INSTITUTE OF POLICY AND PLANNING SCIENCES

Discussion Paper Series

No. 856

Unbiased Tests for Location and Scale Parameters
-Case of Cauchy Distribution-

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April 2000

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JAPAN
Unbiased Tests for Location and Scale Parameters

-----Case of Cauchy Distribution-----

By Yoshiko Nogami

Abstract.

In this paper we deal with Cauchy distribution with the density

\[ f(x|\theta,\gamma) = \gamma^{-1} \left( 1 + (x - \theta)^2 \right)^{-\gamma}, \text{ for } -\infty < x < \infty \]

where \(-\infty < \gamma < \infty\) and \(\gamma > 0\).

We first consider \(\gamma = 1\). Based on a random sample of size \(n\) from \(f(x|\theta,1)\) we consider the problem of testing the null hypothesis \(H_0:\theta = \theta_0\) versus the alternative \(H_1:\theta \neq \theta_0\) for some constant \(\theta_0\). We propose the test with the acceptance region derived from inverting the shortest confidence interval (C. I.) for \(\theta_0\) and check if this test is unbiased.

We secondly consider \(\gamma = 0\). This time we consider the problem of testing \(H_0:\gamma = \gamma_0\) versus \(H_1:\gamma \neq \gamma_0\) for some constant \(\gamma_0\). We again propose the test with acceptance region derived from inverting the C. I. for \(\gamma_0\) and check if this test is unbiased.
§1. Introduction.

In this paper we deal with Cauchy distribution whose density is given as follows:

\[ f(x|\theta, \gamma) = \gamma^{-1} \frac{1}{\pi} \left\{ 1 + (x-\theta)^2 \right\}^{-1} \quad \text{for } -\infty < x < \infty \]

provided that \(-\infty < \theta < \infty\) and \(\gamma > 0\).

Let \(\gamma\) be the defining property. We first consider the density \(f(x|\theta) = f(x|\theta, 1)\). Let \(X_1, \ldots, X_n\) be a random sample of size \(n\) taken from the density \(f(x|\theta)\). We find in Section 2 the confidence interval (C.I.) for the location parameter \(\theta\) with the shortest length using Lagrange’s method. In Section 3 we consider the problem of testing the null hypothesis \(H_0: \theta = \theta_0\) versus the alternative hypothesis \(H_1: \theta \neq \theta_0\) for some constant \(\theta_0\). We propose the test with the acceptance region derived from inverting the shortest C.I. for \(\theta_0\). Let \(s\) be a real number such that \(0 < s < 1\). When \(n = 2m+1\) with \(m\) a nonnegative integer, we show that our test is unbiased and of size \(s\). But, when \(n = 2m\), because we use conventional method to get the C.I. for \(\theta\), we cannot show unbiasedness of our test. (However, for large \(m\) our test becomes almost unbiased as the test in case of \(n = 2m+1\) shows.)

In the second half we consider the density \(f(x|\gamma) = \frac{1}{\pi} \left\{ 1 + (x-\theta)^2 \right\}^{-1}\). Based on a random sample of size \(n\) from the density \(f(x|\gamma)\) we find in Section 4 the C.I. for the scale parameter \(\gamma\). In Section 5 we consider the problem of testing \(H_0: \gamma = \gamma_0\) versus \(H_1: \gamma \neq \gamma_0\) for some constant \(\gamma_0\). Again we propose the test with acceptance region derived from inverting the C.I. for \(\gamma_0\). When \(n = 2m+1\), we show that our test is unbiased and of size \(s\). But, in the same reason as that for \(\theta\) our test is not unbiased when \(n = 2m\). (However, for large \(m\) our test becomes almost unbiased as the test in case of \(n = 2m+1\) shows.)
§2. The Interval Estimation for \( \theta \).

In this section we deal with the density

\[
(2) \quad f(x|\theta) = \pi^{-1/2} \left(1 + (x - \theta)^2\right)^{-1/2}, \quad \text{for } -\infty < x < \infty
\]

where \(-\infty < \theta < \infty\). We find the shortest C. I. for \( \theta \) using Lagrange’s method.

Let \( n = 2m + 1 \) with \( m \) a nonnegative integer, until (15). Let \( X_{(1)} \) be the \( i \)-th smallest observation of \( X_1, \ldots, X_n \). We estimate \( \theta \) by \( \hat{\theta} = X_{(m+1)} \). To get the shortest C. I. for \( \theta \) we first find the density of \( Y \). Let \( F(x|\theta) \) be the cumulative distribution function (c.d.f.) of \( X \). Then, by (2) we get

\[
(3) \quad F(x|\theta) = \pi^{-1/2} \tan^{-1}(x - \theta) + 2^{-1}, \quad \text{for } -\infty < x < \infty.
\]

Hence, the density of \( Y \) is of form

\[
(4) \quad g_Y(y|\theta) = k(F(y))^m(1 - F(y))^n f(y|\theta), \quad \text{for } -\infty < y < \infty
\]

where

\[
(5) \quad k = \Gamma(2m+2)/[\Gamma(m+1)]^2.
\]

Let \( \alpha \) be a real number such that \( 0 < \alpha < 1 \). Let \( r_1 \) and \( r_2 \) be real numbers such that \( r_1 < r_2 \). To find the shortest C. I. for \( \theta \) at confidence coefficient \( 1 - \varepsilon \) we want to minimize \( r_2 - r_1 \) under the condition that

\[
(6) \quad P_\varepsilon[r_1 < Y < r_2] = 1 - \varepsilon.
\]

But, it follows by a variable transformation \( W = F(Y) \) that

the left hand side of (6) = \( P_\varepsilon[r_1 + \theta < Y < r_2 + \theta] \)

\[
= P_\varepsilon[F(r_1 + \theta) < W < F(r_2 + \theta)] = 1 - \varepsilon.
\]
Hence, we want to minimize \( r_1 - r_1 \) under the condition (7). To do so we use Lagrange's method. Let \( \lambda \) be a real number and define

\[
F(x_1 + \theta)
\]

\[
L(x_1, x_2; \lambda) = x_2 - x_1 - \lambda \{ \text{h}_w(w) \, dw = \lambda \theta \}
\]

where \( h_w(w) \) is the density of \( W \) given by

\[
h_w(w) = kw^m(1-w)^n, \quad \text{for } 0 < w < 1
\]

where \( k \) is given by (5). The right hand side of (9) is the probability density function (p.d.f.) of Beta distribution \( \text{Beta}(m+1, m+1) \) with \( (m+1, m+1) \) degrees of freedom. Then, by Lagrange's method we have that

\[
\begin{align*}
\frac{\partial L}{\partial x_1} &= -1 + \lambda h_w(F(x_1 + \theta))f(x_1 + \theta | \theta) = 0 \\
\frac{\partial L}{\partial x_2} &= 1 - \lambda h_w(F(x_2 + \theta))f(x_2 + \theta | \theta) = 0
\end{align*}
\]

By (10) we get

\[
h_w(F(x_1 + \theta))f(x_1 + \theta | \theta) = h_w(F(x_2 + \theta))f(x_2 + \theta | \theta) = \lambda^{-1}, \quad \forall \theta.
\]

Taking

\[
F(x_1 + \theta) = \beta(s/2) \quad \text{and} \quad F(x_2 + \theta) = 1 - \beta(s/2)
\]

where \( \beta(s/2) \) is given by

\[
\beta(s/2) = \int h_w(w) \, dw = s/2
\]

we obtain by (3) that \( x_1 = x_2 - x \) where

\[
x = F^{-1}(1 - \beta(s/2)) - \theta = \tan(2^{-1} \beta(s/2) s).
\]
We also have that \( h_w(F(-r+\theta)) = h_w(F(r+\theta)) \) and \( f(-r+\theta | \theta) = f(r+\theta | \theta) \) with \( r \) given by (14). Thus, (11) and (7) are satisfied for \( r_1 = r_2 = r \) with \( r \) given by (14). Therefore, the shortest C. I. for \( \theta \) at confidence coefficient \( 1-\alpha \) is given by

\[
(Y-r, Y+r) = (Y - \tan[(2^{-1}-\theta(s/2))x], Y+\tan[(2^{-1}-\theta(s/2))x]).
\]

Let \( n=2m \). This time we estimate \( \theta \) by \( Y^2X_{(m)} \). In the similar way to the above we get the density of \( Y \) as follows:

\[
g_Y(y|\theta) = k_1(F(y))^{m-1}(1-F(y))^{n-1}f(y|\theta), \quad \text{for} \ -\infty < y < \infty
\]

where

\[
k_1 = \Gamma(2m+1)/\{\Gamma(m)\Gamma(m+1)\}.
\]

Putting \( W=F(Y) \) we minimize \( r_1-r_2 \) under the condition (7). However, since the density of \( W \) is now of form

\[
h_i(w) = k_i w^{n-1}(1-w)^{m}, \quad \text{for} \ 0 < w < 1
\]

which is the p.d.f. of the Beta\((m, m+1)\) distribution with \( k_i \) defined by (17), it is difficult to get exact values for \( F(r_i+\theta), i=1, 2 \) which satisfy

\[
h_i(F(r_1+\theta))f(r_1+\theta | \theta) = h_i(F(r_2+\theta))f(r_2+\theta | \theta).
\]

Hence, we use conventional values for \( F(r_i+\theta), i=1, 2 \). Those are

\[
F(r_i+\theta) = \beta_{m, m+1}(s/2) \quad \text{and} \quad F(r_2+\theta) = 1 - \beta_{m+1, m}(s/2)
\]

where \( \beta_{m, m+1}(s/2) \) and \( \beta_{m+1, m}(s/2) \) are respectively determined by

\[
\int_0^{\beta_{m, m+1}(s/2)} h_i(w) \, dw = s/2 = \int_0^{\beta_{m+1, m}(s/2)} w^{n-1}(1-w)^{m-i} \, dw.
\]
Thus, by (3) \( r_1 \) and \( r_2 \) are respectively given by

\[
\begin{align*}
  r_1 &= F^{-1}(\beta_{m,n+1}(s/2)) - \theta = -\tan[(2^{-i-1}\beta_{m,n+1}(s/2))\pi], \\
  r_2 &= F^{-1}(1-\beta_{m+1,n}(s/2)) - \theta = \tan[(2^{-i-1}\beta_{m+1,n}(s/2))\pi].
\end{align*}
\]

(22)

Therefore, the C. I. for \( \theta \) at confidence coefficient \( 1-\alpha \) is

(23) \( (Y-r_3, Y-r_1) = (Y - \tan[(2^{-i-1}\beta_{m+1,n}(s/2))\pi], Y + \tan[(2^{-i-1}\beta_{m+1,n}(s/2))\pi]). \)

In the next section we check if the tests with the acceptance regions derived from inverting the C. I.'s (15) for \( n=2m+1 \) and (23) for \( n=2m \) are unbiased and of size \( \alpha \).

§3. Two-Sided Test for \( \theta \).

In this section we consider the problem of testing the null hypothesis \( H_0: \theta = \theta_0 \) versus the alternative hypothesis \( H_1: \theta \neq \theta_0 \) for some constant \( \theta_0 \). We propose the two-sided tests with the acceptance regions derived from inverting the (shortest) C. I.'s for \( \theta_0 \) obtained in Section 2. When \( n=2m+1 \), we show that our test is unbiased and of size \( \alpha \). When \( n=2m \), our test is not unbiased because of usage of conventional method for constructing the C. I. for \( \theta \).

Let \( n=2m+1 \). As in Section 2 we define \( Y = X_{(m+1)} \). By inverting the shortest C. I. (15) for \( \theta_0 \) our test is to reject \( H_0 \) if \( Y \in (-\infty, \theta_0-r) \cup (\theta_0+r, \infty) \) and to accept \( H_0 \) if \( Y \in (\theta_0-r, \theta_0+r) \) where \( r \) is given by (14). Now, we show that this test is unbiased and of size \( \alpha \).

Let \( y_1^0 \) and \( y_2^0 \) be real numbers depending on \( \theta_0 \) such that \( Y_1^0 < Y_2^0 \). Define \( \psi(\theta) \) by

\[
\psi(\theta) = P_\theta[Y < Y_1^0 \text{ or } Y > Y_2^0] = 1 - \int_{Y_1^0}^{Y_2^0} g_Y(y|\theta) \, dy
\]

(24)

where \( g_Y(y|\theta) \) is defined by (4).
To get unbiased size-$\varepsilon$ test with the acceptance region $(y_1^0, y_2^0)$ we choose $y_1^0$ and $y_2^0$ which satisfy

$$\psi(\theta_0) = 1 - P_{\theta_0}[y_1^0 < Y < y_2^0] = \varepsilon$$

and minimize $\psi(\theta)$ at $\theta = \theta_0$; namely

$$\frac{d\psi(\theta)}{d\theta} \bigg|_{\theta = \theta_0} = g_\psi(y_2^0 | \theta_0) - g_\psi(y_1^0 | \theta_0) = 0.$$  

We consider the test with the acceptance region $(\theta_0 - r, \theta_0 + r)$. Since from the construction the equality (11) with $r_1 = -r$, $r_2 = r$ and $\theta = \theta_0$ is satisfied, it follows from (4) and (9) that $g_\psi(\theta_0 - r | \theta_0) = g_\psi(\theta_0 + r | \theta_0)$; (26) is satisfied for $y_1^0$ and $y_2^0$ replaced by $\theta_0 - r$ and $\theta_0 + r$, respectively. (25) with $y_1^0$ and $y_2^0$ replaced by $\theta_0 - r$ and $\theta_0 + r$, respectively is the same as (6) except for $\theta$, $r_1$ and $r_2$ replaced by $\theta_0$, $-r$ and $r$, respectively. Therefore, our test with the acceptance region $(\theta_0 - r, \theta_0 + r)$ is unbiased and of size $\varepsilon$.

Let $n = 2m$. As in Section 2 we define $Y^{\Delta X}_{(n)}$. Again, by inverting the C. I. (23) for $\theta_0$ our test is to reject $H_0$ if $Y \in (\theta_0, \theta_0 + r_1) \cup (\theta_0 + r_2, 2m)$ and to accept $H_0$ if $Y \notin (\theta_0 + r_1, \theta_0 + r_2)$ where $r_1$ and $r_2$ are given by (22). In this case our test depends on the conventional values for $F(x + \theta)$, $i = 1, 2$. Hence, we have that $g_\psi(\theta_0 + r_1 | \theta_0) + g_\psi(\theta_0 + r_2 | \theta_0)$. Furthermore, (25) with $y_1^0$ and $y_2^0$ replaced by $\theta_0 + r_1$ and $\theta_0 + r_2$, respectively is the same as (6) except for $\theta$ replaced by $\theta_0$. Thus, our test is of size $\varepsilon$, but is not unbiased. However, for large $m$ our test becomes almost unbiased as the test in the case of $n = 2m + 1$ shows.

In the next two sections we deal with the scale parameter $\lambda$. In Section 4 we obtain the C. I. for $\lambda$ and in Section 5 we check if two-sided test with acceptance region derived from inverting the C. I. for $\lambda_0$ is unbiased.

§4. The Interval Estimation for $\lambda$.

In this section we consider the density (1) with $\theta = 0$;

$$f(x | \lambda) = f(x | 0, \lambda) = \lambda^{-1} \{\lambda^2 + x^2\}^{-1}, \quad \text{for } -\infty < x < \infty$$

provided that $\lambda > 0$.  

Let \( X_1, \ldots, X_n \) be a random sample of size \( n \) taken from the population with density \( f(x|\xi) \). Again, we first consider the case of \( n=2m+1 \) with \( m \) a nonnegative integer and secondly the case of \( n=2m \). Putting \( \xi^*=\ln \xi \) we have

\[
f(x|\xi) = t^{-1}e^{-t^*}\{1+e^{t\{\ln x-\xi^*\}}\}^{-1}, \quad \text{for } -\infty < x < \infty.
\]

Thus, letting \( Z=\ln|X| \) and \( Z_{(i)} \) be the \( i \)-th smallest observation of \( Z_1, \ldots, Z_n \) we estimate \( \xi^* \) by \( Y^=Z_{(n+1)} \) when \( n=2m+1 \) and by \( Y^=Z_{(m)} \) when \( n=2m \), respectively. We find the C. I.'s for \( \xi \) according to these estimates.

We beforehand derive the distribution of \( Z \). Since \( x=\xi^* \) for \( x>0 \); \( x=-\xi^* \) for \( x<0 \); \( z=0 \) for \( x=0 \), by a variable transformation \( Z=\ln|X| \) the density of \( Z \) is obtained as follows:

\[
q_2(z) = \frac{e^{-x^*}}{2\xi^{-1}e^{x^*}(1+e^{x^*})}, \quad -\infty < z < \infty
\]

where \( -\infty < \xi^* < \infty \). Since \( q_2(2\xi^*-z) = q_2(z) \), \( q_2(z) \) is symmetric about \( z=\xi^* \) and the unimodal function with the mode \( \xi^* \).

Now, we let \( n=2m+1 \) until (37). We estimate \( \xi^* \) by \( Y^=Z_{(n+1)} \). Letting \( Q_2(z) \) be the c.d.f. of \( Z \) we obtain by (28) that

\[
Q_2(z) = Q_2(z|\xi^*) = 2\xi^{-1}\tan^{-1}(e^{z-\xi^*}), \quad \text{for } -\infty < z < \infty.
\]

The p.d.f. \( g_Y(y|\xi^*) \) of \( Y \) is derived as follows:

\[
g_Y(y|\xi^*) = kQ_2(y)^\xi^*(1-Q_2(y))^{1-\xi^*}q_2(y), \quad \text{for } -\infty < y < \infty.
\]

Let \( \alpha \) be a real number such that \( 0<\alpha<1 \). Let \( r_1 \) and \( r_2 \) be real numbers such that \( 0<r_1<r_2 \). To find the C. I. for \( \xi \) at confidence coefficient \( 1-\alpha \) we want to find \( r_1 \) and \( r_2 \) under the condition that

\[
P_\xi[r_1 e^\xi < y < r_2 e^\xi] = 1-\alpha.
\]
But, it follows by a variable transformation $W=Q_2(Y)$ that

\[
\text{the left hand side of (31)} = P_z[-\ln r_2 < Y - \xi' \cdot -\ln r_1]
\]
\[
= P_z \left[ Q_2(\xi' - \ln r_2) \cdot Q_2(\xi' - \ln r_1) \right] = 1 - q.
\]

Hence, we want to find $r_1$ and $r_2$ which minimize $Q_2(\xi' - \ln r_1) - Q_2(\xi' - \ln r_2)$ under the condition (32). To do so we use Lagrange's method. Let $\lambda$ be a real number and define

\[
L = L(\xi' - \ln r_1), Q_2(\xi' - \ln r_2); \lambda)
\]
\[
= Q_2(\xi' - \ln r_1)
\]
\[
\lambda = Q_2(\xi' - \ln r_1) - Q_2(\xi' - \ln r_2) - \lambda \left\{ h_\ast(w) \right\}
\]
\[
Q_2(\xi' - \ln r_2)
\]

where $h_\ast(w)$ is defined by (9). Then, by Lagrange's method we have that

\[
\begin{align*}
\frac{\partial L}{\partial Q_2(\xi' - \ln r_1)} &= 1 - \lambda \quad h_\ast(Q_2(\xi' - \ln r_1)) = 0 \\
\frac{\partial L}{\partial Q_2(\xi' - \ln r_2)} &= -1 + \lambda \quad h_\ast(Q_2(\xi' - \ln r_2)) = 0
\end{align*}
\]

By (34) we get

\[
h_\ast(Q_2(\xi' - \ln r_1)) = h_\ast(Q_2(\xi' - \ln r_2)) = 1 - \lambda, \quad \forall \xi'.
\]

Taking

\[
Q_2(\xi' - \ln r_1) = \varphi(\epsilon/2) \quad \text{and} \quad Q_2(\xi' - \ln r_2) = 1 - \varphi(\epsilon/2)
\]

where $\varphi(\epsilon/2)$ is given by (13), we obtain by (29) that

\[
\begin{align*}
\begin{cases}
 r_1 = [\tan(2^{-1}\chi(1 - \varphi(\epsilon/2)))]^{-1} \\
r_2 = [\tan(2^{-1}\chi\varphi(\epsilon/2))]^{-1}
\end{cases}
\end{align*}
\]
and furthermore (35) and (32) are satisfied for \( r_1 \) and \( r_2 \) given by (36). Therefore, the C. I. for \( \tau \) is given by

\[
(r_1 e^Y, r_2 e^Y) \leq ([\tan(2^{-\iota}(1-\iota/2))]^{-1} e^Y, [\tan(2^{-\iota}(\iota/2))]^{-1} e^Y).
\]

We now consider the case of \( n=2m \). In this case we estimate \( \tau \) by \( Y^{\sim Z_{(\iota)}} \). Then, the p.d.f. of \( Y \) is given by

\[
g_y(y|\iota) = k_1(Q_\iota(y))^{n-1}(1-Q_\iota(y))^{n-1} q_x(y), \quad \text{for } -\infty < y < \infty
\]

where \( k_1 \) is given by (17). To find the C. I. for \( \iota \) at confidence coefficient \( 1-\iota \), we want to find \( r_1 \) and \( r_2 \) with \( 0 < r_1 < r_2 \) under the condition that

\[
P_1[r_1 e^Y \leq \{r_2 e^Y\} = 1-\iota.
\]

But, it follows by a variable transformation \( W = Q_\iota(Y) \) that

the left hand side of (39) = \( P_1[\{\text{ln } r_2 \leq Y \leq \text{ln } r_1\}]
\]

\[
= P_1[Q_\iota(\{\text{ln } r_2\}) \leq W < Q_\iota(\{\text{ln } r_1\})] = 1-\iota.
\]

Hence, we want to find \( r_1 \) and \( r_2 \) which minimize \( Q_\iota(\{\text{ln } r_1\}) - Q_\iota(\{\text{ln } r_2\}) \) under the condition (40). Going through the similar process to (33) through (35), we get

\[
h_1(Q_\iota(\{\text{ln } r_1\})) = h_1(Q_\iota(\{\text{ln } r_2\}) (=1^{-1}), \quad \forall \iota
\]

where \( h_1(w) \) is the density of \( W \) given by (18). However, again it is difficult to get exact values of \( Q_\iota(\{\text{ln } r_1\}), i=1,2 \) which satisfy (41) (and furthermore \( \pi_2(\{\text{ln } r_1\}) = \pi_2(\{\text{ln } r_2\}) \)). Hence, we use conventional values for \( Q_\iota(\{\text{ln } r_1\}), i=1,2 \). Those are

\[
Q_\iota(\{\text{ln } r_1\}) = \beta_{m, m+1}(\iota/2) \quad \text{and} \quad Q_\iota(\{\text{ln } r_1\}) = 1-\beta_{m+1, m}(\iota/2).
\]
where \( \theta_{m+1}(z/2) \) and \( \theta_{m+1,n}(z/2) \) are respectively determined by (21).

Thus, by (29) we obtain

\[
\begin{align*}
    r_1 &= [\tan(2^{-1}x(1-\theta_{m+1,n}(z/2)))]^{-1}, \\
    r_2 &= [\tan(2^{-1}x\theta_{m+m+1}(z/2))]^{-1}.
\end{align*}
\]

Therefore, the C. I. for \( \ell \) is

\[
(44) \quad (r_1 e^y, r_2 e^y)
\]

where \( r_1 \) and \( r_2 \) are given by (43).

\section{5. Two-Sided Test for \( \ell \).}

In this section we consider the problem of testing the hypothesis \( H_0 : \ell = \ell_0 \)
versus the alternative hypothesis \( H_1 : \ell \neq \ell_0 \) for some constant \( \ell_0 \). We propose
the test with the acceptance region derived from inverting the C. I. for \( \ell_0 \). Let \( n \) be the size of the random sample \( X_1, \ldots, X_n \). When \( n = 2m+1 \) with \( m \) a
nonnegative integer, we show that this test is unbiased and of size \( \alpha \). When
\( n = 2m \), our test is of size \( \alpha \), but cannot be unbiased because we use the
conventional device to determine the C. I. for \( \ell \). However, it will be almost
unbiased for large \( m \).

Let \( n = 2m+1 \). As in Section 4 we let \( Z_{-1} \ln |X| \) and \( Z_{(i)} \) be the \( i \)-th smallest
observation of \( Z_1, \ldots, Z_n \). Let \( \ell_0^* = \ln \ell_0 \) and define \( Y \triangleq Z_{(m+1)} \). By inverting the
C. I. (37) for \( \ell_0 \) our test is to reject \( H_0 \) if \( Y \in (-\infty, \ell_0^* - \ln r_2] \cup [\ell_0^* - \ln r_1, +\infty) \)
and to accept \( H_0 \) if \( Y \in (\ell_0^* - \ln r_2, \ell_0^* - \ln r_1) \) where \( r_1 \) and \( r_2 \) are given by (36).

Now, we show that this test is unbiased and of size \( \alpha \).

Let \( y_1^0 \) and \( y_2^0 \) be real numbers depending on \( \ell_0 \) such that \( y_1^0 < y_2^0 \). Define \( \psi(\ell) \) by

\[
\psi(\ell) = P_1[Y < y_1^0 \text{ or } y_2^0 < Y]
\]

\[
(45) \quad y_2^0 = 1 - \int_{y_1^0}^{\infty} g_Y(y|\ell) \, dy
\]

where \( y_1^0 \) and \( y_2^0 \) are given by (43).
where $g_1(y_1|\xi)$ is given by (30). To get unbiased size-$\alpha$ test with acceptance region $(y_1^0, y_2^0)$ we choose $y_1^0$ and $y_2^0$ which satisfy

$$
\psi(\xi_0) = 1 - P_{\xi_0}[Y_1^0 < Y < Y_2^0] = \alpha
$$

and minimize $\psi(\xi)$ at $\xi = \xi_0$; namely

$$
d\psi(\xi)/d\xi |_{\xi = \xi_0} = \psi^{-1}(g_1(y_2^0|\xi_0) - \psi^{-1}(g_1(y_1^0|\xi_0)) = 0
$$

Let $y_1^* = \xi_0^0 - \ln r_2$ and $y_2^* = \xi_0^0 - \ln r_1$. Then, since $q_2(y_1^0|\xi_0) = \psi^{-1}(\sin y_1^0)$, and since, from construction and (35),

$$
h_2(Q_2(y_1^0)) = h_2(Q_2(y_2^0))
$$

we obtain by (30) and (9) that $g_1(y_1^0|\xi_0) = g_1(y_2^0|\xi_0)$. Therefore, $(y_1^*, y_2^*)$ satisfies (47). On the other hand, (46) with $y_1^0$ and $y_2^0$ replaced by $y_1^*$ and $y_2^*$, respectively, is the same as (40) except for $\xi$ replaced by $\xi_0$. Therefore, our test with the acceptance region $(y_1^*, y_2^*)$ is unbiased and of size $\alpha$.

Let $n = 2m$. As in Section 4 we define $Y^2 Z_{(m)}$. Again, by inverting the C. I. (44) for $\xi_0$ our test is to reject $H_0$ if $Y \notin (-\infty, \xi_0^* - \ln r_2] \cup [\xi_0^* - \ln r_1, +\infty)$ and to accept $H_0$ if $Y \in (\xi_0^* - \ln r_2, \xi_0^* - \ln r_1)$ where $r_1$ and $r_2$ are determined by (43). In this case our test depends on the conventional values for $Q_2(\xi_0 - \ln r_1)$, $i = 1, 2$. So, we have $g_1(\xi_0^* - \ln r_2|\xi_0)$ $g_1(\xi_0^* - \ln r_1|\xi_0)$. Furthermore, (46) with $y_1^0$ and $y_2^0$ replaced by $\xi_0^* - \ln r_2$ and $\xi_0^* - \ln r_1$, respectively, is the same as (40) except for $\xi$ replaced by $\xi_0$. Thus, our test is still of size-$\alpha$, but is not unbiased. However, for large $m$ our test becomes almost unbiased as the test in case of $n = 2m + 1$ shows.