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An Unbiased One-sided Test for the Positional Parameter of the Exponential Distribution

by

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An unbiased one-sided test for the positional parameter
of the exponential distribution.

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Abstract.
In this paper the underlined distribution is of form

\[ f(x|\theta) = \begin{cases} e^{-(x-\theta)}, & \text{for } \theta < x < \infty \\ 0, & \text{otherwise} \end{cases} \]

(\(-\infty < \theta < \infty\)) and the author proposes an unbiased one-sided test for testing the hypothesis \(H_0: \theta = \theta_0\) versus the alternative hypothesis \(H_1: \theta > \theta_0\) with some constant \(\theta_0\).
§1. Introduction.

In the paper by Nogami (2000) the author discussed goodness of the two-sided test derived from the Lagrange's method. In this paper we use the same estimate for $\theta$ to derive the one-sided test for testing the hypothesis $H_0 : \theta = \theta_0$ versus the alternative hypothesis $H_1 : \theta > \theta_0$ with some constant $\theta_0$.

Let $X_1, \ldots, X_n$ be a random sample of size $n$ taken from

$$f(x|\theta) = \begin{cases} e^{-(x-\theta)}, & \text{for } x < \theta \\ 0, & \text{otherwise} \end{cases}$$

($-\infty < \theta < \infty$). We use an unbiased estimate $Y = \bar{X} - 1 = (n^{-1})^{\sum_{i=1}^{n} X_i - 1}$ for $\theta$ to get a one-sided test.

§2. Optimal one-sided test.

We shall first find the density of $Y$ and furthermore the density of $T = Y + 1 - \theta$ to obtain the one-sided test.

Finding the joint density of variables $W = X_1 + \ldots + X_n$, $Z_1 = X_1$, $\ldots$, $Z_{n-1} = X_{n-1}$ and taking the marginal density $g_W(w|\theta)$ of $W$ we obtain

$$g_W(w|\theta) = (\Gamma(n))^{-1} (w-n\theta)^{n-1} e^{-(w-n\theta)} I_{(n\theta, \infty)}(w).$$

Noticing $Y = n^{-1}W - 1$ we get the density of $Y$ as follows:

$$h_Y(y|\theta) = g_W(n(y+1)|\theta) n$$

$$= (\Gamma(n))^{-1} n^n (y+1-\theta)^{n-1} e^{-n(y+1-\theta)} I_{(\theta+1-\theta, \infty)}(y).$$

Furthermore, letting $t = y + 1 - \theta$ we have the density of $T$ so that

$$h_T(t) = (\Gamma(n))^{-1} n^n t^{n-1} e^{-n t} I_{(\theta, \infty)}(t)$$

which is the gamma density with parameters $n$ and $n$.

Let $\delta$ be a real number such that $0 < \delta < 1$. We propose the one-sided test which rejects $H_0$ if $\delta - 1 + t_0 \leq Y$ and accepts $H_0$ if $\delta - 1 + t_0 > Y$ where $t_0$ is given by
Using the test function we write this test as

\[ h_T(t) = \begin{cases} 1, & \text{for } y \geq \theta_0 + t_0 - 1 \\ 0, & \text{for } y < \theta_0 + t_0 - 1. \end{cases} \]

To check unbiasedness of this test we obtain the power function as follows:

\[
x(\theta) = \mathbb{E}_\theta (\phi(Y)) = \int_{\theta_0 + t_0 - 1}^{\infty} h_Y(y|\theta) \, dy
\]

\[= \int_{\theta_0 - \theta + t_0}^{\infty} h_T(t) \, dt.
\]

Since \(dx/d\theta = h_T(\theta_0 - \theta + t_0)(20), \forall \theta \) and \( x(\theta_0) = \varepsilon \), our one-sided test is unbiased.

**REFERENCE:**