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<th>著者</th>
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<tbody>
<tr>
<td>原稿</td>
<td>227-232</td>
</tr>
<tr>
<td>年月</td>
<td>2012-06</td>
</tr>
<tr>
<td>出版社</td>
<td>0.1016/j.difgeo.2012.04.003</td>
</tr>
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doi: 10.1016/j.difgeo.2012.04.003
A $tt^*$-bundle associated with a harmonic map from a Riemann surface into a sphere

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Abstract

A $tt^*$-bundle is constructed by a harmonic map from a Riemann surface into an $n$-dimensional sphere. This $tt^*$-bundle is a high-dimensional analogue of a quaternionic line bundle with a Willmore connection. For the construction, a flat connection is decomposed into four parts by a fiberwise complex structure.

Keywords: $tt^*$-bundle, harmonic map, the Clifford algebra  
2010 MSC: 53C43, 81R12

1. Introduction

A $tt^*$-bundle is a real vector bundle equipped with a family of flat connections, parametrized by a circle. The present paper delivers a $tt^*$-bundle derived from a harmonic map from a Riemann surface to an $n$-dimensional sphere.

The notion of $tt^*$-bundles is introduced by Schäfer [10] as a simple solution to a generalized version of the equation of topological-antitopological fusion, introduced by Cecotti and Vafa [2], in terms of real differential geometry. A topological-antitopological fusion of a topological field theory model is a special geometry structure on a Frobenius manifold. As a geometric interpretation of a special geometry structure on a quasi-Frobenius manifold, Dubrovin [6] showed that a solution to the equation is locally a pluriharmonic map from an $n$-dimensional quasi-Frobenius manifold to the symmetric space $GL(n, \mathbb{R})/O(n)$.

Preprint submitted to Differential Geometry and its Application  
July 5, 2011
Schäfer [10] showed that an admissible pluriharmonic map from a simply connected complex manifold $M$ to a symmetric space $GL(r, \mathbb{R})/O(p, q)$, and that to $SL(r, \mathbb{R})/SO(p, q)$ with $p + q = r$, gives rise from a metric $tt^*$-bundle. A harmonic map from a Riemann surface to $SU(1, 1)/SU(1) \times U(1) \cong SL(2, \mathbb{R})/SO(2)$ is obtained by the generalized Weierstrass representation formula by Dorfmeister, Pedit, and Wu [5]. The Gauss map of a spacelike surface of constant mean curvature in the Minkowski space $\mathbb{R}^{2,1}$ is a harmonic map from a Riemann surface to $SL(2, \mathbb{R})/SO(2)$. The Sym-Bobenko formula (Bobenko [1], Dorfmeister and Haak [4]) connects a surface and its Gauss map. Applying these formulae, Dorfmeister, Guest, and Rossman [3] gave the description of the quantum cohomology of $\mathbb{C}P^1$. The quantum cohomology of $\mathbb{C}P^1$ provides a solution to the third Painlevé equation.

A surface of constant mean curvature in $\mathbb{R}^3$ is an interesting research subject in the theory of surfaces. Its Gauss map is a harmonic map from a Riemann surface to the two-dimensional sphere $S^2$. It is impossible to write $S^2$ as a symmetric space $GL(r, \mathbb{R})/O(p, q)$ or $SL(r, \mathbb{R})/SO(p, q)$. This led the authors to find a $tt^*$-bundle for a harmonic map into $S^2$. The theory of a quaternionic line bundle with a Willmore connection by Ferus, Leschke, Pedit, and Pinkall [8] provides a way to construct a $tt^*$-bundle for a harmonic map from a Riemann surface into $S^2$. This method is extended and a $tt^*$-bundle associated with a harmonic map from a Riemann surface into $S^n$ ($n \geq 2$) is obtained (Theorem 4.1).

2. $tt^*$-bundles

We recall a $tt^*$-bundle (Schäfer [10]).

Let $M$ be a complex manifold with complex structure $J^M$. For a one-form $\omega$ on $M$, we define a one-form $*\omega$ on $M$ by $*\omega := \omega \circ J^M$. Let $E$ be a trivial real vector bundle of rank $n$ over $M$, $\nabla$ a connection on $E$, and $S$ a one-form with values in the real endomorphisms of $E$. A one-form $S$ is considered as a one-form with values in $n$-by-$n$ real matrices. Define a family of connections $\{\nabla^\theta\}_{\theta \in \mathbb{R}}$ on $E$ by

$$\nabla^\theta := \nabla + (\cos \theta)S + (\sin \theta) * S.$$
The curvature of $\nabla^\theta$ is

$$d\nabla^\theta \circ \nabla^\theta$$

$$= d\nabla \circ \nabla + (\cos \theta)d\nabla S + (\sin \theta)d\nabla S$$

$$+ ((\cos \theta)S + (\sin \theta)S) \wedge ((\cos \theta)S + (\sin \theta)S)$$

$$= d\nabla \circ \nabla + (\cos \theta)d\nabla S + (\sin \theta)d\nabla S + (\cos \theta)^2 S \wedge S + (\sin \theta)^2 S \wedge S$$

$$+ (\cos \theta)\sin \theta (S \wedge S + *S \wedge S) + (\sin \theta)^2 S \wedge S$$

$$= (\cos \theta)\sin \theta (S \wedge S + *S \wedge S) + (\sin \theta)^2 S \wedge S$$

$$+ (\cos \theta)^2 S \wedge S + (\sin \theta)^2 S \wedge S$$

$$= d\nabla \circ \nabla + (\cos \theta)d\nabla S + (\sin \theta)d\nabla S$$

$$+ \frac{1 + \cos 2\theta}{2} S \wedge S + \frac{\sin 2\theta}{2} (S \wedge S + *S \wedge S) + \frac{1 - \cos 2\theta}{2} *S \wedge *S$$

$$= d\nabla \circ \nabla + (\cos \theta)d\nabla S + (\sin \theta)d\nabla S$$

$$+ \frac{\cos 2\theta}{2} (S \wedge -S \wedge *S) + \frac{\sin 2\theta}{2} (S \wedge S + *S \wedge S).$$

A vector bundle $E$ with $\nabla$ and $S$ is called a $tt^*$-bundle if $\nabla^\theta$ is flat for all $\theta \in \mathbb{R}$. By the preceding calculation, a vector bundle $E$ with $\nabla$ and $S$ is a $tt^*$-bundle, if and only if

$$d\nabla \circ \nabla + S \wedge S = 0,$$  

$$d\nabla S = 0,$$  

$$d\nabla *S = 0,$$  

$$S \wedge S = *S \wedge *S,$$  

$$S \wedge *S = -S \wedge S.$$  

Indeed,

$$S(X)S(Y) - S(Y)S(X) - S(J^M X)S(J^M Y) + S(J^M Y)S(J^M X)$$

$$= S(X)S(J^M Y) + S(J^M Y)S(X)$$

$$- S(J^M X)S(J^M Y) + S(J^M Y)S(J^M X)$$

$$= S(J^M X)S(J^M Y) + S(J^M Y)S(J^M X)$$

$$+ S(J^M Y)S(X) - S(J^M X)S(J^M Y)$$

$$= -(S \wedge S + *S \wedge S)(X, J^M Y)$$

for any tangent vectors $X, Y$ of $M$. Hence, $S \wedge S = *S \wedge *S$ is equivalent to $S \wedge *S = - *S \wedge S$. Then, a vector bundle $E$ with $\nabla$ and $S$ is a $tt^*$-bundle, if and only if

$$d\nabla \circ \nabla + S \wedge S = 0,$$  

$$d\nabla S = 0,$$  

$$d\nabla *S = 0,$$  

$$S \wedge S = *S \wedge *S.$$  

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Assume that $E$ with $\nabla$ and $S$ forms a $tt^*$-bundle. Define $F$ as the complexification of $E$, that is, $F := \mathbb{C} \otimes E$. Denote the complex-linear extensions of $\nabla$ and $S$ by the same notations respectively. Define a family of connections $\{\nabla^\mu\}_{\mu \in \mathbb{C} \setminus \{0\}}$ of $F$ by
\[
\nabla^\mu = \nabla + \frac{1}{\mu} C + \mu \bar{C}, \quad C = \frac{1}{2}(S - i \ast S) .
\]
Then $C$ is a $(1,0)$-form on $M$ with values in complex linear endmorphisms of $F$. The $tt^*$-bundle $E$ with $\nabla$ and $S$ is the real part of $F$ with $\nabla^\mu$ if and only if $|\mu| = 1$.

**Proposition 2.1.** For each $\mu \in \mathbb{C} \setminus \{0\}$, the connection $\nabla^\mu$ is flat.

**Proof.** As $E$ with $\nabla$ and $S$ is a $tt^*$-bundle, it follows that
\[
d^\nabla C = 0, \quad d^\nabla \bar{C} = 0 ,
\]
\[
C \wedge C = \frac{1}{4}(S \wedge S - iS \wedge \ast S - i \ast S \wedge S - \ast S \wedge \ast S) = 0 ,
\]
\[
C \wedge \bar{C} = \frac{1}{4}(S \wedge S + iS \wedge \ast S - i \ast S \wedge S + \ast S \wedge \ast S) = \frac{1}{2}(S \wedge S + iS \wedge \ast S) .
\]
Then
\[
d^{\nabla^\mu} \circ \nabla^\mu = d^\nabla \circ \nabla + \left(\frac{1}{\mu} C + \mu \bar{C}\right) \wedge \left(\frac{1}{\mu} C + \mu \bar{C}\right)
\]
\[
= d^\nabla \circ \nabla + C \wedge \bar{C} + \bar{C} \wedge C
\]
\[
= d^\nabla \circ \nabla + S \wedge S = 0.
\]
Hence $\nabla^\mu$ is flat. \hfill $\square$

Adding the assumption in Proposition 2.1, we assume that there exists a hermitian pseudo-metric $h$ on $F$, and a metric connection $\nabla$ with respect to $h$, such that
\[
h(C(X)a,b) = h(a, \bar{C}(\bar{X})b) ,
\]
where $a, b \in \Gamma(F)$, and $X$ is a vector field of type $(1,0)$ on $M$. Then $(F, \nabla, C, \bar{C}, h)$ becomes a harmonic bundle defined in Schäfer [11].
3. Decomposition of a connection

We obtain a condition for a map from a Riemann surface into a sphere, to become a harmonic map, by decomposing a flat connection into four parts.

Let $\mathcal{C}l_n$ be the Clifford algebra associated with $\mathbb{R}^n$ and the quadratic form $x_1^2 + x_2^2 + \cdots + x_n^2$ (see Lawson and Michelsohn [9]). The Clifford algebra $\mathcal{C}l_n$ is the algebra generated by an orthonormal basis $e_1, \ldots, e_n$ subject to the relation

$$e_ie_j + e_je_i = -2\delta_{ij}.$$  

Then $\mathcal{C}l_n$ is identified with $\mathbb{R}^{2n}$. The set

$$\{a \in \mathbb{R}^n \subset \mathcal{C}l_n \mid a^2 = -1\}$$

is an $(n-1)$-dimensional unit sphere $S^{n-1} \subset \mathbb{R}^n \subset \mathcal{C}l_n \cong \mathbb{R}^{2n}$.

Let $M$ be a Riemann surface with complex structure $J^M$ and $V$ be the trivial associate bundle of a principal $\mathcal{C}l_n$-bundle, with right $\mathcal{C}l_n$ action, over $M$. We denote the set of smooth sections of $V$ by $\Gamma(V)$ and the fiber of $V$ at $p$ by $V_p$. Let $\Omega^m(V)$ be the set of $V$-valued $m$-forms on $M$ for every non-negative integer $m$. Then $\Omega^0(V) = \Gamma(V)$. Let $W$ be another trivial associate bundle of a principal $\mathcal{C}l_n$-bundle, with right $\mathcal{C}l_n$ action, over $M$. We denote by $\text{Hom}(V,W)$ the $\mathcal{C}l_n$-homomorphism bundle from $V$ to $W$. Let $N$ be a smooth section of the Clifford endomorphism bundle $\text{End}(V)$ of $V$ such that $-N_p \circ N_p$ is the identity map $\text{Id}_p$ on $V_p$ for every $p \in M$. The section $N$ is a complex structure at each fiber of $V$. We have a splitting $\text{End}(V) = \text{End}(V)_+ \oplus \text{End}(V)_-$, where

$$\text{End}(V)_+ = \{\xi \in \text{End}(V) : N\xi = \xi N\},$$

$$\text{End}(V)_- = \{\xi \in \text{End}(V) : N\xi = -\xi N\}.$$

This splitting induces a decomposition of $\xi \in \text{End}(V)$ into $\xi = \xi_+ + \xi_-$, where $\xi_+ = (\xi - N\xi N)/2 \in \text{End}(V)_+$ and $\xi_- = (\xi + N\xi N)/2 \in \text{End}(V)_-$.

Let $T^*M \otimes_\mathbb{R} V$ be the tensor bundle of the cotangent bundle $T^*M$ of $M$ and $V$ over real numbers. We set $\ast \omega = \omega \circ J^{TM}$ for every $\omega \in \Omega^1(V)$. We have a splitting $T^*M \otimes_\mathbb{R} V = KV \oplus \bar{KV}$, where

$$KV = \{\eta \in T^*M \otimes_\mathbb{R} V : \ast\eta = N\eta\}, \quad \bar{KV} = \{\eta \in T^*M \otimes_\mathbb{R} V : \ast\eta = -N\eta\}.$$

This splitting induces the type decomposition of $\eta \in T^*M \otimes_\mathbb{R} V$ into $\eta = \eta' + \eta''$, where $\eta' = (\eta - N \ast \eta)/2 \in KV$ and $\eta'' = (\eta + N \ast \eta)/2 \in \bar{KV}$.

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Let $C$ be the right trivial Clifford bundle over $M$ with fiber $C\ell_n$. We identify a smooth map $\phi: M \to C\ell_n$ with a smooth section $p \mapsto (p, \phi(p))$ of $C$. The bundle $\text{End}(C)$ is identified with $C$, by the identification of $\xi_p \in \text{End}(C)_p$ with $P_p \in C_p$ such that $\xi_p(1) = P_p$ for every $p \in M$. We assume that $N$ takes values in $\mathbb{R}^n \subset C\ell_n$. Then $N$ is considered as a map from $M$ to $S^{n-1} \subset \mathbb{R}^n$. Then $T^*M \otimes \mathbb{R} C$ decomposes as

$$T^*M \otimes \mathbb{R} C = (KC)_+ \oplus (KC)_- \oplus (\bar{KC})_+ \oplus (\bar{KC})_-$$

According to this decomposition, a connection $\nabla: \Gamma(C) \to \Omega^1(C)$ of the Clifford bundle $C$ decomposes as

$$\nabla = \partial \nabla + A^\nabla + \bar{\partial} \nabla + Q^\nabla,$$

$$\nabla': \Gamma(C) \to \Gamma(KC), \ \nabla' \phi = (\nabla \phi)'$$

$$\nabla'': \Gamma(C) \to \Gamma(\bar{KC}), \ \nabla'' \phi = (\nabla \phi)'$$

$$\partial \nabla: \Gamma(C) \to \Gamma((KC)_+), \ \partial \nabla \phi = (\nabla \phi)'_+$$

$$A^\nabla \in \Gamma(\text{Hom}(C, (KC)_-)), \ A^\nabla \phi = (\nabla \phi)'_-$$

$$\bar{\partial} \nabla: \Gamma(C) \to \Gamma((\bar{KC})_+), \ \bar{\partial} \nabla \phi = (\nabla \phi)'_+$$

$$Q^\nabla \in \Gamma(\text{Hom}(C, (\bar{KC})_-)), \ Q^\nabla \phi = (\nabla \phi)'_-$$

where $\phi$ is any smooth section of $C$. We see that $A^\nabla$ and $Q^\nabla$ are tensorial, that is, $A^\nabla \in \Gamma(\text{Hom}(C, (KC)_-))$ and $Q^\nabla \in \Gamma(\text{Hom}(C, (\bar{KC})_-))$. The sections $A^\nabla$ and $Q^\nabla$ are called the Hopf fields of $\nabla'$ and $\nabla''$ respectively.

We denote by $d$ the trivial connection on $C$.

**Lemma 3.1.** A map $N: M \to S^{n-1} \subset \mathbb{R}^n \subset C\ell_n$ is a harmonic map, if and only if $d \ast A^d = 0$.

**Proof.** The Hopf field $A^d$ satisfies the equation

$$A^d \phi = \frac{1}{2} [(d' + Jd'J) \phi$$

$$= \frac{1}{4} [d - J \ast d + J(d - J \ast d)J] \phi$$

$$= \frac{1}{4} \{(d\phi) - N \ast (d\phi)$$

$$+ [N(dN)\phi - d\phi] + [*(dN)\phi + N \ast d\phi]\}$$

$$= \frac{1}{4} [N(dN) + *(dN)] \phi$$
for every $\phi \in \Gamma(C)$. Hence
\[
d \ast A^d = \frac{1}{4} (dN \wedge \ast dN + Nd \ast dN).
\]
Hence $d \ast A^d = 0$ if and only if
\[
d N \wedge \ast dN + Nd \ast dN = 0.
\]
For an isothermal coordinate $(x, y)$ such that $x + yi$ is a holomorphic coordinate, a map $N: M \to S^{n-1} \subset \mathbb{R}^n \subset C\ell_n$ is a harmonic map if and only if
\[
\Delta N = -(N_{xx} + N_{yy}) dx \wedge dy = |dN|^2 N
\]
(see Eells and Lemaire [7]). We have
\[
d \ast dN = d \ast (N_x dx + N_y dy) = d(-N_x dy + N_y dx)
\]
\[
= -(N_{xx} + N_{yy}) dx \wedge dy = \Delta N,
\]
\[
d N \wedge \ast dN = (N_x dx + N_y dy) \wedge (-N_x dy + N_y dx) = (-N_x^2 - N_y^2) dx \wedge dy
\]
\[
= (|N_x|^2 + |N_y|^2) dx \wedge dy = |dN|^2,
\]
where the Clifford multiplication is used. Hence, $N$ is a harmonic map if and only if $d \ast A^d = 0$.

4. Harmonic maps into a sphere

We construct a $tt^*$-bundle for a harmonic map from a Riemann surface to an $n$-dimensional sphere.

Let $M$ be a Riemann surface with complex structure $J^M$. For a one-form $\omega$ on $M$, define a one-form $\ast \omega$ on $M$ by $\ast \omega := \omega \circ J^M$. For one-forms $\omega$ and $\eta$ on $M$ with values in $C\ell_n$, we have the relation
\[
\ast \omega \wedge \ast \eta = \omega \wedge \eta.
\]
Indeed, for a basis $E_1, E_2$ of a tangent space of $M$ with $J^M E_1 = E_2$, we have
\[
(\omega \wedge \eta)(qE_1 + rE_2, sE_1 + tE_2)
\]
\[
= (qt - rs)(\omega(E_1)\eta(E_2) - \omega(E_2)\omega(E_1)),
\]
\[
(\ast \omega \wedge \ast \eta)(qE_1 + rE_2, sE_1 + tE_2) = (\omega \wedge \eta)(qE_2 - rE_1, sE_2 - tE_1)
\]
\[
= (qt - rs)(\omega(E_1)\eta(E_2) - \omega(E_2)\omega(E_1)),
\]
\[
\begin{align*}
(\omega \wedge \eta)(qE_1 + rE_2, sE_1 + tE_2) & = (qt - rs)(\omega(E_1)\eta(E_2) - \omega(E_2)\omega(E_1)), \\
(\ast \omega \wedge \ast \eta)(qE_1 + rE_2, sE_1 + tE_2) & = (qt - rs)(\omega(E_1)\eta(E_2) - \omega(E_2)\omega(E_1)),
\end{align*}
\]
where \( q, r, s, t \in \mathbb{R} \).

Let \( F := M \times \mathbb{R}^2 \cong M \times C\ell_n \). For a map \( N : M \to S^{n-1} \subset \mathbb{R}^n \subset C\ell_n \), define a one-form \( S \) on \( M \) with values in \( C\ell_n \) by

\[
S := \frac{1}{4}(\ast dN + N dN).
\]

**Lemma 4.1.** \( N \) is a harmonic map if and only if the one-form \( S \) satisfies \( d \ast S = 0 \).

**Proof.** Since we have

\[
4 d \ast S = d(-dN + N \ast dN) = dN \wedge \ast dN + N d \ast dN = 4d \ast A^d,
\]

this lemma follows from Lemma 3.1. \( \square \)

**Theorem 4.1.** A vector bundle \( F \) with \( \nabla := d - S \) and \( S \) is a \( tt^* \)-bundle.

**Proof.** We see that

\[
4dS = d \ast dN + dN \wedge dN = dN \wedge dN + N dN \wedge \ast dN,
16S \wedge S = (\ast dN + N dN) \wedge (\ast dN + N dN)
= \ast dN \wedge \ast dN + \ast dN \wedge N dN + N dN \wedge \ast dN + N dN \wedge N dN
= dN \wedge dN + N dN \wedge \ast dN + N dN \wedge \ast dN + dN \wedge dN
= 2(dN \wedge dN + N dN \wedge \ast dN).
\]

Hence \( dS = 2S \wedge S \) holds.

Lemma 4.1 and a direct calculation yield

\[
\nabla^\theta = d + (\cos \theta - 1)S + (\sin \theta) \ast S,
\]

\[
d^\nabla^\theta \circ \nabla^\theta
= (\cos \theta - 1)dS + ((\cos \theta - 1)S + (\sin \theta) \ast S) \wedge ((\cos \theta - 1)S + (\sin \theta) \ast S)
= (\cos \theta - 1)dS + (\cos \theta - 1)^2 S \wedge S + (\cos \theta - 1)(\sin \theta)S \wedge \ast S
+ (\sin \theta)(\cos \theta - 1) \ast S \wedge S + (\cos \theta - 1)(\sin \theta)^2 \ast S \wedge \ast S
= (\cos \theta - 1)dS - 2(\cos \theta - 1)S \wedge S = 0.
\]

Hence \( F \) with \( \nabla \) and \( S \) is a \( tt^* \)-bundle. \( \square \)

For a harmonic map from a Riemann surface to \( S^2 \), we have two \( tt^* \)-bundles. One is the \( tt^* \)-bundle in Theorem 4.1. The other is that in the theory of quaternionic holomorphic line bundles (see [8]). These do not coincide directly as the fiber of the former is \( C\ell_3 \) and that of the latter is \( C\ell_2 \).
Acknowledgements.

The authors are very grateful to Claus Hertling for the comments to a preliminary version of this paper.

References


