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Topology and its Applications

volume 158
number 9
page range 1163-1171
year 2011-04

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doi:10.1016/j.topol.2011.04.004

URL http://hdl.handle.net/2241/113451
doi: 10.1016/j.topol.2011.04.004
Dynamical systems which realize given random bi-sequences of points on their orbits

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Abstract. A dynamical system consists of a phase space of possible states, together with an evolution rule that determines all future states and all past states given a state at any particular moment. In this paper, we show that for any countable random infinite bi-sequences of states of some phase space, there exists an evolution rule in $C^0$-topology which realizes precisely the given sequences of states on their orbits and satisfies some regular conditions on the times to realize the states.

1 AMS Subject Classification: Primary 54F11; 54H20; 58FE60, Secondary 58F11; 28A32; 57Q40.
2 Key words and phrases: Dynamical system, orbits, transitive homeomorphism, chaotic, good measure, measure-preserving homeomorphism, flow.
1 Introduction

A dynamical system consists of a phase space of possible states, together with an evolution rule that determines all future states and all past states given a state at any particular moment. In this paper, we consider some kinds of chaotic properties of dynamical systems. We show that in the world admitting $C^0$-topology, for any countable random infinite bi-sequences of states of some phase space there exists an evolution rule which realizes precisely the given bi-sequences of states on their orbits and satisfies some regular conditions on the times to realize the states. In other words, for any countable random infinite itineraries, by making a slight modification on our dynamical system, we have a new dynamical system in $C^0$-topology which realizes the given infinite itineraries and satisfies some regular conditions on the times of itineraries. The ideas of this paper depend on works of Oxtoby-Ulam [7] and Bennett [2]. We need the following terminology and concepts. Let $N = \{1, 2, ..., \}$ be the set of all positive integers and $Z = \{0, \pm 1, \pm 2, ..., \}$ the set of all integers. Also let $R$ be the set of all real numbers and $I = [0, 1]$ the unit interval.

In this paper, we suppose that $f : X \to X$ is a homeomorphism of a compact metric space $(X, d)$, where $d$ is a metric on $X$. We put $Supp(f) = \{x \in X \mid f(x) \neq x\}$. A point $x \in X$ is a periodic point of $f$ if there exists a positive integer $n \in N$ such that $f^n(x) = x$. A point $x \in X$ is recurrent under $f$ if for any neighborhood $U$ of $x$ there exists a positive integer $n \in N$ such that $f^n(x) \in U$. The orbit of a point $x \in X$ under $f$, denoted by $Orb(f; x)$, is the set $\{f^n(x) \mid n \in Z\}$. If $x$ is not a periodic point of $f$, we consider the infinite bi-sequence (=ordered orbit) $(f^n(x) \mid n \in Z) = (..., f^{-2}(x), f^{-1}(x), x, f(x), f^2(x), ...)$ of $x$ under $f$. If $x$ is a periodic point of $f$ with period $n$, we also consider the finite sequence (=ordered orbit) $(f^n(x) \mid 0 \leq i \leq n - 1)$ of $x$ under $f$. For any $x \in X$ and $i, j \in Z$ with $i \leq j$, we put $Orb(f; x)_{[i, j]} = \{f^n(x) \mid i \leq n \leq j\}$. Suppose that $Orb(f; x)$ is not a periodic orbit and $y \in Orb(f; x)$. In this case, we put $Time_f(x \to y) = n$, where $n$ is the (unique) integer satisfying $f^n(x) = y (n \in Z)$.

Let $\varphi : X \times R \to X$ be a flow, i.e., $\varphi$ is a map (=continuous function) such that

1. $\varphi(x, 0) = x$ and

2. $\varphi(x, s + t) = \varphi(\varphi(x, s), t)$ for any $x \in X$ and any $s, t \in R$.

A point $x \in X$ is a periodic point of $\varphi$ if there exists a positive number $t \in R$ such that $\varphi(x, t) = x$. The orbit of a point $x \in X$ under $\varphi$, denoted by $Orb(\varphi; x)$, is the set $\{\varphi(x, t) \mid t \in R\}$. If $x$ is not a periodic point of $\varphi$, we consider the ordered orbit $(\varphi(x, t))_{t \in R}$ of $x$ under $\varphi$. If $x$ is a periodic point of $\varphi$ with period $t_0 > 0$, we consider the ordered orbit $(\varphi(x, t))_{0 \leq t < t_0}$ of $x$ under $\varphi$. If $x$ is not a periodic point and $y \in Orb(\varphi, x)$, we put $Time_{\varphi}(x \to y) = t$ if $\varphi(x, t) = y$.

Let $\Lambda = Z$ or $\Lambda = \{0, 1, 2, ..., s\}$ ($s < \infty$). A sequence $S = (a_n \mid n \in \Lambda)$ of points of $X$ is said to be realized by a homeomorphism $f$ if $S$ is a subsequence of the ordered orbit of $a_0$ under $f$. Similarly $S = (a_n \mid n \in \Lambda)$ is said to be realized by a flow $\varphi$ if $S$ is a subsequence of the ordered orbit of $a_0$ under $\varphi$. A sequence $(x_n \mid n \in \Lambda)$ of points of $X$ is a pseudo $\eta$-orbit ($\eta > 0$) of $f$ if $d(f(x_n), x_{n+1}) < \eta$ for any $n, n + 1 \in \Lambda$. Let $a, b \in X$. A finite sequence $(x_n \mid 0 \leq n \leq s)$ is a pseudo $\eta$-orbit of $f$ from $a$ to $b$ if $x_0 = a, x_s = b$ and $d(f(x_n), x_{n+1}) < \eta$ for any $0 \leq n \leq s - 1$. 
Let \((k_n| n \in \mathbb{Z})\) be an arbitrary increasing bi-sequence of integers with \(k_0 = 0\) i.e., \(k_n < k_{n+1}\) for \(n \in \mathbb{Z}\) and let \(S_i = (a_{kn}^n| n \in \mathbb{Z})\) \((i \in \mathbb{N})\) be infinite bi-sequences of distinct points of \(X\). Then the (countable) family \(\{S_i|i \in \mathbb{N}\}\) is said to be chaotic for \((k_n| n \in \mathbb{Z})\) if the following conditions are satisfied:

1. \(S_i\) and \(S_j\) \((i \neq j)\) are mutually disjoint,
2. the set \(\{a_i^0|i \in \mathbb{N}\}\) is dense in \(X\),
3. the sets \(\{a_i^k|n \in \mathbb{N}\}\) and \(\{a_{k-n}^i|n \in \mathbb{N}\}\) are dense in \(X\) for each \(i\),
4. if \(i, j \in \mathbb{N}\) and \(i \neq j\), then \(S_i\) and \(S_j\) are Li-Yorke pair with respect to \((k_n| n \in \mathbb{Z})\) and the diameter \(\delta(X)\) of \(X\), that is,

\[
\lim_{n \to \pm \infty} \inf d(a_{kn}, a_{kn}^j) = 0,
\]

\[
\lim_{n \to \pm \infty} \sup d(a_{kn}, a_{kn}^j) = \delta(X).
\]

It is easy to see that if \(X\) has no isolated point (i.e., \(X\) is perfect), then we have many kinds of chaotic families \(\{S_i|i \in \mathbb{N}\}\).

An \(m\)-dimensional compact connected polyhedron \(X\) is said to be regularly connected if the set

\[
\text{Int}(X) = \{x \in X| x \text{ has an open neighborhood which is homeomorphic to } \mathbb{R}^m\}
\]

is a connected dense open subset of \(X\). Put \(\partial(X) = X - \text{Int}(X)\).

The theory of Menger manifolds was founded by Anderson and Bestvina (see [1] and [3]) and has been studied by many authors. We also study Menger manifolds from the viewpoint of dynamical systems. Anderson and Bestvina gave characterizations of Menger manifolds as follows: For a compactum \(M\), \(M\) is a \(k\)-dimensional Menger manifold if and only if (1) \(\dim M = k\), (2) \(M\) is locally \((k-1)\)-connected, (3) \(M\) has the disjoint \(k\)-cell property, i.e., for any \(\epsilon > 0\) and any maps \(f, g: I^k \to M\), there are maps \(f', g': I^k \to M\) such that \(d(f, f') < \epsilon, d(g, g') < \epsilon\) and \(f'(I^k)\cap g'(I^k) = \phi\). Note that every 0-dimensional Menger manifold is a Cantor set, and every 1-dimensional Menger connected manifold is a Menger curve. If \(X\) is a Menger manifold, we put \(\text{Int}(X) = X\) and \(\partial(X) = \phi\).

Let \(\mu\) be a probability measure on a compact metric space \((X, d)\) which is nonatomic, locally positive; such a measure is called a good measure. Put

\[
M(X; \text{good}) = \{\mu| \mu \text{ is a good measure on } X\}.
\]

If \(X\) is a regularly connected polyhedron, we consider the following subset of measures:

\[
M_0(X; \text{good}) = \{\mu \in M(X; \text{good})| \mu(\partial X) = 0\}.
\]

Let \(H(X, \mu)\) be the set of all \(\mu\)-measure preserving homeomorphisms of \(X\) with metric

\[
\rho(f, g) = d(f, g) + d(f^{-1}, g^{-1}),
\]

3
where \( d(f, g) = \sup \{ d(f(x), g(x)) \mid x \in X \} \). Also, put
\[
H_\partial(X, \mu) = \{ f \in H(X, \mu) \mid f|\partial X = \text{Id} \}.
\]
Note that \( H(X, \mu) \) and \( H_\partial(X, \mu) \) are complete metric spaces (see [7]). Note that if \( X \) is a regularly connected polyhedron and \( \mu, \mu' \in M_\partial(X; \text{good}) \), then there is a homeomorphism \( h : X \to X \) such that \( h_* \mu = \mu' \) (see [7, Corollary 1]). Also, note that if \( X \) is a \( k \)-dimensional Menger manifold \((k \geq 1)\) and \( \mu, \mu' \in M(X; \text{good}) \), then there is a homeomorphism \( h : X \to X \) such that \( h_* \mu = \mu' \) (see [5, Theorem 3.1]).

2 Homeomorphisms which realize precisely the given sequences of points on their orbits

In this section, we consider the case of discrete dynamical systems. A metric \( d \) on a space \( X \) is a convex metric if for any \( x, y \in X \) there is a point \( z \) of \( X \) such that \( d(x, z) = d(z, y) = (1/2)d(x, y) \). It is well-known that a continuum (compact metric connected space) \( X \) is locally connected (=Peano continuum) if and only if \( X \) admits a convex metric \( d \) on \( X \). First, we need the following lemmas (cf. [7, Lemma 14]).

**Lemma 2.1.** Suppose that \( X \) is a regularly connected polyhedron of dimension \( m \geq 1 \) or a Menger \( k \)-dimensional manifold with \( k \geq 1 \) and \( d \) is a convex metric on \( X \). Let \( \mu \) be a good measure on \( X \) and \( h \in H(X, \mu) \). For any \( \delta > 0 \), there is a natural number \( N \) such that for any \( a, b \in X \) and any \( n \geq N \), there is a pseudo \( \delta \)-orbit \( x_0, x_1, ..., x_n \) of \( h \) from \( a \) to \( b \).

**Proof.** For a subset \( A \) of \( X \), let \( U(A, \delta) \) be the \( \delta \)-neighborhood of \( A \) in \( X \). Put \( U_1 = U(h(a), \delta) \). By induction on \( i \), we define \( U_{i+1} = U(h(U_i), \delta) \). Since \( h \in H(X, \mu) \), by [7, Lemma 14] and [5] there is a natural number \( N \) such that \( U_N = X \). Let \( a, b \in X \) and \( n \) any natural number with \( n \geq N \). We choose the point \( y \in X \) such that \( b = h^n(y) \). Since \( U_N = X \), there is a pseudo \( \delta \)-orbit \( x_0, x_1, ..., x_N \) of \( h \) from \( a \) to \( y \). Then the sequence \( x_0, x_1, ..., x_N(= y), x_{N+1}(= h(y)), x_{N+2}(= h^2(y)), ..., x_n(= h^{n-N}(y) = b) \) is a pseudo \( \delta \)-orbit \( x_0, x_1, ..., x_n \) of \( h \) from \( a \) to \( b \).

The following lemma follows from [7, Lemma 12] and [5, Proposition 4.16]. We omit the proof.

**Lemma 2.2.** Suppose that \( X \) is a regularly connected polyhedron of dimension \( m \geq 2 \) or a Menger \( k \)-dimensional manifold \((k \geq 1)\). Let \( \mu \) be a good measure on \( X \). Suppose that \( U \) is a connected open set of \( \text{Int}(X) \) with \( a, b \in U \). Then there exists \( h \in H(X, \mu) \) such that \( h(a) = b \) and \( \text{Supp}(h) \subseteq U \).

The following lemma is a slight modification of [7, Lemma 13]. For completeness, we give the proof.

**Lemma 2.3.** Suppose that \( X \) is a regularly connected polyhedron of dimension \( m \geq 2 \) or a Menger \( k \)-dimensional manifold \((k \geq 1)\) and \( d \) is a convex metric on \( X \). Let \( F \) be a
finite subset of X and μ a good measure on X. Suppose that \( p_i, q_i \) \((i = 1, 2, ..., l)\) are points of \( \text{Int}(X) - F \) such that \( \{p_i, q_i\} \cap \{p_j, q_j\} = \emptyset \) \((i \neq j)\) with \( d(p_i, q_i) < \delta \). Then there exists \( h \in H(X, \mu) \) such that \( h(p_i) = q_i \) for each \( i \), \( d(h, \text{Id}) < \delta \) and \( \text{Supp}(h) \cap (F \cup \partial X) = \emptyset \).

Proof. We shall prove the case that \( X \) is a regularly connected polyhedron of dimension 2. Since \( F \) is a finite set and \( \text{Int}(X) \) is 2-dimensional manifold, we can choose arcs \( L_i \) from \( p_i \) to \( q_i \) such that the length \( l(L_i) \) of \( L_i \) is less than \( \delta \), \( L_i \cap L_j \) is an at most one point set for \( i \neq j \) and

\[
F \cap L_i = \emptyset, L_i \cap \{p_j, q_j| j = 1, 2, ..., l\} = \{p_i, q_i\}.
\]

Note that if \( X \) is the other cases, we can find arcs \( L_i \) from \( p_i \) to \( q_i \) such that the length \( l(L_i) \) of \( L_i \) is less than \( \delta \), \( F \cap L_i = \emptyset \) and \( L_i \cap L_j = \emptyset \) \((i \neq j)\). Let \( k \) be a sufficiently large natural number. For each \( i \), we can take \( k + 1 \) points \( p_i = p_{i,0} < p_{i,1} < ... < p_{i,k} = q_i \) on \( L_i \) such that the length \( l(L_{i,j}) \) of \( L_{i,j} \) is less than \( \delta/k \) and for each \( 1 \leq j \leq k \), the family \( \{L_{i,j}| i = 1, 2, ..., l\} \) are disjoint, where \( L_{i,j} \) is the sub arc from \( p_{i,j-1} \) to \( p_{i,j} \) in \( L_i \) (see the proof of [7, Lemma 13]). Take a sufficiently small neighborhood \( U_{i,j} \) of \( L_{i,j} \) for each \( i, j \) such that \( \delta(U_{i,j}) < \delta/k, U_{i,j} \cap F = \emptyset \) and for each \( j = 1, 2, ..., k \), \( \{U_{i,j}| i = 1, 2, ..., l\} \) are disjoint. For each \( j = 1, 2, ..., k \), we can choose \( h_j \in H(X, \mu) \) such that \( h_j(p_{i,j-1}) = p_{i,j} \) for each \( i \) and \( h_j(X - \bigcup_{i=1}^l U_{i,j}) = \text{Id} \). Put \( h = h_k \circ ... \circ h_1 \). Since \( d(h_j, \text{Id}) < \delta/k \), we see \( d(h, \text{Id}) < \delta \). Hence \( h \) is a desired homeomorphism.

The main result of this section is the following theorem. This theorem means that in the world admitting \( C^0 \)-topology, random infinite sequences of any prophecies will come true by making a slight change. From now on, we may assume that \( X \) admits a convex metric \( d \) if \( X \) is a Peano continuum.

**Theorem 2.4.** Suppose that \( X \) is a regularly connected polyhedron of dimension \( m \geq 2 \) or a Menger \( k \)-dimensional manifold \((k \geq 1)\), and \( \mu \) is a good measure on \( X \). Let \( h \in H(X, \mu) \), \( \epsilon > 0 \) and let \( (k_n| n \in \mathbb{Z}) \) be an arbitrary increasing bi-sequence of integers with \( k_0 = 0 \). Suppose that \( S_i = (a^n_i| n \in \Lambda_i) \) \((i \in \mathbb{N})\) are arbitrary infinite bi-sequences or finite sequences of distinct points of \( \text{Int}(X) \) and \( S_i, S_j \) \((i \neq j)\) are mutually disjoint. Then there is \( f \in H(X, \mu) \) satisfying the following conditions:

1. \( d(f, h) < \epsilon \) and \( f|\partial(X) = h|\partial(X) \).

2. \( S_i \) is realized by \( f \) for each \( i \in \mathbb{N} \). Moreover if \( S_i = (a^n_i| 0 \leq n \leq s_i) \) is a finite sequence, then \( a^n_0 \) is a periodic point of \( f \) and \( S_i \) is realized by \( f \) on the periodic ordered orbit of \( a^n_0 \).

3. If \( S_i \) and \( S_j \) are infinite bi-sequences, then there is \( n(i,j) \in \mathbb{N} \) such that if \( n \in \mathbb{Z} \) and \( |n| \geq n(i,j) \), then \( \text{Time}_f(a^n_0 \to a^n_j) = \text{Time}_f(a^n_0 \to a^n_i) \).

4. If \( S_i \) is an infinite bi-sequence, then \( (\text{Time}_f(a^n_0 \to a^n_j)| n \in \mathbb{Z}) \) is a bi-subsequence of \( (k_n| n \in \mathbb{Z}) \).
Proof. We may assume that $S_{2i-1} = (a^i_n \mid n \in \mathbb{Z})$ is an infinite bi-sequence and $S_{2i} = (b^i_n \mid 0 \leq n \leq s_i)$ is a finite sequence for each $i \in \mathbb{N}$. We consider the set $S = \bigcup_{i=1}^{\infty} S_i$, where $S_{2i-1} = \{a^i_n \mid n \in \mathbb{Z}\}$ and $S_{2i} = \{b^i_n \mid 0 \leq n \leq s_i\}$. Also, put $S_{2i-1,n} = \{a^i_j \mid -n \leq j \leq n\} \ (n \in \mathbb{N})$. By induction on $n$, we will construct a sequence $(h_n)_{n \in \mathbb{N}}$ of homeomorphisms of $X$ and a bi-subsequence of $(l_n \mid n \in \mathbb{Z})$ of $(k_n \mid n \in \mathbb{Z})$ with $l_0 = 0$ such that for each $n \in \mathbb{N}$, the following conditions are satisfied:

1. $h_n \in H_{\partial}(X, \mu)$.
2. $d(h_n \circ h_{n-1} \circ \ldots \circ h_1 \circ h, h_{n-1} \circ h_{n-2} \circ \ldots \circ h_1 \circ h) < \epsilon/3^n$ and $d((h_n \circ h_{n-1} \circ \ldots \circ h_1 \circ h)^{-1}, (h_{n-1} \circ h_{n-2} \circ \ldots \circ h_1 \circ h)^{-1}) < \epsilon/3^n$.
3. For each $1 \leq i \leq n$, the finite subsequence $(a^i_{-i}, a^i_{-i+1}, \ldots, a^i_{-1}, a^i_0)$ of $S_{2i-1}$ is realized by $h_i \circ h_{i-1} \circ \ldots \circ h_1 \circ h$. Moreover

\[ Time_{(h_i \circ h_{i-1} \circ \ldots \circ h_1)}(a^i_0 \rightarrow a^i_i) = l_i, \]
\[ Time_{(h_i \circ h_{i-1} \circ \ldots \circ h_1)}(a^i_0 \rightarrow a^i_{-i}) = l_{-i} \]

and $(Time_{(h_i \circ h_{i-1} \circ \ldots \circ h_1)}(a^i_0 \rightarrow a^i_j) \mid -i \leq j \leq i)$ is a finite subsequence of $(k_n)_{i \in \mathbb{Z}}$.
4. For each $1 \leq i \leq n$, the point $b^i_0$ is a periodic point of $h_i \circ h_{i-1} \circ \ldots \circ h_1 \circ h$ and the sequence $S_{2i} = (b^i_0, b^i_1, b^i_2, \ldots, b^i_{s_i})$ is realized by $h_i \circ h_{i-1} \circ \ldots \circ h_1 \circ h$.
5. If $1 \leq i \leq j < n$, then for $j < s \leq n$

\[ Supp(h_s) \cap Orb((h_j \circ h_{j-1} \circ \ldots \circ h_1 \circ h); a^i_0)_{[l_j, l_i]} = \phi. \]
6. If $1 \leq i < n$, then for $i < s \leq n$

\[ Supp(h_s) \cap Orb((h_i \circ h_{i-1} \circ \ldots \circ h_1 \circ h); b^i_0) = \phi. \]
7. If $1 \leq i \leq j \leq n$, then $(h_n \circ h_{n-1} \circ \ldots \circ h_1 \circ h)^{l_j}(a^i_0) = a^i_i$ and $(h_n \circ h_{n-1} \circ \ldots \circ h_1 \circ h)^{l_i}(a^i_0) = a^i_{-j}$.
8. For each $1 \leq i \leq n$,

\[ Orb(h_n \circ h_{n-1} \circ \ldots \circ h_1 \circ h; a^i_0)_{[l_n, l_i]} \subset X - S_{2i-1,n} \subset X - S \]

and

\[ Orb(h_n \circ h_{n-1} \circ \ldots \circ h_1 \circ h; b^i_0) \subset X - S_{2i} \subset X - S. \]

Let $n = 1$. Suppose that $\delta > 0$ is a sufficiently small positive number. By Lemma 2.1, we can choose $l_{-1}, l_1 \in \mathbb{Z}$ and a pseudo $\delta$-orbit

\[ a^i_{-1} = x(l_{-1}), x(l_{-1} + 1), \ldots, x(-1), x(0), x(1), \ldots, x(l_1 - 1), x(l_1) = a^i_1 \]

of $h$ from $a^i_{-1}$ to $a^i_1$ in $Int(X)$ such that $a^i_0 = x(0)$. We may assume that

\[ \{x(j) \mid -l_1 \leq j \leq l_1\} \cap S = S_{1,1} \]
and \( l_{-1}, l_1 \) are elements of the sequence \( (k_n \mid n \in \mathbb{Z}) \) such that \( l_{-1} < 0 < l_1 \). Also, we may assume that there is a pseudo \( \delta \)-orbit

\[
b_0^1 = z(0), z(1), ..., z(l_1 - 1), z(l_1) = b_0^1
\]
of \( h \) from \( b_0^1 \) to \( b_0^1 \) in \( \text{Int}(X) \) such that \( (b_0^1, b_1^1, ..., b_{n+1}^1) \) is a subsequence of the sequence

\[
z(0), z(1), ..., z(l_1 - 1)).
\]

Also, we may assume that

\[
\{z(j) \mid 0 \leq j \leq l_1 - 1\} \cap (S \cup \{x(j) \mid l_{-1} \leq j \leq l_1\}) = S_2.
\]

For the sake of simplicity, we may assume \( h \) satisfies that \( h(x(j)) \neq x(j) \) and \( h(z(j)) \neq z(j) \) for each \( j \) (see Lemma 2.2); if necessary, we replace \( h \) with the composition \( h' \circ h \) of \( h \) and \( h' \), where \( h' \in H_\beta(X, \mu) \) and \( d(h', Id) \) is sufficiently small. Then

\[
\{h(x(j)), x(j + 1)\} \cap \{h(x(j')), x(j' + 1)\} = \emptyset,
\]

\[
\{h(z(j)), z(j + 1)\} \cap \{h(z(j')), z(j' + 1)\} = \emptyset \quad (j \neq j').
\]

By Lemma 2.3, there is a homeomorphism \( h_1 \in H_\beta(X, \mu) \) such that \( d(h_1, Id) < \delta \) and \( h_1(h(x(j))) = x(j + 1) \) and \( h_1(h(z(j))) = z(j + 1) \) for each \( j \). If \( \delta \) is sufficiently small, then we may assume that \( d(h_1 \circ h, h) < \epsilon/3 \) and \( d((h_1 \circ h)^{-1}, h)^{-1} < \epsilon/3 \). Note that the sequences \((a_{-1}^1, a_0^1, a_1^1)\) and \((b_0^1, b_1^1, ..., b_{n+1}^1)\) are realized by \( h_1 \circ h \) and \( \text{Time}(h_1 \circ h)(a_0^1 \to a_{+1}^1) = l_{+1} \).

Assume that \( h_1, h_2, ..., h_n \) and \( l_{+1}, l_{+2}, ..., l_{+n} \) have been defined for certain \( n \) and they satisfy the conditions 1-8. We define \( h_{n+1} \) and \( l_{+}(n+1) \) as follows.

Let \( \delta > 0 \) be a sufficiently small positive number. Choose integers \( l_{+1}, l_{-(n+1)} \in \mathbb{Z} \) and a pseudo \( \delta \)-orbit

\[
a_{-(n+1)}^{n+1} = x(l_{-(n+1)}), x(l_{-(n+1)} + 1), ..., x(-1), x(0), x(1), ..., x(l_{(n+1)} - 1), x(l_{(n+1)}) = a_{n+1}^{n+1}
\]
of \( h \circ h_{n-1} \circ ... \circ h_1 \circ h \) from \( a_{-(n+1)}^{n+1} \) to \( a_{+1}^{n+1} \) such that the points \( x_i \) are distinct points of \( \text{Int}(X) \), \( a_0^{n+1} = x(0) \), and \((a_i^{n+1}) - (n + 1) \leq i \leq n + 1\) is a subsequence of the sequence

\[
x(l_{-(n+1)}), x(l_{-(n+1)} + 1), ..., x(-1), x(0), x(1), ..., x(l_{(n+1)} - 1), x(l_{(n+1)}).
\]

Also, by Lemma 2.1, for each \( 1 \leq i \leq n \), we may choose a pseudo \( \delta \)-orbit

\[
a_i^n = y_i(l_n), y_i(l_n + 1), ..., y_i(l_{n+1}) = a_i^{n+1}
\]
of \( h_n \circ h_{n-1} \circ ... \circ h_1 \circ h \) from \( a_i^{n} \) to \( a_i^{n+1} \) and a pseudo \( \delta \)-orbit

\[
a_{i_{-(n+1)}}^i = y_i(l_{-(n+1)}), y_i(l_{-(n+1)} + 1), ..., y_i(l_{-n}) = a_{i_{-n}}^i
\]
of \( h_n \circ h_{n-1} \circ ... \circ h_1 \circ h \) from \( a_{i_{-(n+1)}}^i \) to \( a_{i_{-n}}^i \). Also, we may assume that there is a pseudo \( \delta \)-orbit

\[
b_0^{n+1} = z(0), z(1), ..., z(l_{(n+1)} - 1), z(l_{(n+1)}) = b_0^{n+1}
\]
of \( h_n \circ h_{n-1} \circ ... \circ h_1 \circ h \) from \( b_0^{n+1} \) to \( b_0^{n+1} \) such that the points \( z_j \) are distinct points of \( \text{Int}(X) \), \((b_0^{n+1}, b_1^{n+1}, b_2^{n+1}, ..., b_k^{n+1})\) is a subsequence of \( z(0), z(1), ..., z((l(n+1) - 1)) \). Moreover, we may assume that \( A, B_i \) \((i = 1, 2, ..., n)\), \( C \) and \( D \) are mutually disjoint, where

\[
A = \{x(j) | l_{-(n+1)} \leq j \leq l_{n+1}\},
\]

\[
B_i = \{y'(j) | l_{-(n+1)} \leq j \leq l_n - 1\} \cup \{y'(j) | l_n + 1 \leq j \leq l_{n+1}\},
\]

\[
C = \{z(j) | 0 \leq j \leq l_{n+1}\},
\]

\[
D = \bigcup_{i=1}^{n}((h_n \circ h_{n-1} \circ ... \circ h_1 \circ h)^j(a_0^n) | l_n \leq j \leq l_n) \cup \bigcup_{i=1}^{n}((h_n \circ h_{n-1} \circ ... \circ h_1 \circ h)^j(b_0^n) | 0 \leq j \leq l_i\}.
\]

Also we may assume that \( l_{-(n+1)}, l_{n+1} \) are elements of the sequence \((k_n | n \in \mathbb{Z})\) and the finite sequence

\[
(Time_{(h_n \circ h_{n-1} \circ ... \circ h_1 \circ h)})(a_0^n \to a^n_j) | -(n+1) \leq j \leq n+1
\]

is a subsequence of \((k_n | i \in \mathbb{Z})\). By the same argument as above, we may assume that \( x(j), y'(j), z(j) \) are not fixed points of \( h_n \circ h_{n-1} \circ ... \circ h_1 \circ h \). By Lemma 2.3, there is a homeomorphism \( h_{n+1} \in H_\beta(X, \mu) \) such that \( h_{n+1}|D = \text{Id}, d(h_{n+1}, \text{Id}) < \delta \) and

\[
h_{n+1}(h_n \circ h_{n-1} \circ ... \circ h_1 \circ h(x(j)) = x(j + 1),
\]

\[
h_{n+1}(h_n \circ h_{n-1} \circ ... \circ h_1 \circ h(y'(j))) = y'(j + 1),
\]

\[
h_{n+1}(h_n \circ h_{n-1} \circ ... \circ h_1 \circ h(z(j))) = z(j + 1)
\]

for each \( i, j \). If \( \delta \) is sufficiently small, then we may assume that

\[
d(h_{n+1} \circ h_n \circ h_{n-1} \circ ... \circ h_1 \circ h, h_n \circ h_{n-1} \circ ... \circ h_1 \circ h) < \epsilon/3^{n+1},
\]

\[
d((h_{n+1} \circ h_n \circ h_{n-1} \circ ... \circ h_1 \circ h)^{-1}, (h_n \circ h_{n-1} \circ ... \circ h_1 \circ h)^{-1}) < \epsilon/3^{n+1}.
\]

Also, we may assume that the condition 8 is satisfied for \( h_{n+1} \circ h_n \circ h_{n-1} \circ ... \circ h_1 \circ h \).

By using the sequence \((h_n)_{n \in \mathbb{N}}\) of homeomorphisms of \( X \), we put

\[
f = \lim_{n \to \infty} h_n \circ h_{n-1} \circ ... \circ h_1 \circ h.
\]

Note that if \( i, j \leq n \), then

\[
f(a_i^n) = h_n \circ h_{n-1} \circ ... \circ h_1 \circ h(a_i^j),
\]

\[
f(b_i^n) = h_n \circ h_{n-1} \circ ... \circ h_1 \circ h(b_i^j).
\]

Then we can see that \( f \) is a desired homeomorphism.

Let \( f : X \to X \) be a map of a compact metric space \((X, d)\). Then \( f \) is **chaotic in the sense of Devaney** if \( f \) satisfies the following conditions:

1. \( f \) has sensitive dependence on initial conditions, i.e., there is a positive number \( \tau > 0 \) such that for any \( x \in X \) and any neighborhood \( U \) of \( x \) in \( X \), there is a point \( y \in U \) such that \( d(f^n(x), f^n(y)) \geq \tau \) for some positive integer \( n \in \mathbb{N} \),
2. $f$ is topologically transitive, i.e., the (positive) orbit $\{f^n(x) | n \in \mathbb{N}\}$ is dense in $X$ for some point $x \in X$.

3. the set of all periodic points is dense in $X$.

A subset $S$ of $X$ is a scrambled set of $f$ if there is a positive number $\tau > 0$ such that for any $x, y \in S$ with $x \neq y$,

$$\liminf_{n \to \infty} d(f^n(x), f^n(y)) = 0,$$

$$\limsup_{n \to \infty} d(f^n(x), f^n(y)) \geq \tau.$$  

If there is an uncountable scrambled set $S$ of $f$, we say that $f$ is chaotic in the sense of Li-Yorke. A map $f : X \to X$ is everywhere-chaotic (in the sense of Li-Yorke) if the following conditions are satisfied:

1. there is $\tau > 0$ such that if $U$ and $V$ are any nonempty open subsets of $X$ and $N$ is any natural number, then there is a natural number $n \geq N$ such that $d(f^n(x), f^n(y)) \geq \tau$ for some $x \in U$, $y \in V$, and

2. for any nonempty open subsets $U, V$ of $X$ and any $\epsilon > 0$ there is a natural number $n \geq 0$ such that $d(f^n(x), f^n(y)) < \epsilon$ for some $x \in U$, $y \in V$.

Suppose that $X$ is a regularly connected polyhedron of dimension $m \geq 1$. A space homeomorphic to $I^m$ is an $m$-cell. A 0-dimensional compactum $D$ in $\text{Int}(X)$ is flat if for any neighborhood $V$ of $D$ in $X$, there is a closed neighborhood $U$ of $D$ in $X$ such that $U \subset V$ and $U = B_1 \cup \cdots \cup B_p$, where $B_i$ ($i = 1, 2, \ldots, p$) are mutually disjoint $k$-cells. By Generalized Schoenflies theorem, we see that if $C$ and $C'$ are flat Cantor sets in $\text{Int}(X)$, then any homeomorphism $f : C \cup \partial X \to C' \cup \partial X$ can be extended to a homeomorphism $\overline{f} : X \to X$ (e.g., see the proof of [6, p. 93, Theorem 7]). Also, note that any closed subset of a flat 0-dimensional compactum is also flat.

**Theorem 2.5.** Suppose that $X$ is a regularly connected polyhedron of dimension $m \geq 2$ and $E$ is a dense $F_\sigma$-set of $X$ such that $E$ is a countable union of flat Cantor sets in $\text{Int}(X)$. Let $\mu$ be a good measure on $X$ with $\mu(E) = 1$. Suppose that $h \in H(X, \mu)$, $\epsilon > 0$ and $(k_n | n \in \mathbb{Z})$ is an arbitrary increasing bi-sequence of integers with $k_0 = 0$. Then there is a homeomorphism $f : X \to X$ satisfying the following conditions:

1. $d(f, h) < \epsilon$ and $f|\partial(X) = h|\partial(X)$.

2. $f$ and $f^{-1}$ are chaotic in the sense of Devaney and chaotic in the sense of Li-Yorke such that the set $E$ is a scrambled set of $f$. Moreover, if $a, b \in E$ and $a \neq b$, then

   (a) the sets $\{f^{k_n}(a) | n \in \mathbb{N}\}$ and $\{f^{-k_n}(a) | n \in \mathbb{N}\}$ are dense in $X$,

   (b) $\liminf_{n \to \pm \infty} d(f^{k_n}(a), f^{k_n}(b)) = 0$ and $\limsup_{n \to \pm \infty} d(f^{k_n}(a), f^{k_n}(b)) = \delta(X)$.

To prove the above theorem, we need the following notions: Let $X$ be a space and $R$ be any subset of $X^m$ ($m \geq 2$). A subset $F \subset X$ is said to be independent in $R$ if for any different $m$ points $x_1, \ldots, x_m$ of $F$ (i.e., $x_i \neq x_j$ for $i \neq j$), we have $(x_1, x_2, \ldots, x_m) \in X^m - R$. A countable union of nowhere dense sets is called a set of the first category.
Proposition 2.6. [4, Proposition 2.3] Suppose that $X$ is a regularly connected polyhedron of dimension $\geq 1$ and $R \subset X^m$ $(m \geq 2)$. If $X$ has no isolated point and $R$ is of the first category, then there is a subset $S$ of $X$ such that $S = \bigcup_{n=1}^{\infty} C_n$, where $C_n$ are flat Cantor sets in $X$, $S$ is independent in $R$, and $\text{Cl}(S) = X$.

By modifying the proof of [4, Theorem 2.6], we can prove the following.

Proposition 2.7. Suppose that $X$ is a regularly connected polyhedron of dimension $m \geq 1$. Let $E$ and $S$ be sets which are countable unions of flat Cantor sets of $\text{Int}(X)$. Then for any $\delta > 0$ there is a homeomorphism $u : X \to X$ such that $u(E) = S$ and $d(u, \text{Id}) < \delta$.

Proof of Theorem 2.5. Let $\{S_i | i \in \mathbb{N}\}$ be a countable family which is chaotic for $(k_n | n \in \mathbb{Z})$. By Theorem 2.4, there is $g \in H(X, \mu)$ such that $d(g, h) < \epsilon/2$ and $g$ satisfies the conditions as in Theorem 2.4. Then $g$ and $g^{-1}$ are everywhere-chaotic. Also we may assume that $g$ and $g^{-1}$ are chaotic in the sense of Devanay. We shall show that the set

$$T(g) = \{x \in X | \text{Cl}(\{f^{k_n}(x) | n \in \mathbb{N}\}) = X = \text{Cl}(\{f^{k-n}(x) | n \in \mathbb{N}\})\}$$

is a dense $G_\delta$-set of $X$. Let $\{U_i \}_{i \in \mathbb{N}}$ be an open countable base of $X$. For each $i, j \in \mathbb{N}$, consider the sets

$$T_{i,j}^+ = \{x \in X | g^{k_n}(x) \in (X - U_i) \text{ for } n \geq j\},$$

$$T_{i,j}^- = \{x \in X | g^{k-n}(x) \in (X - U_i) \text{ for } n \geq j\}.$$

Then

$$T(g) = X - \bigcup_{i,j \in \mathbb{N}} (T_{i,j}^+ \cup T_{i,j}^-).$$

Note that each $T_{i,j}^+$ is a closed and nowhere dense set of $X$ and hence we see that $T(g)$ is a dense $G_\delta$-set of $X$. Put

$$R_0 = ((X - T(g)) \times X) \cup (X \times (X - T(g))).$$

Then $R_0$ is of the first category in $X^2$.

Next, we consider the following sets:

$$R_1^+ = \{(x, y) \in X^2 | \limsup_{n \to \infty} d(g^{k_n}(x), g^{k_n}(y)) < \delta(X)\},$$

$$R_2^+ = \{(x, y) \in X^2 | \liminf_{i \to \infty} d(g^{k_i}(x), g^{k_i}(y)) > 0\}.$$

Let $\{\epsilon_i\}$ be a decreasing sequence of positive numbers with $\lim_{i \to \infty} \epsilon_i = 0$. Then $R_1^+ = \bigcup_{i=1}^{\infty} T_i$, where

$$T_i = \{(x, y) \in X^2 | d(g^{k_n}(x), g^{k_n}(y)) \leq \delta(X) - \epsilon_i \text{ for every } n \geq i\}.$$

Also, $R_2^+ = \bigcup_{i=1}^{\infty} W_i$, where

$$W_i = \{(x, y) \in X^2 | d(g^{k_n}(x), g^{k_n}(y)) \geq \epsilon_i \text{ for every } n \geq i\}.$$
Since \( T_i \) and \( W_i \subset X^2 \) are closed, \( R_1^+ \) and \( R_2^+ \) are of the first category in \( X^2 \). Put
\[
R_1^- = \{(x, y) \in X^2 | \limsup_{n \to -\infty} d(g^{kn}(x), g^{kn}(y) < \delta(X))\};
\]
\[
R_2^- = \{(x, y) \in X^2 | \liminf_{n \to -\infty} d(g^{kn}(x), g^{kn}(y) > 0)\}.
\]
Then \( R = R_0 \cup R_1^+ \cup R_2^+ \cup R_1^- \cup R_2^- \) is of the first category. By Proposition 2.6, there is a subset \( S \) of \( X \) such that \( S = \bigcup_{n=1}^{\infty} C_n \), where \( C_n \) are flat Cantor sets in \( Int(X) \), \( S \) is independent in \( R \) and \( \text{Cl}(S) = X \). By Proposition 2.7, there is a homeomorphism \( u : X \to X \) such that \( u(E) = S \) and \( d(u, Id) \) is sufficiently small. Put \( f = u^{-1} \circ g \circ u \). Then \( f : X \to X \) is topologically conjugate to \( g \), \( d(f, g) < \epsilon/2 \) and \( E \) is the scrambled set \( E \) of \( f \). We see that \( f \) is a desired homeomorphism.

### 3 Flows which realize precisely the given sequences of points on their orbits

In this section, we consider the case of continuous dynamical systems. For any \( t \in \mathbb{R} \), we define the integer \( < t > \in \mathbb{Z} \) by \( < t > = [t + 1/2] \), where \([x]\) is the greatest integer that is less than or equal to \( x \in \mathbb{R} \). Note that if \( t \in \mathbb{R} - \{Z + 1/2\} \), then the integer \( < t > \in \mathbb{Z} \) satisfies \( |t - < t > | < 1/2 \). The main result of this section is the following theorem.

**Theorem 3.1.** Suppose that \( X \) is a regularly connected polyhedron of dimension \( m \geq 3 \). Let \( (k_n| n \in \mathbb{Z}) \) be an arbitrary increasing bi-sequence of integers with \( k_0 = 0 \). Suppose that \( S_i = (a^i_n| n \in \Lambda_i) (i \in \mathbb{N}) \) are any infinite bi-sequences or finite sequences of (distinct) points which are contained in some polyhedral \( m \)-cell \( C \) of \( Int(X) \) and \( \Lambda_i \) are mutually disjoint. Then there exist \( \mu \in M_\varphi(X; \text{good}) \) and a \( \mu \)-measure preserving flow \( \varphi : X \times \mathbb{R} \to X \) satisfying the following conditions:

1. Each \( S_i \) (\( i \in \mathbb{N} \)) is realized by \( \varphi \). Moreover if \( S_i = (a^i_n| 0 \leq n \leq s_i) \) is a finite sequence, then \( a^i_0 \) is a periodic point of \( \varphi \) and \( S_i \) is realized by \( \varphi \) on the periodic orbit of \( a^i_0 \).

2. If \( S_i \) and \( S_j \) are infinite bi-sequences, then there is \( n(i, j) \in \mathbb{N} \) such that if \( n \in \mathbb{Z} \) with \( |n| \geq n(i, j) \), then
\[
< \text{Time}_\varphi(a^i_0 \rightarrow a^i_n) >= < \text{Time}_\varphi(a^j_0 \rightarrow a^j_n) >.
\]

3. If \( S_i \) is an infinite bi-sequence, then the bi-sequence \( < \text{Time}_\varphi(a^i_0 \rightarrow a^i_n) | n \in \mathbb{Z} > \) is a subsequence of \( (k_n| n \in \mathbb{Z}) \).

Proof. We use the methods of [7]. By [7, Lemma 1], \( X \) is a continuous image of an \( m \)-cell \( Z \) under a map which is a homeomorphism up to the boundary and which is a simplicial map of a certain subdivision of \( Z \) onto \( X \). Hence we may assume that \( X \) is the \( m \)-dimensional unit cube and \( C \) is an \( m \)-dimensional cube in the interior \( Int(X) \) of \( X \).
Let $B = I^{m-1}$ be the $(m - 1)$-dimensional unit cube and $B_1$ an $(m - 1)$-dimensional cube in the interior of $B$. Also, let $Q$ be the $m$-dimensional cube, that is, the product space of $B$ with $= [0, 1]$ where points $(b, 0)$ and $(b, 1)$ are identified and $p : B \times I \to Q$ denotes the quotient map. By the proof of [7, Theorem 3], there is an onto map $q : Q \to X$ such that $q|\text{Int}(Q)$ is an embedding and $q(\partial Q)$ is an $(m - 1)$-dimensional subpolyhedron of $X$. Hence we may assume that $X = Q$ and $C$ is a subset of $Q$ such that $C \subset p(B_1 \times [0, 1/2])$. Choose a countable subset $D$ of the interior $\text{Int}(B_1)$ of $B_1$ with $\text{Cl}(D) = B_1$. Let $S$ be the set which is the union of all $S_i$. By modifying the proof of Bennett’s theorem [2], we have a homeomorphism $h : Q \to Q$ such that $h|\partial Q = Id$, $h|S : S \to D \times [0, 1/2]$ is an embedding satisfying that $h(S) \cap p(\{d\} \times I)$ is an empty set or a one point set for each each $d \in D$. Consequently, we may assume that $S$ is contained in $p(D \times [0, 1/2])$ and for each each $d \in D$, $S \cap p(\{d\} \times [0, 1/2])$ is an empty set or a one point set. Let $d^n_i$ be the point of $D$ such that $a^n_i \in p(\{d^n_i\} \times [0, 1/2])$ for each $i \in \mathbb{N}, n \in \Lambda_i$. We consider the corresponding sequences $D_i = (d^n_i|n \in \Lambda_i)$ $(i \in \mathbb{N})$ of the sequences $S_i$. We define a measure $\nu$ in $B$ by $\nu(A) = \int_A 1/f(p)dp$, where $f : \text{Int}(B) \to \mathbb{R}$ is a map (=continuous function) such that $\int_B 1/f(p)dp = 1$ and $f(p) > 0$ for $p \in B - \partial B$, $f(B_1) = 1$ and $f(p)$ tends to infinity at the boundary $\partial B$ (see the proof of [7, Theorem 3]). By Theorem 2.4, we have $g \in H_\partial(B, \nu)$ satisfying the conditions of Theorem 2.4 with respect to $h = Id$, the the sequences $D_i = (d^n_i|n \in \Lambda_i)$ $(i \in \mathbb{N})$ and $(k_n| n \in \mathbb{Z})$. Then there is an isotopy $h_t$ of $B$, $0 \leq t \leq 1$, such that $h_t = Id$ $(0 \leq t \leq 1/2)$, $h_1 = g$, $h_t|\partial(B) = Id$. Define a map $\varphi : B \times I \to Q$ by $\varphi(x, t) = h_t(x)$ for $0 \leq t \leq 1$. Consider the mapping torus $Q_1$ of the map $g : B \to B$, i.e., $Q_1$ is obtained from $B \times I$ by identifying points $(x, 1)$ and $(g(x), 0)$ for $x \in B$. Then there is the natural homeomorphism $h : Q_1 \to Q$ such that $h([x, t]) = h_t(x)$. Hence we may assume that $Q = Q_1$. By the proof of [7, Theorem 3], we can define a flow $\varphi$ upward along streamlines perpendicular to $B$, taking the velocity at any point to be $1/f(x)$, where $x$ is the last intersection of the streamline with $B$. Then the flow $\varphi$ preserves $m$-dimensional Lebesgue measure in $Q_1$ (see the proof of [7, Theorem 3]). Note that the velocity at any point $x$ on streamlines perpendicular to $B_1$ is $1/f(x) = 1$. By the construction of $\varphi$, each $S_i$ $(i \in \mathbb{N})$ is realized by the flow $\varphi$. Also, the $m$-dimensional Lebesgue measure in $Q_1$ induces a good measure $\mu$ on $X$ by the map $q : Q \to X$. Let $S_i$ and $S_j$ be infinite bi-sequences. Since $|\text{Time}_q(d^n_0 \to d^n_i) - \text{Time}_q(a^n_0 \to a^n_j)| < 1/2$, we see that

$$< \text{Time}_\varphi(a^n_0 \to a^n_i) >= \text{Time}_g(d^n_0 \to d^n_i) \in \mathbb{Z}.$$ 

Note that $\text{Time}_q(d^n_0 \to d^n_i) = \text{Time}_q(d^n_0 \to d^n_j)$ for $|n| \geq n(i, j)$. Hence we see that if $S_i$ and $S_j$ are infinite bi-sequences, then for $n \in \mathbb{Z}$ with $|n| \geq n(i, j)$,

$$< \text{Time}_\varphi(a^n_0 \to a^n_i) >= < \text{Time}_\varphi(a^n_0 \to a^n_j) > .$$

We can see that $\mu$ and $\varphi$ satisfy the desired conditions of Theorem 3.1.

By a modification of the proof of Theorem 3.1, we can prove the following theorem. We omit the proof.

**Theorem 3.2.** Suppose that $X$ is a regularly connected polyhedron of dimension $m \geq 3$. If $S_i$ $(i \in \mathbb{N})$ are any infinite bi-sequences or finite sequences of distinct points of $\text{Int}(X)$
and \( S_i \), \( S_j \) (\( i \neq j \)) are mutually disjoint, then there exist \( \mu \in M_\theta(X; \text{good}) \) and a \( \mu \)-measure preserving flow \( \varphi : X \times \mathbb{R} \to X \) such that for each \( i \in \mathbb{N} \), \( S_i \) is realized by the flow \( \varphi \), and moreover if \( S_i \) is a finite sequence, then \( S_i \) is realized by \( \varphi \) on the periodic ordered orbit of \( \varphi \).

Note that if a separable metric space \( S \) is a countable set and perfect, then \( S \) is homeomorphic to the set \( \mathbb{Q} \) of all rational numbers. If \( f : X \to X \) is a transitive homeomorphism of a perfect compact metric space \( X \) and \( \text{Orb}(x, f) \) is dense in \( X \), then \( \text{Orb}(x, f) \) is homeomorphic to the set \( \mathbb{Q} \).

**Theorem 3.3.** Suppose that \( X \) is a regularly connected polyhedron of dimension \( n \geq 2 \) or a Menger \( k \)-dimensional manifold with \( k \geq 1 \). Let \( \mu \) be a good measure on \( X \), \( h \in H(X, \mu) \) and \( \epsilon > 0 \). Suppose that \( S_i \) \( (i \in \mathbb{N}) \) is a dense countable subset or a finite set of \( \text{Int}(X) \) such that the family \( \{S_i | i \in \mathbb{N} \} \) are mutually disjoint. Then there is \( f \in H(X, \mu) \) satisfying the following conditions:

1. \( d(f, h) < \epsilon \) and \( f|\partial(X) = h|\partial(X) \).
2. If \( S_i \) is an infinite set, then \( S_i \) coincides with a dense orbit of \( f \), i.e., \( S_i = \text{Orb}(a_i, f) \) for \( a_i \in S_i \), and if \( S_i \) is a finite set, then \( S_i \) is a subset of a periodic orbit of \( f \).

Proof. Let \( T_i \) \( (i \in \mathbb{N}) \) be infinite bi-sequences of points of \( \text{Int}(X) \) such that \( \{T_i | i \in \mathbb{N} \} \) is a chaotic family for \( Z \). Also, we can choose \( \{P_i | i \in \mathbb{N} \} \) which is a family of finite sequences of points of \( \text{Int}(X) \) such that \( \lim_{i \to \infty} P_i = X \) and \( P_i \) \( (i \in \mathbb{N}) \) are mutually disjoint, where \( P_i \) is the set induced by the sequence \( P_i \). By Theorem 2.4, there is \( g \in H(X, \mu) \) such that \( d(h, g) < \epsilon/2 \), \( g|\partial(X) = h|\partial(X) \) and \( T_i \) and \( P_i \) are realized by \( g \). Moreover, we may assume that \( P_i \) is realized on a periodic orbit of \( g \). Hence we can choose a countable family \( \{T_i' | i \in \mathbb{N} \} \) of mutually disjoint dense orbits of \( g \) and a countable family \( \{P_i' | i \in \mathbb{N} \} \) of mutually disjoint periodic orbits of \( g \) such that \( \lim_{i \to \infty} P_i' = X \). By modifying the proof of Bennett [2], we can prove that there is \( u \in H\theta(X, \mu) \) satisfying the following conditions; if \( S_i \) is an infinite set, then \( u(S_i) = T_i' \) and if \( S_i \) is a finite set, then \( u(S_i) \subset P_i' \) for some \( j_i \). Put \( f = u^{-1} \circ g \circ u \). Then \( f \) is a desired homeomorphism.

Finally, we have the following problem.

**Problem 3.4.** Are any versions of the results of this paper true in the smooth category?

Acknowledgement: The author thanks the referee for helpful comments on the problem above.

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