The separation (1) is not unique, and accordingly, the gluon helicity operator (5) and orbital (6) parts has been made possible at the cost of introducing nonlocality: In general, $A_{\text{phys}}^{\mu}$ and $A_{\text{pure}}^{\mu}$ are nonlocally related to the total $A^{\mu}$. The key ingredient in their construction is the separation of the gauge field into physical and “pure gauge” parts. We suggest a simple separation scheme which transform differently under gauge transformations and show that the resulting gluon helicity coincides with the first moment of the conventional polarized gluon distribution measurable in high energy experiments.

\[ M^{\mu\nu}_{\text{quark-spin}} = \frac{1}{2} \epsilon^{\mu\nu\lambda\sigma} \bar{\psi} \gamma_5 \gamma_\sigma \psi, \]
\[ M^{\mu\nu}_{\text{quark-orbit}} = \bar{\psi} \gamma^\mu (x^\nu iD^- - x^\lambda iD^\nu) \psi, \]
\[ M^{\mu\nu}_{\text{glue-orbit}} = F^{\mu\nu}_{\text{phys}} A_{\text{phys}}^\mu - F^{\mu\nu}_{\text{pure}} A_{\text{pure}}^\mu, \]
\[ M^{\mu\nu}_{\text{glue-orbit}} = F^{\mu\nu}_{\text{phys}} (x^\tau (D^{\lambda}_{\text{phys}} A^\lambda_{\text{phys}})_{\alpha} - x^\lambda (D^\nu_{\text{pure}} A^\lambda_{\text{phys}})_{\alpha}) + (D^\nu_{\text{pure}} A^\mu_{\text{phys}} - x^\lambda A^\mu_{\text{pure}}), \]

where $D^\mu_{\text{pure}} \equiv \partial^\mu + igA^\mu_{\text{pure}}$ and $a, b = 1, 2, \cdots, 8$ are the color indices. We use the convention $\epsilon_{0123} = +1$. The second term on the right hand side of (6) is gauge–invariant on its own, and Chen et al. included it in the quark–orbital part. (This would result in the change $D^\nu \rightarrow D^\nu_{\text{pure}}$ in (4).) Following Wakamatsu [8], we have relocated it to the gluon–orbital part. With this modification the quark part agrees with Ji’s decomposition and can be extracted from GPD analyses [3]. The decomposition of the gluon spin into the helicity (5) and orbital (6) parts has been made possible at the cost of introducing nonlocality: In general, $A_{\text{phys}}^{\mu}$ and $A_{\text{pure}}^{\mu}$ are nonlocally related to the total $A^{\mu}$.

Let us focus on the gluon helicity operator (5). We go to the infinite momentum frame and use the light–cone coordinates $x^z = \frac{1}{2} (x^0 \pm x^1)$. In the framework of Chen et al., the gluon helicity contribution is given by the nucleon

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matrix element of the $\mu \nu \lambda = +12$ tensor component of (5)
\[ -\frac{1}{2P^+} \langle PS| F_{a}^{\nu} (0) \epsilon_{\mu \nu \lambda}^{\text{phys}} (0)|PS \rangle. \] (7)

On the other hand, the conventional and experimentally accessible gluon helicity is given by the first moment of the polarized gluon distribution (see, e.g., [10,11])
\[ \Delta G = \int_{0}^{1} dx_{B} \Delta g(x_{B}) = \frac{1}{4P^+} \int_{-\infty}^{\infty} dy^- \, e(y^-) \langle PS| F_{a}^{+ \mu} (0) \times \mathcal{P} \exp \left( -ig \int_{y^-}^{0} A^+ (y'^-) dy'^- \right) \times e_{+ \mu} F_{b}^{+ \nu} (y^-)|PS \rangle, \] (8)
where $x_{B}$ is the usual Bjorken variable and the Wilson line is in the adjoint representation. ($\mathcal{P}$ denotes path–ordering.

If we insist that the two definitions (7) and (8) are equivalent, we must have that, using the notation $x^\mu = (x^-, \vec{x})$, where $\vec{x} = (x^+, x^1, x^2),$
\[ A_{\mu \nu}^{\text{phys}} (x) = \frac{1}{2} \int_{-\infty}^{\infty} dy^- \, e(y^- - x^-) \times \mathcal{P} \exp \left( -ig \int_{y^-}^{x^-} A^+ (y'^-) d\vec{x}'^- \right) \, F_{b}^{+ \mu} (y^-, \vec{x}). \] (9)

Does this identification make sense? The right hand side obeys the gauge transformation law (2) as expected for $A_{\mu \nu}^{\text{phys}}$, but it is far from obvious that the difference $A_{\mu \nu}^{\text{pure}} = A_{\mu \nu}^{\text{phys}}$ is pure gauge. Remarkably, however, there exists a special, but very simple scheme of separation (1) in which (9) becomes an identity rather than a definition [12].

In order to find such a scheme, we first observe that (9) immediately implies that
\[ A_{\mu \nu}^{\text{phys}} = 0. \] (10)

This motivates us to write, denoting fields as matrices in the adjoint representation,
\[ A_{\mu \nu}^{\text{pure}} = -\frac{i}{g} VW \partial^\nu (VW)^\dagger = \frac{i}{g} \partial^\nu (VW)(VW)^\dagger, \] (11)
where
\[ V(x) = \mathcal{P} \exp \left( -ig \int_{\infty}^{x^-} A^+ (x'^-) d\vec{x}'^- \right), \]
\[ W(\vec{x}) = \mathcal{P} \exp \left( -ig \int_{\infty}^{\vec{x}} \vec{A} (\pm \infty, \vec{x}') \cdot d\vec{x}' \right). \] (12)

Note that $W$ is evaluated at $x^- = \pm \infty$ where the plus (minus) sign corresponds to the choice $x^- = +\infty (x^- = -\infty)$ in the lower limit of the integration in $V$. The path to spatial infinity (denoted as $'x = \infty \vec{n}'$ with $\vec{n}$ being a constant vector) is arbitrary assuming that the field strength vanishes at $x^- = \pm \infty$.

Promoting (11) to a four–dimensional relation, we define
\[ A_{\mu \nu}^{\text{pure}} = -\frac{i}{g} VW \partial^\nu (VW)^\dagger, \] (13)
which guarantees that $F_{\mu \nu}^{\text{pure}} = 0$, and
\[ A_{\mu \nu}^{\text{phys}} = A_{\mu \nu}^{\text{pure}} - A_{\mu \nu}^{\text{pure}}. \] (14)

In order for $A_{\mu \nu}^{\text{pure}}$ to transform according to (2) under gauge transformation, we require that
\[ \lim_{x^- \rightarrow -\infty} \partial^\nu (U(x^-) \vec{x}), \vec{x} = 0, \] (15)
that is, we allow only for global gauge rotations as $(x^-, \vec{x}) \rightarrow (\pm \infty, \vec{n})$. $A_{\mu \nu}^{\text{pure}}$ should vanish (by a gauge choice) in the same limit and this is already implied by (13). Except for this minor qualification, our separation scheme is independent of the gauge choice.

Still, it will be very convenient in the following to consider the light–cone gauge which has a special status in our scheme and which can be accessed by setting $U = VW$ (consistently with (15)). Denoting fields in the light–cone gauge with a tilde, we find
\[ \tilde{A}_{\mu \nu}^{\text{phys}} = (VW)^\dagger A_{\mu \nu}^{\text{phys}} VW, \] (16)
so that $\tilde{A}_{\mu \nu} = \tilde{A}_{\mu \nu}^{\text{phys}}$ in this gauge. The residual $(x^-\text{–independent})$ gauge symmetry in the light–cone gauge is essentially contained in $W(\vec{x})$. It may seem more natural to let $\tilde{A}_{\mu \nu}^{\text{pure}}$ carry these degrees of freedom. However, we have absorbed them in $\tilde{A}_{\mu \nu}^{\text{phys}}$ for our purpose. The point is that, by using these degrees of freedom, one can fix the boundary condition for $\tilde{A}_{\mu \nu} = \tilde{A}_{\mu \nu}^{\text{phys}}$ as $x^- \rightarrow \pm \infty$.

We are now ready to prove (9). The last factor can be written as, suppressing color indices,
\[ \mathcal{P} \exp \left( -ig \int_{y^-}^{x^-} A^+ (y'^-) d\vec{x}'^- \right) F_{b}^{+ \mu} (y^-, \vec{x}). \] (16)
where in the second equality we have used the fact that $F_{\mu \nu}^{\text{phys}} = F_{\mu \nu}^{\text{phys}}$ in the light–cone gauge. [Remember that matrices are in the adjoint representation so that, for instance, the following identity holds: $(VW)^\dagger F_{b}^{+ \mu} = [(VW)^\dagger F_{b}^{+ \mu} (VW)]_a = \tilde{F}_{a}^{+ \mu}$.] The right hand side of (9) then becomes
- \frac{1}{2} \int dy^- e(y^- - x^-) VW(x) \frac{\partial}{\partial y^-} \tilde{A}^\mu_{\text{phys}}(y^-, \bar{x})

= VW(x) \tilde{A}^\mu_{\text{phys}}(x) - \frac{1}{2} VW(x) (\tilde{A}^\mu_{\text{phys}}(\infty, \bar{x}) + \tilde{A}^\mu_{\text{phys}}(-\infty, \bar{x}))

= A^\mu_{\text{phys}}(x) - \frac{1}{2} VW(x) (\tilde{A}^\mu_{\text{phys}}(\infty, \bar{x}) + \tilde{A}^\mu_{\text{phys}}(-\infty, \bar{x}))

\text{(18)}

\text{where we integrated by parts. Equation (18) differs from} A^\mu_{\text{phys}} \text{by the surface terms at } x^- = \pm \infty. \text{However, these surface terms can be consistently eliminated. To see this, suppose that (9) is valid. Then}

\tilde{A}^\mu_{\text{phys}}(\infty, \bar{x}) = (VW)^\dagger(\infty, \bar{x}) A^\mu_{\text{phys}}(\infty, \bar{x})

= \frac{1}{2} W^\dagger(\bar{x}) \int dy^- \mathcal{P} \exp (-ig \int_{y^-}^{-\infty} A^+ (y^-, \bar{x}) dy^-) F^+ \mu(y^-, \bar{x})

= W_\mu^\dagger(\bar{x}) A^\mu_{\text{phys}}(-\infty, \bar{x})

= -\tilde{A}^\mu_{\text{phys}}(-\infty, \bar{x})

\text{(20)}

\text{where, for definiteness, we have chosen } x^- = -\infty \text{ as the lower limit of the integration in (12). (The other case } x^- = \infty \text{ is a trivial modification.) Therefore, the surface terms in (18) cancel and this completes the proof of (9).}

\text{Note that the cancellation we have just observed is nothing but the well-known antisymmetric boundary condition of the gauge field in the light-cone gauge. Thus, in the above proof we have implicitly chosen this boundary condition by adjusting } W(\bar{x}). \text{This conforms to the sign function } e(y^-) \text{ in (8), or equivalently, the principal value prescription for the } 1/x_B \text{ pole in } \Delta g(x_B)

\int_{-\infty}^{\infty} dx_B p^+ v \left( \frac{1}{x_B} \right) e^{iy^- p^+ x_B} = i\pi \epsilon(y^-)

\text{(21)}

\text{Different prescriptions for the } 1/x_B \text{ pole lead to different boundary conditions for } \tilde{A}^\mu = \tilde{A}^\mu_{\text{phys}} \text{ at } x^- \to \pm \infty, \text{just like the prescription for the } 1/k^+ \text{ pole of the gluon propagator in the light-cone gauge [13]. It does not matter which prescription one uses, since the difference is proportional to } \delta(x_B) \text{ and vanishes under the assumption that the } x_B-\text{integral converges as } x_B \to 0. \text{However, it does change the appearance of } \Delta G. \text{Had we chosen a different prescription, say, } 1/(x_B - i\epsilon), \text{we would have obtained a formula for } \Delta G \text{ similar to (8), but with the step function } 2\Theta(y^-) \text{ in place of the sign function } \epsilon(y^-). \text{In the light-cone gauge, this corresponds to the advanced boundary condition } \tilde{A}^\mu(\infty, \bar{x}) = 0. \text{For each different prescription, the surface terms will be different. But they always vanish under the corresponding boundary condition.}

\text{In conclusion, the gauge-invariant decomposition of the gluonic contribution to the nucleon spin into helicity and orbital parts is not possible if one restricts to } local \text{ operators [3]. Once one allows for } nonlocal \text{ operators, it becomes possible [4]. We have shown that the traditional definition of the gluon helicity, } \Delta G, \text{can be nicely accommodated in this latter approach, thereby dispelling any concerns about the physical meaning of } \Delta G. \text{After all, } \Delta G \text{ is measurable, gauge invariant, and meets the criterion by Chen et al. for a proper definition of the gluon helicity in QCD.}

\text{By using the explicit relation between } A^\mu_{\text{phys}} \text{ and } A^\mu, \text{one can write down the all-order expression for the gluon orbital angular momentum (6) as well. While it is measurable as the difference between the total gluon contribution (from the GPD) and } \Delta G, \text{more direct access to the orbital component would of course be desirable.}

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[12] Wakamatsu [9] discussed the equivalence of the matrix elements (7) and (8) in the light–cone gauge. Here we intend to show (9) as an operator identity in generic gauges.