

## Thermodynamic Bethe Ansatz Equations for Minimal Surfaces in $AdS_3$

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### Abstract

We study classical open string solutions with a null polygonal boundary in  $AdS_3$  in relation to gluon scattering amplitudes in  $\mathcal{N} = 4$  super Yang–Mills at strong coupling. We derive in full detail the set of integral equations governing the decagonal and the dodecagonal solutions and identify them with the thermodynamic Bethe ansatz equations of the homogeneous sine-Gordon models. By evaluating the free energy in the conformal limit we compute the central charges, from which we observe general correspondence between the polygonal solutions in  $AdS_n$  and generalized parafermions.

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## 1. Introduction

Recently there has been much interest in computing gluon scattering amplitudes in  $\mathcal{N} = 4$  super Yang–Mills theory at strong coupling by using AdS/CFT correspondence. The amplitude is dual to the Wilson loop with light-like segments [1], which corresponds to the area of minimal surfaces in AdS with the same boundary [2].

In [2], the minimal surface for the 4-point amplitude has been obtained by solving the Euler-Lagrange equation in the static gauge. The minimal surfaces in AdS are further studied in [3,4] but it is a very difficult problem to extend the 4-point solution to the general  $n$ -point amplitudes. It is important to evaluate the area for the  $n$ -point amplitudes in order to determine the remainder function, which represents deviation from the conjectured BDS formula of the multi-loop amplitudes [5]. The remainder functions, which are functions of cross-ratios of momenta, are shown to exist in the 6-point amplitudes at two-loop level [6] and evaluated numerically [7] and explicitly [8].

Recently, there has been remarkable progress in obtaining exact solutions of minimal surfaces with a null polygonal boundary in AdS. It is shown that the equations for the minimal surface in  $AdS_3$  can be reduced to the  $SU(2)$  Hitchin equations [9]. The minimal area is obtained by finding the Stokes data of the associated linear problem, which is studied in detail by Gaiotto, Moore and Neitzke [10, 11]. The explicit formula for the area of the minimal surface for the 8 sided polygon has been obtained [9]. This is further generalized to the  $AdS_4$  [12], [13] and the  $AdS_5$  case [12]. In the  $AdS_5$  case, the minimal area problem is shown to be equivalent to solving the  $SU(4)$  Hitchin system. Motivated by the connection between the solution of the associated linear problem and the Thermodynamic Bethe Ansatz (TBA) integral equations [10], Alday, Gaiotto and Maldacena found that the minimal area of the 6 sided polygon is evaluated by the free energy of the TBA equations of the  $A_3$  integrable theory [12]. TBA equations also appear in the study of the spectral problem in AdS/CFT correspondence [14].

The TBA equations [15] have been studied extensively in the investigation of the massive integrable field theory and its relevant perturbed conformal field theory (CFT). It will be a quite interesting problem to study the role of the TBA equations in the minimal area problem in AdS.

In this paper we study minimal surfaces with a null polygonal boundary in AdS. We focus on the minimal surfaces with a  $2n$  sided polygonal boundary in  $AdS_3$ . We determine the integral equations explicitly in the case of the decagon and the

dodecagon. We find that the integral equations fit precisely in the general form proposed by Gaiotto, Moore and Neitzke [10]. We identify the present integral equations with the TBA equations of the homogeneous sine-Gordon model [16, 17]. The free energy of the TBA system is related to the regularized area [12]. We then evaluate the free energy and compute the central charges for the decagon and dodecagon. We find that the regularized areas precisely match those obtained from the central charges in the CFT limit of the TBA system. We generalize these results to the general  $2n$  sided polygons in  $AdS_3$  and argue that relevant CFTs are identified with generalized parafermion theories for  $SU(n-2)_2/U(1)^{n-3}$ . We comment on the case of  $AdS_5$ .

This paper is organized as follows. In sect. 2, we review the construction of open string solution in  $AdS_3$  [9] and discuss the Stokes data of the associated linear problem. In sect. 3, we analyze the Riemann–Hilbert problem and introduce the functional variables with simple asymptotics for the decagon and the dodecagon cases. We derive the integral equations and identify them with the TBA equations of the homogeneous sine-Gordon models. In sect. 4, we study the free energy of the homogeneous sine-Gordon models and their CFT limit. We compare the central charges and the regularized areas. In sect. 5, we present conclusions and discussion.

## 2. Minimal surfaces with a null polygonal boundary in $AdS_3$

### 2.1. The linear problem

In this paper we consider classical open string solutions in  $AdS_3$  with a Euclidean world-sheet.<sup>1</sup> Let  $z$  be a complex coordinate parametrizing the world-sheet. Let  $\vec{Y} = (Y_{-1}, Y_0, Y_1, Y_2)^T \in \mathbb{R}^{2,2}$  denote the global coordinate parametrizing the  $AdS_3$  spacetime. The  $AdS_3$  is given as a hypersurface

$$\vec{Y} \cdot \vec{Y} := -Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 = -1 \quad (2.1)$$

in  $\mathbb{R}^{2,2}$ . Inner product of two vectors  $\vec{A}, \vec{B} \in \mathbb{R}^{2,2}$  is defined as

$$\vec{A} \cdot \vec{B} = \eta_{ij} A^i B^j, \quad \eta_{ij} = \text{diag}(-1, -1, +1, +1). \quad (2.2)$$

The solution is given by a function  $\vec{Y}(z, \bar{z})$  satisfying the constraint (2.1), the classical equations of motion

$$\vec{Y}_{z\bar{z}} - (\vec{Y}_z \cdot \vec{Y}_{\bar{z}}) \vec{Y} = 0 \quad (2.3)$$

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<sup>1</sup>In this subsection we basically follow the notation of [9].

and the Virasoro constraints

$$\vec{Y}_z^2 = \vec{Y}_{\bar{z}}^2 = 0. \quad (2.4)$$

Here we abbreviate the world-sheet derivatives as  $\partial_z \vec{Y} = \vec{Y}_z$ ,  $\partial_{\bar{z}} \vec{Y} = \vec{Y}_{\bar{z}}$ .

Let us introduce the following notations

$$e^{2\alpha} := \frac{1}{2} \vec{Y}_z \cdot \vec{Y}_{\bar{z}}, \quad (2.5)$$

$$N_i := \frac{1}{2} e^{-2\alpha} \epsilon_{ijkl} Y^j Y_z^k Y_{\bar{z}}^l, \quad \epsilon_{(-1)012} = +1, \quad (2.6)$$

$$p := -\frac{1}{2} \vec{N} \cdot \vec{Y}_{zz}. \quad (2.7)$$

The pseudo vector  $\vec{N}$  is chosen so that

$$\vec{N} \cdot \vec{Y} = \vec{N} \cdot \vec{Y}_z = \vec{N} \cdot \vec{Y}_{\bar{z}} = 0, \quad \vec{N} \cdot \vec{N} = 1. \quad (2.8)$$

Using (2.1), (2.3), (2.4), one can show that

$$\bar{\alpha} = \alpha, \quad \overline{\vec{N}} = -\vec{N}, \quad (2.9)$$

$$p_{\bar{z}} = 0, \quad (2.10)$$

*i. e.*,  $\alpha(z, \bar{z})$  is real-valued and  $p(z)$  is holomorphic for the string solutions.

One can consider a moving frame basis spanned by the following vectors

$$\vec{q}_1 = \vec{Y}, \quad \vec{q}_2 = e^{-\alpha} \vec{Y}_{\bar{z}}, \quad \vec{q}_3 = e^{-\alpha} \vec{Y}_z, \quad \vec{q}_4 = \vec{N}. \quad (2.11)$$

We recast them into the following form

$$W_{\alpha\dot{\alpha},a\dot{a}} = \begin{pmatrix} W_{11,a\dot{a}} & W_{12,a\dot{a}} \\ W_{21,a\dot{a}} & W_{22,a\dot{a}} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (q_1 + q_4)_{a\dot{a}} & (q_2)_{a\dot{a}} \\ (q_3)_{a\dot{a}} & (q_1 - q_4)_{a\dot{a}} \end{pmatrix}, \quad (2.12)$$

where  $\alpha, \dot{\alpha} = 1, 2$  denote internal spinor indices while  $a, \dot{a} = 1, 2$  denote spacetime spinor indices. In the r.h.s. of (2.12) the 4-vectors  $\vec{q}_j$  are expressed in the spinor notation. In this notation  $\vec{Y}$ , for example, is expressed as

$$Y_{a\dot{a}} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} = \begin{pmatrix} Y_{-1} + Y_2 & Y_1 - Y_0 \\ Y_1 + Y_0 & Y_{-1} - Y_2 \end{pmatrix}. \quad (2.13)$$

Evolution of  $W_{\alpha\dot{\alpha},a\dot{a}}$  is described by the following set of linear equations

$$\partial_z W_{\alpha\dot{\alpha},a\dot{a}} + (B_z^L)_\alpha{}^\beta W_{\beta\dot{\alpha},a\dot{a}} + (B_z^R)_{\dot{\alpha}}{}^{\dot{\beta}} W_{\alpha\dot{\beta},a\dot{a}} = 0, \quad (2.14)$$

$$\partial_{\bar{z}} W_{\alpha\dot{\alpha},a\dot{a}} + (B_{\bar{z}}^L)_\alpha{}^\beta W_{\beta\dot{\alpha},a\dot{a}} + (B_{\bar{z}}^R)_{\dot{\alpha}}{}^{\dot{\beta}} W_{\alpha\dot{\beta},a\dot{a}} = 0, \quad (2.15)$$

where

$$B_z^L = B_z(1), \quad B_z^R = UB_z(i)U^{-1}, \quad (2.16)$$

$$B_{\bar{z}}^L = B_{\bar{z}}(1), \quad B_{\bar{z}}^R = UB_{\bar{z}}(i)U^{-1} \quad (2.17)$$

with

$$B_z(\zeta) = \frac{\alpha_z}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{1}{\zeta} \begin{pmatrix} 0 & e^\alpha \\ e^{-\alpha} p & 0 \end{pmatrix}, \quad (2.18)$$

$$B_{\bar{z}}(\zeta) = -\frac{\alpha_{\bar{z}}}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \zeta \begin{pmatrix} 0 & e^{-\alpha \bar{p}} \\ e^\alpha & 0 \end{pmatrix}, \quad (2.19)$$

$$U = \begin{pmatrix} 0 & e^{\frac{\pi i}{4}} \\ e^{\frac{3\pi i}{4}} & 0 \end{pmatrix}. \quad (2.20)$$

The equations of motion (2.3) as well as the constraints (2.1), (2.4) have been used in deriving the particular form of  $B_z, B_{\bar{z}}$ . In other words, the linear equations (2.14), (2.15) with the connections given by (2.16)–(2.20) encode the conditions for  $\vec{Y}$ .

The above evolution equations exhibit a peculiar structure that the connection decomposes into the left and the right parts. Moreover, since each entry of the matrix (2.12) is a null vector,  $W_{\alpha\dot{\alpha},a\dot{a}}$  can be expressed as a product of two spinors

$$W_{\alpha\dot{\alpha},a\dot{a}} = \psi_{\alpha,a}^L \psi_{\dot{\alpha},\dot{a}}^R. \quad (2.21)$$

Therefore, once we fix such decomposition at one point on the  $z$ -plane,  $\psi_{\alpha,a}^L$  and  $\psi_{\dot{\alpha},\dot{a}}^R$  evolve separately over the whole  $z$ -plane, obeying

$$\partial_z \psi_{\alpha,a}^L + (B_z^L)_\alpha{}^\beta \psi_{\beta,a}^L = 0, \quad \partial_{\bar{z}} \psi_{\alpha,a}^L + (B_{\bar{z}}^L)_\alpha{}^\beta \psi_{\beta,a}^L = 0, \quad (2.22)$$

$$\partial_z \psi_{\dot{\alpha},\dot{a}}^R + (B_z^R)_{\dot{\alpha}}{}^{\dot{\beta}} \psi_{\dot{\beta},\dot{a}}^R = 0, \quad \partial_{\bar{z}} \psi_{\dot{\alpha},\dot{a}}^R + (B_{\bar{z}}^R)_{\dot{\alpha}}{}^{\dot{\beta}} \psi_{\dot{\beta},\dot{a}}^R = 0. \quad (2.23)$$

The original string solutions are constructed from solutions of these equations. Note that the solutions have to satisfy not only the evolution equations (2.14), (2.15), but also the normalization condition

$$\epsilon^{ab} \epsilon^{\dot{a}\dot{b}} W_{\alpha\dot{\alpha},a\dot{a}} W_{\beta\dot{\beta},b\dot{b}} = \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} \quad (2.24)$$

and the reality condition

$$\overline{W_{\alpha\dot{\alpha},a\dot{a}}} = W_{\alpha\dot{\alpha},a\dot{a}}. \quad (2.25)$$

Such solutions are obtained if one slightly generalize (2.21) as

$$W_{\alpha\dot{\alpha},a\dot{a}} = M_{a\dot{a},b\dot{b}} \psi_{\alpha,b}^L \psi_{\dot{\alpha},\dot{b}}^R \quad (2.26)$$

and determine the constant  $M_{a\dot{a},b\dot{b}}$  accordingly.<sup>2</sup> For later purpose we adopt the normalization conditions

$$\psi_a^L \wedge \psi_b^L \equiv \epsilon^{\beta\alpha} \psi_{\alpha,a}^L \psi_{\beta,b}^L = \epsilon_{ab}, \quad \psi_{\dot{a}}^R \wedge \psi_{\dot{b}}^R \equiv \epsilon^{\beta\dot{\alpha}} \psi_{\dot{\alpha},\dot{a}}^R \psi_{\beta,\dot{b}}^R = \epsilon_{\dot{a}\dot{b}}, \quad (2.27)$$

where  $\psi_a^L$  for  $a = 1, 2$  and  $\psi_{\dot{a}}^R$  for  $\dot{a} = 1, 2$  are two linearly independent solutions of (2.22) and (2.23), respectively.

The linear problems (2.22) and (2.23) can be promoted to a family of linear problems with the general spectral parameter  $\zeta$

$$\left(\partial_z + B_z(\zeta)\right)\psi(z, \bar{z}; \zeta) = 0, \quad \left(\partial_{\bar{z}} + B_{\bar{z}}(\zeta)\right)\psi(z, \bar{z}; \zeta) = 0, \quad (2.28)$$

where the connections are given by (2.18), (2.19). From now on we use matrix notations for indices  $\alpha, \dot{\alpha}$ , where  $B_z, B_{\bar{z}}$  denote  $2 \times 2$  matrices as well as  $\psi$  a 2-component column vector.

By using the variable such that

$$dw = \sqrt{p(z)}dz \quad (2.29)$$

with the redefinition

$$\hat{\alpha} = \alpha - \frac{1}{4} \log p\bar{p} \quad (2.30)$$

and the gauge transformation

$$\hat{\psi} = g\psi, \quad (2.31)$$

$$g = e^{i\frac{\pi}{4}\sigma^3} e^{i\frac{\pi}{4}\sigma^2} e^{\frac{1}{8} \log \frac{p}{\bar{p}} \sigma^3} = \begin{pmatrix} \frac{1+i}{2} & \frac{1+i}{2} \\ -\frac{1+i}{2} & \frac{1-i}{2} \end{pmatrix} \begin{pmatrix} \left(\frac{p}{\bar{p}}\right)^{\frac{1}{8}} & 0 \\ 0 & \left(\frac{p}{\bar{p}}\right)^{-\frac{1}{8}} \end{pmatrix}, \quad (2.32)$$

one can completely remove  $p(z)$  from the equations (at the price of having complicated branch cut structure). One then obtains

$$\left(\partial_w + \hat{B}_w\right)\hat{\psi} = 0, \quad \left(\partial_{\bar{w}} + \hat{B}_{\bar{w}}\right)\hat{\psi} = 0, \quad (2.33)$$

where

$$\hat{B}_w(\zeta) = \frac{\hat{\alpha}_w}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \frac{1}{\zeta} \begin{pmatrix} \cosh \hat{\alpha} & i \sinh \hat{\alpha} \\ i \sinh \hat{\alpha} & -\cosh \hat{\alpha} \end{pmatrix}, \quad (2.34)$$

$$\hat{B}_{\bar{w}}(\zeta) = -\frac{\hat{\alpha}_{\bar{w}}}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \zeta \begin{pmatrix} \cosh \hat{\alpha} & -i \sinh \hat{\alpha} \\ -i \sinh \hat{\alpha} & -\cosh \hat{\alpha} \end{pmatrix}. \quad (2.35)$$

Note that the linear differential operators are transformed under the gauge transformation as

$$\left[\partial_w + \hat{B}_w\right] = p^{-\frac{1}{2}} g \left[\partial_z + B_z\right] g^{-1}, \quad \left[\partial_{\bar{w}} + \hat{B}_{\bar{w}}\right] = \bar{p}^{-\frac{1}{2}} g \left[\partial_{\bar{z}} + B_{\bar{z}}\right] g^{-1}. \quad (2.36)$$

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<sup>2</sup>The discussion is similar to that in the case of finite-gap solutions [4].

## 2.2. Bases of solutions

We are interested in open string solutions whose boundary consists of light-like segments forming  $2n$ -gons on the AdS boundary [9]. We consider solutions such that  $p(z)$  is a polynomial of degree  $n - 2$  and  $\hat{\alpha}$  behaves as

$$\hat{\alpha} \rightarrow 0 \quad \text{for} \quad |w| \rightarrow \infty. \quad (2.37)$$

For solutions with such  $\hat{\alpha}$ , equations (2.33) are approximated at large  $|w|$  as

$$(\partial_w - \zeta^{-1}\sigma_3)\hat{\psi} = 0, \quad (\partial_{\bar{w}} - \zeta\sigma_3)\hat{\psi} = 0, \quad (2.38)$$

which have two independent solutions

$$\hat{\eta}_+ = \begin{pmatrix} e^{\left(\frac{w}{\zeta} + \bar{w}\zeta\right)} \\ 0 \end{pmatrix}, \quad \hat{\eta}_- = \begin{pmatrix} 0 \\ e^{-\left(\frac{w}{\zeta} + \bar{w}\zeta\right)} \end{pmatrix}. \quad (2.39)$$

These describe the asymptotic behavior of the big and the small solutions of (2.33) in each Stokes sector. In particular, the small solutions are uniquely specified by the asymptotic behavior. Let us introduce the notation  $\hat{s}_{2k-1}(w, \bar{w}; \zeta)$  for the small solutions in each Stokes sector. They are characterized by the following asymptotic behavior at large  $|w|$

$$\hat{s}_{2k-1} \simeq (-1)^{k-1}\hat{\eta}_- \quad \text{for} \quad w \in \hat{W}_{2k-1}, \quad \hat{s}_{2k} \simeq (-1)^k\hat{\eta}_+ \quad \text{for} \quad w \in \hat{W}_{2k}, \quad (2.40)$$

where  $\hat{W}_j$ 's denote the Stokes sectors

$$\hat{W}_j : \quad \left(j - \frac{3}{2}\right)\pi + \arg \zeta < \arg w < \left(j - \frac{1}{2}\right)\pi + \arg \zeta. \quad (2.41)$$

We have determined the normalizations of  $\hat{s}_j$ 's so that

$$\begin{aligned} \hat{s}_j \wedge \hat{s}_{j+1} &\equiv \det(\hat{s}_j \ \hat{s}_{j+1}) \\ &= 1. \end{aligned} \quad (2.42)$$

(2.40)–(2.42) uniquely determine the small solutions  $\hat{s}_j$ .

One can take  $\hat{s}_{j-1}$  and  $\hat{s}_j$  as the basis of the solutions. Then  $\hat{s}_{j+1}$  is expressed as

$$\hat{s}_{j+1} = -\hat{s}_{j-1} + b_j\hat{s}_j, \quad (2.43)$$

where the coefficient of  $\hat{s}_{j-1}$  is determined by (2.42) and  $b_j(\zeta)$  is a coefficient independent of  $w, \bar{w}$ . It can be expressed as

$$b_j(\zeta) = \hat{s}_{j-1} \wedge \hat{s}_{j+1}. \quad (2.44)$$

Next let us consider the periodicity constraint. First let us introduce pull-back of the basis solutions  $s_j(z, \bar{z}; \zeta)$  by

$$s_j = g^{-1} \hat{s}_j. \quad (2.45)$$

Let us also introduce the notation

$$W_j : \quad \frac{(2j-3)\pi}{n} + \frac{2}{n} \arg \zeta < \arg z < \frac{(2j-1)\pi}{n} + \frac{2}{n} \arg \zeta \quad (2.46)$$

for the Stokes sectors on the  $z$ -plane.  $s_j(z, \bar{z}; \zeta)$  are solutions to the original differential equations (2.28). Since differential operators in (2.28) have no singularities at  $|z| < \infty$ , solutions  $\psi(z, \bar{z}; \zeta)$  to (2.28) are also regular over the whole  $z$ -plane. This means that the Stokes sectors  $W_{j+n}$  and  $W_j$  are identified for  $2n$ -gon solutions. Therefore  $s_{j+n}$  and  $s_j$ , which are both the small solution in this sector, coincide up to a normalization factor

$$s_{j+n} = \mu_j s_j. \quad (2.47)$$

This factor is identified with the formal monodromy as follows.  $\hat{s}_j$  becomes a large solution in the Stokes sector  $\hat{W}_{j-1}$  and  $\hat{W}_{j+1}$ . In the Stokes sector  $\hat{W}_j$ , both  $s_{j-1}$  and  $s_{j+1}$  grow with the same largest exponent. In this way we see that  $s_j$  with even  $j$  grow with the same largest exponent while  $s_j$  with odd  $j$  grow with the other exponent. Therefore by going around the  $z$ -plane twice, one can evaluate  $\mu_j$  as

$$\mu_j^2 = \exp(S_j(e^{4\pi i} z) - S_j(z)), \quad (2.48)$$

where  $S_j$  denotes the corresponding largest exponent. The explicit form of the exponents can be read from the components of  $s_j$

$$\begin{aligned} \begin{pmatrix} s_{j,1} \\ s_{j,2} \end{pmatrix} &= \begin{pmatrix} \left(\frac{p}{\bar{p}}\right)^{-\frac{1}{8}} & 0 \\ 0 & \left(\frac{p}{\bar{p}}\right)^{\frac{1}{8}} \end{pmatrix} \begin{pmatrix} \frac{1-i}{2} & \frac{-1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{pmatrix} \begin{pmatrix} \hat{s}_{j,1} \\ \hat{s}_{j,2} \end{pmatrix} \\ &= \begin{pmatrix} \left(\frac{p}{\bar{p}}\right)^{-\frac{1}{8}} \left(\frac{1-i}{2} \hat{s}_{j,1} - \frac{1+i}{2} \hat{s}_{j,2}\right) \\ \left(\frac{p}{\bar{p}}\right)^{\frac{1}{8}} \left(\frac{1-i}{2} \hat{s}_{j,1} + \frac{1+i}{2} \hat{s}_{j,2}\right) \end{pmatrix}. \end{aligned} \quad (2.49)$$

As  $p(z)$  is a polynomial of degree  $n-2$ , the factor  $(p/\bar{p})^{1/8}$  contributes to the monodromy by

$$\left(\frac{p}{\bar{p}}\right)^{\frac{1}{8}} \rightarrow e^{(n-2)\pi i} \left(\frac{p}{\bar{p}}\right)^{\frac{1}{8}} \quad \text{as } z \rightarrow e^{4\pi i} z \quad (2.50)$$



at large  $|z|$ . For  $n$  even, the phase factor in the above equation is trivial, but the formal monodromy receives a contribution from the residue appearing in the  $1/z$  expansion of  $\ln(p/\bar{p})^{1/8}$ . For  $n$  odd, the phase factor in (2.50) gives  $-1$ . In this case no contribution comes from the series expansion. Hence one obtains

$$\mu_j = \pm i \quad \text{for } n : \text{ odd.} \quad (2.51)$$

In order to figure out the relation among  $b_j(\zeta)$  and  $\mu$ , it is convenient to introduce the notations

$$\hat{S}_j = (\hat{s}_j \ \hat{s}_{j+1}), \quad B_j = \begin{pmatrix} 0 & -1 \\ 1 & b_j \end{pmatrix}, \quad M_j = \begin{pmatrix} \mu_j & 0 \\ 0 & \mu_{j+1} \end{pmatrix}. \quad (2.52)$$

Then (2.43) and (2.47) are expressed as

$$\hat{S}_{j+1} = \hat{S}_j B_{j+1}, \quad (2.53)$$

$$\hat{S}_{j+n} = \hat{S}_j M_j. \quad (2.54)$$

As  $\hat{S}_j$  is invertible, it follows that

$$B_{j+1} B_{j+2} \cdots B_{j+n} = M_j. \quad (2.55)$$

Since  $\det B_j = 1$ , we see that  $\det M_j = 1$ , namely

$$\mu_{j+1} = \mu_j^{-1}. \quad (2.56)$$

Therefore one can set

$$\mu_{2k-1} = \mu_{2k}^{-1} = \mu \quad k = 1, 2, 3, \dots \quad (2.57)$$

It also follows from (2.55) that

$$B_{j+1} M_{j+1} = M_j B_{j+n+1}, \quad (2.58)$$

which gives

$$b_{j+n} = \mu_j^{-2} b_j. \quad (2.59)$$

### 2.3. Constraints from involutions

It can be easily checked that the pair of Dirac operators satisfies a holomorphic involution<sup>3</sup>

$$\sigma_2 [\partial_w + \hat{B}_w(\zeta)] \sigma_2 = [\partial_w + \hat{B}_w(-\zeta)], \quad \sigma_2 [\partial_{\bar{w}} + \hat{B}_{\bar{w}}(\zeta)] \sigma_2 = [\partial_{\bar{w}} + \hat{B}_{\bar{w}}(-\zeta)] \quad (2.60)$$

<sup>3</sup>This  $\mathbb{Z}_2$  symmetry corresponds to the  $\mathbb{Z}_4$  symmetry appearing in the case of  $AdS_5$ .

and an antiholomorphic involution

$$\overline{\partial_w + \hat{B}_w(\zeta)} = \partial_{\bar{w}} + \hat{B}_{\bar{w}}(\bar{\zeta}^{-1}), \quad \overline{\partial_{\bar{w}} + \hat{B}_{\bar{w}}(\zeta)} = \partial_w + \hat{B}_w(\bar{\zeta}^{-1}). \quad (2.61)$$

The implication of these involutions is the following: If  $\hat{\psi}(\zeta)$  is a solution to the equations (2.33), so are  $\sigma_2 \hat{\psi}(-\zeta)$  and  $\hat{\psi}(\bar{\zeta}^{-1})$ .

Let us first examine the constraints arising from the holomorphic involution. As  $\hat{s}_1(w, \bar{w}; \zeta)$  is a solution to (2.33), so is  $\sigma_2 \hat{s}_1(w, \bar{w}; e^{\pi i} \zeta)$ . It exhibits the asymptotic behavior as

$$\sigma_2 \hat{s}_1(w, \bar{w}; e^{\pi i} \zeta) \simeq -i \hat{\eta}_+(w, \bar{w}; \zeta) \quad \text{for } w \in \hat{W}_2. \quad (2.62)$$

Note that the asymptotic behavior appears in the Stokes sector  $\hat{W}_2$  for  $\hat{s}_1$  with the spectral parameter  $e^{\pi i} \zeta$ . As the above solution is the small solution in  $\hat{W}_2$ , it should be identified with  $\hat{s}_2$  as

$$\sigma_2 \hat{s}_1(w, \bar{w}; e^{\pi i} \zeta) = i \hat{s}_2(w, \bar{w}; \zeta). \quad (2.63)$$

Similarly, one can show that

$$\sigma_2 \hat{s}_j(w, \bar{w}; e^{\pi i} \zeta) = i \hat{s}_{j+1}(w, \bar{w}; \zeta). \quad (2.64)$$

It then follows that

$$b_j(e^{\pi i} \zeta) = b_{j+1}(\zeta). \quad (2.65)$$

The antiholomorphic involution implies that for each  $j$ ,  $\overline{\hat{s}_j(w, \bar{w}; \bar{\zeta}^{-1})}$  is a solution. Analysis of the asymptotic behavior tells us that it is the small solution in the Stokes sector  $\hat{W}_j$ . Thus one can identify it as

$$\overline{\hat{s}_j(w, \bar{w}; \bar{\zeta}^{-1})} = \hat{s}_j(w, \bar{w}; \zeta). \quad (2.66)$$

It then follows that

$$\overline{b_j(\bar{\zeta}^{-1})} = b_j(\zeta). \quad (2.67)$$

The constraints (2.65) and (2.67) are peculiar to the current Hitchin system which originates from the classical strings in  $AdS_3$ .

### 3. Thermodynamic Bethe ansatz equations

In this section we derive integral equations which characterize the minimal surfaces with a null polygonal boundary in  $AdS_3$ . As demonstrated in [12] in the case of hexagonal solutions in  $AdS_5$ , the functional form of  $b_j(\zeta)$  is fully determined by a

Riemann–Hilbert problem. To construct the Riemann–Hilbert problem, one needs boundary conditions in addition to the relations and the constraints for  $b_j$  derived in the last section. The boundary conditions are given by asymptotic behavior for  $|\zeta| \rightarrow \infty$  and for  $|\zeta| \rightarrow 0$ . What makes the story nontrivial is that each  $b_j(\zeta)$  exhibits simple asymptotics only in some particular angular sectors in the  $\zeta$  plane. In order to write down the Riemann–Hilbert problem in a simple form, one therefore introduces new functional variables  $\chi_j(\zeta)$ , which have a simple asymptotics in all the angular sectors but have discontinuities along some of the semi-infinite border lines. A reasonable definition of  $\chi_j(\zeta)$  was presented in [11] as the Fock–Goncharov coordinates [18]. The coordinates are defined for every WKB triangulation, which is uniquely determined given the value of  $\zeta$ . One can then figure out the explicit relations between  $\chi_j$  and  $b_j$ . The behavior of  $\chi_j$  at discontinuities is described by the periodicity condition of  $b_j$  discussed in the last section. Below we illustrate using some simple examples how the Riemann–Hilbert problem is constructed from the data of  $p(z)$  and the constraints of  $b_j$ .

The Riemann–Hilbert problem for  $\chi_j$  is written in the form of integral equations. It was pointed out that the integral equations possess the structure of TBA equations [10].<sup>4</sup> However, it was not clear what models are described by these equations in practice. We find that the TBA equations in the present cases are identified with those of the homogeneous sine-Gordon model.

### 3.1. Decagon solutions ( $n = 5$ )

#### 3.1.1. Periodicity condition

Let us now focus on the case of decagon ( $n = 5$ ). The condition (2.55) for  $j = 0$  is written down for each component as

$$\mu^{-1} = b_2 + b_4 - b_2 b_3 b_4, \quad (3.1)$$

$$0 = -1 + b_2 b_3 + b_2 b_5 + b_4 b_5 - b_2 b_3 b_4 b_5, \quad (3.2)$$

$$0 = 1 - b_1 b_2 - b_1 b_4 - b_3 b_4 + b_1 b_2 b_3 b_4, \quad (3.3)$$

$$\mu = b_1 + b_3 + b_5 - b_1 b_2 b_3 - b_1 b_2 b_5 - b_1 b_4 b_5 - b_3 b_4 b_5 + b_1 b_2 b_3 b_4 b_5. \quad (3.4)$$

---

<sup>4</sup>A connection between TBA systems and certain ordinary differential equations has been found [19], where cross-ratios turn out to play an interesting role [19, 20]. For generalizations, see for example [21].

These can be simplified as

$$b_1 b_2 = 1 - \mu b_4, \quad (3.5)$$

$$b_2 b_3 = 1 - \mu^{-1} b_5, \quad (3.6)$$

$$b_3 b_4 = 1 - \mu^{-1} b_1, \quad (3.7)$$

$$b_4 b_5 = 1 - \mu b_2. \quad (3.8)$$

Similar relations are obtained from (2.55) with other  $j$ 's. By introducing a new notation by

$$\beta_{2k-1} = \mu^{-1} b_{2k-1}, \quad \beta_{2k} = \mu b_{2k}, \quad (3.9)$$

and using (2.51), these relations can be concisely written as

$$\beta_j \beta_{j+1} = 1 - \beta_{j+3}. \quad (3.10)$$

Note that in terms of  $\beta_j$ 's, with the help of (2.51), the relation (2.59) is simplified as

$$\beta_{j+5} = \beta_j. \quad (3.11)$$

We are now in a position to consider the integral equations. The form of the integral equations is characterized by the polynomial  $p(z)$  and the connectivity condition (3.10). In the case of decagon,  $p(z)$  is a cubic polynomial. We choose it as

$$p(z) = z^3 - 3A^2 z + u = (z - z_1)(z - z_2)(z - z_3). \quad (3.12)$$

It is important to notice that we should essentially consider two cases for the configurations of the roots  $z_i$  ( $i = 1, 2, 3$ ) or for the location of  $u$  in the moduli space. This is called the wall-crossing phenomenon in the literature.

### 3.1.2. Inside the wall of marginal stability

Let us first consider the case where  $u$  is located inside the wall of marginal stability (see Sec. 9.4.4 “ $N = 3$ ” in [11]). Let  $\gamma_1, \gamma_2$  denote cycles which encircle the pair of branch points  $[z_1, z_2], [z_3, z_2]$ , respectively (see Figure 1 (A)). Given the phase of  $\zeta$ , one can draw the WKB lines and determine the WKB triangulation. Figure 1 schematically shows the evolution of the WKB triangulations as the phase of  $\zeta$  increases. The WKB triangulations jumps discontinuously when  $\zeta$  crosses semi-infinite lines (so-called BPS rays). We see that for generic  $\zeta$  there always exist two tetragons each of which respectively surround the edges  $[z_1, z_2], [z_3, z_2]$ . Once the WKB trian-

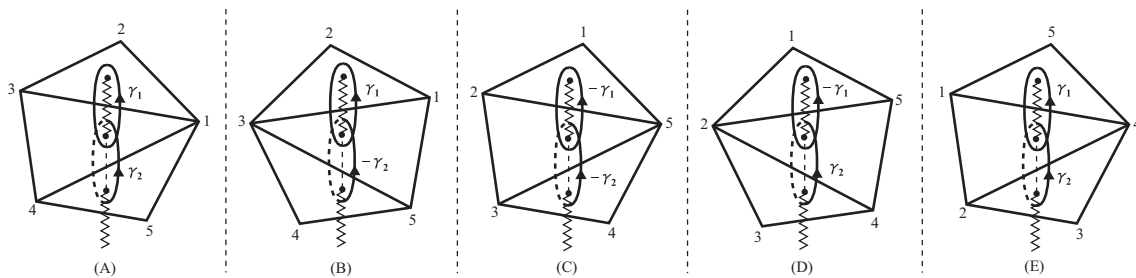


Figure 1: Transition of WKB triangulations for the decagon when  $u$  is inside the wall of marginal stability. We show the evolution of the WKB triangulations from  $\arg \zeta = 0$  to  $\arg \zeta = 2\pi$ . There are four jumps in this case.

Table 1: The definition of  $\chi_{\gamma_1}$  and  $\chi_{\gamma_2}$  for the decagon when  $u$  is inside the wall of marginal stability. The regions of  $\arg \zeta$  correspond to those of Figure 1.

$\arg \zeta$	(A)	(B)	(C)	(D)	(E)
$\chi_{\gamma_1}$	$-\beta_5^{-1}$	$-\beta_4$	$-\beta_4$	$-\beta_3^{-1}$	$-\beta_3^{-1}$
$\chi_{\gamma_2}$	$-\beta_2$	$-\beta_2$	$-\beta_1^{-1}$	$-\beta_1^{-1}$	$-\beta_5$

gulation is given, we can define functions  $\chi_{\gamma_i}(\zeta)$  as the Fock–Goncharov coordinates for each tetragon. For example, let us consider the case where  $\arg \zeta$  is in the region (B) of Figure 1. We define  $\chi_{\gamma_i}$  ( $i = 1, 2$ ) by

$$\chi_{\gamma_1} = -\frac{(s_1 \wedge s_2)(s_3 \wedge s_5)}{(s_2 \wedge s_3)(s_5 \wedge s_1)} = -\beta_4, \quad (3.13)$$

$$\chi_{\gamma_2}^{-1} = \chi_{-\gamma_2} = -\frac{(s_5 \wedge s_1)(s_3 \wedge s_4)}{(s_1 \wedge s_3)(s_4 \wedge s_5)} = -\beta_2^{-1}, \quad (3.14)$$

where we used the relation  $\chi_{-\gamma}(\zeta) = 1/\chi_{\gamma}(\zeta)$ . Similarly one can define  $\chi_{\gamma_i}$  ( $i = 1, 2$ ) for other regions. The definition of  $\chi_{\gamma_i}$  ( $i = 1, 2$ ) is summarized in Table 1.

Note that  $\chi_{\gamma_i}$  defined in this way has the asymptotic form [11]

$$\chi_{\gamma_i}(\zeta) \simeq \exp\left(\frac{Z_i}{\zeta} + \bar{Z}_i \zeta\right) \quad (3.15)$$

for large  $|\zeta|$  where

$$Z_i = \oint_{\gamma_i} \sqrt{p(z)} dz. \quad (3.16)$$

From the discontinuity data of  $\chi_{\gamma_i}(\zeta)$  and the asymptotic form (3.15), we can imme-

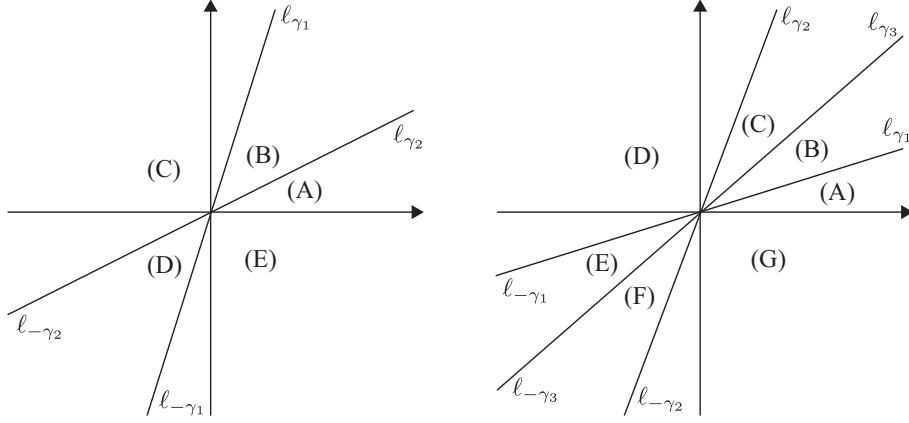


Figure 2: The BPS rays for the decagon solutions. The left (right) figure shows the case where  $u$  is located inside (outside) the wall of marginal stability.

diately write down the integral equations for  $\chi_{\gamma_i}$ ,

$$\begin{aligned} \log \chi_{\gamma_1}(\zeta) &= \frac{Z_1}{\zeta} + \bar{Z}_1 \zeta + \frac{1}{4\pi i} \int_{l_{\gamma_2}} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log(1 + \chi_{\gamma_2}(\zeta')) \\ &\quad - \frac{1}{4\pi i} \int_{l_{-\gamma_2}} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log(1 + \chi_{-\gamma_2}(\zeta')), \end{aligned} \quad (3.17)$$

$$\begin{aligned} \log \chi_{\gamma_2}(\zeta) &= \frac{Z_2}{\zeta} + \bar{Z}_2 \zeta - \frac{1}{4\pi i} \int_{l_{\gamma_1}} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log(1 + \chi_{\gamma_1}(\zeta')) \\ &\quad + \frac{1}{4\pi i} \int_{l_{-\gamma_1}} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log(1 + \chi_{-\gamma_1}(\zeta')), \end{aligned} \quad (3.18)$$

where the contours  $l_{\gamma'}$  is chosen as (see Figure 2)

$$l_{\gamma'} : \frac{Z_{\gamma'}}{\zeta'} \in \mathbb{R}_-. \quad (3.19)$$

Following Appendix E in [10], we can rewrite (3.17) and (3.18) as the following TBA equations,

$$\epsilon_1(\theta) = 2|Z_1| \cosh \theta - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta - \theta' - i\tilde{\alpha})} \log(1 + e^{-\epsilon_2(\theta')}), \quad (3.20)$$

$$\epsilon_2(\theta) = 2|Z_2| \cosh \theta - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta - \theta' + i\tilde{\alpha})} \log(1 + e^{-\epsilon_1(\theta')}), \quad (3.21)$$

where we have introduced  $\theta$  and  $\epsilon_k(\theta) \equiv \epsilon_{\gamma_k}(\theta)$  as  $Z_k = |Z_k|e^{i\alpha_k}$ ,  $\zeta = -e^{\theta+i\alpha_k}$ ,  $\chi_{\gamma_k}(\zeta = -e^{\theta+i\alpha_k}) = e^{-\epsilon_{\gamma_k}(\theta)}$ , and  $\tilde{\alpha} \equiv \pi/2 - (\alpha_1 - \alpha_2)$ . We used the relations  $\epsilon_{-\gamma_k}(\theta) = \epsilon_{\gamma_k}(\theta)$ . These relations hold from the  $\mathbb{Z}_2$ -symmetry (2.60).

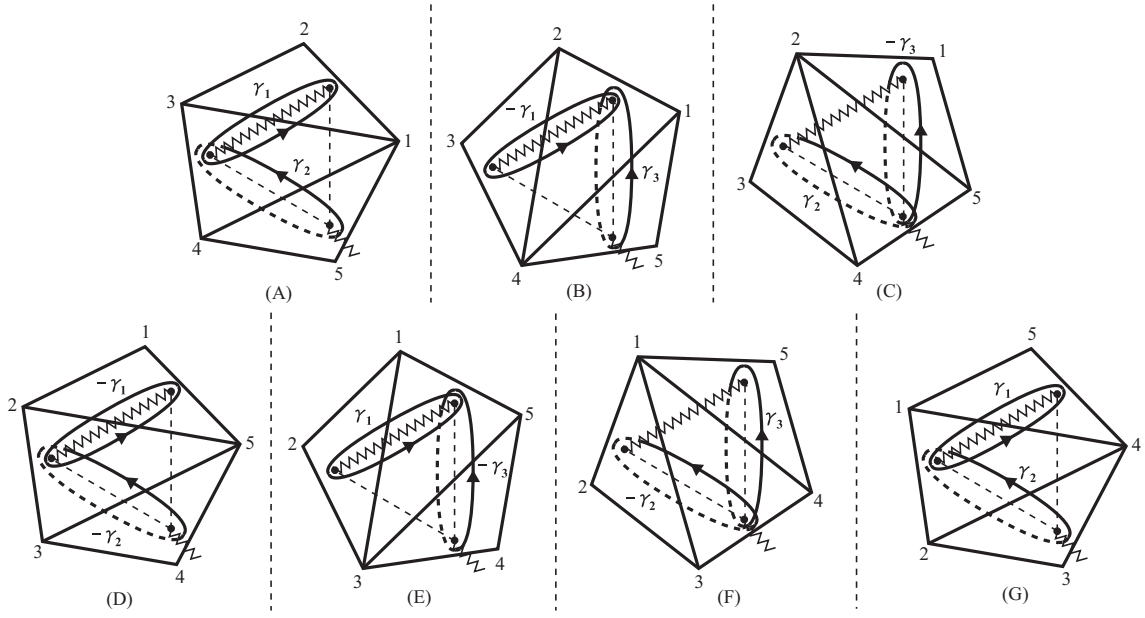


Figure 3: Transition of WKB triangulations for the decagon when  $u$  is outside the wall of marginal stability. There are six jumps as  $\arg \zeta$  varies from 0 to  $2\pi$ .

If  $u = 0$ ,  $Z_1$  and  $Z_2$  are simply related to  $\Lambda$ ,

$$Z_1 = -2 \int_{-\sqrt{3}\Lambda}^0 dz \sqrt{p(z)} = -\frac{\sqrt{\pi}\Gamma(\frac{3}{4})}{2\Gamma(\frac{9}{4})}(\sqrt{3}\Lambda)^{5/2}, \quad (3.22)$$

$$Z_2 = -2 \int_0^{\sqrt{3}\Lambda} dz \sqrt{p(z)} = iZ_1. \quad (3.23)$$

Since  $\alpha_1 - \alpha_2 = \pi/2$ , (3.20) and (3.21) yield

$$\epsilon_1(\theta) = 2|Z| \cosh \theta - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi \cosh(\theta - \theta')} \log(1 + e^{-\epsilon_2(\theta')}), \quad (3.24)$$

$$\epsilon_2(\theta) = 2|Z| \cosh \theta - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi \cosh(\theta - \theta')} \log(1 + e^{-\epsilon_1(\theta')}). \quad (3.25)$$

### 3.1.3. Outside the wall of marginal stability

Next let us consider the case where  $u$  is located outside the wall of marginal stability. In this case, as  $\arg \zeta$  increases, the WKB triangulations change as shown in Figure 3. In the same way as the previous case, we define three functions  $\chi_{\gamma_i}(\zeta)$  ( $i = 1, 2, 3$ ) from the WKB triangulations. These are summarized in Table 2. Note that for all regions these functions satisfy the relation  $\chi_{\gamma_3}(\zeta) = \chi_{\gamma_1}(\zeta)\chi_{\gamma_2}(\zeta)$ . We can write down the integral equations for  $\chi_{\gamma_i}(\zeta)$  ( $i = 1, 2, 3$ ), and rewrite them as the following

Table 2: The definition of  $\chi_{\gamma_i}(\zeta)$  ( $i = 1, 2, 3$ ) for the decagon when  $u$  is outside the wall of marginal stability.

$\arg \zeta$	(A)	(B)	(C)	(D)	(E)	(F)	(G)
$\chi_{\gamma_1}$	$-\beta_5^{-1}$	$-\beta_5^{-1}$	$\beta_1/\beta_3$	$-\beta_4$	$-\beta_4$	$\beta_2/\beta_5$	$-\beta_3^{-1}$
$\chi_{\gamma_2}$	$-\beta_2$	$\beta_5/\beta_3$	$-\beta_1^{-1}$	$-\beta_1^{-1}$	$\beta_2/\beta_4$	$-\beta_5$	$-\beta_5$
$\chi_{\gamma_3}$	$\beta_2/\beta_5$	$-\beta_3^{-1}$	$-\beta_3^{-1}$	$\beta_4/\beta_1$	$-\beta_2$	$-\beta_2$	$\beta_5/\beta_3$

TBA equations,

$$\begin{aligned} \epsilon_1(\theta) = 2|Z_1| \cosh \theta &- \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta - \theta' - i\tilde{\alpha}_{12})} \log(1 + e^{-\epsilon_2(\theta')}) \\ &- \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta - \theta' - i\tilde{\alpha}_{13})} \log(1 + e^{-\epsilon_3(\theta')}), \end{aligned} \quad (3.26)$$

$$\begin{aligned} \epsilon_2(\theta) = 2|Z_2| \cosh \theta &- \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta - \theta' + i\tilde{\alpha}_{12})} \log(1 + e^{-\epsilon_1(\theta')}) \\ &- \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta - \theta' + i\tilde{\alpha}_{32})} \log(1 + e^{-\epsilon_3(\theta')}), \end{aligned} \quad (3.27)$$

$$\begin{aligned} \epsilon_3(\theta) = 2|Z_3| \cosh \theta &- \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta - \theta' + i\tilde{\alpha}_{13})} \log(1 + e^{-\epsilon_1(\theta')}) \\ &- \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta - \theta' - i\tilde{\alpha}_{32})} \log(1 + e^{-\epsilon_2(\theta')}), \end{aligned} \quad (3.28)$$

where  $Z_3 = Z_1 + Z_2$  and  $\tilde{\alpha}_{ab} \equiv \pi/2 - (\alpha_a - \alpha_b)$ .

### 3.2. Dodecagon solutions ( $n = 6$ )

In the dodecagonal case  $n = 6$ , the degree of the polynomial  $p(z)$  is four, and we choose it as

$$p(z) = z^4 + 4\Lambda^2 z^2 + 2mz + u = (z - z_1)(z - z_2)(z - z_3)(z - z_4). \quad (3.29)$$

From (2.55), the relations among  $b$ 's are given by

$$b_{j+2}b_{j+3} = 1 + \mu_j - \mu_j b_{j+5}b_{j+6}, \quad (3.30)$$

$$b_{j+1} + b_{j+3} - b_{j+1}b_{j+2}b_{j+3} = \mu_{j+1}b_{j+5}. \quad (3.31)$$

As mentioned above, the wall-crossing phenomenon also occurs in this case. Here we focus on the simplest case, *i.e.*, we only consider the region where the number of BPS rays is the smallest (three+three). The WKB triangulations evolve as in Figure 4. The definition of  $\chi_{\gamma_i}$  ( $i = 1, 2, 3$ ) is summarized in Table 3. The discontinuity of



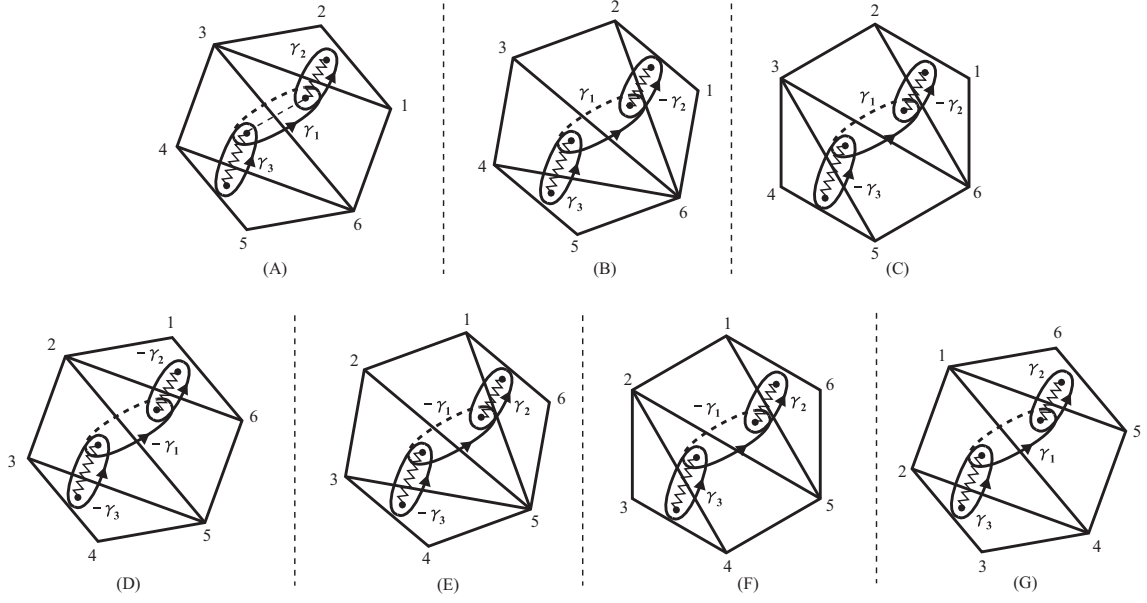


Figure 4: Transition of WKB lines for the dodecagonal case.

$\chi_{\gamma_i}$  can be expressed by the remaining two functions using (3.30) and (3.31). When  $\arg \zeta$  crosses the BPS ray from (C) to (D) in Table 3, for example, the ratio of the discontinuity of  $\chi_{\gamma_2}$  is evaluated as

$$\frac{\chi_{\gamma_2}^+}{\chi_{\gamma_2}^-} = \frac{1}{(1 - b_3 b_4)(1 - b_4 b_5)} = \frac{1}{1 - \mu^{-1} b_1 b_4} = (1 + \chi_{\gamma_1})^{-1}. \quad (3.32)$$

Table 3: The definition of  $\chi_{\gamma_i}(\zeta)$  ( $i = 1, 2, 3$ ) for the dodecagonal case.

$\arg \zeta$	(A)	(B)	(C)	(D)
$\chi_{\gamma_1}$	$-1/\mu b_2 b_5$	$-b_1/\mu b_5$	$-b_1 b_4/\mu$	$-b_1 b_4/\mu$
$\chi_{\gamma_2}$	$\mu(1 - b_4 b_5)$	$\mu(1 - b_4 b_5)$	$\mu(1 - b_4 b_5)$	$\mu/(1 - b_3 b_4)$
$\chi_{\gamma_3}$	$-(1 - b_4 b_5)$	$-(1 - b_4 b_5)$	$-(1 - b_4 b_5)$	$-1/(1 - b_3 b_4)$
$\arg \zeta$	(E)	(F)	(G)	
$\chi_{\gamma_1}$	$\mu b_4/b_6$	$-\mu/b_3 b_6$	$-\mu/b_3 b_6$	
$\chi_{\gamma_2}$	$\mu/(1 - b_3 b_4)$	$\mu/(1 - b_3 b_4)$	$\mu(1 - b_2 b_3)$	
$\chi_{\gamma_3}$	$-1/(1 - b_3 b_4)$	$-1/(1 - b_3 b_4)$	$-(1 - b_2 b_3)$	

Thus we finally obtain the TBA equations,

$$\begin{aligned} \epsilon_1(\theta) = 2|Z_1| \cosh \theta &- \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta - \theta' - i\tilde{\alpha}_{12})} \log(1 + e^{-\epsilon_2(\theta')}) \\ &- \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta - \theta' - i\tilde{\alpha}_{13})} \log(1 + e^{-\epsilon_3(\theta')}), \end{aligned} \quad (3.33)$$

$$\epsilon_2(\theta) = 2|Z_2| \cosh \theta - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta - \theta' + i\tilde{\alpha}_{12})} \log(1 + e^{-\epsilon_1(\theta')}), \quad (3.34)$$

$$\epsilon_3(\theta) = 2|Z_3| \cosh \theta - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta - \theta' + i\tilde{\alpha}_{13})} \log(1 + e^{-\epsilon_1(\theta')}). \quad (3.35)$$

If the moduli parameter  $u$  crosses the wall of marginal stability, the integral equations should be modified as well as the decagonal case. Although we can derive the TBA equations for such cases from the WKB data in the same way, we do not write them down explicitly here.

### 3.3. Integral equations for general $2n$ -gon

Until now, we have focused on the two special polygons: decagon and dodecagon. The integral equations derived there have the same forms as those in [10]. Thus it is natural to expect that these are true for general  $2n$ -gons. Here we rewrite the integral equations in [10] for our interested situations. We will identify them with the TBA equations for the homogeneous sine-Gordon models associated with the coset CFTs later.

Our starting equations are the followings,

$$\log \chi_{\gamma_k}(\zeta) = \frac{Z_{\gamma_k}}{\zeta} + \bar{Z}_{\gamma_k} \zeta - \frac{1}{4\pi i} \sum_{\gamma'} \langle \gamma_k, \gamma' \rangle \int_{\ell_{\gamma'}} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log(1 + \chi_{\gamma'}(\zeta')). \quad (3.36)$$

For the  $2n$ -gon,  $\gamma'$  in the sum runs over  $\pm\gamma_1, \pm\gamma_2, \dots, \pm\gamma_{n-3}$ .<sup>5</sup> The contour  $\ell_{\gamma'}$  is chosen as

$$\ell_{\gamma'} : \frac{Z_{\gamma'}}{\zeta'} \in \mathbb{R}_-. \quad (3.37)$$

By combining the terms for  $\gamma' = \gamma_k$  and  $\gamma' = -\gamma_k$  and using the  $\mathbb{Z}_2$ -symmetry, which is inherent in the present  $AdS_3$  system, we obtain the following simple integral equations,

$$\epsilon_k(\theta) = 2|Z_k| \cosh \theta - \sum_{l=1}^{n-3} \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{-i\langle \gamma_k, \gamma_l \rangle}{\sinh(\theta - \theta' + i\alpha_k - i\alpha_l)} \log \left( 1 + e^{-\epsilon_l(\theta')} \right), \quad (3.38)$$

where  $Z_k \equiv Z_{\gamma_k}$ . Note that one can reproduce (3.20)–(3.21), (3.33)–(3.35) from (3.38). The discussion outside the wall of marginal stability is similar.

<sup>5</sup>We focus on the simplest region of the moduli space.

### 3.4. TBA equations of the homogeneous sine-Gordon model

The homogeneous sine-Gordon models [16, 17] are a class of two-dimensional integrable models generalizing the sine-Gordon model. They are obtained by integrable perturbations of conformal field theories<sup>6</sup> corresponding to  $G_k$ -parafermions, or cosets  $G_k/[U(1)]^{r_{\mathfrak{g}}}$  [23], where  $G$  is a simple compact Lie group with Lie algebra  $\mathfrak{g}$ , and  $r_{\mathfrak{g}}$  is the rank of  $\mathfrak{g}$ . An integer  $k$  is the level of affine Lie algebra  $\hat{\mathfrak{g}}$ . The S-matrices describing the models for simply laced  $G$ 's are proposed in [24].

For the minimal surfaces in  $AdS_3$ , it turns out that the case of the  $SU(N)_2/[U(1)]^{N-1}$  coset is relevant, which is discussed in detail in [25]. An explicit form of the non-trivial part of the S-matrix in this case is given by

$$S_{ab}(\theta) = (-1)^{\delta_{ab}} \left[ c_a \tanh \frac{1}{2} \left( \theta + \sigma_{ab} - \frac{\pi}{2} i \right) \right]^{I_{ab}}. \quad (3.39)$$

Here,  $a = 1, \dots, N-1$  labels the particles corresponding to each simple root with mass  $m_a$ ,  $I_{ab}$  is the incidence matrix,  $c_a$  are constants and  $\sigma_{ab} = -\sigma_{ba}$  are some parameters.

Following the standard procedure, one finds the TBA equations from this S-matrix with inverse temperature  $R$ :

$$\epsilon_a(\theta) = m_a R \cosh \theta - \sum_b \int \frac{d\theta'}{2\pi} \frac{i I_{ab}}{\sinh(\theta - \theta' + \sigma_{ab} + \frac{\pi}{2} i)} \log(1 + e^{-\epsilon_b}). \quad (3.40)$$

Now it is clear that the TBA equations from the  $SU(N)_2/[U(1)]^{N-1}$  homogeneous sine-Gordon model coincide with those in (3.38) under the identifications  $n-2 \leftrightarrow N$ ,  $2|Z_a| \leftrightarrow m_a R$ ,  $\langle \gamma_a, \gamma_b \rangle \leftrightarrow \epsilon_{ab} I_{ab}$ , and  $i(\alpha_a - \alpha_b) \leftrightarrow \sigma_{ab} + \frac{\pi}{2} i$ . Here  $\epsilon_{ab} = -\epsilon_{ba} = \pm 1$ .

The TBA equations for the general homogeneous sine-Gordon models can be derived from the S-matrices in [24]. It would be of interest to see the relevance to the minimal surfaces in  $AdS_5$  and  $AdS_4$ . We comment on this point in the next section.

## 4. Regularized area and free energy

From the solutions to the TBA equations in sect. 3.1, 3.2, one can extract the physical quantities following [9, 12]. As we argue shortly, we expect that this is the case for the TBA equations (3.38) or (3.40) with general  $n$ . First, the cross-ratios of the  $AdS_3$  boundary coordinates  $x_{ij}^{\pm} x_{kl}^{\pm} / x_{ik}^{\pm} x_{jl}^{\pm}$  are given by  $(s_i \wedge s_j)(s_k \wedge s_l) / (s_i \wedge s_k)(s_j \wedge s_l)$

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<sup>6</sup>Regarding these, see also [22].

evaluated at  $\zeta = 1$  and  $\zeta = i$ , respectively. They are in turn read off from  $\chi$ 's and  $\beta$ 's. Second, the area of the minimal surface, representing the scattering amplitude, is decomposed as

$$A = A_{\text{sinh}} + 4 \int d^2w, \quad A_{\text{sinh}} = 4 \int d^2z (e^{2\alpha} - \sqrt{p\bar{p}}), \quad (4.1)$$

where  $\int d^2w$  is divergent and should be regularized. As for the finite piece  $A_{\text{sinh}}$ , the Poisson brackets among the Fock–Goncharov coordinates [10] imply that the relation [9] between  $A_{\text{sinh}}$  and the free energy of the TBA system, if any, generically holds:

$$A_{\text{sinh}} = F + c_n. \quad (4.2)$$

The constant term  $c_n$  is fixed by considering the limit where the zeros of  $p(z)$  become far apart from each other [9]: In this limit, the solution is regarded as a superposition of single-zero solutions. Since each single-zero solution corresponds to the hexagon solution, one has  $A_{\text{sinh}} \rightarrow (n-2)A_{\text{sinh}}(n=3)$  with  $A_{\text{sinh}}(n=3) = 7\pi/12$ . Therefore,

$$c_n = \frac{7}{12}(n-2)\pi. \quad (4.3)$$

#### 4.1. CFT limit and coset models

It is often the case that massive integrable models are obtained by perturbing conformal field theories. The identification of the conformal model is useful to analyze the TBA system.

In the previous section, we found that the TBA equations for the decagonal and the dodecagonal solutions are identified with those of the homogeneous sine-Gordon model which are obtained by perturbing the conformal field theory associated with the coset [26]

$$\frac{\text{SU}(n-2)_2}{[\text{U}(1)]^{n-3}} \simeq \frac{[\text{SU}(2)_1]^{n-2}}{\text{SU}(2)_{n-2}}, \quad (4.4)$$

with  $n = 5, 6$ . From the further identification with the TBA-like equations in (3.38), we expect that the general  $2n$ -gon solutions in  $AdS_3$  are described by the TBA equations of the above homogeneous sine-Gordon model with general  $n$ .

In fact, as obvious from the right-hand side of (4.4), the number of the degrees of freedom in this systems is  $(n-2) - 1 = (2n-6)/2$ , which matches the number of the independent cross ratios. We note here that the left and the right sectors are described by the same Hitchin system in the  $AdS_3$  case.

Moreover, one finds a precise agreement in the conformal limit between  $A_{\text{sinh}}$  from the minimal surfaces and the free energy  $F$  from the TBA equations of the homogeneous sine-Gordon model. On the minimal surface side, the solution in the conformal limit reduces to the regular polygon solution, where  $A_{\text{sinh}}$  is evaluated as [9]

$$A_{\text{sinh}} = \frac{\pi}{4n}(3n^2 - 8n + 4). \quad (4.5)$$

On the TBA side, the free energy in the conformal limit is obtained by setting  $m_a = 0$  and is given by the ground state energy of the corresponding conformal model [15]. Since the coset model

$$\frac{\text{SU}(K)_k}{[\text{U}(1)]^{K-1}} \simeq \frac{[\text{SU}(k)_1]^K}{\text{SU}(k)_K} \quad (4.6)$$

has the central charge  $c = (k - 1)K(K - 1)/(k + K)$ , the free energy in our case is

$$F = \frac{\pi}{6}c = \frac{\pi}{6n}(n - 2)(n - 3). \quad (4.7)$$

Taking into account the constant term in (4.2), we find that

$$F + c_n = \frac{\pi}{4n}(3n^2 - 8n + 4), \quad (4.8)$$

which is in precise agreement with (4.5). One can also derive (4.7) directly by starting from the TBA equations of the homogeneous sine-Gordon model in (3.40) [27, 28].<sup>7</sup>

Finally, we would like to comment on the case of  $AdS_5$ . In [12], it was shown that the hexagon solution in  $AdS_5$  is described by the  $A_3$  TBA system, which corresponds to  $k = 4, K = 2$  in (4.6). From a consideration on the degrees of freedom and the symmetry of the Hitchin system, we expect that the  $m$ -gon solution in  $AdS_5$  is described by the TBA equations of the homogeneous sine-Gordon model corresponding to the coset

$$\frac{\text{SU}(m - 4)_4}{[\text{U}(1)]^{m-5}} \simeq \frac{[\text{SU}(4)_1]^{m-4}}{\text{SU}(4)_{m-4}}, \quad (4.9)$$

in a region of marginal stability. The coset model has the central charge  $3(m - 4)(m - 5)/m$ . This should be reproduced from the regular polygon solutions.

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<sup>7</sup>For  $n = 5$  one can also check that by starting from (3.26)–(3.28).

## 5. Conclusions

In this paper we studied the classical open string solutions with a null polygonal boundary in  $AdS_3$ . We derived in full detail the set of integral equations for the decagonal and the dodecagonal solutions. These integral equations were identified with the TBA equations of the homogeneous sine-Gordon model, whose CFT limits are the generalized parafermion models. We also observed general correspondence between the polygonal solutions in  $AdS_n$  and generalized parafermions.

Since the deformations around the CFT point of homogeneous sine-Gordon models have rich structure [25, 29, 30], it is interesting to study the remainder functions around the CFT point. It is also interesting to study supersymmetric extensions and quantum corrections.

**Note Added:** During the preparation of this paper, we have noticed the paper by Alday, Maldacena, Sever and Vieira [31], which considerably overlaps with the present work. In particular, they also present the TBA equations for general  $2n$ -gons in the  $AdS_3$  case at the end of sect. 3.5, which coincide with ours (3.38) under appropriate identification of functional variables. According to their results, the regular  $m$ -gon solutions in  $AdS_5$  with  $\mu = 1$  is found to be consistent with the central charge of the generalized parafermions. The structure of the Y-system is also in accord with the spectrum of the homogeneous sine-Gordon model.

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