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On Optimization over the Doubly Nonnegative Cone

by

Yasuaki MATSUKAWA and Akiko YOSHISE

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UNIVERSITY OF TSUKUBA
Tsukuba, Ibaraki 305-8573
JAPAN

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Yasuaki Matsukawa* and Akiko Yoshise†

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Abstract

In recent years, many studies have focused on semidefinite relaxation for combinatorial optimization problems and their usefulness. While those studies force the solution matrix of the relaxation problem to be symmetric and positive semidefinite, we often see that each element of the solution matrix is meant to be nonnegative. A positive semidefinite matrix whose elements are nonnegative is called a *doubly nonnegative matrix*. It would be natural to obtain a better relaxation by considering an optimization problem over the set of such matrices (we call it the *doubly nonnegative cone*). In this paper, we will show that the doubly nonnegative relaxation gives significantly tight bounds for a class of quadratic assignment problems, while the computational time may not be affordable as long as we solve the relaxation as usual, i.e., as an optimization problem over the symmetric cone given by the direct sum of the semidefinite cone and the nonnegative orthant. Aiming to develop new and efficient algorithms for solving the doubly nonnegative optimization problems, we provide some basic properties of the doubly nonnegative cone focusing on barrier functions on its interior.

1 Introduction

In recent years, many studies have focused on semidefinite relaxation (SDP relaxation) for combinatorial optimization problems and their usefulness (see, e.g., [4], [7], [19], [12], [8], [5] and [13]). While those studies force the solution matrix of the relaxation problem to be symmetric and positive semidefinite, we often see that each element of the solution matrix is meant to be nonnegative.

A positive semidefinite matrix whose elements are nonnegative is called a *doubly nonnegative matrix*. We call the set of doubly nonnegative matrices the *doubly nonnegative cone* (DNN cone). In this paper, we first observe how the DNN relaxation gives tighter bounds than the SDP relaxation for a class of quadratic assignment problems which are NP-hard combinatorial optimization problems. In our computational experiments, we represent the DNN cone as a symmetric cone given by the direct sum of the SDP cone and the nonnegative orthant adding many slack variables, and solve the converted DNN relaxation by adopting existing conic optimization solvers. Resultantly, the size of the problem grows too much large. Besides the high quality of the DNN relaxation, our computational results show that this symmetric cone representation approach is not promising enough due to it being too time consuming. This is the

*Graduate School of Systems and Information Engineering, University of Tsukuba, Tsukuba, Ibaraki 305-8573, Japan. e-mail: matsuk50@sk.tsukuba.ac.jp

†Corresponding author. Graduate School of Systems and Information Engineering, University of Tsukuba, Tsukuba, Ibaraki 305-8573, Japan. e-mail: yoshise@sk.tsukuba.ac.jp Research supported in part by Grants-in-Aid for Scientific Research ((C)20560052, (C)19510137, (B)18310101, (B)20310082) of the Ministry of Education, Culture, Sports, Science and Technology of Japan.

motivation of this paper. Aiming to develop another approach, we provide some basic properties of the DNN cone focusing on a barrier function on its interior.

The paper is organized as follows. In Section 2, we show that our computational results of the DNN relaxation for a class of quadratic assignment problems according to the symmetric cone representation approach. In order to provide another side of the DNN cone, we review some important definitions and properties related to the hyperbolic polynomial in Section 3. In Section 4, we observe that the DNN cone is given by the closure of a hyperbolic cone and define the primal and dual optimization problems over the DNN cone. The hyperbolicity of (the interior of) the DNN cone implies that the so-called self-concordant barrier function can be defined on its interior. In Section 5, we briefly summarize a result on an algorithm based on the self-concordant barrier function approach [15] which can be adopted to solve DNN optimization problems. Concluding remarks are given in Section 6.

2 The DNN relaxation for quadratic assignment problems

In this section, we show that the DNN relaxation gives significantly tight bounds for a class of quadratic assignment problems. Here we define some special sets of matrices which appear in the paper.

Definition 2.1 (Five cones of matrices). *1. The cone $\mathcal{S}_n^+$ of $n \times n$ symmetric positive semi-definite matrices (the SDP cone) is the set given by*

$$\mathcal{S}_n^+ := \left\{ X \mid X = X^T,\ \forall d \in \mathbb{R}^n,\ d^T X d \geq 0 \right\}.$$

2. The cone $\mathcal{D}_n$ of $n \times n$ doubly nonnegative matrices (the DNN cone) is the set given by

$$\mathcal{D}_n^+ := \left\{ X \mid X = X^T,\ \forall d \in \mathbb{R}^n,\ d^T X d \geq 0,\ X \geq O \right\}.$$

3. The cone $\mathcal{C}_n$ of copositive matrices (the copositive cone) is given by

$$\mathcal{C}_n := \left\{ X \mid X = X^T,\ \forall d \in \mathbb{R}^n_+,\ d^T X d \geq 0 \right\}$$

where $\mathbb{R}^n_+$ denotes the nonnegative orthant in $\mathbb{R}^n$.

4. The dual cone $\mathcal{C}_n^*$ of $\mathcal{C}_n$ is the cone of completely positive matrices (the CP cone) given by

$$\mathcal{C}_n^* := \left\{ X \,\middle|\, \exists K,\ z^1, \cdots, z^K \in \mathbb{R}^n_+,\ X = \sum_{k=1}^{K} (z^k)(z^k)^T \right\}.$$

We can easily see that the following inclusive relation holds:

$$\mathcal{C}_n^* \subset \mathcal{D}_n \subset \mathcal{S}_n^+ \subset \mathcal{C}_n \tag{1}$$

It is known that the CP cone $\mathcal{C}_n^*$, the minimum cone among the above five cones, has high ability to express some combinatorial optimization problems including quadratic assignment problems (QAPs) ([7, 13],etc.). Let A and B be given $n \times n$ matrices. Then the QAP is expressed as follows:

$$\begin{array}{ll} \text{Minimize} & \displaystyle\sum_{i=1,\ldots,n} \sum_{j=1,\ldots,n} a_{ij} b_{\pi(i)\pi(j)} \\ \text{subject to} & \pi \text{ is a permutation.} \end{array}$$

The QAP is an NP-hard problem and it is still considered a computationally nontrivial task to solve modest size problems, say of size $n = 25$ [3]. Introducing a permutation matrix X, the QAP can be written equivalently as follows:

$$\begin{array}{ll} \text{Minimize} & \langle B \otimes A, xx^T \rangle \\ \text{subject to} & x = \text{vec}(X), \;\; X \in \Pi \end{array}$$

where $\text{vec}(X)$ is the vector in $\mathbb{R}^{n^2}$ obtained from X columnwise, $\langle C, D \rangle := \text{Tr}(C^T D)$ for $n^2 \times n^2$ matrices C and D, and Π is the set of all permutation matrices. Note that the set Π can be completely characterized as

$$\Pi = \{X \in \mathbb{R}^{n \times n} \mid X^T X = I, \; X \geq O\}.$$

By adding two seemingly redundant constraints $XX^T = I$ (see [1]) and $\sum_{i=1}^n \sum_{j=1}^n x_{ij} = n$ (see [13]), Povh and Rendle [13] show the following theorem which implies a close relationship between the QAP and an optimization problem over the CP cone:

Theorem 2.2 (Theorem 3 and Corollary 4 of [13]). *The convex hull of the set*

$$\{xx^T \mid x = \text{vec}(X), \quad X \in \Pi\}$$

is equal to the set of feasible solutions of the following problem QAP-CP:

$$\begin{array}{ll} \textit{Minimize} & \langle B \otimes A, Y \rangle \\ \textit{subject to} & \displaystyle\sum_{i=1,\ldots,n} Y^{ii} = I, \\ & \langle I, Y^{ij} \rangle = \delta_{ij} \;\; (i, j = 1, \ldots, n) \\ & \langle E_{n^2}, Y \rangle = n^2, \\ & Y = \begin{pmatrix} Y^{11} & \cdots & Y^{1n} \\ \vdots & \ddots & \vdots \\ Y^{n1} & \cdots & Y^{nn} \end{pmatrix} \in \mathcal{C}^*_{n^2} \end{array} \qquad (2)$$

where E_{n^2} is the $n^2 \times n^2$ matrix whose elements are 1s. Therefore, the optimal value of the QAP is equal to the optimal value of the QAP-CP.

To obtain a tractable relaxation of the QAP-CP, Povh and Rendle [13] consider the problem QAP-SDP where the constraint (2) is replaced by the SDP constraint $Y \in \mathcal{S}^+_{n^2}$.

We consider a tighter relaxation of the QAP-CP in terms of the inclusive relation (1) among the cones. Note that the DNN cone $\mathcal{D}_n$ is close to the CP cone $\mathcal{C}^*_n$ when n is small. In fact, $\mathcal{D}_n = \mathcal{C}^*_n$ holds for $n \leq 4$ [2]. An aim of this paper is to examine the DNN relaxation (the QAP-DNN) where the constraint (2) is replaced by the DNN constraint $Y \in \mathcal{D}_{n^2}$.

The DNN constraint can be represented equivalently as a symmetric cone constraint as follows

$$(Y, Z) \in \mathcal{S}^+_{n^2} \times \mathbb{R}^{n^2 \times n^2}_+, \; Y = Z. \qquad (3)$$

According to the above symmetric cone representation, we can adopt primal-dual interior-point algorithms to solve the QAP-DNN. In what follows, we compare the two relaxations, the QAP-SDP and the QAP-DNN, of the QAP-CP in terms of the accuracy and computational effort.

We use SeDuMi [18] on a PC at 2.4GHz for solving the QAP-SDP. In contrast, the QAP-DNN is a really tough problem, and to solve the problem, we have to use SDPA Online Solver [17] and TSUBASA

Table 1: QAP-SDP: obtained value and CPU time (sec)

Instance	SDP val	QAP	SDP cpusec
chr12c	−23479.928*	11156	(28.73)
esc16a	46.365*	68	(174.86)
esc16b	249.892	292	(213.11)
esc16c	94.386	160	(193.02)
had12	1578.573	1652	(2.20)
had14	2605.154	2724	(72.42)
nug12	476.173	578	(22.00)
nug15	982.890	1150	(109.88)
rou12	202833.519*	235528	(26.42)
scr12	8040.685	31410	(22.41)
scr15	10145.392*	51140	(101.14)
tai12a	193514.727	224416	(28.33)
tai15a	325576.933*	388214	(98.55)

Table 2: QAP-DNN: obtained value and CPU time (sec)

Instance	DNN p-val	DNN d-val	QAP	DNN cpusec
chr12c	11155.999	11156.206	11156	(1261.020)
esc16a	49.542	67.520	68	(24257.230)
esc16b	288.360	290.129	292	(28313.260)
esc16c	152.972	154.312	160	(34148.000)
had12	1651.992	1653.034	1652	(1544.670)
had14	2723.988	2724.512	2724	(12564.870)
nug12	567.677	568.204	578	(1508.380)
nug15	1140.136	1141.567	1150	(23992.140)
nug16a	1598.861	1599.982	1610	(148300.015)*
nug20	2505.770	2508.345	2570	(15444.69)*
rou12	235527.997	235528.373	235528	(2118.500)
scr12	31409.968	31410.775	31410	(1672.280)
scr15	51139.963	51142.118	51140	(25938.950)
tai12a	224415.998	224416.435	224416	(1660.260)
tai15a	377097.595	377101.813	388214	(21634.140)

at CompView [6] of Tokyo Institute of Technology with the help of Katsuki Fujisawa and Makoto Yamashita.

All instances are taken from QAPLIB website [3], a library of QAP test problems, and their optimal values have been found as in the "QAP" column in two tables, TABLE 1 and TABLE 2.

In TABLE 1, the "SDP val" and the "SDP cpusec" columns show the obtained values and the consumed CPU times of the SDP relaxation with SeDuMi. The values marked with $*$ in the "SDP val" column indicate that a numerical error is reported, i.e., the parameter ϵ does not achieve the default value 10^{-8}.

In TABLE 2, the "DNN p-val" and the "DNN d-val" columns show the obtained primal and dual values of the DNN relaxation, respectively. The "DNN cpusec" shows the CPU times consumed by the DNN relaxation with SDPA Online Solver. The values marked with $*$ in the "DNN cpusec" column indicate that those problems are solved on TSUBASA.

From the two tables, we can see that the DNN relaxation gives significantly tighter bounds than the SDP relaxation for those instances. Such a strong evidence also can be seen in terms of extracted solutions from the QAP-DNN problems. The following matrix $\bar{X}$ is given by $\bar{X} = \mathrm{mat}(y_{\max})$ where $\mathrm{mat}(y)$ is the $n \times n$ matrix Y obtained from the vector $y \in \mathbb{R}^{n^2}$ satisfying $y = \mathrm{vec}(Y)$, and $y_{\max}$ is the eigenvector corresponding to the maximum eigenvalue of the obtained solution Y by the DNN relaxation for the instance had12.

$$\bar{X} = \begin{pmatrix}
.000 & .000 & .985 & .000 & .000 & .000 & .000 & .004 & .001 & .000 & .000 & .000 \\
.000 & .000 & .000 & .001 & .000 & .000 & .000 & .000 & .000 & .988 & .000 & .000 \\
.000 & .003 & .000 & .000 & .002 & .000 & .000 & .000 & .000 & .001 & .982 & .001 \\
.000 & .984 & .000 & .000 & .001 & .000 & .000 & .000 & .000 & .000 & .004 & .001 \\
.000 & .000 & .000 & .000 & .000 & .001 & .001 & .000 & .000 & .000 & .001 & .987 \\
.000 & .000 & .000 & .000 & .985 & .002 & .001 & .000 & .000 & .000 & .002 & .000 \\
.000 & .001 & .000 & .000 & .002 & .487 & .500 & .000 & .000 & .000 & .001 & .001 \\
.004 & .000 & .000 & .001 & .000 & .498 & .486 & .000 & .000 & .000 & .000 & .000 \\
.000 & .000 & .004 & .001 & .000 & .000 & .000 & .985 & .000 & .000 & .000 & .000 \\
.985 & .001 & .000 & .000 & .000 & .002 & .001 & .000 & .000 & .000 & .000 & .000 \\
.000 & .001 & .000 & .987 & .000 & .000 & .000 & .000 & .002 & .000 & .000 & .000 \\
.000 & .000 & .001 & .002 & .000 & .000 & .000 & .001 & .987 & .000 & .000 & .000
\end{pmatrix}$$

It is known that the instance had12 has the following two optimal solutions, X_1^* and X_2^*. The observation

$$\bar{X} \approx (X_1^* + X_2^*)/2$$

implies that almost optimal solution of had12 can be obtained by the DNN relaxation. The same phenomena are also seen for the instances scr12, rou12, tai12a and had14.

$$X_1^* = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix},$$

$$X_2^* = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}$$

On the other hand, in terms of the CPU times, we have to say that the DNN relaxation is too much computationally expensive and not practical as long as we solve it using the symmetric cone representation (3) of the DNN cone. As we will explore in the succeeding sections, another approach should be adopted to the DNN relaxation.

3 Preliminaries on hyperbolic polynomials

In this section, in order to explore another approach to the DNN optimization problem, we review some definitions and properties related to the hyperbolic polynomial based on the papers [10] and [16].

Let $\mathcal{E}$ be a finite-dimensional Euclidean space.

Definition 3.1 (Hyperbolic polynomial). *A homogenous polynomial $p : \mathcal{E} \to \Re$ is said to be* hyperbolic *if there exists a direction $e \in \mathcal{E}$, $p(e) > 0$, with the property that, for each $x \in \mathcal{E}$, the univariate polynomial $t \mapsto p(x + te)$ has only real roots (i.e., each root has no imaginary part). The polynomial is said to be hyperbolic* in direction e.

Here we raise two examples of hyperbolic polynomials [16].

Example 3.2 (Linear programming 1). *$\mathcal{E} = \Re^n$, $p(x) = x_1 \cdots x_n$ and e is any vector with only positive coordinates.*

Example 3.3 (Semidefinite Programming 1). *$\mathcal{E} = \mathcal{S}_n$ (the set of $n \times n$ symmetric matrices), $p(x) = \det(x)$ and e is any symmetric matrix with only positive eigenvalues.*

The univariate functional $\lambda \mapsto p(\lambda e + x)$ is the characteristic polynomial of x (with respect to p, in direction e). The roots of the characteristic polynomial are the eigenvalues of x. Let r denote the degree of p. We write the eigenvalues of x as

$$\lambda_1(x) \leq \lambda_2(x) \cdots \leq \lambda_r(x),$$

counting multiplicities, and $\lambda_{\min}(x) := \lambda_1(x)$.

Definition 3.4 (Hyperbolic cone). *The set*

$$\Lambda_{++}(p, e) := \{x \in \mathcal{E} \mid \lambda_{\min}(x) > 0\}$$

is the hyperbolicity cone for p in direction e.

If a set K is the hyperbolicity cone for some p in some direction e, then we say that K is a hyperbolic cone.

Example 3.5 (Linear programming 2). *The set $\Re^n_{++} = \{x \in \Re^n \mid x > 0\}$ is a hyperbolic cone given by $\Re^n_{++} = \Lambda_{++}(p, e)$ where $p(x) = x_1 \cdots x_n$ and e is any vector with only positive coordinates.*

Example 3.6 (Semidefinite Programming 2). *The set S^n_{++} of $n \times n$ symmetric positive definite matrices is a hyperbolic cone given by $S^n_{++} = \Lambda_{++}(p, e)$ where $p(x) = \det(x)$ and e is any symmetric positive definite matrix.*

We summarize some properties of hyperbolic cones.

Proposition 3.7 (Hyperbolic cone). *Let $K \subset \mathcal{E}$, $K_1 \subset \mathcal{E}$ and $K_2 \subset \mathcal{E}$ be hyperbolic cones given by*

$$K = \Lambda_{++}(p, e), \ K_1 = \Lambda_{++}(p_1, e), \ K_2 = \Lambda_{++}(p_2, e)$$

for some $p : \mathcal{E} \to \Re$ with degree r, $p_1 : \mathcal{E} \to \Re$ with degree r_1, $p_2 : \mathcal{E} \to \Re$ with degree r_2 and for a common $e \in \mathcal{E}$.

(i) *The hyperbolic cone K is the connected component of $\{x \mid p(x) \neq 0\}$ containing e.*

(ii) *The hyperbolic cone K is convex.*

(iii) *The hyperbolic barrier function $F(x) = -\log p(x)$ is a selfconcordant barrier function on K with barrier parameter equal to r. Therefore F is convex on K.*

(iv) *The intersection $K_1 \cap K_2$ of K_1 and K_2 is a hyperbolic cone given by $K_1 \cap K_2 = \Lambda_{++}(p_1 p_2, e)$ where $p_1 p_2(x) := p_1(x)p_2(x)$.*

(v) *The function $-\log(p_1 p_2(x)) = -\log p_1(x) - \log p_2(x)$ is a logarithmically homogeneous self-concordant barrier for the hyperbolic cone $K_1 \cap K_2 = \Lambda_{++}(p_1 p_2, e)$ with barrier parameter less than $r_1 + r_2$.*

Proof. (i) and (ii): See Proposition 1 and Theorem 2 of [16].

(iii): See Theorem 4.1 of [10].

(iv): We can see that $p_1 p_2(e) = p_1(e)p_2(e) > 0$ and for each $x \in \mathcal{E}$, the univariate polynomial $t \mapsto p_1 p_2(x + te) = p_1(x + te)p_2(x + te)$ has only real roots. Thus $p_1 p_2$ is hyperbolic in direction e. Since $t \mapsto p_1 p_2(x + te) = p_1(x + te)p_2(x + te)$ has a nonpositive root if and only if $p_1(x + te)$ or $p_2(x + te)$ has a nonpositive root, this yields that

$$\begin{aligned} K_1 \cap K_2 &= \Lambda_{++}(p_1 p_2, e) \cap \Lambda_{++}(p_1 p_2, e) \\ &= \Lambda_{++}(p_1 p_2, e) \end{aligned}$$

(v): It follows from the assertions (iii), (iv) and Theorem 2.3.1 of [15]. □

Define

$$\Lambda_+(p, e) := \{x \mid \lambda_{\min}(x) \geq 0\}.$$

Then we can see that $\Lambda_+(p, e)$ is the closure of $\Lambda_{++}(p_1, e)$. Since $\Lambda_{++}(p_1, e)$ is convex, so is $\Lambda_+(p, e)$ (Theorem 2.33 of [14]).

4 Properties of the DNN cone

In this section, we observe basic properties of the DNN cone. Before proceeding, we introduce some notations and definitions: For a given set C, $\mathrm{con}\,(C)$ and $\mathrm{pos}\,(C)$ denote the convex hull of C and the positive hull of C, respectively, which are given by

$$\begin{array}{l} \mathrm{con}\,(C) := \left\{ \sum_{i=1}^p \alpha_i x_i \;\middle|\; x_i \in C,\ \alpha_i \geq 0 \right\}, \\ \mathrm{pos}\,(C) := \{0\} \cup \{\alpha x \mid x \in C,\ \alpha > 0\}. \end{array} \tag{4}$$

Given a cone $K \subset \Re^n$, we define its dual cone by

$$K^* := \{z \in \Re^n \mid \forall x \in K,\ \langle z, x \rangle \geq 0\}.$$

In what follows, we denote by K_1 and K_2 the set of n-dimensional symmetric positive matrices and the set of n-dimensional symmetric positive definite matrices, respectively, i.e.,

$$K_1 := \{X \in \Re^{n \times n} \mid X = X^T,\ X > O\}, \tag{5}$$

$$K_2 := \{X \in \Re^{n \times n} \mid X = X^T,\ X \succ O\}, \tag{6}$$

It can be easily seen that $\mathrm{cl}\,K_1$ is the set of n-dimensional symmetric nonnegative matrices and $\mathrm{cl}\,K_2$ is the set of n-dimensional symmetric positive semidefinite matrices $\mathcal{S}_n^+$, respectively, i.e.,

$$\begin{array}{rcl} \mathrm{cl}\,K_1 & := & \{X \in \Re^{n\times n} \mid \ X = X^T,\ X \geq O\}, \\ \mathrm{cl}\,K_2 & := & \{X \in \Re^{n\times n} \mid \ X = X^T,\ X \succeq O\}, \end{array}$$

The set $\mathrm{cl}\,K_1 \cap \mathrm{cl}\,K_2$ is the DNN cone. The DNN cone $\mathrm{cl}\,K_1 \cap \mathrm{cl}\,K_2$ has the following properties.

Proposition 4.1 (Properties of the DNN cone)**.** **(i)** $\mathrm{int}\,(\mathrm{cl}\,K_1) = K_1$ *and* $\mathrm{int}\,(\mathrm{cl}\,K_2) = K_2$*.*

(ii) $\mathrm{cl}\,K_1 \cap \mathrm{cl}\,K_2$ *is a closed convex cone and* $\mathrm{int}\,(\mathrm{cl}\,K_1 \cap \mathrm{cl}\,K_2) = K_1 \cap K_2 \neq \emptyset$*.*

(iii) $\mathrm{cl}\,K_1 + \mathrm{cl}\,K_2$ *is a closed convex cone and* $\mathrm{int}\,(\mathrm{cl}\,K_1 + \mathrm{cl}\,K_2) = K_1 + K_2 \neq \emptyset$*.*

(iv) $(\mathrm{cl}\,K_1 \cap \mathrm{cl}\,K_2)^* = \mathrm{cl}\,K_1 + \mathrm{cl}\,K_2$

Proof. (i): The equations follow from the fact that the sets K_1 and K_2 are open and convex (Theorem 2.33 of [14]).

(ii): By (i) above, it suffices to show that $K_1 \cap K_2 \neq \emptyset$. Consider the matrix $I + E$ where E denotes the $n \times n$ matrix whose elements are 1s. Then we have $I + E \in K_1 \cap K_2$.

(ii) For a given positive value $\alpha > 0$, define the two sets

$$\begin{array}{rcl} \Delta_1(\alpha) & := & \left\{ X \in \mathrm{cl}\,K_1 \ \middle|\ \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij} = \alpha \right\}, \\ \Delta_2(\alpha) & := & \{X \in \mathrm{cl}\,K_2 \mid \ \mathrm{Tr}\,(X) = \alpha\}. \end{array}$$

Since the set $\Delta_2(\alpha)$ is compact, we can define the value

$$\beta := \min \left\{ \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij} \ \middle|\ X \in \Delta_2(1) \right\} \tag{7}$$

which may be negative. Consider the sets $\Delta_1(|\beta|+1)$ and $\Delta_2(1)$. Since the sets $\Delta_1(|\beta|+1)$ and $\Delta_2(1)$ are compact and convex, so is the set $\Delta_1(|\beta|+1) + \Delta_2(1)$ (cf. Exercise 3.12 of [14]) and we have

$$\Delta_1(|\beta|+1) + \Delta_2(1) = \mathrm{con}\,\Big(\Delta_1(|\beta|+1) + \Delta_2(1)\Big).$$

Below, we will show that

$$0 \notin \mathrm{con}\,\Big(\Delta_1(|\beta|+1) + \Delta_2(1)\Big) \tag{8}$$

and

$$\begin{array}{rl} & \mathrm{cl}\,K_1 + \mathrm{cl}\,K_2 \\ = & \mathrm{pos}\,\Big(\Delta_1(|\beta|+1) + \Delta_2(1)\Big) \\ = & \mathrm{pos}\,\Big(\mathrm{con}\,\Big(\Delta_1(|\beta|+1) + \Delta_2(1)\Big)\Big) \end{array} \tag{9}$$

hold.

To show (8), suppose that $X \in \Delta_1(|\beta|+1)+\Delta_2(1)$. Then $X = X_1 + X_2$ for some $X_1 \in \Delta_1(|\beta|+1)$ and $X_2 \in \Delta_2(1)$, and by the definition (7) of β, we can see that

$$\begin{aligned}\sum_{i,j} x_{ij} &= \sum_{i,j}((x_1)_{ij}+(x_2)_{ij}) = \sum_{i,j}(x_1)_{ij} + \sum_{i,j}(x_2)_{ij} \\ &= |\beta|+1+\sum_{i,j}(x_2)_{ij} \geq |\beta|+1+\beta \geq 1.\end{aligned}$$

This yields (8).

To show (9), suppose that $X \in \mathrm{cl}\, K_1 + \mathrm{cl}\, K_2$. Then $X = X_1 + X_2$ for some $X_1 \in \mathrm{cl}\, K_1$ and $X_2 \in \mathrm{cl}\, K_2$ with the following four possible cases

(a): $X_1 = O$ and $X_2 = O$,

(b): $X_1 \neq O$ and $X_2 = O$,

(c): $X_1 = O$ and $X_2 \neq O$,

(d): $X_1 \neq O$ and $X_2 \neq O$.

If case (a) occurs then

$$X = O \in \mathrm{pos}\left(\Delta_1(|\beta|+1)+\Delta_2(1)\right)$$

by the definition (4). If case (d) occurs, then since $\gamma_1 := \sum_{i,j}(x_1)_{ij} > 0$ and $\gamma_2 := \mathrm{Tr}\,(X_2) > 0$ hold, X is given by

$$\begin{aligned}X &= \left(\gamma_1/(|\beta|+1)+\gamma_2\right) \\ &\quad \left\{\frac{\gamma_1/(|\beta|+1)}{\gamma_1/(|\beta|+1)+\gamma_2}\bar{X}_1 + \frac{\gamma_2}{\gamma_1/(|\beta|+1)+\gamma_2}\bar{X}_2\right\}\end{aligned}$$

where

$$\bar{X}_1 := \frac{|\beta|+1}{\gamma_1}X_1 \in \Delta_1(|\beta|+1) \quad \text{and} \quad \bar{X}_2 := \frac{1}{\gamma_2}X_2 \in \Delta_2(1),$$

which implies that

$$\begin{aligned}X &\in \mathrm{pos}\left(\mathrm{con}\left(\Delta_1(|\beta|+1)+\Delta_2(1)\right)\right) \\ &= \mathrm{pos}\left(\Delta_1(|\beta|+1)+\Delta_2(1)\right).\end{aligned}$$

We can easily find that $X \in \mathrm{pos}\ (\Delta_1(|\beta|+1)+\Delta_2(1))$ for cases (b) and (c) as well as case (d). Therefore,

$$\mathrm{cl}\, K_1 + \mathrm{cl}\, K_2 \subseteq \mathrm{pos}\left(\Delta_1(|\beta|+1)+\Delta_2(1)\right)$$

holds. The converse inclusion is obvious.

Since $\mathrm{pos}\,(C)$ is closed for any compact set $C \not\ni 0$ (cf. Exercise 3.48 of [14]), by (8) and (9), we conclude that the set $\mathrm{cl}\, K_1 + \mathrm{cl}\, K_2$ is also closed.

The convexity of the sets K_1 and K_2 guarantees that the equation $\mathrm{int}\,(K_1+K_2) = \mathrm{int}\, K_1 + \mathrm{int}\, K_2 = K_1 + K_2$ holds (cf. Proposition 2.24 and Exercise 2.45 of [14]). It is easy to see that $E + I \in K_1 + K_2$ where E is the matrix of 1s.

(iii): The assertion follows from $(\mathrm{cl}\, K_1 \cap \mathrm{cl}\, K_2)^* = \mathrm{cl}\,(\mathrm{cl}\, K_1 + \mathrm{cl}\, K_2)$ (cf. Corollary 11.25 of [14]) and the fact that $\mathrm{cl}\, K_1 + \mathrm{cl}\, K_2$ is closed as we have shown above. □

By (ii) of Proposition 4.1 and (iv) and (v) of Proposition 3.7, we obtain the following result.

Proposition 4.2. *The DNN cone $\mathcal{D}_n = \mathrm{cl}\, K_1 \cap \mathrm{cl}\, K_2 = \mathbb{R}^n_+ \cap \mathcal{S}^+_n$ is the closure of the hyperbolic cone $\mathrm{int}\, \mathcal{D}_n = K_1 \cap K_2$ with the logarithmically homogeneous self-concordant barrier function of the form*

$$-\Pi_{ij} \log x_{ij} - \log\det X = -\sum_{i,j} \log x_{ij} - \log\det X.$$

Note that in the proof of Proposition 4.1, we utilize only the following properties of K_1 and K_2:

(i) K_1 and K_2 are open and convex cones.

(ii) $K_1 \cap K_2 \neq \emptyset$.

(iii) $\mathrm{cl}\, K_1$ and $\mathrm{cl}\, K_2$ are given by $\mathrm{cl}\, K_1 = \mathrm{con}\,(C_1)$ and $\mathrm{cl}\, K_2 = \mathrm{con}\,(C_2)$ for some compact and convex sets $C_1 \not\ni 0$ and $C_2 \not\ni 0$.

For example, the closure of any symmetric cone K in a Euclidean Jordan algebra $(V, \circ)$ is given by

$$\mathrm{cl}\, K = \mathrm{pos}\ (\Delta)$$

where

$$\Delta = \{x \in \mathrm{cl}\, K \mid \mathrm{tr}\,(x) = 1\}$$

and $\mathrm{tr}\,(x)$ denotes the first coefficient of minimal polynomial of $x \in V$. By the spectral decomposition theorem (cf. Theorem III.1.2 of [9]), any $x \in V$ is given by

$$x = \sum_{i=1}^{r} \lambda_i c_i$$

for some eigenvalues $\lambda_1, \cdots, \lambda_r$ and Jordan frame $c_1, \cdots, c_r$. Combining this with the boundedness of any nonzero idempotent (cf. (iv) of Proposition of 2.7 in [20]), there exists a positive $\omega > 0$ such that

$$\|x\| \leq \sum_{i=1}^{r} |\lambda_i| \|c_i\| \leq \sum_{i=1}^{r} |\lambda_i| \omega \tag{10}$$

holds for any $x \in V$. Since $x \in \Delta$ implies that $x \in \mathrm{cl}\, K$ and $\mathrm{tr}\,(x) = 1$, we can see that

$$\lambda_i \geq 0 \ (i = 1, \ldots, r) \quad \text{and} \quad 1 = \sum_{i=1}^{r} \lambda_i = \sum_{i=1}^{r} |\lambda_i|, \tag{11}$$

which implies that the set Δ is compact and $0 \notin \Delta$. Therefore, we obtain the following proposition which is an extension of Proposition 4.1 to symmetric cones.

Proposition 4.3. *Let K_1 and K_2 be two symmetric cones having the nonempty intersection $K_1 \cap K_2 \neq \emptyset$. Then the assertions (i)-(iv) in Proposition 4.1 hold.*

Using the results in Proposition 4.1, we can define the primal and dual linear optimization problems over the DNN cone as follows:

$$
\begin{array}{lll}
\text{(P)} & \text{Minimize} & \langle C, X\rangle \\
 & \text{subjectto} & \langle A_i, X\rangle = b_i \ (i = 1, \dots, m), \\
 & & X \in \mathrm{cl}\, K_1 \cap \mathrm{cl}\, K_2, \\
\text{(D)} & \text{Maximize} & b^T y \\
 & \text{subjectto} & C - \sum_{i=1}^m y_i A_i \in \mathrm{cl}\, K_1 + \mathrm{cl}\, K_2
\end{array}
$$

where $\langle X, Z\rangle := \mathrm{Tr}\,(X^T Z) = \mathrm{Tr}\,(XZ)$.

Since the set $\mathrm{cl}\, K_1 \cap \mathrm{cl}\, K_2$ is a closed convex cone with nonempty interior and its dual cone is given by $(\mathrm{cl}\, K_1 \cap \mathrm{cl}\, K_2)^* = \mathrm{cl}\, K_1 + \mathrm{cl}\, K_2$, the following duality theorem can be obtained (cf. Theorem 3.2.8 of [15]).

Theorem 4.4 (Duality theorem of the DNN optimization)**.** *If the dual problem (D) is strongly feasible and the primal problem (P) is feasible, then the primal problem (P) has an optimal solution. Similarly, if the primal problem (P) is strongly feasible and the dual problem (D) is feasible, then the dual problem (D) has an optimal solution.*

5 A barrier function approach for solving the DNN optimization problems

As we have seen that the DNN cone $\mathcal{D}_n$ is the closure of a hyperbolic cone (Proposition 4.2), we can adopt the primal barrier method where Newton's method is used to minimize the associated logarithmically homogeneous self-concordant barrier function. See Section 2.4.2 of the book [15] for a detailed description of the algorithm. For the method, the following result can be derived from Theorem 2.4.1 of [15].

Theorem 5.1. *Let K_1 and K_2 be given by (5) and (6), respectively. Define the set*

$$
\begin{aligned}
D_F \quad := \quad & \{X \mid \langle A_i, X\rangle = b_i \ (i = 1, \dots, m), \\
& X \in \mathrm{cl}\, K_1 \cap \mathrm{cl}\, K_2\}
\end{aligned} \tag{12}
$$

which is the feasible region of the primal DNN optimization problem (P) defined in Section 4. Suppose that the set D_F is bounded and we have an initial point $X^0 \in \mathrm{int}\, D_F$. For a given $\epsilon \in (0, 1)$, the primal barrier method will terminate within

$$
\mathcal{O}\left(n \log\left(\frac{n^2}{\epsilon \mathrm{sym}\,(X^0, D_F)}\right)\right)
$$

iterations of the algorithm, all points X computed thereafter satisfy

$$
\frac{\langle C, X\rangle - p_*}{p^* - p_*} \le \epsilon.
$$

Here $\mathrm{sym}\,(X, D)$ *is the* symmetry of D about X *defined by*

$$
\mathrm{sym}\,(x, D) := \inf\{r(L) \mid \ L \in \mathcal{L}(x, D)\}
$$

where $\mathcal{L}(x, D)$ is the set of all lines through x which intersect D in an interval of positive length and $r(L)$ is the ratio of the length of the smaller to the larger of the two intervals in $L \cap (D \setminus \{X\})$. The values p_* and p^* are given by

$$
\begin{aligned}
p_* \quad &:= \quad \inf\{\langle C, X\rangle \mid \ X \in D_F\} \quad \textit{and} \\
p^* \quad &:= \quad \sup\{\langle C, X\rangle \mid \ X \in D_F\}.
\end{aligned}
$$

6 Concluding remarks

Our computational experiments show that the quality of the DNN relaxation is much higher than the SDP relaxation for solving a class of QAPs. Simultaneously, those show that the DNN relaxation is not practical as long as we represent the DNN cone as a symmetric cone $\mathcal{S}^+_{n^2} \times \mathbb{R}^{n^2 \times n^2}_+$ in the space $\mathbb{R}^{n^2 \times n^2} \times \mathbb{R}^{n^2 \times n^2}$ as in (3). In order to develop another approach, we have investigated the DNN cone and shown that the DNN cone is the closure of a hyperbolic cone. Thus we can adopt the primal barrier function method for solving the DNN optimization problems. However, the result is not enough to conduct an experimental study of the primal barrier function method. The DNN relaxation problem does not satisfy the assumption in Theorem 5.1 in usual. In particular, the feasible region of the DNN relaxation problem QAP-DNN has no interior. We have to consider an artificial problem and/or an infeasible algorithm as provided to the primal-dual interior point method for solving symmetric cone optimization problems. This is the focus of our future research.

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