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Sequential Interval Estimation of a Location Parameter with the Fixed Width in the Non-regular Case

Dedicated to Professor Masafumi Akahira on his 60th birthday

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Abstract: For a location-scale parameter family of distributions with a finite support, a sequential confidence interval with a fixed width is obtained for the location parameter, and its asymptotic consistency and efficiency are shown. Some comparisons with the Chow-Robbins procedure are also done.

Keywords: Coverage probability; Extreme value; Non-regular case; Sequential interval estimation.

Subject Classifications: 62L12; 62F25.

1. INTRODUCTION

Suppose that we are to estimate a location parameter \( \theta \) of a sequence of random observations \( X_1, X_2, \ldots, X_n, \ldots \) with unknown scale \( \xi \). We would like to obtain sequentially a confidence interval of fixed width \( 2d \) with confidence coefficient \( 1 - \alpha \). Obviously we can not obtain a fixed sample size procedure if \( \xi \) is unknown. There are many works on the fixed-width interval estimation of normal mean (see, e.g. Ghosh et al. (1997)).
Suppose that $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of independent and identically distributed (i.i.d.) random variables according to the uniform distribution on the interval $(\theta - (\xi/2), \theta + (\xi/2))$, where $\theta (\in \mathbb{R}^1)$ and $\xi (> 0)$ are unknown. Let $X_{(1)} := \min_{1 \leq i \leq n} X_i, X_{(n)} := \max_{1 \leq i \leq n} X_i$. Then the midrange and the range are $M_n := (X_{(1)} + X_{(n)})/2, R_n := X_{(n)} - X_{(1)}$, respectively. Akahira and Koike (2005) considered a stopping rule:

$$\tau_1 := \inf \left\{ n \geq n_0 \left| \frac{R_n}{n - 1} \leq -\frac{2d}{\log \alpha} \right. \right\},$$

where $n_0(\geq 2)$ is an initial size of sample. They showed the asymptotic consistency and efficiency of the estimation procedure $(\tau_1, [M_{\tau_1} - d, M_{\tau_1} + d])$.

In this paper, we consider the case of a location-scale parameter family of distributions with a finite support on the interval $(\theta - \xi a, \theta + \xi a)$, where $\theta$ and $\xi$ are unknown, and obtain a sequential confidence interval of $\theta$ with fixed width $2d$ and confidence coefficient $1 - \alpha$, and show its asymptotic consistency and efficiency. Some comparisons with the Chow-Robbins procedure are also done.

2. ASYMPTOTIC DISTRIBUTIONS OF THE EXTREME VALUES

In this section we consider the asymptotic distributions of the extreme values for distributions with a finite support, in a similar way to Akahira (1991) and Akahira and Takeuchi (1995).

Let $Z_1, Z_2, \ldots, Z_{\infty}$ be a sequence of independent and identically distributed (i.i.d.) random variables according to the density function $f_0(x - \theta)$ ($\theta \in \mathbb{R}^1$) with respect to the Lebesgue measure. We assume the following conditions:

(A1) $f_0(x)$ has a finite support $(-a, a)$, i.e., $f_0(x) > 0$ for $-a < x < a$, and $f_0(x) = 0$ otherwise.

(A2) $f_0(x)$ is continuously differentiable in the open interval $(-a, a)$ and

$$\lim_{x \to -a^+} f_0(x) = c, \quad \lim_{x \to a^-} f_0(x) = c', \quad \text{if} \quad f_0(-a) = 0.$$

1If the support of $f_0$ is $(-a, b)$ ($a \neq b$), then the normalized midrange does not converge to $\theta$ in probability as $n \to \infty$. 

2
where $c$ and $c'$ are some positive constants.

(A3) $f_0(x)$ satisfies

\[ f_0(x) \approx g(x + a)^\gamma \quad (x \rightarrow -a + 0), \]
\[ f_0(x) \approx g'|x - a| \gamma \quad (x \rightarrow a - 0), \]

where $\gamma, g$ and $g'$ are some positive constants. \(^2\)

Putting $Z(1) := \min_{1 \leq i \leq n} Z_i$, $Z(n) := \max_{1 \leq i \leq n} Z_i$, $U := n(Z(1) + a - \theta)$ and $V := n(Z(n) - a - \theta)$, we have the following lemma (cf. Akahira (1991), Akahira and Takeuchi (1995)).

**Lemma 1.** Under the conditions (A1) and (A2), the joint (j.) p.d.f. $f_{U,V}(u, v)$ of $(U, V)$ satisfies

\[ f_{U,V}(u, v) \rightarrow \begin{cases} c c' \exp\{c'v - cu\} & (v < 0 < u), \\ 0 & \text{otherwise}, \end{cases} \tag{2.1} \]

as $n \rightarrow \infty$.

**Proof.** The j.p.d.f. $f_{U,V}^{(n)}(u, v)$ of $(U, V)$ is

\[
\begin{align*}
\frac{f_{U,V}^{(n)}(u, v)}{n} &= \begin{cases} 
\frac{n - 1}{n} \left\{ F\left(a + \frac{v}{n}\right) - F\left(-a + \frac{u}{n}\right) \right\}^{n-2} f_0\left(-a + \frac{u}{n}\right) f_0\left(a + \frac{v}{n}\right) & (v < 0 < u), \\
0 & \text{otherwise},
\end{cases}
\end{align*}
\]

where $F(x) = \int_{-\infty}^{x} f_0(u)du$. Hence, by its expansion, we have the desired result. \(\square\)

Next, we consider the location-scale parameter family of distributions with a finite support $(\theta - \xi a, \theta + \xi a)$. Suppose that $X_1, X_2, \ldots, X_n$ is a sequence of i.i.d. random variables with the p.d.f. $(1/\xi) f_0((x - \theta)/\xi)$, where $\theta \in \mathbb{R}$ and $\xi > 0$. Put $Y_i := (X_i - \theta)/\xi$ for each $i = 1, 2, \ldots$, and $Y(1) := \min_{1 \leq i \leq n} Y_i$, $Y(n) := \max_{1 \leq i \leq n} Y_i$. Letting $S := n(Y_1 + Y_n)/2$ and $T = n(Y_1 - Y_n + 2a)/2$, we have the asymptotic (as.) j.p.d.f. of $(S, T)$

\[
f_{S,T}(s, t) = \begin{cases} 
2cc' \exp\{-c' t\} & (t > |s|), \\
0 & \text{otherwise},
\end{cases}
\]

\(^2\)If the converging order $\gamma$ is different, then the normalized midrange does not converge to $\theta$ in probability as $n \rightarrow \infty.$
Then the as. marginal(m.) p.d.f.’s of $S$ and $T$ are given by

\[
\begin{align*}
  f_S(s) &= \begin{cases} 
    Ke^{-2cs} & (s \geq 0), \\
    Ke^{2cs} & (s < 0), 
  \end{cases} \\
  f_T(t) &= \begin{cases} 
    \frac{2c'}{c-c'} \left(e^{-2ct} - e^{-2ct'}\right) & (t > 0 \text{ and } c \neq c'), \\
    4c^2te^{-2ct} & (t > 0 \text{ and } c = c'), \\
    0 & (\text{otherwise}),
  \end{cases}
\end{align*}
\]

respectively, where $K = \frac{2cc'}{(c+c')}$. In the case when $\lim_{x \to -a+0} f_0(x) = \lim_{x \to -a-0} f_0(x) = 0$, we need another lemma. Putting $U' := n^{1/(\gamma+1)}(Z_1 + a - \theta)$ and $V' := n^{1/(\gamma+1)}(Z_n - a - \theta)$, we have the following lemma in a similar way to Lemma 1.

**Lemma 2.** Under the conditions (A1) and (A3), the j.p.d.f. $f_{U',V'}(u,v)$ of $(U', V')$ satisfies

\[
\begin{align*}
  f_{U',V'}^{(n)}(u,v) \to \begin{cases} 
    gg'(-uv)^\gamma \exp\left\{-\frac{g'}{\gamma+1}(-v)^{\gamma+1} - \frac{g}{\gamma+1}u^{\gamma+1}\right\} & (v < 0 < u), \\
    0 & (\text{otherwise}).
  \end{cases}
\end{align*}
\]

as $n \to \infty$.

The proof is omitted since it is similar to the one of Lemma 1.

From Lemma 2, $U'$ and $(-V')$ are asymptotically, independently distributed according to Weibull distributions.

### 3. CONSTRUCTING CONFIDENCE INTERVAL

In this section we construct a sequential confidence interval for $\theta$. In the first place, we consider the case under the conditions (A1) and (A2). For $0 < \alpha < 1$, let $l_0$ be the solution\(^3\) of $l$ for the equation

\[
\frac{c+c'}{cc'}\alpha = \frac{e^{-2cl}}{c} + \frac{e^{-2cl'}}{c'}.
\]

\(^3\)It can be shown easily that such $l_0$ exists uniquely.
If \( \xi \) is known, we have from (2.2) that
\[
P\{ |M_n - \theta| \leq d \} = P\{ n|M_n - \theta|/\xi \leq dn/\xi \}
\approx \int_{-dn/\xi}^{dn/\xi} f_S(s)ds
\]
\[
= 1 - \frac{cc'}{c + c'} \left( \frac{e^{-2cn_d/\xi}}{c} + \frac{e^{-2cn_d/\xi}}{c'} \right),
\]
where \( \approx \) means that the distribution of \( n|M_n - \theta|/\xi \) is approximated by the asymptotic distribution. Letting \( n^* = l_0 \xi/d \), we have for \( n \geq n^* \)
\[
1 - \frac{cc'}{c + c'} \left( \frac{e^{-2cn_d/\xi}}{c} + \frac{e^{-2cn_d/\xi}}{c'} \right) \geq 1 - \alpha.
\]
\( n^* \) is referred as the asymptotically optimal size of samples if \( \xi \) is known.

Note that \( n(M_n - \theta)/\xi = S \) and \( R_n/\xi = -(T/n) + 2a \). Now we take as the stopping rule
\[
\tau_2 := \inf \left\{ n \geq n_0 \mid \frac{R_n}{n-1} \leq \frac{2ad}{l_0} \right\},
\]
where \( n_0 (\geq 2) \) is an initial size of sample. Then we obtain the asymptotic properties of the estimation procedure \( (\tau_2, [M_{\tau_2} - d, M_{\tau_2} + d]) \) as follows.

**Theorem 1.** For the sequential estimation procedure \( (\tau_2, [M_{\tau_2} - d, M_{\tau_2} + d]) \), the following hold.

(i) \( \lim_{d \to 0+} P\{ |M_{\tau_2} - \theta| \leq d \} = 1 - \alpha \) (asymptotic consistency).

(ii) \( \tau_2/n^* \xrightarrow{a.s.} 1 \quad (d \to 0+) \).

(iii) \( E(\tau_2)/n^* \to 1 \quad (d \to 0+) \) (asymptotic efficiency).

**Proof.** (i) From Lemma 1 of Chow and Robbins (1965), the stopping rule \( \tau_2 \) given by (3.1) satisfies
\[
\lim_{d \to 0+} \frac{d\tau_2}{\xi l_0} = 1 \quad \text{a.s.}
\]
Since \( S = n(M_n - \theta)/\xi \) converges in distribution to a distribution with the density given by (2.2) as \( n \to \infty \), it follows from Theorem 1 of Anscombe (1952) that \( \tau_2(M_{\tau_2} - \theta) \) converges in distribution to the same distribution as \( d \to 0+ \). Hence, since \( d\tau_2/\xi \xrightarrow{a.s.} l_0 \) as \( d \to 0+ \) from (3.2), it follows that
\[
\lim_{d \to 0+} P\{ |M_{\tau_2} - \theta| \leq d \} = \lim_{d \to 0+} P\{ \tau_2|M_{\tau_2} - \theta|/\xi \leq d\tau_2/\xi \}
\]
\[
= \int_{-l_0}^{l_0} f_S(s)ds = 1 - \alpha.
\]
(ii) From (3.2) and the definition of $l_0$, we have $\tau_2/n^* = \tau_2 d/(l_0 \xi)^{a/s} - 1$ as $d \to 0+$.

(iii) From Lemma 2 of Chow and Robbins (1965), we have the desired result.

\[ \square \]

Remark. In particular, if $c = c'$, then $l_0 = -\log \alpha/(2c)$ and $\tau_2$ given in (3.1) is expressed as

\[ \tau_2 = \inf \left\{ n \geq n_0 \left| \frac{R_n}{n - 1} \leq -\frac{4 acd}{\log \alpha} \right. \right\}, \]

which is equal to $\tau_1$ when the underlying distribution is uniform distribution on the interval $(\theta - (\xi/2), \theta + (\xi/2))$.

In the second place, we compare this with the Chow-Robbins procedure. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with the mean $\theta$ and the variance $\sigma^2$. Let $\bar{X}_n := \sum_{i=1}^{n} X_i / n$, $s_n^2 = \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 / (n - 1)$. Chow and Robbins (1965) considered a stopping rule defined by

\[ \tau_{CR} := \inf \left\{ n \geq n_0 \left| n \geq u_{\alpha/2}^2 \sigma^2 / d^2 \right. \right\}, \]

where $u_{\alpha/2}$ is the upper $\alpha/2$ point of $N(0, 1)$ and $n_0(\geq 2)$ is an initial size of samples. They showed the asymptotic consistency and efficiency of the estimation procedure ($\tau_{CR}, [\bar{X}_{\tau_{CR}} - d, \bar{X}_{\tau_{CR}} + d]$).

Since, from Theorem 2.2 of Akahira and Koike (2005), Theorem 1 and Theorem of Chow and Robbins (1965),

\[ \tau_1 \approx \frac{\log \alpha}{\log (1 - (2d/\xi))} \approx \frac{-\xi \log \alpha}{2d}, \quad \tau_2 \approx l_0 \xi / d, \quad \tau_{CR} \approx u_{\alpha/2}^2 \sigma^2 / d^2, \]

as $d \to 0+$, we have $\tau_1/\tau_{CR}, \tau_2/\tau_{CR} \to 0 (d \to 0+)$. Therefore $\tau_1, \tau_2$ is asymptotically better than $\tau_{CR}$ in the sense of the average size of sample.

Furthermore, we consider the case under the conditions (A1) and (A3). By putting $S' := n^{1/(\gamma+1)}(Y(1) + Y(n))/2$ and $T' := n^{1/(\gamma+1)}(Y(1) - Y(n) + 2a)/2$, the as.j.p.d.f. of $(S', T')$ and the as.m.p.d.f.’s of $S'$ and $T'$ are obtained from Lemma 2. In a similar way to (3.3), we take $l_0$ satisfying $\int_{-l_0}^{l_0} f_{S'}(s) ds = 1 - \alpha$ for the as.m.p.d.f. $f_{S'}(s)$ of $S'$. 
If $\xi$ is known, we have

$$
P\{\left| M_n - \theta \right| \leq d \} = P\{n^{1/(\gamma+1)}|M_n - \theta|/\xi \leq dn^{1/(\gamma+1)}/\xi \}
$$

$$
\approx \int_{-dn^{1/(\gamma+1)}/\xi}^{dn^{1/(\gamma+1)}/\xi} f_{S'}(s) ds,
$$

where \(\approx\) means that the distribution of \(n^{1/(\gamma+1)}|M_n - \theta|/\xi\) is approximated by the asymptotic distribution. The optimal size of sample required for attaining the preassigned coverage probability \(1 - \alpha\) is the smallest positive integer \(\geq (l_0\xi/d)^{\gamma+1} =: n^{**}\) (say). Define a stopping rule as

$$
\tau_3 := \inf \left\{ n \geq n_0 \left| \frac{R_n}{n^{1/(\gamma+1)}} \leq \frac{2ad}{l_0} \right. \right\},
$$

where \(n_0(\geq 2)\) is an initial size of samples. Then the next theorem follows.

**Theorem 2.** For the sequential estimation procedure \(\tau_3, [M_{\tau_3} - d, M_{\tau_3} + d]\), the following hold.

(i) \(\lim_{d \to 0^+} P\{\left| M_{\tau_3} - \theta \right| \leq d \} = 1 - \alpha\) (asymptotic consistency).

(ii) \(\tau_3/n^{**} \xrightarrow{a.s.} 1\) \((d \to 0^+)\).

(iii) \(E(\tau_3)/n^{**} \to 1\) \((d \to 0^+)\) (asymptotic efficiency).

**Proof.** The proof for (i) is similar to the one of Theorem 1 (i). (ii) follows from \((\tau_3/n^{**})^{1/(\gamma+1)} \xrightarrow{a.s.} 1\) as \(d \to 0^+\).

(iii) From (ii), by Fatou’s lemma,

$$
\liminf_{d \to 0^+} \frac{E(\tau_3)}{n^{**}} \geq E \left( \liminf_{d \to 0^+} \frac{\tau_3}{n^{**}} \right) = 1. \quad (3.4)
$$

On the other hand, since \(0 \leq R_n \leq 2a\xi\) with probability 1 for any \(n \in \mathbb{N}\), we have \(0 \leq (R_n l_0/(2ad))^{\gamma+1} \leq (2a\xi l_0/(2ad))^{\gamma+1} = (l_0\xi/d)^{\gamma+1}\) with probability 1 for any \(n \in \mathbb{N}\). So, \(0 \leq (R_n l_0/(2ad))^{\gamma+1} \leq n\) with probability 1 for \(n\) satisfying \(n \geq (l_0\xi/d)^{\gamma+1} + 1\). Therefore, since \(\tau_3 =\)

$$
\inf \left\{ n \geq n_0 \left| (R_n l_0/(2ad))^{\gamma+1} \leq n \right. \right\},
$$

we have \(\tau_3 \leq (l_0\xi/d)^{\gamma+1} + 1\). Then, using the definition of \(n^{**}\), we have

$$
\frac{E(\tau_3)}{n^{**}} \leq \left\{ \left( \frac{l_0\xi}{d} \right)^{\gamma+1} + 1 \right\} \left( \frac{l_0\xi}{d} \right)^{-(\gamma+1)} = 1 + \left( \frac{d}{l_0\xi} \right)^{\gamma+1},
$$
hence
\[ \limsup_{d \to 0^+} \frac{E(\tau_3)}{n^{**}} \leq 1. \] (3.5)

Combining (3.4) and (3.5), we obtain (iii).

From Theorem 2 and Theorem of Chow and Robbins (1965), \( \tau_3 \approx (l_0\xi/d)^{\gamma+1} \) and \( \tau_{CR} \approx u_2^2\sigma^2/d^2 \) as \( d \to 0^+ \). Therefore,
\[
\frac{\tau_3}{\tau_{CR}} \begin{cases} = o(1) & (0 < \gamma < 1), \\ = O(1) & (\gamma = 1), \\ \to \infty & (\gamma > 1) \end{cases}
\]
as \( d \to 0^+ \). Therefore, \( \tau_3 \) is asymptotically better than \( \tau_{CR} \) in the sense of the average size of sample if \( 0 < \gamma < 1 \).

In this paper, we considered the cases when the values at the endpoints of the support of the p.d.f. are positive simultaneously, or tend to 0 at the same speed. In the meantime, if the either value at the endpoints of the support of the p.d.f. is positive, or tend to 0 at a different speed, then the coefficients of \( n^\gamma (X_{(1)} - a - \theta) \) and \( n^\delta (X_{(n)} - b - \theta) \) converging to nontrivial random variables are different and estimation by using the midrange \( M_n \) is inappropriate.

4. NUMERICAL EXAMPLE

In this section we examine the coverage probability of the procedure \([M_{r_2} - d, M_{r_2} + d]\) by simulation based on 100000 repetitions. Suppose that \( X_1, X_2, \ldots, X_n, \ldots \) is a sequence of i.i.d. random variables with the p.d.f. \((1/\xi)f_0((x - \theta)/\xi)\), where \( \theta \in \mathbb{R} \), \( \xi > 0 \) and \( f_0(\cdot) \) is a trapezoid-shape p.d.f. given by
\[
f_0(x) = \begin{cases} \left(\frac{1}{2} - c\right)x + \frac{1}{2} & (x \in (-1, 1)), \\ 0 & \text{(otherwise)} \end{cases}
\] with \( 0 < c < 1 \). Note that, \( f_0 \) is the p.d.f. of the uniform distribution over \((-1, 1)\) and an asymmetric p.d.f. over \((-1, 1)\) for \( c = 0.5 \) and a sufficiently small \( c > 0 \), respectively. Since \( M_{r_2} \) is location equivariant, we may assume \( \theta = 0 \) without loss of generality.

When \( \alpha = 0.10, d = 0.01(0.01)0.05, \xi = 1(1)5 \) and \( n_0 = 5 \), Tables 1 and 2 show the values of coverage probabilities of the sequential estimation
procedure \((\tau_2, [M_{\tau_2} - d, M_{\tau_2} + d])\) for \(c = 0.1\) and \(c = 0.5\), respectively. The result suggests that the estimation procedure is consistent for this case.

**Table 1.** Coverage probabilities of \([M_{\tau_2} - d, M_{\tau_2} + d]\) for \(c = 0.1\)

<table>
<thead>
<tr>
<th>(\xi \setminus d)</th>
<th>0.01</th>
<th>0.02</th>
<th>0.03</th>
<th>0.04</th>
<th>0.05</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.90637</td>
<td>0.91545</td>
<td>0.92348</td>
<td>0.93092</td>
<td>0.93758</td>
</tr>
<tr>
<td>2</td>
<td>0.89830</td>
<td>0.90544</td>
<td>0.90960</td>
<td>0.91424</td>
<td>0.92017</td>
</tr>
<tr>
<td>3</td>
<td>0.90123</td>
<td>0.90313</td>
<td>0.90713</td>
<td>0.90832</td>
<td>0.91030</td>
</tr>
<tr>
<td>4</td>
<td>0.89926</td>
<td>0.90117</td>
<td>0.90333</td>
<td>0.90615</td>
<td>0.90804</td>
</tr>
<tr>
<td>5</td>
<td>0.89817</td>
<td>0.89952</td>
<td>0.90318</td>
<td>0.90421</td>
<td>0.90561</td>
</tr>
</tbody>
</table>

**Table 2.** Coverage probabilities of \([M_{\tau_2} - d, M_{\tau_2} + d]\) for \(c = 0.5\)

<table>
<thead>
<tr>
<th>(\xi \setminus d)</th>
<th>0.01</th>
<th>0.02</th>
<th>0.03</th>
<th>0.04</th>
<th>0.05</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>0.91183</td>
<td>0.91328</td>
<td>0.91988</td>
</tr>
<tr>
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<td>0.90131</td>
<td>0.90330</td>
<td>0.90628</td>
<td>0.91176</td>
</tr>
<tr>
<td>3</td>
<td>0.89849</td>
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<td>0.90235</td>
<td>0.90525</td>
</tr>
<tr>
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<td>0.89729</td>
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<tr>
<td>5</td>
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<td>0.8998</td>
<td>0.89906</td>
<td>0.89862</td>
<td>0.90054</td>
</tr>
</tbody>
</table>

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