting

$$T^*M \otimes_{\mathbb{R}} V = KV \oplus \bar{K}V,$$

where

$$KV = \{\eta \in T^*M \otimes_{\mathbb{R}} V : *\eta = N\eta\}, \quad \bar{K}V = \{\eta \in T^*M \otimes_{\mathbb{R}} V : *\eta = -N\eta\}.$$  

This splitting induces the type decomposition of $\eta \in T^*M \otimes_{\mathbb{R}} V$ into $\eta = \eta' + \eta''$, where $\eta' = (\eta - N * \eta)/2 \in KV$ and $\eta'' = (\eta + N * \eta)/2 \in \bar{K}V$. 

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Let $C$ be the right trivial Clifford bundle over $M$ with fiber $C\ell_n$. We identify a smooth map $\phi: M \to C\ell_n$ with a smooth section $p \mapsto (p, \phi(p))$ of $C$. The bundle $\text{End}(C)$ is identified with $C$, by the identification of $\xi_p \in \text{End}(C)_p$ with $P_p \in C_p$ such that $\xi_p(1) = P_p$ for every $p \in M$. We assume that $N$ takes values in $\mathbb{R}^n \subset C\ell_n$. Then $N$ is considered as a map from $M$ to $S^{n-1} \subset \mathbb{R}^n$. Then $T^*M \otimes_{\mathbb{R}} C$ decomposes as

$$T^*M \otimes_{\mathbb{R}} C = (KC)_+ \oplus (KC)_- \oplus (\bar{K}C)_+ \oplus (\bar{K}C)_-.$$

According to this decomposition, a connection $\nabla: \Gamma(C) \to \Omega^1(C)$ of the Clifford bundle $C$ decomposes as

$$\nabla = \partial^\nabla + A^\nabla + \bar{\partial}^\nabla + Q^\nabla,$$

$$\nabla': \Gamma(C) \to \Gamma((KC)_+), \quad \nabla' \phi = (\nabla \phi)' = (\nabla \phi)^+,$$

$$\nabla'': \Gamma(C) \to \Gamma((\bar{K}C)_+), \quad \nabla'' \phi = (\nabla' \phi)' = (\nabla' \phi)^+,$$

$$\partial^\nabla: \Gamma(C) \to \Gamma(\text{Hom}(C, (KC)_-)), \quad A^\nabla \phi = (\nabla' \phi)_-, \quad Q^\nabla \phi = (\nabla'' \phi)_-,$$

where $\phi$ is any smooth section of $C$. We see that $A^\nabla$ and $Q^\nabla$ are tensorial, that is, $A^\nabla \in \Gamma(\text{Hom}(C, (KC)_-))$ and $Q^\nabla \in \Gamma(\text{Hom}(C, (\bar{K}C)_-))$. The sections $A^\nabla$ and $Q^\nabla$ are called the Hopf fields of $\nabla'$ and $\nabla''$ respectively.

We denote by $d$ the trivial connection on $C$.

**Lemma 3.1.** A map $N: M \to S^{n-1} \subset \mathbb{R}^n \subset C\ell_n$ is a harmonic map, if and only if $d * A^d = 0$.

**Proof.** The Hopf field $A^d$ satisfies the equation

$$A^d \phi = \frac{1}{2} [(d' + Jd'J) \phi] = \frac{1}{4} [d - J \ast d + J(d - J \ast d)J] \phi = \frac{1}{4} \{ (d \phi) - N \ast (d \phi) + [N(dN) \phi - d \phi] + [*(dN) \phi + N \ast d \phi] \} = \frac{1}{4} [N(dN) + *(dN)] \phi$$
for every $\phi \in \Gamma(C)$. Hence
\[
d \ast A^d = \frac{1}{4}(dN \wedge dN + Nd \ast dN).
\]
Hence $d \ast A^d = 0$ if and only if
\[
dN \wedge dN + Nd \ast dN = 0.
\]
For an isothermal coordinate $(x, y)$ such that $x + yi$ is a holomorphic coordinate, a map $N: M \to S^{n-1} \subset \mathbb{R}^n \subset \mathbb{C}^n$ is a harmonic map if and only if
\[
\Delta N = -(N_{xx} + N_{yy})dx \wedge dy = |dN|^2 N
\]
(see Eells and Lemaire [7]). We have
\[
d \ast dN = d \ast (N_x dx + N_y dy) = d(-N_x dy + N_y dx)\\= - (N_{xx} + N_{yy})dx \wedge dy = \Delta N,
\]
\[
dN \wedge dN = (N_x dx + N_y dy) \wedge (-N_x dy + N_y dx) = (-N_x^2 - N_y^2)dx \wedge dy\\= (|N_x|^2 + |N_y|^2)dx \wedge dy = |dN|^2,
\]
where the Clifford multiplication is used. Hence, $N$ is a harmonic map if and only if $d \ast A^d = 0$.

4. Harmonic maps into a sphere

We construct a $tt^*$-bundle for a harmonic map from a Riemann surface to an $n$-dimensional sphere.

Let $M$ be a Riemann surface with complex structure $J^M$. For a one-form $\omega$ on $M$, define a one-form $\ast \omega$ on $M$ by $\ast \omega := \omega \circ J^M$. For one-forms $\omega$ and $\eta$ on $M$ with values in $C\ell_n$, we have the relation
\[
\ast \omega \wedge \ast \eta = \omega \wedge \eta.
\]
Indeed, for a basis $E_1, E_2$ of a tangent space of $M$ with $J^M E_1 = E_2$, we have
\[
(\omega \wedge \eta)(qE_1 + rE_2, sE_1 + tE_2)\\= (qt - rs)(\omega(E_1)\eta(E_2) - \omega(E_2)\omega(E_1)),
\]
\[
(\ast \omega \ast \eta)(qE_1 + rE_2, sE_1 + tE_2) = (\omega \wedge \eta)(qE_2 - rE_1, sE_2 - tE_1)\\= (qt - rs)(\omega(E_1)\eta(E_2) - \omega(E_2)\omega(E_1)),$
where \( q, r, s, t \in \mathbb{R} \).

Let \( F := M \times \mathbb{R}^2 \cong M \times \mathbb{C}^\ell \). For a map \( N: M \to S^{n-1} \subset \mathbb{R}^n \subset \mathbb{C}^\ell \), define a one-form \( S \) on \( M \) with values in \( \mathbb{C}^\ell \) by

\[
S := \frac{1}{4}(\ast dN + N dN).
\]

**Lemma 4.1.** N is a harmonic map if and only if the one-form \( S \) satisfies

\[
d \ast S = 0.
\]

**Proof.** Since we have

\[
4 d \ast S = d(-dN + N \ast dN) = dN \wedge \ast dN + N d \ast dN = 4d \ast A^d,
\]

this lemma follows from Lemma 3.1. \( \square \)

**Theorem 4.1.** A vector bundle \( F \) with \( \nabla := d - S \) and \( S \) is a \( tt^* \)-bundle.

**Proof.** We see that

\[
4dS = d \ast dN + dN \wedge dN = dN \wedge dN + N dN \wedge \ast dN,
\]

\[
16 S \wedge S = (\ast dN + N dN) \wedge (\ast dN + N dN)
\]

\[
= \ast dN \wedge \ast dN + \ast dN \wedge N dN + N dN \wedge \ast dN + N dN \wedge N dN
\]

\[
= dN \wedge dN + N dN \wedge \ast dN + N dN \wedge \ast dN + N dN \wedge dN
\]

\[
= 2(dN \wedge dN + N dN \wedge \ast dN).
\]

Hence \( dS = 2S \wedge S \) holds.

Lemma 4.1 and a direct calculation yield

\[
\nabla^\theta = d + (\cos \theta - 1)S + (\sin \theta) \ast S,
\]

\[
d\nabla^\theta \circ \nabla^\theta
\]

\[
= (\cos \theta - 1)dS + ((\cos \theta - 1)S + (\sin \theta) \ast S) \wedge ((\cos \theta - 1)S + (\sin \theta) \ast S)
\]

\[
= (\cos \theta - 1)dS + (\cos \theta - 1)^2S \wedge S + (\cos \theta - 1)(\sin \theta)S \wedge \ast S
\]

\[
+ (\sin \theta)(\cos \theta - 1) \ast S \wedge S + (\sin \theta)^2 \ast S \wedge \ast S
\]

\[
= (\cos \theta - 1)dS - 2(\cos \theta - 1)S \wedge S = 0.
\]

Hence \( F \) with \( \nabla \) and \( S \) is a \( tt^* \)-bundle. \( \square \)

For a harmonic map from a Riemann surface to \( S^2 \), we have two \( tt^* \)-bundles. One is the \( tt^* \)-bundle in Theorem 4.1. The other is that in the theory of quaternionic holomorphic line bundles (see \([8]\)). These do not coincide directly as the fiber of the former is \( \mathbb{C}^\ell_3 \) and that of the latter is \( \mathbb{C}^\ell_2 \).
Acknowledgements.

The authors are very grateful to Claus Hertling for the comments to a preliminary version of this paper.

References


