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Abstract

This paper examines a model to locate a facility within a given convex polygonal region taking two criteria of equity and efficiency into account. Equity, or rather inequality in the inhabitant-facility distances, is measured by minimizing the sum of the absolute differences between all pairs of squared Euclidean distances from inhabitants to the facility. This measure meets the Pigou-Dalton principle of transfers, and can easily be minimized. Efficiency is measured through Weber-type sum of squared inhabitant-facility distances, either to be minimized or maximized) for an attracting or repellent facility respectively. Geometric localization results are obtained for the whole set of Pareto optimal solutions for each of the resulting bicriteria problems. A polynomial procedure is developed to obtain the full bicriteria plot and both trade-off curves.

Subject classification: Facilities planning: inequality continuous location. Programming: bicriteria

Area of review: Optimization

1 Introduction

Over the last decade, many studies in location science have been made concerning equity in a geographical setting. In order to quantify the inequalities based on facility-inhabitant distances, many inequality measures, for example, *range*, *variance*, *mean absolute deviation*, *sum of absolute deviations*, *Gini coefficient* have been incorporated into location models as objective functions. Comprehensive reviews of such inequality studies can be found in Erkut(1993), Marsh and Schilling(1994), Eiselt and Laporte(1995).

This study formulates inequality location models in a Euclidean plane using the *sum of absolute deviations*. That is, it seeks for the facility locations in such a way that the sum of absolute differences between all pairs of facility-inhabitant distances is minimized. We show that this measure is strictly improved if a unit distance from a longer facility-inhabitant distance is transferred to a shorter facility-inhabitant distance. Accordingly, this fulfills the *Pigou-Dalton principle of transfers*, which is regarded as a necessary condition for measures of inequality by many economists.

Then, we consider the two bicriteria models generated by this inequality model combined with Weber (resp. anti-Weber) like efficiency measures, minimizing (resp. maximizing) the sum of functional distances from the inhabitants to the facility. Therefore, we consider bicriteria formulations applicable not only to attractive facilities but also to obnoxious facilities, for which Weber problems with negative weights have been examined e.g. in Hansen et al.(1981a), Drezner and Wesolowsky(1990).

In this paper, the *squared* Euclidean distances between inhabitants and the facility are used rather than simple Euclidean distances, as in Drezner and Wesolowsky(1990), Ohsawa(1999). Such a quadratic formulation are applicable to the problems of locating emergency service agencies, and may be regarded as approximation to simple Euclidean cases. However, this squared distance view has the advantage that an exact analysis and solution can be obtained by geometrical means, both for the single objective equity model as for the biobjective ones.

The current research differs from existing inequality location works in several respects. First, a continuous plane is explicitly considered. Most existing models, for example, Mandell(1991), have been formulated within the discrete framework of candidate for optimal location, as pointed out by Eiselt and Laporte(1995). Since the candidate set is finite, standard mathematical programming methods are applicable to solve these discrete models. One exception is Drezner et al.(1986), who search for the location in the plane which minimizes the *range*, i.e., the difference between the maximum and minimum facility-inhabitant distances. Another one is Carrizosa (1999), who minimizes the *variance* of all Euclidean facility-inhabitant distances. Unfortunately neither the range nor the variance satisfy the Pigou-Dalton principle of transfers.

Second, we examine the conflict between equity and efficiency through bicriteria models. We introduce the sum of the facility-inhabitant distances as efficiency criterion. In the minisum case (Weber problem), the bicriteria problem is applicable to attractive facilities. In the maxisum case (anti-Weber problem), the bicriteria one is applicable to obnoxious facilities. The set of Pareto-optimal solutions associated with these problems offers a reduced number of the more interesting alternatives.

Furthermore, we introduce the new idea of the bicriteria plot to continuous location problems. Most real-world selections of an appropriate facility location on a continuous region require a careful evaluation of alternatives. The tradeoff curve corresponding to the Pareto-optimal solutions enables us to compare only the alternatives along this frontier, but the objective space is much better described by considering all alternatives. In other words, although the tradeoff curves provide only myopic information, the bicriteria plot corresponding to the whole feasible region offers the global bicriteria picture. Hence, the comparison of the tradeoff curves with the bicriteria plot helps to visually evaluate these Pareto-optimal solutions from a global point of view.

As pointed out by Carrizosa and Plastria(1999), determining the Pareto-set in a continuous plane is usually a rather difficult problem because standard procedures of convex analysis are not always directly helpful. However, Ohsawa(2000) gave a polynomial algorithm to compute the Pareto-set when maximin and minimax Euclidean distances are used as push and pull objectives. Ohsawa and Tamura(2003) extended this work to bicriteria models combining on the one hand maximin elliptic and minisum rectilinear distances, and maximin or minimax rectilinear distances on the other hand. Ohsawa, Plastria and Tamura(2005) introduced partial covering in the bicriteria model of Ohsawa (2000). Melachrinoudis and Xanthopulos(2003) proposed a numerical approach for a bicriteria problem with maximin and minisum Euclidean distances, based on Voronoi diagrams and optimality conditions in nonlinear programming. Our general localization results are somewhat comparable to this latter work, but from the algorithmic point of view, our solution method is rather similar to the one by Ohsawa(2000). We present a polynomial algorithm by using the line tessellation generated by the perpendicular bisectors of all pairs of inhabitant locations, and show that this allows the construction of the bicriteria plot. Its leftward envelope then determines the two types of the Pareto-sets. This shows that the framework developed by Ohsawa(2000) may be extended to other bicriteria location models.

The remainder of this paper is structured as follows. Section 2 explores the single-objective inequality location models, first the equity objective, then the efficiency ones. Section 3 discusses the bicriteria models generated by combining the equity model with either Weber or anti-Weber models, giving proofs of the localization theorems for the Pareto-sets, then describing polynomial algorithms to construct the bicriteria plot and the Pareto-sets. Section 4 contains our conclusions.

2 Single-Objective Problems

2.1 Equity model

We are given inhabitants on an Euclidean plane. Let I and $\{\mathbf{p}_1, \cdots, \mathbf{p}_{|I|}\}$ be their index and location sets. Let $\{\omega_1, \cdots, \omega_{|I|}\}$ be their positive weights, i.e., $\omega_i > 0$. Define $W \equiv \sum_{i \in I} \omega_i$. Suppose that a facility can be built within the convex polygon Ω. Let $\partial\Omega$ and $|\partial\Omega|$ be the boundary and the number of sides of Ω, respectively. We may allow that some $\mathbf{p}_i \notin \Omega$, i.e., some inhabitants may be situated outside the feasible region. To avoid unnecessary complication we assume that $|I| \geq 3$, and that all $\mathbf{p}_i$ and the vertices of $\partial\Omega$ are in general position.

Our inequality problem seeks for a location minimizing the sum of absolute difference between all pairs of facility-inhabitant squared Euclidean distances. Its formulation is

$$\min_{\mathbf{x} \in \Omega} \qquad F(\mathbf{x}) \equiv \sum_{i \in I} \sum_{j \in I} \omega_i \omega_j \left| \|\mathbf{x} - \mathbf{p}_i\|^2 - \|\mathbf{x} - \mathbf{p}_j\|^2 \right|. \tag{1}$$

Substitution of $|\|\mathbf{x} - \mathbf{p}_i\|^2 - \|\mathbf{x} - \mathbf{p}_j\|^2| = \|\mathbf{x} - \mathbf{p}_i\|^2 + \|\mathbf{x} - \mathbf{p}_j\|^2 - 2\min\{\|\mathbf{x} - \mathbf{p}_i\|^2, \|\mathbf{x} - \mathbf{p}_j\|^2\}$ into (1) yields

$$F(\mathbf{x}) = 2\left(W \sum_{i \in I} \omega_i \|\mathbf{x} - \mathbf{p}_i\|^2 - \sum_{i \in I} \sum_{j \in I} \omega_i \omega_j \min\{\|\mathbf{x} - \mathbf{p}_i\|^2, \|\mathbf{x} - \mathbf{p}_j\|^2\} \right). \tag{2}$$

This expression shows that $F(\mathbf{x})$ decreases whenever a small part from a longer facility-inhabitant distance is transferred to a shorter facility-inhabitant distance. Hence the Pigou-Dalton principle of transfers holds, see Erkut(1993), Marsh and Schilling(1994). This measure is also the numerator of the well known Gini index, so it is intimately related to the Lorenz curve which is frequently used to measure inequality of income in economics: see Sen(1973).

It should be noted that whenever $\mathbf{x}$ moves elsewhere inside the convex hull of $\mathbf{p}_1, \cdots, \mathbf{p}_{|I|}$ at least one inhabitant will always be worse off. Hence, if the feasible region Ω is a subset of the convex hull, $F(\mathbf{x})$ also fulfills the principle of Pareto optimality: see Erkut(1993), Marsh and Schilling(1994).

Let $i_1, \cdots, i_{|I|}$ be an ordering of the inhabitant set I. Define $V_{i_1, i_2, \cdots, i_{|I|}}$ to be the ordered order-$|I|$ Voronoi polygon associated with the sequence $\mathbf{p}_{i_1}, \mathbf{p}_{i_2}, \cdots, \mathbf{p}_{i_{|I|}}$: see Okabe et al.(1999). That is,

$$V_{i_1, i_2, \cdots, i_{|I|}} \equiv \{\mathbf{x} | \|\mathbf{x} - \mathbf{p}_{i_1}\| \leq \|\mathbf{x} - \mathbf{p}_{i_2}\| \leq \cdots \leq \|\mathbf{x} - \mathbf{p}_{i_{|I|}}\|\}.$$

This set will be empty for many orderings $i_1, i_2, \cdots, i_{|I|}$ of I. The union of all non-empty $V_{i_1, i_2, \cdots, i_{|I|}}$'s is called the *ordered order-$|I|$ Voronoi diagram*. Observe that the boundary of this diagram coincides with the line tessellation generated by all the perpendicular bisectors l_{ij} of pairs $\mathbf{p}_i, \mathbf{p}_j$. It follows that the number of nonempty Voronoi polygons is $O(|I|^4)$. Let ∂V be the collection of the boundaries of $V_{i_1, i_2, \cdots, i_{|I|}}$'s within Ω. The ordered order Voronoi diagram (the line tessellation) is illustrated in Figure 1, where five inhabitants $\mathbf{p}_1, \cdots, \mathbf{p}_5$ are indicated by filled circles, and ∂V and $\partial\Omega$ are shown as thick and thin lines, respectively.

Proposition 1 *The function $F(\mathbf{x})$ is convex, piecewise linear with pieces the Voronoi polygons $V_{i_1, i_2, \cdots, i_{|I|}}$.*

Proof For two points $\mathbf{p}_i$ and $\mathbf{p}_j$ $(i \neq j)$, and writing scalar product as $\langle \cdot\, ; \cdot \rangle$, we have

$$\begin{aligned} \|\mathbf{x} - \mathbf{p}_i\|^2 - \|\mathbf{x} - \mathbf{p}_j\|^2 &= \langle\, \mathbf{x} - \mathbf{p}_i \ ; \ \mathbf{x} - \mathbf{p}_i \,\rangle - \langle\, \mathbf{x} - \mathbf{p}_j \ ; \ \mathbf{x} - \mathbf{p}_j \,\rangle & (3) \\ &= 2\langle\, \frac{\mathbf{p}_i + \mathbf{p}_j}{2} - \mathbf{x} \ ; \ \mathbf{p}_i - \mathbf{p}_j \,\rangle. & (4) \end{aligned}$$

This shows that this difference of squared distances is an affine function taking, of course, value zero along the bisector of $\mathbf{p}_i$ and $\mathbf{p}_j$, i.e. the line l_{ij} of equation $\langle\, \frac{\mathbf{p}_i + \mathbf{p}_j}{2} - \mathbf{x} \ ; \ \mathbf{p}_i - \mathbf{p}_j \,\rangle = 0$. Its absolute value is therefore convex and piecewise linear, with pieces the two halfplanes bounded by l_{ij}.$F(\mathbf{x})$ is the sum of such functions over all $i, j \in I$, and therefore also convex and piecewise linear, with pieces the intersections of halfplanes bounded by the l_{ij}'s, which are exactly the Voronoi polygons $V_{i_1,i_2,\cdots,i_{|I|}}$. ∎

The piecewise linearity of $F(\mathbf{x})$ leads to the important fact that the level curves of $F(\mathbf{x})$ within a Voronoi polygon consist of parallel lines. Some resulting level curves of $F(\mathbf{x})$ for our problem are shown in Figure 2. In addition, the minimum of $F(\mathbf{x})$ within each Voronoi polygon restricted to Ω is reached at some vertex. We will now show that the global solution $\mathbf{z}^*$ is usually unique.

As proved in Sen(1973), for any $\mathbf{x} \in V_{i_1,i_2,\cdots,i_{|I|}}$, the measure (2) can be rewritten as

$$F(\mathbf{x}) = 2\left(u_{i_1}\|\mathbf{x}-\mathbf{p}_{i_1}\|^2 + u_{i_2}\|\mathbf{x}-\mathbf{p}_{i_2}\|^2 + \cdots + u_{i_s}\|\mathbf{x}-\mathbf{p}_{i_s}\|^2 + \cdots + u_{i_{|I|}}\|\mathbf{x}-\mathbf{p}_{i_{|I|}}\|^2\right), \qquad (5)$$

where $u_{i_1} \equiv \omega_{i_1}(\omega_{i_1} - W)$, $u_{i_2} \equiv \omega_{i_2}(2\omega_{i_1} + \omega_{i_2} - W)$, $u_{i_s} \equiv \omega_{i_s}(\sum_{k=1}^{s}\omega_{i_k} - \sum_{k=s}^{|I|}\omega_{i_k})$ and $u_{i_{|I|}} \equiv \omega_{i_{|I|}}(W - \omega_{i_{|I|}})$. Since $\sum_{k=1}^{|I|} u_k = 0$, the minimization of $F(\mathbf{x})$ subject to $\mathbf{x}$ within a Voronoi polygon is a squared Euclidean Weber problem with positive and negative weights summing up to zero. By convexity we know that the optimal solution set must be convex. Suppose then that the optimal solution would not be unique, then all points along the non trivial segment joining any two optima would also be optimal and thus at least one of following three cases would happen for some optimal $\mathbf{z}^*$: *case 1*: $\mathbf{z}^*$ is on the interior of some $V_{i_1,i_2,\cdots,i_{|I|}}$; *case 2*: $\mathbf{z}^*$ is on the relative interior of some edge of ∂V; *case 3*: $\mathbf{z}^*$ is on the relative interior of some edge of $\partial\Omega$. In the first case, by linearity, $F(\mathbf{x})$ has to be constant over the whole Voronoi polygon, so $\sum_{k=1}^{|I|} u_{i_k}\mathbf{p}_{i_k} = 0$, as shown in Drezner and Wesolowsky(1990), which is not a general position of the $\mathbf{p}_i$'s. In the second case, $F(\mathbf{x})$ is constant along an edge of some $V_{i_1,i_2,\cdots,i_{|I|}}$, so its gradient in this cell $-\sum_{k=1}^{|I|} u_{i_k}\mathbf{p}_{i_k}$ must be orthogonal to one of its edges, so to some line l_{ij}, i.e. parallel to $\mathbf{p}_i - \mathbf{p}_j$; so $\sum_{k=1}^{|I|} u_{i_k}\mathbf{p}_{i_k} = \lambda(\mathbf{p}_i - \mathbf{p}_j)$ for some λ, which is again not a general position of the $\mathbf{p}_i$'s since $|I| \geq 3$. In the third case, $F(\mathbf{x})$ is constant along an edge of $\partial\Omega$, so $\sum_{k=1}^{|I|} u_{i_k}\mathbf{p}_{i_k}$ is normal to one of the given edges of $\partial\Omega$, contradicting the general position of $\mathbf{p}_i$'s and the vertices of $\partial\Omega$.

Proposition 2 *A solution $\mathbf{z}^*$ exists at either a vertex of ∂V (i.e., an intersection of the line tessellation) or a vertex of $\partial\Omega$ or an intersection of ∂V and $\partial\Omega$. It is unique in the general case.*

This Proposition yields directly the following algorithm:

Step 1. Define the planar graph $\partial V \cup \partial\Omega$.

Step 2. Find the minimum solution of $F(\mathbf{x})$ from the nodes of the graph $\partial V \cup \partial\Omega$.

The line tessellation consists of $O(|I|^2)$ lines, so we can construct the graph ∂V in $O(|I|^4)$ time. Hence, Step 1 takes $O(|I|^4 + |\partial\Omega|)$ time since $\partial\Omega$ is a closed polygonal line. To compute $F(\mathbf{x})$ in (5), which is a less demanding expression than (1) and (2), the $\mathbf{p}_i$'s are arranged according to the facility-inhabitant distances. Once the facility-inhabitant distances within one Voronoi polygon have been sorted in $O(|I|\log|I|)$, we can compute $F(\mathbf{x})$ at any vertex of the polygon in $O(|I|)$ using (5). But based on this computation we can also calculate $F(\mathbf{x})$ at vertices within any of its surrounding polygons in only constant time. This is because moving from one polygon to an adjacent polygon across some Voronoi edge interchanges only the order of the two consecutive inhabitant points at equal distance of all points on the edge: see e.g. $V_{2,3,4,1,5}$ with $V_{3,2,4,1,5}$ in Figure 1. Since the planar graph $\partial V \cup \partial\Omega$ has $O(|I|^4 + |\partial\Omega|)$ edges, Step 2 runs in $O(|I|^4 + |\partial\Omega|)$ time by traversing at most twice each edge of the graph $\partial V \cup \partial\Omega$. Accordingly, we have

Proposition 3 *A solution $\mathbf{z}^*$ can be found in $O(|I|^4 + |\partial\Omega|)$ time.*

Alternatively, Step 2 may also be done by grouping the calculations along each bisector as follows. Consider a bisector $l_{i_0j_0}$ with parametric equation $\mathbf{x} = \mathbf{c} + t\mathbf{q}$ where $\mathbf{c} = (\mathbf{p}_{i_0} + \mathbf{p}_{j_0})/2$ and $\mathbf{q}$ is orthogonal to $\mathbf{p}_{i_0} - \mathbf{p}_{j_0}$. $l_{i_0j_0}$ either contains no points of $\partial\Omega$, or just 1, or exactly 2, or, exceptionally a full line segment, side of Ω, which may in this first stage be reduced to its two endpoints. In the two last cases we have to search further along $l_{i_0 j_0}$. Using expression (4) one easily obtains the following new expression for $F(\mathbf{x})$ along $l_{i_0 j_0}$

$$F(\mathbf{x}) = F(\mathbf{c}+t\mathbf{q}) = \sum_{i \neq j \in I \setminus i_0, j_0} \omega_i \omega_j \alpha_{ij} |t_{ij} - t| + \sum_{i \in I \setminus i_0} \omega_{i_0} \omega_j \alpha_{i_0 j} |t_{i_0 j} - t| + \sum_{j \in I \setminus j_0} \omega_i \omega_{j_0} \alpha_{i j_0} |t_{i j_0} - t|, \tag{6}$$

where

$$\alpha_{ij} \equiv |\langle\, 2\mathbf{q} \;;\; \mathbf{p}_i - \mathbf{p}_j \,\rangle|, \qquad t_{ij} \equiv 2\langle\, \frac{\mathbf{p}_i + \mathbf{p}_j}{2} - \mathbf{c} \;;\; \mathbf{p}_i - \mathbf{p}_j \,\rangle / \alpha_{ij}.$$

Note that the t_{ij} corresponds to the two types of Voronoi vertices on $l_{i_0 j_0}$: the points $\mathbf{c} + t_{ij}\mathbf{q}$ of order 4, which are intersections of type $l_{i_0 j_0} \cap l_{ij}$ with $|\{i_0, j_0, i, j\}| = 4$, and those of form $\mathbf{c} + t_{i_0 j}\mathbf{q}$ or $\mathbf{c} + t_{i j_0}\mathbf{q}$ of order 6, which are those of type $l_{i_0 j_0} \cap l_{i_0 i} \cap l_{i j_0}$ with $|\{i_0, j_0, i\}| = 3$.

Expression (6) shows that $F(\mathbf{c}+t\mathbf{q})$ as a function of t along $l_{i_0 j_0}$ is a weighted sum of distances to the points t_{ij}, and its minimization is a one-dimensional Weber problem, well known to be solved at a (weighted) median point (see e.g. Francis et al., 1992), which may be obtained in time linear in the number of points by the procedure of Balas and Zemel (1980). Thus we can obtain the unconstrained minimum point of $F(\mathbf{x})$ along each bisector in $O(|I|^2)$, and then in constant time its constrained minimum by simple comparison with the boundary points.

In fact a descent procedure, which first finds the median along one bisector, then iteratively moves to another bisector which passes at this point and along which there is a decrease direction, until no such direction exists, will work much quicker in practice. (Note that the median finding technique applies along any line, so may also be used, if necessary, along Ω's boundary.) It is not clear, however, if a better worst case complexity may be shown for this method.

The solution $\mathbf{z}^*$ for our sample problem with $\omega_1 = \omega_2 = \ldots = \omega_{|I|}$ lies on a vertex of ∂V, as shown in Figure 2.

2.2 Efficiency models

As efficiency criterion, we consider the following typical single-objective function:

$$G(\mathbf{x}) \equiv \sum_{i \in I} \omega_i \|\mathbf{x} - \mathbf{p}_i\|^2. \tag{7}$$

The problems to either minimize or maximize $G(\mathbf{x})$ are called (quadratic distance) *Weber* or *anti-Weber problems*, respectively, as in Hansen et al.(1981a). The minimization of $G(\mathbf{x})$ may be used for locating a purely attractive facility such as a fire station or police station, whereas the maximization of $G(\mathbf{x})$ may be applicable to a purely repellent facility in the sense that the total distance to inhabitants is minimized. Therefore, this is applicable to undesirable facilities which emit contaminants such as gas, dust and smoke.

It is straightforward to check (see e.g. Francis et al., 1992) that

$$G(\mathbf{x}) = W\|\mathbf{x} - \overline{\mathbf{p}}\|^2 + \sum_{i \in I} \omega_i \|\mathbf{p}_i\|^2 - W\|\overline{\mathbf{p}}\|^2 \tag{8}$$

where $\overline{\mathbf{p}} \equiv \frac{1}{W} \sum_{i \in I} \omega_i \mathbf{p}_i$ is the center of gravity (centroid) of the points $\mathbf{p}_i$. This shows that $G(\mathbf{x})$ is a strictly convex function of $\mathbf{x}$.

It also immediately leads to the well-known result that the unconstrained minimum of $G(\mathbf{x})$ is given by $\overline{\mathbf{p}}$ and that the level sets of $G(\mathbf{x})$ consist of circular disks with center $\overline{\mathbf{p}}$. The constrained minimum $\mathbf{m}^*$ of $G(\mathbf{x})$ on Ω is unique, due to strict convexity, and is evidently the projection of $\overline{\mathbf{p}}$ on Ω, i.e. the point of Ω closest to $\overline{\mathbf{p}}$.

The strict convexity of $G(\mathbf{x})$ also implies that $G(\mathbf{x})$ can be maximum on Ω only at a vertex of Ω which we will denote by $\mathbf{a}^*$, as discussed in Hansen et al.(1981a). Note that the minimizer $\mathbf{m}^*$ exists uniquely. The solutions $\mathbf{m}^*$ and $\mathbf{a}^*$ for the sample problem are also shown in Figure 2. Note that in this example $\mathbf{m}^* = \overline{\mathbf{p}}$, because $\overline{\mathbf{p}} \in \Omega$.

3 Biobjective Problem

3.1 Formulation

In order to examine the tradeoff between equity and efficiency, we formulate the following two bicriteria problems combining the inequality model (1) with either minimizing or maximizing $G(\mathbf{x})$ in (7):

$$\min_{\mathbf{x}\in\Omega} \{F(\mathbf{x}), G(\mathbf{x})\}\,, \tag{9}$$

$$\min_{\mathbf{x}\in\Omega} \{F(\mathbf{x}), -G(\mathbf{x})\}\,. \tag{10}$$

As usual, the *Pareto-optimal solutions* are simultaneously at least as good for both objective functions, and strictly better for at least one objective function than any other feasible location. Let E_+^* and E_-^* be the Pareto-set associated with the problems (9) and (10), respectively. We call *bicriteria plot* the set of pairs $(F(\mathbf{x}), G(\mathbf{x}))$ in the objective space for the feasible region, and *tradeoff curve* that for the Pareto-set only. For notational purpose, for any subset S of the plane, let $(F,G)(S) \equiv \{(F(\mathbf{x}), G(\mathbf{x}))|\mathbf{x} \in S\}$. Hence, the bicriteria plot is given by $(F,G)(\Omega)$. Likewise, the tradeoff curves associated with the problems (9) and (10) are the sets $(F,G)(E_+^*)$ and $(F,G)(E_-^*)$, respectively.

In what follows we always consider the objective space with the horizontal (vertical) axis measuring the values of $F(\mathbf{x})$ $(G(\mathbf{x}))$. For the attractive facility location (9), since the left and lower directions on the objective space are better in terms of $F(\mathbf{x})$ and $G(\mathbf{x})$, respectively, a Pareto-optimal solution has no alternative in southwesternly quadrant direction. Graphically, therefore, the Pareto-set E_+^* is given by the set of locations corresponding to the lower-left envelope of the bicriteria plot $(F,G)(\Omega)$. In the same way, the Pareto-set E_-^* is given by the set of locations corresponding to the upper-left envelope of the bicriteria plot $(F,G)(\Omega)$.

3.2 Localization results

Within each constrained ordered Voronoi cell $V_{i_1,i_2,\cdots,i_{|I|}} \cap \Omega$, $-F(\mathbf{x})$ is a linear function, and $G(\mathbf{x})$ is a strictly convex function. Thus problem (10) is a bicriterion convex maximization problem on a convex polygonal region. It was shown by Carrizosa and Plastria (2000) that all weakly Pareto-optimal solutions are then found on the edges of this region. Thus we obtain

Proposition 4 *The Pareto-set E_-^* is a subset of $\partial V \cup \partial\Omega$.*

Both $F(\mathbf{x})$ and $G(\mathbf{x})$ are convex functions all over the plane, and thus problem (9) is a bicriteria convex minimization problem constrained to a convex polygonal region. For this type of problems Plastria and Carrizosa (1996) derived the following weakly Pareto-optimality condition at a point $\mathbf{x}$: the convex hull of the subdifferentials of both objectives at $\mathbf{x}$ must intersect the opposite of the normal cone of the constraint region at $\mathbf{x}$. When $\mathbf{x}$ in an interior point of some constrained ordered Voronoi cell $V_{i_1,i_2,\cdots,i_{|I|}} \cap \Omega$, this condition is very easily checked since both objectives are differentiable, so their subdifferentials consist of their gradient only, and in the interior of the constraint set the normal cone is reduced to the zero vector. The gradient of $F(\mathbf{x})$ at $\mathbf{x}$ is orthogonal to $F(\mathbf{x})$'s level curves, while the gradient of $G(\mathbf{x})$ at $\mathbf{x}$ is given by $\mathbf{x} - \overline{\mathbf{p}}$. Thus such an interior point $\mathbf{x}$ can only be (weakly) Pareto-optimal if these two gradients are parallel and of opposite sign, i.e. $\mathbf{x} - \overline{\mathbf{p}}$ is orthogonal to $F(\mathbf{x})$'s level curves in the direction of decrease of $F(\mathbf{x})$, and, by (5), must be of the form $-\sum_{k=1}^{|I|} u_{i_k}\mathbf{p}_{i_k}$ when in $V_{i_1,i_2,\cdots,i_{|I|}}$. Define therefore $L_{i_1,i_2,\cdots,i_{|I|}}$ as the intersection of $V_{i_1,i_2,\cdots,i_{|I|}} \cap \Omega$ with the halfline issued from $\overline{\mathbf{p}}$ in the direction of $-\sum_{k=1}^{|I|} u_{i_k}\mathbf{p}_{i_k}$. Note that many of these intersections will be empty, and let L be the collection of nonempty $L_{i_1,i_2,\cdots,i_{|I|}}$'s. This set is illustrated in Figure 3, where L consists of eleven broken line-segments.

Proposition 5 *The Pareto-set E_+^* is a subset of $\partial V \cup \partial\Omega \cup L$.*

3.3 Constructing the bicriteria plot

Consider now any region W of the line tessellation defined by $\partial V \cup \partial\Omega \cup L$. Any such region W is simply-connected. By continuity of $F(\mathbf{x})$ and $G(\mathbf{x})$, this means that the set of the plots $(F, G)(W)$ in the objective space also is simply-connected.

Within W and along any line, $F(\mathbf{x})$ is a linear function and $G(\mathbf{x})$ is a quadratic function, so we can express $G(\mathbf{x})$ as a quadratic function of $F(\mathbf{x})$, the exact form of which is easily obtained from (8) and an expression similar to (6), suitably adapted to the line under consideration. Furthermore, this relationship between $F(\mathbf{x})$ and $G(\mathbf{x})$ is monotonic because within W the scalar product of the gradients of both functions has constant sign. This implies that the set $(F, G)(W)$ is the bounded domain surrounded by the plots of the line-segments of the boundary of W, and each of these latter is a piece of an upright parabola.

Proposition 6 *The bicriteria plot $(F, G)(\Omega)$ is given by the union of the domains bounded by the plots of ∂V, $\partial\Omega$ and L.*

We obtain the following algorithm whose validity follows from Proposition 6.

Step 1. Establish the planar graph $\partial V \cup L \cup \partial\Omega$.

Step 2. For each face W of the graph, hatch the regions bounded by the loci $(F, G)((\partial V \cup L) \cap W)$ in the objective space.

The planar graph $\partial V \cup \partial\Omega$ can be constructed in $O(|I|^4 + |\partial\Omega|)$ time, as we have seen. Similarly as for Step 2 of the previous algorithm, once the facility-inhabitant distances within one Voronoi polygon $V_{i_1,i_2,\cdots,i_{|I|}}$ is sorted in $O(|I| \log |I|)$, the line segment $L_{i_1,i_2,\cdots,i_{|I|}}$ is identified in $O(|I|)$ time. After this, $\partial\Omega$ within any of its surrounding polygons can be obtained in constant time. Since the graph $\partial V \cup \partial\Omega$ has $O(|I|^4 + |\partial\Omega|)$ edges, the planar graph $\partial V \cup L \cup \partial\Omega$ can be constructed in $O(|I|^4 + |\partial\Omega|)$ time by traversing at most twice each edge of the graph $\partial V \cup \partial\Omega$. Hence, Step 1 takes $O(|I|^4 + |\partial\Omega|)$. Similarly as for Step 1, by traversing at most twice each edge of the graph, all plots corresponding to these edges can be constructed in $O(|I|^4 + |\partial\Omega|)$ time using expression (5). This leads to an $O(|I|^4 + |\partial\Omega|)$ time for Step 2. Thus, we get

Proposition 7 *The bicriteria plot $(F, G)(\Omega)$ can be drawn in $O(|I|^4 + |\partial\Omega|)$ time.*

The bicriteria plot $(F, G)(\Omega)$ corresponding to our sample problem is indicated by a shadowed region in Figure 4, where the plots corresponding to ∂V, L and $\partial\Omega$ are also indicated by thin, broken and thick lines, respectively. This figure shows that roughly speaking, the relationship between the equity and efficiency criteria slope upward to the right, though the bicriteria plot has a complicated shape. Clearly this bicriteria plot provides an explicit representation for any proposed location of the simultaneous gains and losses that may be obtained for both objectives within the feasible region. It therefore forms a powerful evaluation instrument for decision makers.

3.4 Determination of the Pareto-sets

The Pareto-set E_+^* can be found through the following multiobjective convex minimization problems: see e.g. Warburton (1983):

$$\min_{\mathbf{x}\in\Omega} \theta F(\mathbf{x}) + (1-\theta) G(\mathbf{x}). \tag{11}$$

As we have shown, $F(\mathbf{x})$ is convex and $G(\mathbf{x})$ is strictly convex of $\mathbf{x}$, so the multiobjective convex minimization problem (11) possesses a unique solution for any $0 < \theta < 1$. In addition, the solutions $\mathbf{z}^*$ and $\mathbf{m}^*$ uniquely exist. Therefore, E_+^* is a simple continuous curve connecting $\mathbf{z}^*$ and $\mathbf{m}^*$, as discussed in Hansen et al.(1981b), Warburton (1983) and Ohsawa(1999). This leads to the following algorithm:

Step 1. Construct the planar graph $\partial V \cup L \cup \partial\Omega$.

Step 2. Identify $\mathbf{z}^*$ and $\mathbf{m}^*$.

Step 3. Find the steepest descent path connecting $\mathbf{z}^*$ and $\mathbf{m}^*$ on the planar graph.

As we have seen, Steps 1 and 2 take $O(|I|^4 + |\partial\Omega|)$ time. Starting from $\mathbf{m}^*$, we can move towards $\mathbf{z}^*$ along edges of the graph $\partial V \cup L \cup \partial\Omega$ in the steepest decent direction of $F(\mathbf{x})$. Since such direction at a vertex of the graph can be determined in constant time and the graph $\partial V \cup L \cup \partial\Omega$ has $O(|I|^4 + |\partial\Omega|)$ edges, Step 3 take at most $O(|I|^4 + |\partial\Omega|)$ time.

Proposition 8 *The Pareto-set E^*_+ can be found in $O((|I|^4 + |\partial\Omega|)$ time.*

The tradeoff curve of the obnoxious facility problem coincides with the upper-left envelope of the bicriteria plot. Hence, the Pareto-set E^*_- can be identified by the following algorithm, which generalizes slightly the method by Ohsawa(2000).

Step 1. Build up the planar graph $\partial V \cup \partial\Omega$.

Step 2. Plot the parabolic loci of $(F, G)(\mathbf{x})$ for the planar graph in objective space.

Step 3. Find the upper-left envelope of the loci.

Step 4. Determine the subedges corresponding to the envelope in the geographical space.

As we have shown higher, Steps 1 and 2 take $O(|I|^4 + |\partial\Omega|)$ time. One parabolic locus corresponding to one edge within the planar graph intersects any other parabolic locus in at most two points. Since both graph have $O(|I|^4+|\partial\Omega|)$ edges, Step 3 requires $O((|I|^4+|\partial\Omega|)\log(|I|^4+|\partial\Omega|))$, which dominates the rest.

Proposition 9 *The Pareto-set E^*_- can be determined in $O((|I|^4 + |\partial\Omega|)\log(|I|^4 + |\partial\Omega|))$ time.*

Note that the Pareto-set E^*_+ can be identified by tracing out the lower-left envelope of the bicriteria plot in $O((|I|^4 + |\partial\Omega|)\log(|I|^4 + |\partial\Omega|))$ time, which is larger than $O((|I|^4 + |\partial\Omega|)$ in Proposition 8. The tradeoff curves $(F, G)(E^*_+)$ and $(F, G)(E^*_-)$ are visualized collectively in Figure 4 by thick curves. The tradeoff curve $(F, G)(E^*_+)$ consists of a continuous portion which corresponds to the lower-left envelop of $F(\partial V \cup L \cup \partial\Omega), G(\partial V \cup L \cup \partial\Omega))$, but the other tradeoff curve $(F, G)(E^*_-)$ includes three continuous portions. Figure 4 enables us to perceive clearly and quickly to what extent the Pareto-solutions are really better than other candidates within Ω by comparing the bicriteria plot $(F, G)(\Omega)$ corresponding to all alternatives. Thus, our solution supports well-informed decision making.

The Pareto-sets E^*_+ and E^*_- are traced out in Figure 3 by using the corresponding lines in Figure 4, respectively. The Pareto-set E^*_+ is a connected piecewise linear curve from $\mathbf{m}^*$ to $\mathbf{z}^*$, as was to be expected. The other Pareto-set E^*_- is composed of a piecewise linear curve joining $\mathbf{z}^*$ with $\mathbf{s}_1$, a second connecting $\mathbf{s}_2$ with $\mathbf{s}_3$ and a third connecting $\mathbf{s}_4$ with $\mathbf{a}^*$. Thus, the Pareto-optimal solutions for a repellent facility E^*_-, spread out more over the feasible region than for an attractive facility E^*_+. This result is intuitively reasonable, because the Pareto-optimal locations for a repellent facility directly depend on the shape of the feasible region Ω.

4 Conclusions

This paper has formulated a new location problem using an equity measure of facility-inhabitant distances which satisfies the Pigou-Dalton principle of transfers. On the one hand, we have characterized the optimal solution(s) for the single-objective equity problem, the Pareto-sets and the bicriteria plot associated with the bicriterion problem which combines this equity objective with Weber-type objectives. Second, we have presented a polynomial-time graphical solution to trace out the solution, the Pareto-sets, and the bicriteria plot by using line tessellations.

Some further remarks on our formulation and results are in order.

First, any union of non-convex polygons can be partitioned into convex subareas in polynomial time. Therefore, the solution, the bicriteria plot, and even the Pareto-sets can be obtained in polynomial time by separate treatment of each subarea and merging of the results.

Next, the Pareto-sets remain the same in case both social threat and benefit due to the facilities are arbitrary increasing functions of the sum of the squared Euclidean distance.

Lastly, the arguments used in Propositions 4 and 5 are sufficiently general to remain applicable for any other convex efficiency objectives, e.g. minimizing the sum of distances, or maximizing the maximum distance, etc. This will lead to similar localization results, which will, however, be harder to exploit algorithmically since the nice quadratic relationships we have used here will not be present anymore.

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References

[1] E. Balas and E. Zemel, An algorithm for large zero-one knapsack problems, *Operations Research*, 28 (1980) 1130–1154.

[2] E. Carrizosa, Minimizing the variance of Euclidean distances, *Studies in Locational Analysis*, 12(1999), 101–118.

[3] E. Carrizosa and F. Plastria, Location of semi-obnoxious facilities, *Studies in Locational Analysis*, 12(1999), 1–27.

[4] E. Carrizosa and F. Plastria, Dominators for multiple-objective quasiconvex maximization problems, *Journal of Global Optimization*, 18 (2000)35–58.

[5] Z. Drezner and G.O. Wesolowsky, The Weber problem on the plane with some negative weights, *INFOR*, 29(1990), 87–99.

[6] Z. Drezner, J.-F. Thisse and G.O. Wesolowsky, The minimax-min location problem, *Journal of Regional Science*, 26(1986), 87–101.

[7] H.A. Eiselt and G. Laporte, Objectives in location problems in *Facility Location*, ed. by Z.Drezner. Springer, Berlin, (1995) 151–180.

[8] E. Erkut, Inequality measures for location problems, *Location Science*, 1(1993), 199–217.

[9] R.L.Francis, L.F.McGinnis and J.A.White, *Facility layout and location: an analytical approach* (2nd edition) Prentice Hall, Englewood Cliffs (NJ), (1992).

[10] P. Hansen, D. Peeters and J.F. Thisse, On the location of an obnoxious facility, *Sistemi Urbani*, 3(1981a), 299–317.

[11] P. Hansen, D. Peeters and J.F. Thisse, Constrained location and the Weber-Rawls problem, *Annals of Operations Research*, 11(1981b), 147–166.

[12] M.B. Mandell, Modelling effectiveness-equity trade-offs in public service delivery systems, *Management Science*, 37(1991), 467–482.

[13] M.T. Marsh and D.A. Schilling, Equity measurement in facility location analysis: a review and framework, *European Journal of Operational Research*, 74(1994), 1–17.

[14] E. Melachrinoudis and Z. Xanthopulos, Semi-obnoxious single facility location in Euclidean space, *Computers and Operations Research*, 30(2003), 2191–2209.

[15] Y. Ohsawa, A geometrical solutions for quadratic bicriteria location models, *European Journal of Operational Research*, 114(1999), 380–388.

[16] Y. Ohsawa, Bicriteria Euclidean location associated with maximin and minimax criteria, *Naval Research Logistics*, 47(2000), 581–592.

[17] Y. Ohsawa, F.Plastria and K. Tamura, Euclidean push-pull partial covering problems, *Computers and Operations Research*, (2005), to appear.

[18] Y. Ohsawa and K. Tamura, Efficient location for a semi-obnoxious facility, *Annals of Operations Research*, 123(2003), 173–188.

[19] A. Okabe, B. Boots, K. Sugihara and S.N. Chiu, *Spatial Tessellations*. Wiley, New York, 1999.

[20] F. Plastria and E. Carrizosa, Geometrical characterization of weakly efficient points, *Journal of Optimization Theory and Applications*, 90 (1996), 217–223.

[21] A. Sen, *On Economic Inequality*. Oxford University Press, Oxford, 1973.

[22] A.R. Warburton, Quasiconcave vector maximization, *Journal of Optimization Theory and Applications*, 40 (1983), 537-557.

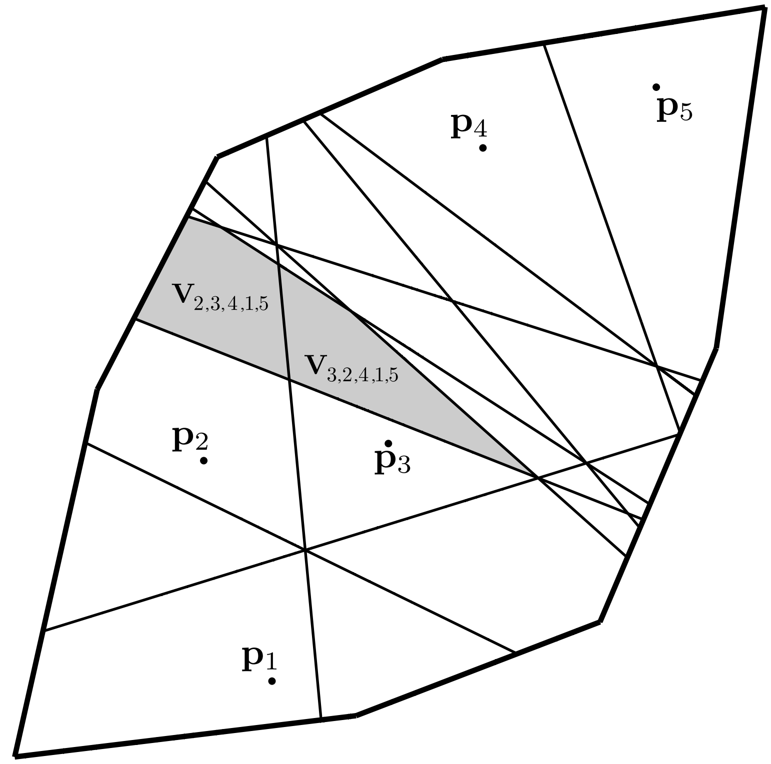

Figure 1: Line tessellation

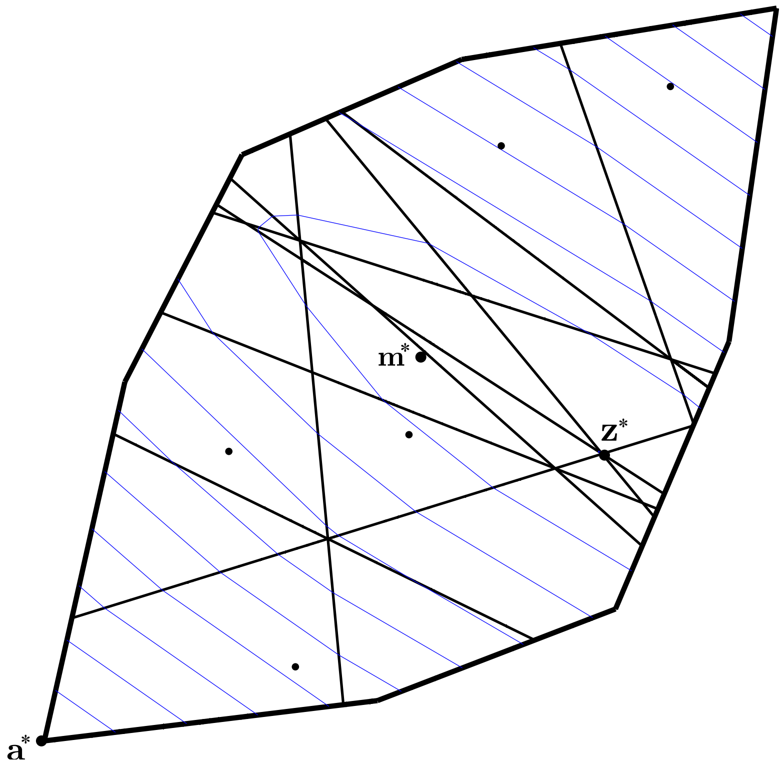

Figure 2: Level sets of $F(\mathbf{x})$ and solutions $\mathbf{a}^*$, $\mathbf{m}^*$, $\mathbf{z}^*$

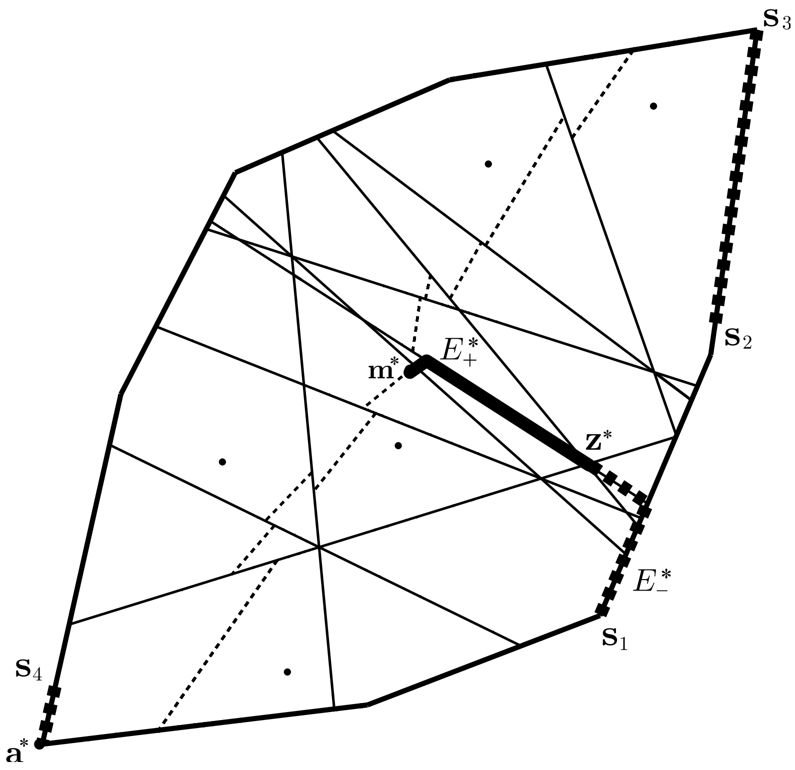

Figure 3: Pareto sets E_-^* and E_+^*

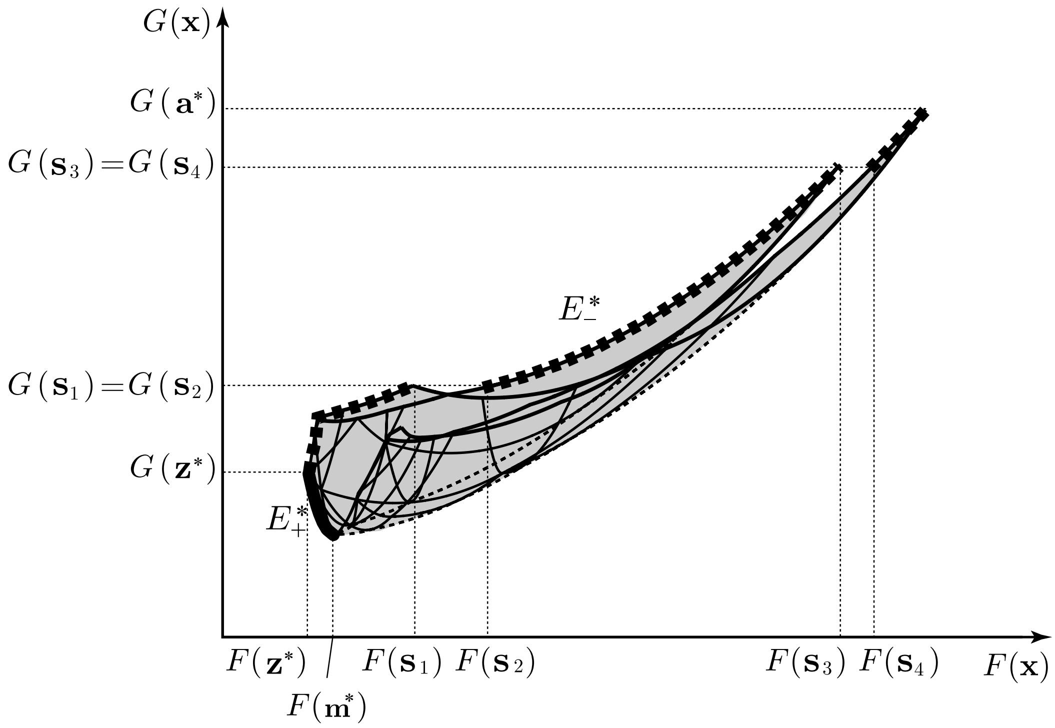

Figure 4: Bicriteria plot and Tradeoff curves