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Short-distance repulsion in three-nucleon forces from perturbative quantum chromodynamics

Sinya Aoki¹, Janos Balog² and Peter Weisz³

¹ Graduate School of Pure and Applied Sciences, University of Tsukuba, Tsukuba, Ibaraki 305-8571, Japan

² Research Institute for Particle and Nuclear Physics, 1525 Budapest 114, Pf. 49, Hungary

³ Max-Planck-Institut für Physik, Föhringer Ring 6, D-80805 München, Germany

E-mail: saoki@het.ph.tsukuba.ac.jp, balog@rmki.kfki.hu and pew@mpp.mpg.de

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Abstract. We investigate the short-distance behavior of three-nucleon forces (3NF) defined through the Nambu–Bethe–Salpeter (NBS) wave functions using the operator product expansion and calculating anomalous dimensions of nine-quark operators in perturbative quantum chromodynamics (QCD). As in the case of NN forces considered previously, we show that 3NF have repulsions at short distance at one-loop, which becomes exact in the short-distance limit thanks to the asymptotic freedom of QCD. Moreover, these behaviors are universal in the sense that they do not depend on the energy of the NBS wave function for three nucleons.

Contents

1. Introduction	2
2. Renormalization group (RG) analysis and operator product expansion for three-nucleon forces (3NF)	5
2.1. Operator product expansion (OPE) and RG equations	5
2.2. Perturbative solution to the RG equation and factorization of OPE	6
2.3. Three-nucleon forces from the OPE	7
3. Anomalous dimensions for three nucleons (3N) at one-loop	8
3.1. OPE for 3N operators at tree level	8
3.2. General formula for the divergent part at one-loop	9
3.3. Results of anomalous dimensions for nine-quark operators at one-loop	10
4. Short-distance repulsion of 3NF	13
Acknowledgments	15
References	15

1. Introduction

Realistic nuclear potentials between two nucleons (2N), determined precisely from 2N scattering data together with the deuteron binding energy, have often been used to study nuclear many-body problems. These two-nucleon forces (2NF), however, generally underestimate the experimental binding energies of light nuclei [1, 2] and this fact indicates the necessity of taking into account three-nucleon forces (3NF). In addition, an indication of 3NF is observed in high-precision deuteron–proton elastic scattering data at intermediate energies [3–5]. Various signatures of 3NF in experimental/theoretical studies are reviewed in [6] (the role of 3NF in the context of low-momentum nuclear forces, see Bogner *et al* [6]).

The 3NF may also play an important role for various phenomena in nuclear physics and astrophysics, which include (i) the backward scattering cross sections in nucleus–nucleus elastic scattering [7], (ii) the anomaly in the oxygen isotopes near the neutron drip-line [8] and (iii) the nuclear equation of state at high density relevant to the physics of neutron stars [9]. Universal short-distance repulsion for three baryons (nucleons and hyperons) is also suggested to explain the observed maximum mass of neutron stars [10].

Despite its phenomenological importance, a microscopic understanding of 3NF is still limited, due to difficulties in studying 3NF experimentally. Pioneered by Fujita and Miyazawa [11], the long-range part of 3NF has been modeled by two-pion exchange [12], which is known to be attractive at long distance. In addition, a repulsive component of 3NF at short distance is introduced in a purely phenomenological way [13].

To go beyond phenomenology, it is most desirable to determine 3NF directly from the fundamental degrees of freedom, the quarks and the gluons, on the basis of quantum chromodynamics (QCD). Recently, the first investigation of this kind was attempted using lattice QCD simulations, where 3NF have been extracted from the Nambu–Bethe–Salpeter (NBS) wave function for a specific alignment of three nucleons [14–16]. The method used there had been previously employed to extract nucleon–nucleon potentials (i.e. 2NF) [17–20] as follows. The NBS wave function for two nucleons is defined by

$$\varphi_E(\vec{r}) = \langle 0 | N(\vec{x} + \vec{r}, 0) N(\vec{x}, 0) | 2N, W \rangle, \quad (1)$$

where $|2N, W\rangle$ is a QCD eigenstate with energy $W = 2\sqrt{m_N^2 + k^2}$ with m_N being the nucleon mass, $E = k^2/m_N$ represents the non-relativistic kinetic energy and N is a nucleon interpolating operator made of three quarks such as $N(x) = \epsilon^{abc} q^a(x) q^b(x) q^c(x)$. The non-local but energy-independent potential (more precisely the half off-shell T -matrix) is extracted from this NBS wave function as

$$(E - H_0)\varphi_E(\vec{r}) = \int U(\vec{r}, \vec{r}')\varphi_E(\vec{r}')d^3r', \quad (2)$$

where $H_0 = -\nabla^2/m_N$. The non-local potential can be expanded in terms of the velocity (derivative) with local function as

$$U(\vec{r}, \vec{r}') = V(\vec{r}, \vec{\nabla})\delta^3(\vec{r} - \vec{r}'), \quad (3)$$

which becomes

$$V(\vec{r}, \vec{\nabla}) = V_0(r) + V_\sigma(r)\vec{\sigma}_1 \cdot \vec{\sigma}_2 + V_T(r)S_{12} + V_{LS}(r)\vec{L} \cdot \vec{S} + O(\nabla^2) \quad (4)$$

at the lowest few orders, where $r = |\vec{r}|$, $\vec{\sigma}_i$ represents the Pauli matrices acting on the spin index of the i th nucleon, $\vec{S} = (\vec{\sigma}_1 + \vec{\sigma}_2)/2$ is the total spin, $\vec{L} = \vec{r} \times \vec{p}$ is the angular momentum and

$$S_{12} = 3 \frac{(\vec{r} \cdot \vec{\sigma}_1)(\vec{r} \cdot \vec{\sigma}_2)}{r^2} - \vec{\sigma}_1 \cdot \vec{\sigma}_2 \quad (5)$$

is the tensor operator. This method has been shown to work well. The central potentials at leading order in the expansion have qualitatively reproduced the common features of phenomenological 2N potentials: the force is attractive at medium to long distance, while it has a characteristic repulsive core at short distance. See also [21, 22] for a summary of the results and recent developments.

The present authors have investigated short-distance behaviors of the 2NF defined in the framework mentioned above using the operator product expansion (OPE) and perturbation theory thanks to asymptotic freedom (AF) of QCD [23–25]. (See also a similar attempt for the solvable models in two dimensions [26].) The behavior of the NBS wave function $\varphi_E(\vec{r})$ at short distance ($r \rightarrow 0$) is encoded in the OPE of the 2N operators:

$$N(\vec{x}/2, 0) N(-\vec{x}/2, 0) \approx \sum_k D_k(\vec{x}) \mathcal{O}_k(\vec{0}, 0), \quad (6)$$

where $\{\mathcal{O}_k\}$ is a set of local color singlet six-quark operators with 2N quantum numbers. Asymptotically, the \vec{x} -dependence and energy dependence of the wave function are factorized into

$$\varphi_E(\vec{x}) \approx \sum_k D_k(\vec{x}) \langle 0 | \mathcal{O}_k(\vec{0}, 0) | 2N, W \rangle. \quad (7)$$

Standard renormalization group (RG) analysis gives [24] the leading short-distance behavior of the OPE coefficient function as

$$D_k(\vec{x}) \approx \left(\ln \frac{r_0}{r} \right)^{\nu_k} c_k, \quad (8)$$

where ν_k is related to the one-loop coefficient of the anomalous dimension of the operator \mathcal{O}_k , c_k is the tree-level contribution of $D_k(\vec{0})$ and finally r_0 is some typical non-perturbative QCD

scale. Assuming that its matrix element does not vanish, the operator with largest RG power ν_k dominates the wave function (7) at short distances. We denote the largest power by ν_1 and the second largest one by ν_2 .

If ν_1 is nonzero, this leads to the leading asymptotics of the s-wave potential of the form

$$V(r) \approx -\frac{\nu_1}{r^2 \left(\ln \frac{r_0}{r}\right)}, \quad (9)$$

which is attractive for $\nu_1 > 0$ and repulsive for $\nu_1 < 0$.

If $\nu_1 = 0$, the situation is more complicated. The relative sign of the ratio R between the leading and the subleading contributions becomes important and we find that

$$V(r) \approx -R \frac{\nu_2}{r^2 \left(\ln \frac{r_0}{r}\right)^{1-\nu_2}}. \quad (10)$$

If R is positive, the potential is repulsive, while it is attractive for negative R . A system of 2N corresponds to this degenerate case. Unfortunately, in this case R depends on the energy E . In [24] it is argued that in the relevant energy range the relative coefficient R is positive, so that the short-distance limit of the nucleon potential is repulsive.

In this paper, we extend the above OPE analysis to the 3NF, in order to investigate the short-distance behavior of the 3NF in the same scheme as used in lattice QCD [14–16]. The corresponding equal time NBS wave function for three nucleons is given by

$$\psi_{3N}(\vec{r}, \vec{\rho}) = \langle 0 | N(\vec{x}_1, 0) N(\vec{x}_2, 0) N(\vec{x}_3, 0) | E_{3N} \rangle, \quad (11)$$

where E_{3N} and $|E_{3N}\rangle$ denote the energy and the 3N state. We introduce the Jacobi coordinates $\vec{R} = (\vec{x}_1 + \vec{x}_2 + \vec{x}_3)/3$, $\vec{r} = (\vec{x}_1 - \vec{x}_2)/2$, $\vec{\rho} = (\vec{x}_3 - (\vec{x}_1 + \vec{x}_2)/2)/\sqrt{3}$. From this wave function, the three nucleon potential is defined by

$$\left[-\frac{1}{2\mu_r} \nabla_r^2 - \frac{1}{2\mu_\rho} \nabla_\rho^2 + \sum_{i < j} V_{2N}(\vec{r}_{ij}) + V_{3NF}(\vec{r}, \vec{\rho}) \right] \psi_{3N}(\vec{r}, \vec{\rho}) = E_{3N} \psi_{3N}(\vec{r}, \vec{\rho}), \quad (12)$$

where $V_{2N}(\vec{r}_{ij})$ with $\vec{r}_{ij} = \vec{x}_i - \vec{x}_j$ denotes 2NF between (i, j) -pair, $V_{3NF}(\vec{r}, \vec{\rho})$ the 3NF and $\mu_r = \mu_\rho = m_N/2$ the reduced masses.

In section 2, we start with RG considerations and OPE, which are relevant for 3NF. The anomalous dimensions of nine-quark operators are computed in section 3. Finally, we discuss the short-distance behavior of 3NF in section 4. For the convenience of the reader we give a brief summary of our results here. The OPE analysis shows that the 3N central potential at short distance behaves as

$$V_{3NF}(\vec{r}, \vec{\rho}) \simeq \frac{1}{m_N} \frac{-4\beta_A^{\text{tree}}}{s^2 (-\log(\Lambda s))}, \quad (13)$$

as $s = \sqrt{(\vec{r})^2 + (\vec{\rho})^2} \rightarrow 0$, where β_A^{tree} is given by

$$\beta_A^{\text{tree}} = -14/(33 - 2N_f), \quad (14)$$

with N_f being the number of dynamical quarks, and Λ is the perturbative Λ -parameter in the MS scheme [24].

Unlike the 2NF where the situation was not completely determined by PT alone, it is shown that the 3N potential always has a repulsive core. Furthermore, it is universal in the sense that

it does not depend on the details of the 3N state used to define the NBS wave function such as the energy of the state. This theoretical determination of the asymptotic behavior of the NBS 3NF at short distance not only provides a good boundary condition for the 3NF computed at all lengths from lattice QCD [14–16] in the near future, but also gives a deeper understanding of the short-distance repulsion of the 3NF.

2. Renormalization group (RG) analysis and operator product expansion for three-nucleon forces (3NF)

2.1. Operator product expansion (OPE) and RG equations

Let us recall the OPE for $\rho, r \rightarrow 0^4$:

$$O_1(\vec{r} - \vec{\rho}/\sqrt{3})O_2(-\vec{r} - \vec{\rho}/\sqrt{3})O_3(2\vec{\rho}/\sqrt{3}) \simeq D_B(\vec{r}, \vec{\rho}) O_B(0). \quad (15)$$

We will need it in the special case when the operators O_1, O_2, O_3 on the left-hand side are nucleon operators and the set of operators O_B on the right-hand side are local nine-quark operators of engineering dimension $27/2$ and higher. All operators in (15) are renormalized ones. As is well known, the nucleon operators are renormalized diagonally and we recall the perturbative expansion of the corresponding RG gamma function:

$$\gamma_N(g) = \gamma_N^{(1)} g^2 + O(g^4), \quad \gamma_N^{(1)} = 24d, \quad d = \frac{1}{32N_c\pi^2} = \frac{1}{96\pi^2}. \quad (16)$$

Our subsequent analysis, which is completely analogous to our discussion in [24], is based on the RG equations satisfied by the coefficient functions:

$$\mathcal{D}D_B(g, \mu, \vec{r}, \vec{\rho}) + D_A(g, \mu, \vec{r}, \vec{\rho}) \tilde{\gamma}_{AB}(g) = 0, \quad (17)$$

where the RG operator, expressing the independence of physics from the renormalization scale parameter μ , is

$$\mathcal{D} = \mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \quad (18)$$

with the beta function

$$\beta(g) = -\beta_0 g^3 + O(g^5), \quad \beta_0 = \frac{1}{16\pi^2} \left\{ \frac{11}{3} N_c - \frac{2}{3} N_f \right\}. \quad (19)$$

In (17) $\tilde{\gamma}_{AB}$ is the effective gamma function matrix

$$\tilde{\gamma}_{AB}(g) = \gamma_{AB}(g) - [3\gamma_N(g)] \delta_{AB}, \quad (20)$$

where

$$\gamma_{AB}(g) = \gamma_{AB}^{(1)} g^2 + O(g^4) \quad (21)$$

is the mixing matrix of nine-quark operators.

⁴ Summation of repeated indices is assumed throughout this paper unless indicated otherwise.

2.2. Perturbative solution to the RG equation and factorization of OPE

We want to solve the vector partial differential equation (17), and for this purpose it is useful to introduce $\hat{U}_{AB}(g)$, the solution to the matrix ordinary differential equation

$$\beta(g) \frac{d}{dg} \hat{U}_{AB}(g) = \tilde{\gamma}_{AC}(g) \hat{U}_{CB}(g) \quad (22)$$

and its matrix inverse $U_{AB}(g)$. We will assume that the coefficient has the perturbative expansion

$$\begin{aligned} D_A(g, \mu, \vec{r}, \vec{\rho}) &= \sum_{\alpha_1 + \alpha_2 = \tilde{d}_A} r^{\alpha_1} \rho^{\alpha_2} D_A^{\alpha_1 \alpha_2}(g; \mu s, \omega, \Omega_r, \Omega_\rho) \\ &= \sum_{\alpha_1 + \alpha_2 = \tilde{d}_A} r^{\alpha_1} \rho^{\alpha_2} [D_{A;0}^{\alpha_1 \alpha_2} + g^2 D_{A;1}^{\alpha_1 \alpha_2}(\mu s, \omega, \Omega_r, \Omega_\rho) + O(g^4)], \end{aligned} \quad (23)$$

where $s^2 = r^2 + \rho^2$ and $\tan \omega = \rho/r$ with $r = |\vec{r}|$, $\rho = |\vec{\rho}|$, and Ω_r, Ω_ρ are solid angles of the vectors \vec{r} and $\vec{\rho}$, respectively. Here $\tilde{d}_A = d_A - (d_1 + d_2 + d_3)$ is the dimension of the coefficient function. Note that in the massless theory, operators of different dimensions do not mix. In the full theory quark mass terms are also present, but they correspond to higher powers in r and ρ and can therefore be neglected.

We will also assume that the basis of operators has been chosen such that the one-loop mixing matrix is diagonal:

$$\tilde{\gamma}_{AB}(g) = 2\beta_0 \beta_A g^2 \delta_{AB} + O(g^4). \quad (24)$$

In such a basis the solution of (22) in perturbation theory takes the form

$$\hat{U}_{AB}(g) = \{\delta_{AB} + R_{AB}(g)\} g^{-2\beta_B}, \quad (25)$$

where $R_{AB}(g) = O(g^2)$, with possible multiplicative $\log g^2$ factors, depending on the details of the spectrum of one-loop eigenvalues β_A .

Having solved (22) we can write down the most general solution of (17):

$$D_B^{\alpha_1 \alpha_2}(g; \mu s, \omega, \Omega_r, \Omega_\rho) = F_A^{\alpha_1 \alpha_2}(\Lambda s, \omega, \Omega_r, \Omega_\rho) U_{AB}(g). \quad (26)$$

Here the vector $F_A^{\alpha_1 \alpha_2}$ is RG-invariant. Introducing the running coupling \bar{g} by the usual definition

$$\ln \frac{\Lambda}{\mu} = \int_{\bar{g}}^g \frac{dx}{\beta(x)}, \quad (27)$$

$F_B^{\alpha_1 \alpha_2}$ can be rewritten as

$$F_B^{\alpha_1 \alpha_2}(\Lambda s, \omega, \Omega_r, \Omega_\rho) = D_A^{\alpha_1 \alpha_2}(\bar{g}; 1, \omega, \Omega_r, \Omega_\rho) \hat{U}_{AB}(\bar{g}). \quad (28)$$

Since, because of AF, for $s \rightarrow 0$ also $\bar{g} \rightarrow 0$ as

$$\bar{g}^2 \approx -\frac{1}{2\beta_0 \ln(\Lambda s)}, \quad (29)$$

$F_B^{\alpha_1 \alpha_2}$ can be calculated perturbatively using (23) and (25).

Putting everything together, we find that the right-hand side of the OPE (15) can be rewritten as

$$O_1(\vec{r} - \vec{\rho}/\sqrt{3}) O_2(-\vec{r} - \vec{\rho}/\sqrt{3}) O_3(2\vec{\rho}/\sqrt{3}) \simeq \sum_{\alpha_1 + \alpha_2 = \tilde{d}_B} r^{\alpha_1} \rho^{\alpha_2} F_B^{\alpha_1 \alpha_2}(\Lambda s, \omega, \Omega_r, \Omega_\rho) \tilde{O}_B(0), \quad (30)$$

where

$$\tilde{O}_B = U_{BC}(g) O_C. \quad (31)$$

There is a factorization of the operator product into perturbative and non-perturbative quantities: $F_B^{\alpha_1 \alpha_2}(\Lambda s, \omega, \Omega_r, \Omega_\rho)$ is perturbative and calculable (for $s \rightarrow 0$) thanks to AF, whereas the matrix elements of \tilde{O}_B are non-perturbative (but s -independent). Note that $d_C = d_B$ in (31).

2.3. Three-nucleon forces from the OPE

Using the results in the previous subsection, the NBS wave function for 3N can be written at short distance as

$$\psi_{3N}(\vec{r}, \vec{\rho}) \simeq \sum_{A,B} \sum_{\alpha_1 + \alpha_2 = \tilde{d}_A} r^{\alpha_1} \rho^{\alpha_2} D_A^{\alpha_1 \alpha_2}(\vec{g}, 1, \omega, \Omega_r, \Omega_\rho) \hat{U}_{AB}(\vec{g}) \langle 0 | \tilde{O}_B(0) | E_{3N} \rangle.$$

Since an $r^{\alpha_1} \rho^{\alpha_2}$ term produces angular momenta $l_1 \leq \alpha_1$ and $l_2 \leq \alpha_2$, we can write

$$D_A^{\alpha_1 \alpha_2}(\vec{g}, 1, \omega, \Omega_r, \Omega_\rho) \simeq \sum_{m_1 m_2} D_A^{\alpha_1 m_1 \alpha_2 m_2}(\vec{g}, 1, \omega) Y_{\alpha_1 m_1}(\Omega_r) Y_{\alpha_2 m_2}(\Omega_\rho), \quad (32)$$

up to less singular terms at short distances. Then the sum of 2N and 3NF potentials V_{2N+3NF} is extracted as

$$\begin{aligned} V_{2N+3NF}(\vec{r}, \vec{\rho}) &\equiv \sum_{i < j} V_{2N}(\vec{r}_{ij}) + V_{3NF}(\vec{r}, \vec{\rho}) \\ &= E_{3N} + \frac{1}{m_N} \frac{(\nabla_r^2 + \nabla_\rho^2) \psi_{3N}(\vec{r}, \vec{\rho})}{\psi_{3N}(\vec{r}, \vec{\rho})}, \end{aligned} \quad (33)$$

where

$$\nabla_r^2 = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{\hat{L}_r^2}{r^2} \equiv d_r^2 - \frac{\hat{L}_r^2}{r^2}, \quad \nabla_\rho^2 = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \rho^2 \frac{\partial}{\partial \rho} - \frac{\hat{L}_\rho^2}{\rho^2} \equiv d_\rho^2 - \frac{\hat{L}_\rho^2}{\rho^2}.$$

Since

$$\nabla_r^2 r^l Y_{lm}(\Omega_r) = \nabla_\rho^2 \rho^l Y_{lm}(\Omega_\rho) = 0, \quad (34)$$

we have

$$\begin{aligned} (\nabla_r^2 + \nabla_\rho^2) \psi_{3N}(\vec{r}, \vec{\rho}) &\simeq \sum_{AB} \sum_{\alpha_1 + \alpha_2 = \tilde{d}_A} r^{\alpha_1} Y_{\alpha_1 m_1}(\Omega_r) \rho^{\alpha_2} Y_{\alpha_2 m_2}(\Omega_\rho) \\ &\times \langle 0 | \tilde{O}_B(0) | E_{3N} \rangle (d_r^2 + d_\rho^2) \left[D_A^{\alpha_1 m_1 \alpha_2 m_2}(\vec{g}, 1, \omega) \hat{U}_{AB}(\vec{g}) \right]. \end{aligned} \quad (35)$$

Since terms with $\tilde{d}_A = 0$ dominate in Ψ_{3N} at short distance, contributions from $\tilde{d}_A \neq 0$ terms to 2N+3NF potentials are suppressed by an $r^{\alpha_1} \rho^{\alpha_2}$ factor, so that they do not contribute at short

distance. Therefore we consider terms with $\tilde{d}_A = 0$ ($\alpha_1 = \alpha_2 = 0$) hereafter and do not write α_i dependence in coefficients. We then have

$$(\nabla_r^2 + \nabla_\rho^2)\psi_{3N}(\vec{r}, \vec{\rho}) \simeq \sum_{AB} \langle 0 | \bar{O}_B(0) | E_{3N} \rangle \times (d_r^2 + d_\rho^2) \left[D_A(\bar{g}, 1, \omega) \hat{U}_{AB}(\bar{g}) \right]. \quad (36)$$

In terms of s and ω we write

$$d_r^2 + d_\rho^2 = \frac{1}{s^5} \frac{\partial}{\partial s} s^5 \frac{\partial}{\partial s} + \frac{1}{s^2} \left[\frac{\partial^2}{\partial \omega^2} + 4 \frac{\cos(2\omega)}{\sin(2\omega)} \frac{\partial}{\partial \omega} \right], \quad (37)$$

so that the ω -dependent part gives a $1/s^2$ contribution unless $\omega = 0, \pi/2, \pi, 3\pi/2$, where either $r = 0$ or $\rho = 0$. We assume $r \neq 0$ and $\rho \neq 0$ in our calculation. Since an ω dependence appears only at one-loop or higher orders in D_A , we can neglect it unless an operator O_A which appears at ℓ_A loop has large anomalous dimension such that $\beta_A - \ell_A$ is larger than other β_B corresponding to operators O_B appearing at tree level. As we will see later, such operators are absent; it is then enough to consider the tree-level contribution in D_A , so that ω -dependent terms in D_A can be neglected. The largest eigenvalue among operators appearing at tree level is thus denoted by β_A , which corresponds to ν_1 discussed in the introduction for 2N forces.

We then obtain

$$(d_r^2 + d_\rho^2) \left[D_A(\bar{g}, 1, \omega) \hat{U}_{AB}(\bar{g}) \right] \simeq D_{A:0} (d_r^2 + d_\rho^2) \bar{g}^{-2\beta_A} \simeq D_{A:0} \frac{-4\beta_A}{s^2(-\log(\Lambda s))} (-2\beta_0 \log(\Lambda s))^{\beta_A}. \quad (38)$$

The NBS wave function is dominated by the term with largest β_A . If we assume that β_A is nonzero, we finally obtain

$$V_{2N+3NF}(\vec{r}, \vec{\rho}) \simeq \frac{1}{m_N} \frac{-4\beta_A}{s^2(-\log(\Lambda s))}. \quad (39)$$

3. Anomalous dimensions for three nucleons (3N) at one-loop

3.1. OPE for 3N operators at tree level

The general form of a gauge invariant local three-quark operator is given by

$$B_\Gamma^F(x) \equiv B_{\alpha\beta\gamma}^{fgh}(x) = \varepsilon^{abc} q_\alpha^{a,f}(x) q_\beta^{b,g}(x) q_\gamma^{c,h}(x), \quad (40)$$

where α, β, γ are spinor, f, g, h are flavor and a, b, c are color indices of the (renormalized) quark field q . The color index runs from 1 to $N_c = 3$, the spinor index from 1 to 4, and the flavor index from 1 to N_f . Note that $B_{\alpha\beta\gamma}^{fgh}$ is symmetric under any interchange of pairs of indices (e.g. $B_{\alpha\beta\gamma}^{fgh} = B_{\beta\alpha\gamma}^{gfh}$) because the quark fields anticommute. For simplicity, we sometimes use notation such as $F = fgh$ and $\Gamma = \alpha\beta\gamma$ as indicated in (40).

The usual nucleon operator that is employed in lattice simulations is constructed from the above operators as

$$B_\alpha^f(x) = (P_{+4})_{\alpha\alpha'} B_{\alpha'\beta\gamma}^{fgh} (C\gamma_5)_{\beta\gamma} (i\tau_2)^{gh}, \quad (41)$$

where $P_{+4} = (1 + \gamma_4)/2$ is the projection to the large spinor component, $C = \gamma_2\gamma_4$ is the charge conjugation matrix, and τ_2 is the Pauli matrix in the flavor space (for $N_f = 2$) given by $(i\tau_2)^{fg} = \varepsilon^{fg}$. Both $C\gamma_5$ and $i\tau_2$ are anti-symmetric under the interchange of two indices, so

that the nucleon operator has spin-1/2 and isospin-1/2. Although an explicit form of the γ matrices is unnecessary in principle, we find it convenient to use a (chiral) convention given by

$$\gamma_k = \begin{pmatrix} 0 & i\sigma_k \\ -i\sigma_k & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}.$$

As discussed in the previous section, the OPE at the tree level (generically) dominates at short distance. The OPE of 3N operators given above at tree level becomes

$$B_\alpha^f(x+y-z/\sqrt{3})B_\beta^g(x-y-z/\sqrt{3})B_\gamma^h(x+2z/\sqrt{3}) = B_\alpha^f(x)B_\beta^g(x)B_\gamma^h(x) + \dots, \quad (42)$$

where \dots denote higher-dimensional operators, which do not contribute at short distance. The leading operators couple only to the states with $L = 0$ (the S -state).

If we construct local 3N operators at $L = 0$ from $B_\alpha^f(x)$ for nucleons (which only involve two different flavors), there is only one such operator, which has $I = 1/2$ and $S = 1/2$, due to the Pauli statistics of nucleons. Explicitly, it is given by

$$(B_{\text{tree}}^{(3)})_{\alpha\beta\alpha}^{ffg} \equiv B_\alpha^f B_\beta^f B_\alpha^g, \quad B_\alpha^f = B_{\alpha+\hat{\alpha}, [2, 1] + [\hat{2}, \hat{1}]}^{ffg}, \quad (43)$$

where $f \neq g$, $f, g = u, d$ and $\alpha \neq \beta$, $\alpha, \beta = 1, 2$, $\hat{1} = 3$, $\hat{2} = 4$ for the explicit form of the γ matrices. Above, no summation is taken for f and α .

3.2. General formula for the divergent part at one-loop

As shown in [24], using dimensional regularization in $D = 4 - 2\epsilon$ dimensions, the gauge invariant part of the divergence from diagrams involving exchange of a gluon between any pair of quark fields is given by

$$\left[q_\alpha^{a,f}(x) q_\beta^{b,g}(x) \right]^{1\text{-loop, div}} = \frac{g^2 d}{\epsilon} [\mathbf{T}_0 \cdot q^a(x) \otimes q^b(x)]_{\alpha,\beta}^{fg}, \quad (44)$$

where

$$(\mathbf{T}_0)_{\alpha\alpha_1, \beta\beta_1}^{ff_1, gg_1} = \delta^{ff_1} \delta^{gg_1} [\delta_{\alpha\alpha_1} \delta_{\beta\beta_1} - 2\delta_{\beta\alpha_1} \delta_{\alpha\beta_1}] + N_c \delta^{gf_1} \delta^{fg_1} [\delta_{\beta\alpha_1} \delta_{\alpha\beta_1} - 2\delta_{\alpha\alpha_1} \delta_{\beta\beta_1}], \quad (45)$$

for either $\alpha_1, \beta_1 \in \{1, 2\}$ (right-handed) or $\alpha_1, \beta_1 \in \{3, 4\}$ (left-handed), and it vanishes for other combinations.

The operator in equation (43) can be written as a linear combination of simple operators $[BBB]_{\Gamma_1\Gamma_2\Gamma_3}^{F_1F_2F_3}$. According to this one-loop formula for divergences, such a simple operator mixes only with operators $[BBB]_{\Gamma_A\Gamma_B\Gamma_C}^{F_A F_B F_C}$, which preserve the set of flavor and Dirac indices in the chiral basis as

$$F_1 \cup F_2 \cup F_3 = F_A \cup F_B \cup F_C, \quad \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 = \Gamma_A \cup \Gamma_B \cup \Gamma_C. \quad (46)$$

Note, however, that such operators are not all linearly independent. In the case of a 2N operator, we have the following constraint:

$$3[BB]_{\Gamma_1, \Gamma_2}^{F_1, F_2} + \sum_{i,j=1}^3 [BB]_{(\Gamma_1, \Gamma_2)[i, j]}^{(F_1, F_2)[i, j]} = 0, \quad (47)$$

which comes from the general identity

$$N_c \varepsilon^{a_1 \dots a_{N_c}} \varepsilon^{b_1 \dots b_{N_c}} = \sum_{j,k=1}^{N_c} \varepsilon^{a_1 \dots a_{j-1} b_k a_{j+1} \dots a_{N_c}} \varepsilon^{b_1 \dots b_{k-1} a_j b_{k+1} \dots b_{N_c}}. \quad (48)$$

Here $[i, j]$ means a simultaneous exchange between the i th indices of F_1, Γ_1 and the j th indices of F_2, Γ_2 . This identity can be generalized to

$$N_c \varepsilon^{a_1 \dots a_{N_c}} \varepsilon^{b_1 \dots b_{N_c}} \varepsilon^{c_1 \dots c_{N_c}} = \sum_{i,j,k=1}^{N_c} \varepsilon^{a_1 \dots a_{i-1} b_j a_{i+1} \dots a_{N_c}} \varepsilon^{b_1 \dots b_{j-1} c_k b_{j+1} \dots b_{N_c}} \varepsilon^{c_1 \dots c_{k-1} a_i c_{k+1} \dots c_{N_c}}, \quad (49)$$

from which we have

$$3[BBB]_{\Gamma_1, \Gamma_2, \Gamma_3}^{F_1, F_2, F_3} - \sum_{i,j,k=1}^3 [BBB]_{(\Gamma_1, \Gamma_2, \Gamma_3)[i,j,k]}^{(F_1, F_2, F_3)[i,j,k]} = 0, \quad (50)$$

where the i th index of ABC , the j th index of DEF and the k th index of GHI are cyclically interchanged in $(ABC, DEF, GHI)[i, j, k]$. For example,

$$(\Gamma_1, \Gamma_2, \Gamma_3)[1, 1, 2] = (\beta_3 \beta_1 \gamma_1, \alpha_1 \beta_2 \gamma_2, \alpha_3 \alpha_2 \gamma_3),$$

$$(\Gamma_1, \Gamma_2, \Gamma_3)[1, 2, 3] = (\gamma_3 \beta_1 \gamma_1, \alpha_2 \alpha_1 \gamma_2, \alpha_3 \beta_3 \beta_2).$$

Note that the cyclic interchange of indices occurs simultaneously for both Γ_i and F_i in the above formula. Both 2N and 3N identities are incorporated into our calculation.

3.3. Results of anomalous dimensions for nine-quark operators at one-loop

In tables 1–4, we give all eigenvalues of the matrix $\gamma_{AB}^{(1)}$ in units of $2d$, which were calculated and checked independently by Mathematica and Maple programs, for $F_1 \cup F_2 \cup F_3 = (fffffgggg)$ with $f \neq g$ and all independent combinations of $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. The five digits n_1, n_3, n_5, n_7, n_9 in the isospin column give the number of representations with isospins $1/2, 3/2, 5/2, 7/2, 9/2$, respectively. For example, 32100 means 3 operators with $I = 1/2$, 2 with $I = 3/2$, 1 with $I = 5/2$ and 0 with $I = 7/2, 9/2$.

The results in the tables show some notable patterns. Firstly, the eigenvalues $\gamma_j/2d$ are all even integers; this is nontrivial since there appears to be considerable mixing in our initial operator bases. Secondly, there is a tendency for the operators with larger isospin to have smaller (more negative) eigenvalues. Thirdly, there are relations between the entries in the tables for different Dirac indices; for example, the isospin degeneracies for the indices 111113344 and 111122223 in table 1 are identical and, furthermore, all the corresponding eigenvalues are related by a common shift of -32 . These observations suggest that there is an underlying algebraic structure that we have unfortunately not yet been able to reveal. Note that a combination obtained from another one by

- (i) the interchange $(1, 2) \leftrightarrow (3, 4)$ or
- (ii) the simultaneous interchange of $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$ or
- (iii) the interchange $1 \leftrightarrow 2$

has obviously the same spectrum of anomalous dimensions and is for this reason not listed separately.

The star symbol * next to the eigenvalues means that there is a corresponding operator that overlaps with the tree operator in equation (43). Since the tree operator has $I = 1/2$, this can happen only if the corresponding n_1 is different from zero. The tree operator is invariant under symmetry (i) above, whereas its two-spin components are exchanged under symmetry (ii). This means that whenever the tree operator overlaps with a particular operator, it also overlaps with

Table 1. Eigenvalues γ_j of the anomalous dimension matrix γ and isospins of the corresponding eigenvectors for the case 5f4g. The five digits n_1, n_3, n_5, n_7, n_9 in the isospins column give the number of representations with isospins $1/2, 3/2, 5/2, 7/2, 9/2$, respectively.

Dirac indices	$\gamma_j/(2d)$	Isospins	Dirac indices	$\gamma_j/(2d)$	Isospins
111111222	-84	01000	111122223	-76	01100
111111333	-36	01000		-60	11000
111112222	-84	01000		-52	11000
	-60*	10000		-48*	11000
111113333	-28	11000		-40*	10000
111111223	-60	11000	111122233	-76	01110
111111334	-36	01000		-52	12100
	-12	10000		-46	12100
111111332	-44	11000		-44	11000
111112223	-76	01100		-34	21000
	-60	11000		-28 [‡]	21000
	-52	11000	111133344	-60	01110
	-40 [‡]	10000		-36	12100
111113334	-44	01100		-30	12100
	-28	11000		-28	11000
	-20	11000		-18	21000
	-8	10000		-12	21000
111113332	-36	23100	111133322	-52	12210
	-24	10000		-36	23100
111112233	-52	12100		-28	22000
	-44	11000		-24	10000
	-34	21000		-16	11000
111113344	-44	01100	111222333	-84	01111
	-28	11000		-52	12210
	-20	11000		-48	01100
	-16	11000		-36	23100
	-8	10000		-28	22000
111133332	-36	12100		-24	32100
	-28	11000		-16	11000
	-18	21000	111111234	-44	11000
				-28	10000

these transformed ones. The meaning of the symbol [‡] is an overlap between the tree operator and the transform of the operator under symmetry (iii). It is intriguing that the tree operator generally overlaps with the largest eigenvalue for given Dirac indices; again a fact for which we do not yet have a simple explanation.

As a simple example let us consider the Dirac index distribution 111112222 (the third entry in table 1). The part of the tree operator relevant for this case is

$$T_1 = (B_{112}^{ffg} - B_{121}^{ffg}) (B_{212}^{ffg} - B_{221}^{ffg}) (B_{112}^{gfg} - B_{121}^{gfg}). \quad (51)$$

There are altogether 53 local nine-quark operators with this distribution of Dirac indices, but the number of independent ones is reduced to 2 after imposing all the gauge identities (47) and (50).

Table 2. As in Table 1. (Continued.)

Dirac indices	$\gamma_j/(2d)$	Isospins	Dirac indices	$\gamma_j/(2d)$	Isospins
111112234	-60	11100	111133324	-52	11110
	-52	12100		-44	01100
	-44	11000		-40	11100
	-42	11000		-36	12100
	-36	01000		-34	12100
	-34	21000		-28	22100
	-28	10000		-22	21000
	-18	10000		-20	11000
111113324	-48	11100	-18	21000	
	-36	24100	-16	01000	
	-30	11000	-10	11000	
	-24	10000	-8	10000	
	-18	10000	2	10000	
	-12	21000	111122334	-64	11110
111122234	-76	01110		-52	12210
	-60	11200		-48	11100
	-54	11100		-46	01100
	-52	12100		-40	33200
	-46	12100		-36	24100
	-44	11000		-30	11000
	-42	11000		-28	24100
	-36*	12000		-24	10000
	-34	21000		-22	22000
	-30	01000		-18	10000
	-28*	31000		-16	11000
	-18	10000	-12	21000	
-12*	10000	-10 ^b	10000		
		8 ^b	10000		

A possible choice is

$$O_1 = B_{111}^{fff} B_{221}^{ffg} B_{122}^{ggg}, \quad O_2 = B_{112}^{fff} B_{221}^{ffg} B_{112}^{ggg}. \quad (52)$$

Using the gauge identities we have

$$T_1 = \frac{5}{6} O_1 - 5 O_2, \quad (53)$$

which is proportional to the operator (of $I = 1/2$) corresponding to anomalous dimension -60 in our units. The other combination

$$O_1 + 3 O_2 \quad (54)$$

has anomalous dimension -84 and $I = 3/2$ and has no overlap with T_1 . As explained above, among the operators with the Dirac index distribution 333334444 there is one with anomalous dimension -60 which also overlaps with the spin = 1 component of the tree operator and among the ones corresponding to 222221111 one which overlaps with its spin = 2 component. The space of operators corresponding to other Dirac index distributions is considerably larger than

Table 3. As in Table 1. (Continued.)

Dirac indices	$\gamma_j/(2d)$	Isospins	Dirac indices	$\gamma_j/(2d)$	Isospins
111133442	-60	11110	111222334	-84	01111
	-52	11110		-64	11110
	-44	01100		-60	11220
	-42	01100		-52	12210
	-40	11100		-48	12200
	-36	12100		-46	01100
	-34	12100		-42	01100
	-30	11000		-40	33200
	-28	22100		-36	35200
	-24	12100		-30	11000
	-22	21000		-28	24100
	-20	11000		-24*	42100
	-18	21000		-22	22000
	-16	12000		-18*	21000
	-10	11000		-16	11000
	-8	10000		-12	21000
-6	21000	-10*	10000		
2	10000	0*	11000		
			8*	10000	

in this example. For example, the case 111223344 (the last entry in table 4) has 1369 operators before and 117 operators after imposing the gauge identities.

As can be seen from tables 1–4, the largest eigenvalue of $\gamma_{AB}^{(1)}$ is $16d$ (occurring already at the tree level). Therefore, the largest eigenvalue of $\tilde{\gamma}_{AB}^{(1)}$ becomes $2d \times (8 - 36)$, which is negative, so that $\beta_A^{\text{tree}} = -14/(33 - 2N_f)$. Therefore, in conclusion, the operators at the tree level in OPE dominate at short distance in the 3N NBS wave function.

4. Short-distance repulsion of 3NF

As discussed before, the 3N potential at short distance is given by

$$V_{2N+3NF}(\vec{r}, \vec{\rho}) \simeq \frac{1}{m_N} \frac{-4\beta_A^{\text{tree}}}{s^2(-\log(\Lambda s))}, \quad (55)$$

where β_A^{tree} is given in equation (14). Since this result dominates over the one appearing in the 2N potential, which is of the form (10), the above behavior of $V_{2N+3NF}(\vec{r}, \vec{\rho})$ at short distance must come solely from $V_{3NF}(\vec{r}, \vec{\rho})$. Unlike similar computations for the 2NF, no additional nonperturbative considerations are required in this case and the analysis shows that the NBS 3NF is always repulsive at short distance, a conclusion that is universal in the sense that it does not depend on the details of the 3N states used to define the NBS wave function, such as its energy.

We would like to point out that as listed in tables 1–4, the total number of nine-quark local operators is several hundreds and the spectrum of anomalous dimensions is rather dense. We

Table 4. As in Table 1. (Continued.)

Dirac indices	$\gamma_j/(2d)$	Isospins	Dirac indices	$\gamma_j/(2d)$	Isospins
111333224	-76	11111	111223344	-76	11111
	-60	01110		-60	12220
	-52	12220		-54	11110
	-46	22210		-52	12220
	-44	01100		-48	01110
	-40	11200		-46	22210
	-36	12100		-44	01100
	-34	24200		-42	12200
	-30	12100		-40	11200
	-28	34200		-36	13200
	-22	32100		-34	24200
	-20	11000		-30	23100
	-18	21000		-28	34200
	-16 [†]	22000		-24*	23100
	-12	21000		-22	32100
	-10	12000		-20	11000
	-8	10000		-18	33100
-4	11000	-16*	33000		
2	10000	-12	21000		
8 [†]	10000	-10	12000		
		-8	10000		
		-6	21000		
		-4	11000		
		0*	21000		
		2	10000		
		8*	10000		

singled out the ones with largest anomalous dimensions which dominate at short distance, but it is not evident at what distance scale this leading behavior sets in. Even if it turns out that these individual operators really dominate at extremely short distances only, our main conclusion may remain valid due to the fact that all the eigenvalues of the effective gamma matrix are negative (corresponding to short-distance repulsion). We think a simple explanation of this fact should exist (maybe related to the Pauli principle).

An interesting and important extension of the present analysis is to investigate the short-distance behavior of the three-baryon force (3BF) by the same method. Its results can tell us whether there is a universal short-distance repulsion also in the 3BF, which has been suggested to explain the observed maximum mass of neutron stars [10].

As a final comment we again stress that equation (55) is *a priori* applicable only to the specific definition of the 3NF using the NBS wave function, since notions of 2NF or 3NF are scheme dependent (cf definitions of running couplings in quantum field theories). Indeed, unless one fixes the scheme for the definition of the potential, it is meaningless to ask whether the 3NF has a repulsive core or not. Our analysis is, however, meaningful and relevant since the extraction of the 3NF in lattice QCD in the same scheme is now in progress [14–16]. Our results

can be compared to the short-distance behavior of the 3NF and that of the associated primary wave functions, once the lattice QCD data have been extrapolated to the continuum limit. As remarked above, equation (55) does not depend on the details of the 3N state; in particular it holds even at an energy of the 3N state larger than the inelastic threshold, although the physical interpretation of the 3NF at such a high energy needs to be reconsidered⁵.

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⁵ For a recent proposal to extract hadronic interactions above inelastic thresholds in lattice QCD, see [27].

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