A CLASSIFICATION OF

ORTHOGONAL TRANSFORMATION GROUPS

OF LOW COHOMOGENEITY

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THESIS

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A classification of orthogonal transformation groups of low cohomogeneity

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Dedicated to Professor Ichiro Yokota on his 60th birthday

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1. Introduction

A Lie transformation group on a smooth manifold $M$ is a pair $(G, M)$ of a Lie group $G$ which acts smoothly on $M$. This paper is concerned with the cohomogeneity (abbrev. coh) of $(G, M)$, which is defined by

$$\text{coh}(G, M) = \dim M - \dim G + \min \{ \dim G_x; x \in M \},$$

where $G_x$ is the isotropy subgroup of $G$ at $x$. Then

$$\text{coh}(G, M) \geq \dim M - \dim G \; (=: \text{doh}(G, M)),$$

$x \in M; \text{coh}(G, M) = \text{doh}(G, M) + \dim G_x$ is an open subset of $M$, and

$$\text{coh}(G^0, M) = \text{coh}(G, M)$$

where $G^0$ is the identity connected component of $G$. 
An orthogonal transformation group (abbrev. o.t.g.) on an \( N \) dimensional Euclidean space \( E^N \) is defined as a pair \((G,E^N)\) of a connected Lie subgroup \( G \) of the full orthogonal group \( O(N) \) on \( E^N \). \((G,E^N)\) is said to be contained in another o.t.g. \((G',E^N)\) on \( E^N \) if there is a real linear isometry \( \iota: E^N \rightarrow E^N \) and a Lie group monomorphism \( \tau: G \rightarrow G' \) such that
\[
\tau(g) = \iota g \quad \text{for all } g \text{ in } G.
\]
If moreover \( \tau \) is a Lie group isomorphism, \((G,E^N)\) is said to be equivalent to \((G',E^N)\).

Let \( \rho \) be a linear representation on \( \mathbb{R}^N \) over the field \( \mathbb{R} \) of all real numbers of a Lie group \( G \). We say \((G,\rho,\mathbb{R}^N)\) an orthogonal linear triple and \( \rho \) an orthogonal representation of \( G \) if there is a positive definite inner product on \( \mathbb{R}^N \) which is invariant under the action of \( \rho(G) \). Suppose \( \rho' \) is another orthogonal representation of \( G \). We call \((G,\rho',\mathbb{R}^N)\) and \((G,\rho,\mathbb{R}^N)\) are equivalent as real representation if \( \rho' \) and \( \rho \) are equivalent as real representations of \( G \).

An orthogonal linear triple \((G,\rho,\mathbb{R}^N)\) naturally induces an o.t.g. \((\rho(G^0),E^N)\) which is well defined up to equivalences and denoted by \( O(G,\rho,\mathbb{R}^N) \). We denote
\[
\text{coh}(G,\rho,\mathbb{R}^N) = \text{coh}(O(G,\rho,\mathbb{R}^N)),
\]
\[
\text{doh}(G,\rho,\mathbb{R}^N) = \text{doh}(O(G,\rho,\mathbb{R}^N)).
\]
If \( G \) is compact, then any real representation of \( G \) is
an orthogonal linear representation, and the corresponding o.t.g. is called a compact linear group.

An o.t.g. is called maximal if it does not properly contain an o.t.g. of the same cohomogeneity. Suppose \((G,E^N)\) is a maximal o.t.g. If it contains a compact linear group of the same cohomogeneity, then itself is a compact linear group.

In fact the closure \(\hat{G}\) of \(G\) in \(O(N)\) is compact and \(\text{coh}(\hat{G},E^N) = \text{coh}(G,E^N)\). Since \(\{x \in E^N; G(x) \text{ is compact} \} = \{x \in E^N; \hat{G}(x) = G(x)\}\), \(\text{coh}(G,E^N) = N \cdot \text{dim}G + \text{dim}_Gx\) is an open dense subset of \(E^N\).

Hsiang-Lawson[11] gave a classification theorem of all compact linear groups of cohomogeneity 2 (resp. 3) and maximal by means of the classification of compact linear groups which has a non trivial isotropy subgroup at a point of a principal orbit (cf. Kramer[15], Hsiang[10] and Hsiang-Hsiang[9]). As a result, all (resp. most) of them can be induced from the linear isotropy representations of Riemannian symmetric pairs of rank 2 (resp. 3).

Conversely, the linear isotropy representation of each Riemannian symmetric pair of rank \(r\) induces a compact linear group of cohomogeneity \(r\) (cf. Takagi-Takahashi[19]). Any of its orbit in the representation space is an R-space in the meaning of Takeuchi[20] (cf. Takeuchi-Kobayashi[21]). We define a principal R-space as an R-space of the highest dimension among all R-spaces associated with a given Riemannian symmetric pair.

From tables of Takagi-Takahashi[19, Table I and II], it appears that two principal R-spaces associated with

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two distinct Riemannian symmetric pairs of rank 2 are not equivalent as Riemannian manifolds nor Riemannian submanifolds of a hypersphere of the representation space. Especially if two maximal o.t.g.'s of cohomogeneity 2 contain o.t.g.'s from two distinct Riemannian symmetric pairs of rank 2 respectively, then they are not equivalent (cf. Ozeki-Takeuchi[17; Theorem 1, Theorem 2]).

However it is well known that the o.t.g. from the Riemannian symmetric pair \( (G_2, SO(4)) \) of rank 2 is missed in a theorem of Hsiang-Lawson[11; Theorem 5] (cf. Uchida[23]). More than before, Uchida[23] pointed out many examples of real reducible (i.e., non irreducible) compact linear groups of coh 3 which shows that another theorem of Hsiang-Lawson[11; Theorem 6] should be properly modified. Uchida[23; Theorem] also gave a modified classification theorem of real reducible compact linear groups of coh 3 and maximal in a correct form by the use of a classification of compact Lie groups which act transitively on spheres (cf. Montgomery-Samelson[16], Borel[3],[4]).

In this paper, we study the classification of real irreducible o.t.g.'s of coh at most 3 by a direct method (cf. Sato-Kimura[18], Yokota[25]). We have the list of them in Section 4, which shows that the other theorem of Hsiang-Lawson [11; Theorem 7] should be properly modified and also gives a modified classification of real irreducible compact linear groups of coh 3 and maximal in a correct form (cf. Theorem 4.8, Remark 4.10).
Our results also give a proof of the fact that a compact linear group of coh 2 and maximal is equivalent to an o.t.g. which is induced from the linear isotropy representation of a Riemannian symmetric pair of rank 2. Topologically, Asoh[2] has already completed the classification of compact Lie groups acting on spheres with an orbit of codimension one, which properly modified the result of H.C. Wang[26] (cf. Hsiang-Hsiang[8]). Recently, Dadok[5] classified real irreducible compact linear groups with certain property, so-called 'polar', which is satisfied by each compact linear group of coh 2.
2. Preliminaries

For each type of compact simple Lie algebra of dimension \( g \) and rank \( k \), we shall investigate (cf. Goto-Grosshans[6])

1. 'Real' complex irreducible representations of degree \( m \) such that

\[
d_0 := m - g \leq 3,
\]

2. Complex irreducible representations of degree \( m \) such that

\[
d_1 := 2m - g \leq 4,
\]

3. 'Quaternion' complex irreducible representations of degree \( 2m \) such that

\[
d_2 := 4m - g \leq 6.
\]

We denote a compact simple Lie algebra of type \( X_k \) by \( X_k (X=A,B,C,D,E,F,\text{or } G) \) and the corresponding compact simply connected Lie group by \( \hat{X}_k \) (abbrev. \( X_k \)). A complex irreducible representation of the highest weight \( \Lambda \) is denoted by \( \Lambda \). Especially the trivial representation is denoted by \( O \). The fundamental weights with respect to the simple roots \( \alpha_1, \alpha_2, \ldots, \alpha_k \) are denoted by

\[\Lambda_1, \Lambda_2, \ldots, \Lambda_k\]

(A)

The simple roots of \( A_k \) are given by

\[
\alpha_1 \rightarrow \alpha_2 \rightarrow \ldots \rightarrow \alpha_k \quad (k \geq 1).
\]

1. 'Real' complex irreducible representations of \( A_k \) are given by

\[
\Lambda = 2\lambda_1 \Lambda_1 \quad (\text{if } k = 1), \quad \Lambda = \sum_{i=1}^{h+1} \lambda_i (\Lambda_1 + \Lambda_{k-i+1}) \quad (\text{if } k = 2h+2),
\]

\[
\Lambda = \sum_{i=1}^{2h+1} \lambda_i (\Lambda_1 + \Lambda_{k-i+1}) \quad (\text{if } k = 4h+3), \quad \text{or}
\]

\[
\Lambda = \sum_{i=1}^{2h+2} \lambda_i (\Lambda_1 + \Lambda_{k-i+1}) \quad (\text{if } k = 4h+5),
\]

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where \( h \) and \( \lambda_i \) (\( i=1, \ldots, [(k+1)/2] \)) are non-negative integers, and \([p]\) denotes the maximal integer at most \( p\).

**Proposition 2.1** If \( d_0 := \deg \Lambda - k^2 - 2k \leq 3 \), then \( \Lambda \) is equivalent as a complex representation of \( A_k(k \geq 1) \) to one of the followings:

\[
\begin{align*}
 d_0 < 0: & \quad \Lambda_2(k=3), \ O(k \geq 1), \\
 d_0 = 0: & \quad 2\Lambda_1(k=1), \ \Lambda_1 + \Lambda_k(k \geq 2), \\
 d_0 = 2: & \quad 4\Lambda_1(k=1).
\end{align*}
\]

**Proof:** If \( \lambda_1 \geq 1 \) for some \( i=4, \ldots, [(k+1)/2] \), then \( k \geq 7 \) and \( d_0 \geq \deg \Lambda_4 - k^2 - 2k \geq k+1 + 4 - k^2 - 2k \geq 7 \). If \( [(k+1)/2] \geq 3 \) and \( \lambda_3 \geq 1 \), then \( k \geq 5 \) and \( d_0 \geq \deg (\Lambda_3 + \Lambda_{k-2}) - k^2 - 2k = (k+2)(k+1)^2 k^2(k-4)/36 - k^2 - 2k \geq 140 \).

If \( \lambda_2 \geq 1 \) and \( k \geq 4 \), then \( d_0 \geq \deg (\Lambda_2 + \Lambda_{k-1}) - k^2 - 2k = (k+1)^2 (k^2 - 4)/4 - k^2 - 2k \geq 151 \). Therefore \( \Lambda = 0(k \geq 1) \), \( 2\lambda_1 \Lambda_1(k=1) \), \( \lambda_1 (\Lambda_1 + \Lambda_k)(k \geq 2) \), or \( \lambda_2 \Lambda_2 + \lambda_1 (\Lambda_1 + \Lambda_3)(k=3) \). If \( k=1 \) and \( \lambda_1 \geq 3 \), then \( d_0 \geq \deg 6 \Lambda_1 - 3=4 \).

If \( k \geq 2 \) and \( \lambda_1 \geq 2 \), then \( d_0 \geq \deg 2 (\Lambda_1 + \Lambda_k) - k^2 - 2k = k(k+1)^2(k+4)/4 - k^2 - 2k \geq 18 \). If \( k=3 \) and \( \lambda_2 \geq 2 \), then \( d_0 \geq \deg 2 \Lambda_2 - 15=5 \). If \( k=3 \) and \( \lambda_1 = \lambda_2 = 1 \), then \( d_0 \geq \deg (\Lambda_1 + \Lambda_2 + \Lambda_3) - 15=49 \). Q.E.D.

(2) Complex irreducible representations of \( A_k(k \geq 1) \) are given by \( \Lambda = \sum_{i=1}^{k} \lambda_i A_i \) where \( \lambda_i \) (\( i=1, \ldots, k \)) are non-negative integers.

**Proposition 2.2** If \( d_1 := 2\deg \Lambda - k^2 - 2k \leq 4 \), then \( \Lambda \) is equivalent as a complex representation of \( A_k(k \geq 1) \) to one of the followings:

\[
0(k \geq 1), \ \Lambda_1(k \geq 1), \ 2\Lambda_1(k=1,2), \ \Lambda_2(k \geq 2),
\]

\[
2\Lambda_2(k=2), \ \Lambda_{k-1}(k \geq 4), \ \Lambda_k(k \geq 3).
\]

**Proof:** If \( k=1 \) and \( \lambda_1 \geq 3 \), then \( \deg \Lambda \geq \deg 3 \Lambda_1 = 4 \) and \( d_1 \geq 5 \). If \( k=2 \)
and $\lambda_1 (or \lambda_2 ) \geq 3$, then $\deg \Lambda_2 \geq \deg 3 \Lambda_1 = 10$ and $d_1 \geq 12$. If $k \geq 2$, $\lambda_1 \geq 1$ and $\lambda_k \geq 1$, then $\deg \Lambda_2 \geq \deg (\Lambda_1 + \Lambda_k ) = k (k + 2)$ and $d_1 \geq 8$. If $k \geq 3$ and $\lambda_1 (or \lambda_k ) \geq 2$, then $\deg \Lambda_2 \geq \deg 2 \Lambda_1 = (k + 1) (k + 2) / 2$ and $d_1 \geq 5$. If $\lambda_i \geq 1$ for some $i = 3, \ldots, k - 2$, then $\deg \Lambda_2 \geq \deg \Lambda_i = k (k^2 - 1) / 6$, $k \geq 5$ and $d_1 \geq 5$.

If $\lambda_2 (or \lambda_{k-1}) \geq 2$ and $2 \leq k - 1$, then $\deg \Lambda_2 \geq \deg 2 \Lambda_2 = k (k + 1) (k + 2) / 12$, $k \geq 3$ and $d_1 \geq 25$. If $\lambda_2 \geq 1$, $\lambda_{k-1} \geq 1$ and $2 < k - 1$, then $\deg \Lambda_2 \geq \deg (\Lambda_2 + \Lambda_{k-1}) = (k + 1) (k^2 - 4) / 4$, $k \geq 4$ and $d_1 \geq 126$. If $\lambda_1 \geq 1$, $\lambda_{k-1} \geq 1$ and $1 < k - 1$, then $\deg \Lambda_2 \geq \deg (\Lambda_1 + \Lambda_{k-1}) = (k + 2) (k^2 - 1) / 2$, $k \geq 3$ and $d_1 \geq 15$.

If $\lambda_2 \geq 1$, $\lambda_k \geq 1$ and $2 < k$, then $d_1 \geq 15$. If $\lambda_1 \geq 1$, $\lambda_2 \geq 1$ (or $\lambda_{k-1} \geq 1$, $\lambda_k \geq 1$) and $2 < k - 1$, then $\deg \Lambda_2 \geq \deg (\Lambda_1 + \Lambda_2 ) = 2k (k + 1) (k + 2) / 3$, $d_1 \geq 56$. Q.E.D.

Remark 2.3 $2 \Lambda_1 (k = 1)$, $\Lambda_2 (k = 3)$ are 'real'. $\Lambda_1 (k = 1)$ is 'quaternion'. $\Lambda_1 , \Lambda_k (k \geq 2)$ (resp. $\Lambda_2 , \Lambda_{k-1} (k \geq 4)$, resp. $2 \Lambda_1 , 2 \Lambda_2 (k = 2)$) are conjugate from each other.

(3) 'Quaternion' complex irreducible representations of $\Lambda_k (k \geq 1)$ are given as $\Lambda = (2 \lambda_2 h + 1) \Lambda_2 h + 1 + \sum_{i=1}^{2h} \lambda_i (\Lambda_i + \Lambda_{k-i+1})$ where $k = 4h + 1$, $\lambda_i$ and $h$ are non-negative integers.

Proposition 2.4 If $d_2 := 2 \deg \Lambda - k^2 - 2k \leq 8$, then $\Lambda$ is equivalent as a complex representation of $\Lambda_k (k \geq 1)$ to one of the followings:

$$d_2 = 1: \quad \Lambda_1 (k = 1),$$

$$d_2 = 5: \quad 3 \Lambda_1 (k = 1), \Lambda_3 (k = 5).$$

Proof: If $k = 4h + 1 \geq 6$, then $k \geq 9$ and $d_2 \geq 2 \deg \Lambda_2 h + 1 - k^2 - 2k \geq 2 \deg \Lambda_5 - k^2 - 2k \geq 405$. So $k = 1$ or 5. Suppose $k = 1$. If $\lambda_1 \geq 2$, then $d_2 = 2 \deg (2 \lambda_1 + 1) \Lambda_1 - 3 \geq 2 \deg 5 \Lambda_1 - 3 = 9$. So $\Lambda = \Lambda_1$ or $3 \Lambda_1$. Next suppose $k = 5$.

If $\lambda_2 \geq 1$, then $d_2 \geq 2 \deg (\Lambda_2 + \Lambda_4 ) - 35 = 343$. If $\lambda_1 \geq 1$,
then $d_2 \geq 2\text{deg}(\Lambda_1 + \Lambda_5) - 35 = 35$. If $\lambda_3 \geq 1$, then $d_2 \geq 2\text{deg}(3\Lambda_3) - 35 = 1925$. So $\Lambda = \Lambda_3$. Q.E.D.

(C)

The simple roots of $C_k$ are given by

\[ \alpha_1 - \alpha_2 - \ldots - \alpha_{k-1} = \alpha_k \quad (k \geq 2). \]

(1) 'Real' complex irreducible representations of $C_k (k \geq 2)$ are given by $\Lambda = \sum_{i=1}^{k} \lambda_i \Lambda_i$ where $\sum_{i: \text{odd}} \lambda_i$ is even and $\lambda_i (i = 1, \ldots, k)$ are non-negative integers.

**Proposition 2.5** If $d_0 := \text{deg} \Lambda - k(2k+1) \leq 3$, then $\Lambda$ is equivalent as a complex representation of $C_k (k \geq 2)$ to one of the followings:

- $d_0 < 0$: $0 (k \geq 2), \Lambda_2 (k \geq 2)$,
- $d_0 = 0$: $2\Lambda_1 (k \geq 2)$.

**Proof:** Suppose $k \geq 5$. Then $\text{deg} \Lambda_3 < \text{deg} \Lambda_1$ for $i = 4, \ldots, k$ and $\text{deg} \Lambda_3 - \text{dim} C_k = 4k(k^2 - 3k - 7) \geq 20$. $\text{deg}(\Lambda_1 + \Lambda_2) - \text{dim} C_k = k(2k+1)(4k+1)/3 \geq 165$. $\text{deg}(\Lambda_1 + \Lambda_2) - \text{dim} C_k = k(8k^2 - 6k - 11)/3 \geq 265$. $\text{deg}(\Lambda_1 + \Lambda_2) - \text{dim} C_k = (4k^2 - 13k + 35)/3 \geq 725$. So $\Lambda = 0, \Lambda_2$ or $2\Lambda_1$. Suppose $k = 4$. Then the assertion holds since $\text{deg}(\Lambda_1 + \Lambda_2) - \text{dim} C_4 = 12$, $\text{deg}(\Lambda_1 + \Lambda_2) - \text{dim} C_4 = 6$, $\text{deg}\Lambda_3 - \text{dim} C_4 = 272$, $\text{deg}\Lambda_3 - \text{dim} C_4 = 84$ and $\text{deg}(\Lambda_1 + \Lambda_2) - \text{dim} C_4 = 124$. Suppose $k = 3$. Then the assertion holds since $\text{deg}(\Lambda_1 + \Lambda_2) - \text{dim} C_3 = 35$, $\text{deg}(\Lambda_1 + \Lambda_2) - \text{dim} C_3 = 43$, $\text{deg}(\Lambda_1 + \Lambda_2) - \text{dim} C_3 = 49$, $\text{deg}(\Lambda_1 + \Lambda_2) - \text{dim} C_3 = 63$ and $\text{deg}(\Lambda_1 + \Lambda_2) - \text{dim} C_3 = 69$. Suppose $k = 2$. Then the assertion holds since $\text{deg}(\Lambda_1 + \Lambda_2) - \text{dim} C_2 = 25$, $\text{deg}(\Lambda_1 + \Lambda_2) - \text{dim} C_2 = 25$, $\text{deg}(\Lambda_1 + \Lambda_2) - \text{dim} C_2 = 25$. Q.E.D.
(2) Complex irreducible representations of $C_k(k \geq 2)$ are given by $\Lambda = \sum_{i=1}^{k} \lambda_i A_i$ where $\lambda_i (i = 1, \ldots, k)$ are non-negative integers.

Proposition 2.6 If $d_1 := 2\deg \Lambda - k(2k+1) \leq 6$, then $\Lambda$ is equivalent as a complex representation of $C_k(k \geq 2)$ to one of the followings:

$$0(k \geq 2), \Lambda_1(k \geq 2), \Lambda_2(k = 2).$$

Proof: Suppose $k \geq 3$. If $\Lambda$ is not equivalent to $0$ nor $\Lambda_1$, then $\deg \Lambda \geq \deg \Lambda_2$, so $d_1 \geq 2\deg \Lambda_2 - \text{dim} C_k = 2k^2 - 3k - 2 \geq 7$. Suppose $k = 2$. The assertion holds since $2\deg 2\Lambda_1 - \text{dim} C_2 = 10$, $2\deg (\Lambda_1 + \Lambda_2) - \text{dim} C_2 = 22$ and $2\deg 2\Lambda_2 - \text{dim} C_2 = 18$. Q.E.D.

(3) 'Quaternion' complex irreducible representations of $C_k(k \geq 2)$ are given by $\Lambda = \sum_{i=1}^{k} \lambda_i A_i$ where $\Sigma_i: \text{odd} \lambda_i$ is odd and $\lambda_i (i = 1, \ldots, k)$ are non-negative integers.

Proposition 2.7 If $d_2 := 2\deg \Lambda - k(2k+1) \leq 6$, then $\Lambda$ is equivalent as a complex representation of $C_k(k \geq 2)$ to one of the followings:

$\Lambda_1(k \geq 2)$. 

Proof: Suppose $k \geq 3$. If $\Lambda$ is not equivalent to $\Lambda_1$, then $\deg \Lambda \geq \deg \Lambda_2$, so $d_2 \geq 2\deg \Lambda_2 - \text{dim} C_k = 2k^2 - 3k - 2 \geq 7$. Suppose $k = 2$. If $\Lambda$ is not equivalent to $\Lambda_1$, then $\deg \Lambda \geq \deg (\Lambda_1 + \Lambda_2) = 16$, so $d_2 \geq 22$. Q.E.D.

(B)

The simple roots of $B_k$ are given by

$$\alpha_1 \rightarrow \alpha_2 \rightarrow \ldots \rightarrow \alpha_{k-1} \rightarrow \alpha_k (k \geq 3).$$
(1) 'Real' complex irreducible representations of $B_k(k \geq 3)$ are given by $\Lambda = \sum_{i=1}^{k} \lambda_i A_i$ (if $k = 4h + 3$ or $4h + 4$), $2\lambda_k A_k + \sum_{i=1}^{k-1} \lambda_i A_i$ (otherwise) where $h$ and $\lambda_i (i = 1, ..., k)$ are non-negative integers.

**Proposition 2.8** If $d_0 := \deg \Lambda - k(2k+1) \leq 5$, then $\Lambda$ is equivalent as a complex representation of $B_k(k \geq 3)$ to one of the followings:

- $d_0 < 0$: $\Lambda_1(k \geq 3), \Lambda_k(k = 3 \text{ or } 4), 0(k \geq 3)$,
- $d_0 = 0$: $\Lambda_2(k \geq 3)$.

**Proof:** If $\lambda_i \geq 1$ for some $i = 3, ..., k-1$, then $k \geq 4$ and $d_0 \geq \deg \Lambda_1 - \dim B_k = k(2k+1)(2k-4)/3 \geq 48$. If $\lambda_i \geq 2$, then $d_0 \geq \deg 2\Lambda_1 - \dim B_k = 2k \geq 6$. If $\lambda_i \geq 2$, then $d_0 \geq \deg 2\Lambda_2 - \dim B_k = (2k+3)(2k+1)(k+1)/(k-3)(2k+1) \geq 147$. If $\lambda_i \geq 1$ and $\lambda_i \geq 1$, then $d_0 \geq \deg(\Lambda_1 + \Lambda_2) - \dim B_k = (2k+1)(k+1)(4k-3) \geq 84$. Then $\Lambda = \Lambda_1, \Lambda_2, \Lambda_k$, or $\Lambda_2 + \Lambda_k$ (if $k = 4h + 3$ or $4h + 4$), $\Lambda_1$ or $\Lambda_2$ (otherwise) since $\deg 2\Lambda_k - \dim B_k = 2k+1 C_{k+1} - k(2k+1) \geq 14$ and $\deg(\Lambda_1 + \Lambda_k) - \dim B_k = k^2(k+1) - k(2k+1) \geq 27$. If $k = 4h + 3$ or $4h + 4$, $k \geq 4$ and $\lambda_i \geq 1$, then $k \geq 6$ and $d_0 \geq \deg \Lambda_k - \dim B_k = 2k - k(2k+1) \geq 120$.

If $k = 3$ (resp. 4), then $\deg(\Lambda_1 + \Lambda_k) - \dim B_k = 91$ (resp. 396). Q.E.D.

(2) Complex irreducible representations of $B_k(k \geq 3)$ are given by $\Lambda = \sum_{i=1}^{k} \lambda_i A_i$ where $\lambda_i (i = 1, ..., k)$ are non-negative integers.

**Proposition 2.9** If $d_1 := 2\deg \Lambda - k(2k+1) \leq 8$, then $\Lambda$ is equivalent as a complex representation of $B_k(k \geq 3)$ to one of the followings:

- $d_1 < 0$: $\Lambda_1(k \geq 3), \Lambda_k(k = 3 \text{ or } 4), 0(k \geq 3)$,
- $d_1 = 0$: $\Lambda_2(k \geq 3)$.

**Proof:** If $\lambda_i \geq 1$ for some $i = 2, ..., k-1$, then $d_1 \geq 2\deg \Lambda_2 - k(2k+1) = k(2k+1) \geq 21$. If $\lambda_1 \geq 2$, then $d_1 \geq 2\deg 2\Lambda_1 - k(2k+1) = k(2k+5) \geq 33$. 

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If $\lambda_k \geq 2$, then $d_i \geq 2\deg 2\Lambda_k - k(2k+1) = 2\cdot 2^{k+1} - k(2k+1) \geq 49$. If $\lambda_i \geq 1$ and $\lambda_k \geq 1$, then $d_1 \geq 2\deg (\Lambda_1 + \Lambda_k) - k(2k+1) = 2^{k+2} - k(2k+1) \geq 75$. If $k \geq 5$, then $2\deg \Lambda_k - k(2k+1) = 2^{k+1} - k(2k+1) \geq 9$. Q.E.D.

(3) 'Quaternion' complex irreducible representations of $B_k$ ($k \geq 3$) are given by $\Lambda = \Sigma_{i=1}^{k-1} \lambda_i \Lambda_i + (2\lambda_k + 1) \Lambda_k$ where $k = 4h + 5$ or $4h + 6$, $h$ and $\lambda_i$ ($i = 1, \ldots, k$) are non-negative integers. Then $k \geq 5$.

Proposition 2.10 There is no 'quaternion' complex irreducible representation of $B_k$ such that $d_2 = 2\deg \Lambda - k(2k+1) \leq 8$.

Proof: Since $k \geq 5$, $d_2 \geq 2\deg \Lambda_k - k(2k+1) = 2^{k+1} - k(2k+1) \geq 9$. Q.E.D.

The simple roots of $D_k$ are given by

\[ \alpha_1 - \alpha_2 - \ldots - \alpha_{k-2} - \alpha_{k-1} \quad (k \geq 4). \]

(1) 'Real' complex irreducible representations of $D_k$ ($k \geq 4$) are given by $\Lambda = \Sigma_{i=1}^{k-2} \lambda_i \Lambda_i + \lambda_k - 1(\Lambda_{k-1} + \Lambda_k)$ (if $k = 2h + 5$), $\Sigma_{i=1}^{k-2} \lambda_i \Lambda_i$ (if $k = 4h + 4$), or $\Sigma_{i=1}^{k-2} \lambda_i \Lambda_i + \lambda^*_k - 1(\Lambda_{k-1} + \lambda^*_k \Lambda_k)$ (if $k = 4h + 6$), where $\lambda^*_k - 1 + \lambda^*_k$ is even, $h$ and $\lambda^*_i$ ($i = 1, \ldots, k$) are non-negative integers.

Proposition 2.11 If $d_0 = \deg \Lambda - k(2k-1) \leq 6$, then $\Lambda$ is equivalent as a complex representation of $D_k$ ($k \geq 4$) to one of the followings:

- $d_0 < 0$: $0 (k \geq 4)$, $\Lambda_1 (k \geq 4)$, $\Lambda_4 (k = 4)$, $\Lambda_3 (k = 4)$
- $d_0 = 0$: $\Lambda_2 (k \geq 4)$.

Proof: If $\lambda_i \geq 1$ for some $i = 3, \ldots, k - 2$, then $k \geq 5$ and $d_0 \geq \deg \Lambda_3 - k(2k-1) = k(2k-1)(2k-5)/3 \geq 75$. So $\lambda_i = 0$ for $i = 3, \ldots, k - 2$. 

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Since \( \deg 2A_1- k(2k-1) = 2k-1 \geq 7 \), \( \deg 2A_2 - k(2k-1) = k^2 (4k^2-13) \geq 272 \) and \( \deg (A_1 + A_2) - k(2k-1) = k(4k-5)(2k+1)/3 \geq 132 \), we have \( \lambda_1 + \lambda_2 \leq 1 \).

Suppose \( \lambda_k^{(*)} \geq 1 \). If \( k \geq 8 \), then \( d_0 \geq 2^{k-1} - k(2k-1) \geq 8 \). If \( k=7 \), then \( d_0 \geq \deg (A_6 + A_7) = 91 = 2912 \). If \( k = 6 \), then \( d_0 \geq \deg (A_5 + A_6) - 66 = 726 \) or \( d_0 \geq \deg (2A_5) - 66 = \deg (2A_6) - 66 = 11 \cdot 66 = 396 \). If \( k = 5 \), then \( d_0 \geq \deg (A_4 + A_5) - 45 = 165 \). If \( k = 4 \) and \( \lambda_1 \geq 1 \), then \( d_0 \geq \deg (A_1 + A_4) - 28 = \deg (A_1 + A_3) - 28 = 28 \). If \( k = 4 \) and \( \lambda_2 \geq 1 \), then \( d_0 \geq \deg (A_2 + A_4) - 28 = \deg (A_2 + A_3) - 28 = 132 \). So \( k = 4 \) and \( \Lambda = A_4 \) or \( A_3 \). Q.E.D.

(2) Complex irreducible representations of \( D_k (k \geq 4) \) are given by \( \Lambda = \sum_{i=1}^{k} \lambda_i A_i \) where \( \lambda_i \) (i=1,...,k) are non-negative integers.

**Proposition 2.12** If \( d_1 := 2 \deg \Lambda - k(2k-1) \leq 36 \), then \( \Lambda \) is equivalent as a complex representation of \( D_k (k \geq 4) \) to one of the followings:

\[
\begin{align*}
\text{if } d_1 < 0: & \quad 0(k \geq 4), A_1(k \geq 4), A_3(k=4), A_4(k=4), \\
\end{align*}
\]

\[
\begin{align*}
A_4(k=5), A_5(k=5), A_5(k=6), A_6(k=6).
\end{align*}
\]

**Proof:** If \( \lambda_i > 1 \) for some \( i = 2, \ldots, k-2 \), then \( d_1 \geq 2 \deg A_2 - k(2k-1) = k(2k-1) \geq 28 \). So that \( \lambda_i = 0 \) for \( i = 2, \ldots, k-2 \). Since \( 2 \deg 2A_1 - k(2k-1) = (k+2)(2k-1) \geq 42 \), we have \( \lambda_1 \leq 1 \). Suppose \( \lambda_{k-1} + \lambda_k \geq 1 \). Then \( k \leq 6 \) since \( d_1 \geq 2 \deg A_k - k(2k-1) = 2 \deg A_{k-1} - k(2k-1) = 2^{k-2} - k(2k-1) \geq 37 \) if \( k \geq 7 \).

We have that \( \lambda_1 + \lambda_{k-1} + \lambda_k \leq 1 \) since \( 2 \deg (A_1 + A_k) - k(2k-1) = 2 \deg (A_1 + A_{k-1}) - k(2k-1) = (2^{k-2} - k)(2k-1) \geq 84 \), \( 2 \deg (A_{k-1} + A_k) - k(2k-1) = k(2k-1)[4(2k-2)!/(k-1)!(k+1)!] - 1 \geq 84 \) and \( 2 \deg 2A_k - k(2k-1) = 2 \deg 2A_{k-1} - k(2k-1) = k(2k-1)(2(2k-2)!/(k!)^2 - 1) \geq 42 \). Q.E.D.

**Remark 2.13** \( A_4(k=5) \) and \( A_5(k=5) \) are conjugate. \( A_3(k=4) \) and \( A_4(k=4) \) are 'real', and there are outer automorphisms \( \tau_i (i=1 \)
2) of $D_4$ such that $\Lambda_3 \circ \tau_1$ and $\Lambda_4 \circ \tau_2$ are equivalent as complex representations of $D_4$ to $\Lambda_1$. There is also an outer automorphism $\tau_3$ (resp. $\tau_4$) of $D_6$ (resp. $D_5$) such that $\Lambda_5 \circ \tau_3$ (resp. $\Lambda_4 \circ \tau_4$) and $\Lambda_6$ (resp. $\Lambda_5$) are equivalent as complex representations of $D_6$ (resp. $D_5$).

(3) 'Quaternion' complex irreducible representations of $D_k (k \geq 4)$ are given by $\Lambda = \sum_{i=1}^{k} \lambda_i \Lambda_i$ where $\lambda_k - 1 + \lambda_k$ is odd, $k = 4h + 6$, and $h, \lambda_i (i=1, \ldots, k)$ are non-negative integers.

Proposition 2.14 If $d_2 = 2 \deg \Lambda - k (2k - 1) \leq 36$, then $\Lambda$ is equivalent as a complex representation of $D_k (k \geq 4)$ to one of the followings:

$$d_2 = -2: \quad \Lambda_5 (k=6), \quad \Lambda_6 (k=6).$$

Proof: The assertion follows from Proposition 2.12 and Remark 2.13. Q.E.D.

(E)

The simple roots of exceptional Lie algebras are given by

$$G_2: \quad \alpha_1 \longrightarrow \alpha_2$$

$$F_4: \quad \alpha_1 \longrightarrow \alpha_2 \longrightarrow \alpha_3 \longrightarrow \alpha_4$$

$$E_6: \quad \alpha_1 \longrightarrow \alpha_2 \longrightarrow \alpha_3 \longrightarrow \alpha_4 \longrightarrow \alpha_5 \longrightarrow \alpha_6$$

$$E_7: \quad \alpha_1 \longrightarrow \alpha_2 \longrightarrow \alpha_3 \longrightarrow \alpha_4 \longrightarrow \alpha_5 \longrightarrow \alpha_6 \longrightarrow \alpha_7$$

$$E_8: \quad \alpha_1 \longrightarrow \alpha_2 \longrightarrow \alpha_3 \longrightarrow \alpha_4 \longrightarrow \alpha_5 \longrightarrow \alpha_6 \longrightarrow \alpha_7 \longrightarrow \alpha_8$$
Proposition 2.15 Suppose \( \Lambda \) is a complex irreducible representation of an exceptional Lie algebra of dimension \( g \). If \( d_0 := \text{deg} \Lambda - g \leq 12 \), then \( \Lambda \) is equivalent as a complex representation to one of the followings:

\[
\begin{align*}
\text{d}_0 < 0: & \quad \Lambda_2(G_2), \Lambda_4(F_4), \Lambda_1(E_6), \Lambda_5(E_6), \Lambda_6(E_7), \\
\text{d}_0 = 0: & \quad \Lambda_1(G_2), \Lambda_1(F_4), \Lambda_6(E_6), \Lambda_1(E_7), \Lambda_7(E_8).
\end{align*}
\]

Proof: Case \( G_2 \) If \( \Lambda \) is not equivalent to \( \Lambda_1 \) nor \( \Lambda_2 \), then \( d_0 \geq 13 \) since \( \text{deg} 2 \Lambda_1 = 77, \text{deg} 2 \Lambda_2 = 27 \) and \( \text{deg}(\Lambda_1 + \Lambda_2) = 64 \). Case \( F_4 \) If \( \Lambda \) is not equivalent to \( \Lambda_1 \) nor \( \Lambda_2 \), then \( d_0 \geq 221 \) since \( \text{deg} 2 \Lambda_1 = \text{deg} (\Lambda_1 + \Lambda_4) = 1053, \text{deg} 2 \Lambda_4 = 324, \text{deg} \Lambda_2 = 1274 \) and \( \text{deg} \Lambda_3 = 273 \). Case \( E_6 \) If \( \Lambda \) is not equivalent to \( \Lambda_1, \Lambda_5 \), nor \( \Lambda_6 \), then \( d_0 \geq 273 \) since \( \text{deg} 2 \Lambda_1 = \text{deg} 2 \Lambda_5 = \text{deg} \Lambda_2 = \text{deg} \Lambda_4 = 351, \text{deg} \Lambda_3 = 2925, \text{deg} 2 \Lambda_6 = 2430, \text{deg}(\Lambda_1 + \Lambda_5) = 650 \) and \( \text{deg}(\Lambda_1 + \Lambda_6) = \text{deg}(\Lambda_5 + \Lambda_6) = 1728 \). Case \( E_7 \) If \( \Lambda \) is not equivalent to \( \Lambda_1 \) nor \( \Lambda_6 \), then \( d_0 \geq 779 \) since \( \text{deg} \Lambda_2 = 8645, \text{deg} \Lambda_3 = 365750, \text{deg} \Lambda_4 = 27664, \text{deg} \Lambda_5 = 1539, \text{deg} \Lambda_7 = 912, \text{deg} 2 \Lambda_1 = 7371, \text{deg} 2 \Lambda_6 = 1463 \) and \( \text{deg}(\Lambda_1 + \Lambda_6) = 3920 \). Case \( E_8 \) If \( \Lambda \) is not equivalent to \( \Lambda_7 \), then \( d_0 \geq 3627 \) since \( \text{deg} \Lambda_1 = 3825, \text{deg} \Lambda_2 = 6696000, \text{deg} \Lambda_3 = 6899079264, \text{deg} \Lambda_4 = 146325270, \text{deg} \Lambda_5 = 2450240, \text{deg} \Lambda_6 = 30380, \text{deg} \Lambda_7 = 147250, \) and \( \text{deg} 2 \Lambda_7 = 27000 \). Q.E.D.

Remark 2.16 \( \Lambda_2(G_2) \) is 'real' of degree 7. \( \Lambda_4(F_4) \) is 'real' of degree 26. \( \Lambda_1(E_6) \) and \( \Lambda_5(E_6) \) are conjugate from each other and of degree 27. \( \Lambda_6(E_7) \) is 'quaternion' of degree 56. \( \Lambda_1(G_2), \Lambda_1(F_4), \Lambda_6(E_6), \Lambda_1(E_7) \) and \( \Lambda_7(E_8) \) are the adjoint representations, especially 'real', of degree 14,52,78,144,248 respectively. Any \( \Lambda \) of \( d_1 \) or \( d_2 \leq 12 \) is contained in the above list since \( d_1 - d_2 > d_0 \).
Next propositions are also useful in section 3 and 4.

**Proposition 2.17** Each non trivial 'real' complex irreducible representation of degree at most 3 of a compact simple Lie algebra is equivalent as a complex representation to one of the followings:

- degree 3: $2\Lambda_1(A_1)$.

**Proof:** The assertion follows from Prop.2.1, 2.5, 2.8, 2.11 and 2.15 since $d_0$ is less than the degree which is at most 3. Q.E.D.

**Proposition 2.18** Each non trivial complex irreducible representation of degree at most 3 of a compact simple Lie algebra is equivalent as a complex representation to one of the followings:

- degree 2: $\Lambda_1(A_1)$,
- degree 3: $2\Lambda_1(A_1), \Lambda_1(A_2), \Lambda_2(A_2)$.

**Proof:** The assertion follows from Prop.'s 2.2, 2.6, 2.9, 2.12 and 2.15 since $d_1 = 2$ degree - $g \leq 2 \cdot 3 - 3 = 3$. Q.E.D.

**Remark 2.19** $\Lambda_2(A_2)$ is conjugate to $\Lambda_1(A_2)$.

**Proposition 2.20** Each non trivial 'quaternion' complex irreducible representation of degree at most 6 of a compact simple Lie algebra is equivalent as a complex representation to one of the followings:

- degree 2: $\Lambda_1(A_1)$,
- degree 4: $3\Lambda_1(A_1), \Lambda_1(C_2)$,
- degree 6: $5\Lambda_1(A_1), \Lambda_1(C_3)$.

**Proof:** The assertion is trivial in the case of $A_1$. 

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Otherwise, it follows from Prop.'s 2.4, 2.7, 2.10, 2.14 and 2.15 since $d_2 = 2 \text{degree} - g \leq 2 \cdot 6 - 8 = 4$. Q.E.D.
3. Basic classification
by cohomogeneity

Let \((G, M)\) be a Lie transformation group. For \(x\) in \(M\), we
denote \(G(x)\) the orbit of \(G\) through \(x\), and \(G_x\) the isotropy
subgroup of \(G\) at \(x\).

**Lemma 3.1** Let \((G, M), (G, N)\) be Lie transformation groups
and \(f\) be a \(G\)-equivariant submersion from \(M\) onto \(N\) with
the property:

\[ f^{-1}(f(x)) = G_f(x)(x) \]

at a fixed \(x\) in \(M\). Then we have that

\[ \dim M - \dim G + \dim G_x = \dim N - \dim G + \dim G_f(x). \]

**Proof:** \(\dim M = \dim N + \dim f^{-1}(f(x)) = \dim N + \dim G_f(x)(x) = \)
\(\dim N + \dim G_f(x) - \dim G_x\) since \((G_f(x))(x) = G_x\). Q.E.D.

Let \(R\) and \(H\) be the set of real numbers, complex numbers
and quaternions respectively. Naturally \(H\) contains \(C\), and \(C\)
contains \(R\). The **conjugate** \(\overline{u+jv}\) of \(u+jv\) in \(H\) is defined by

\[ \overline{u+jv} = \overline{u} - jv \]

where \(\overline{u}\) is the complex conjugate of \(u\), \(u\) and \(v\) are in \(C\). For
\(u+jv\), \(u'+jv'\) in \(H\), the **product** \((u+jv)(u'+jv')\) of them are
defined by

\[ (u+jv)(u'+jv') = (uu'-\overline{v}v') + j(vu'+\overline{u}v'). \]

Let \(F\) be \(R\), \(C\), or \(H\). The set of all \((n_1, n_2)\)-matrixes
with coefficients \(F\) is denoted by \(F(n_1, n_2)\). For \(X\) in \(F(n_1, n_2)\),
we denote the **conjugate** of \(X\) with respect to the coefficients
by \(X\), and the **transposed** matrix of \(X\) by \(X^t\). We write \(F^{n} = F(n, 1)\)
, \(F(n) = F(n, n)\), and denote the identity matrix of \(F(n)\) by \(I_n\).
We denote \( hF(n) = \{ X \in F(n); \; tX = X \} \), \( pF(n) = \{ X \in hF(n); \; X \text{ is positive definite} \} \), and use the following notations for classical groups:

\[
GF(n) = \{ X \in F(n); \; tXX = X^t = I_n \}.
\]

If \( F = \mathbb{R} \) or \( \mathbb{C} \), denote

\[
SF(n) = \{ X \in GF(n); \; \det X = 1 \}.
\]

Then \( GR(n) = 0(n), \; GC(n) = U(n), \; GH(n) = Sp(n), \; SR(n) = SO(n) \) and \( SC(n) = SU(n) \) in usual notations. Any subgroup of \( GF(n) \) acts on \( F^n \) linearly over right multiplications of \( F \) by usual manner and acts on \( hF(n) \) (resp. \( pF(n) \)) by

\[
A \cdot X = AX^tA \quad (3.1)
\]

for \( A \) in \( GF(n) \), \( X \) in \( hF(n) \) (resp. \( pF(n) \)). Each matrix of \( hF(n) \) can be transformed to a diagonal form by the action of \( GF(n) \) (resp. \( SF(n) \)). Similarly any subgroup of \( GF(n_1) \times GF(n_2) \) acts on \( F(n_1, n_2) \) by

\[
(A, B) \cdot X = AX^tB \quad (3.2)
\]

for \( (A, B) \) in \( GF(n_1) \times GF(n_2) \), \( X \) in \( F(n_1, n_2) \).

We use mappings \( k, k': H(n_1, n_2) \longrightarrow C(2n_1, 2n_2), \)

\[
h: H(n_1, n_2) \longrightarrow C(2n_1, 2n_2) \quad \text{and} \quad h': H(n_1, n_2) \longrightarrow C(n_1, 2n_2)
\]

such that

\[
k(U + jV) = \begin{bmatrix} U & -V \\ V & U \end{bmatrix}, \quad k'(U + jV) = \begin{bmatrix} U & V \\ -V & U \end{bmatrix}, \quad h(U + jV) = \begin{bmatrix} U \\ V \end{bmatrix},
\]

\[
h'(U + Vj) = (U, V) \quad \text{for} \; U, V \; \text{in} \; C(n_1, n_2).
\]

Then \( k, k' \) are real linear injections such that

\[
tk(P) = k(tP), \quad tk'(P) = k'(tP), \quad k(PQ) = k(P)k(Q), \quad k'(PQ) = k'(P)k'(Q) \quad \text{for} \; P \; \text{in} \; H(n_1, n_2), \; Q \; \text{in} \; H(n_2, n_3),
\]

and \( h(\text{resp.} \; h') \) is a linear bijection over right (resp. left) multiplications of \( C \) such that \( h(PQ) = k(P)h(Q) \) (resp. \( h'(PQ) = h'(P)k(Q) \)).
For $P$ in $H(n_1, n_2)$, we see that column-rank$_H(P):= n_2 - \dim_H\{Q \in H^{n_2}; PQ=0\} = (2n_2 - \dim_C\{Q \in H^{n_2}; PQ=0\})/2 = (\text{rank}_C k'(P))/2 = (2n_1 - \dim_C\{Q \in H(l, n_1); QP=0\})/2 = n_1 - \dim_H\{Q \in H(l, n_1); QP=0\}$: row-rank$_H(P)$. Note that the linear independence in $H^{n_2}$, $H(l, n_1)$ over right multiplications of $H$ is equivalent to one over left multiplications of $H$ respectively owing to $\overline{pq}= \overline{q} \cdot \overline{p}$ ($p, q$ in $H$). Therefore rank$_H(P):= \text{column-rank}_H(P)= \text{row-rank}_H(P)$ is well-defined. Denote $MF(n_1, n_2) = \{X \in F(n_1, n_2); \text{rank}_F(X) = \max(n_1, n_2)\}$. Then $k(MH(n_1, n_2)) = MC(2n_1, 2n_2) \cap k(H(n_1, n_2))$.

Assume $n_1 > n_2$. Denote $f: MF(n_1, n_2) \rightarrow pF(n_2)$ such that $f(X) = t^X X f$ for $X$ in $MF(n_1, n_2)$. Then $f$ is $GF(n_1) \times GF(n_2)$-equivariant with respect to the action (3.2) on $MF(n_1, n_2)$ and the following action on $pF(n_2)$:

\[(A, B) \cdot Y = BY^tB\]

for $(A, B)$ in $GF(n_1) \times GF(n_2)$, $Y$ in $pF(n_2)$.

Lemma 3.2 (1) $f$ is a submersion.

(2) $f^{-1}(f(X)) = (GF(n_1) \times \{I_{n_2}\}) \cdot X$ for $X$ in $MF(n_1, n_2)$.

(3) If $n_1 > n_2$, then $f^{-1}(f(X)) = (SF(n_1) \times \{I_{n_2}\}) \cdot X$ for $X$ in $MF(n_1, n_2)$ where $F= R$ or $C$.

Proof: (1) Since any diagonal matrix in $pF(n_2)$ is in the image of $f$, it follows that $f$ is onto from the diagonalizability by the action (3.3). To prove $df_{X_0}: F(n_1, n_2) \rightarrow hF(n_2); X \mapsto t^X X_0 + t^X_0 X$ is onto at $X_0$ in $MF(n_1, n_2)$, if we use the action (3.2) of $GF(n_1) \times GF(n_2)$, we may assume that $X_0$ has the following form for some non-zero $x_i$ in $R$ ($i=1, \ldots, n_2$):

\[
X_0 = \begin{pmatrix}
x_1 \\
\vdots \\
x_{n_2}
\end{pmatrix}
\]
In fact, the action (3.3) of \( \{I_{n_1}\} \times GF(n_2) \) transforms \( t_{-x_0} x_0 \) to a diagonal form and the action (3.2) of \( GF(n_1) \times \{I_{n_2}\} \) gives a required form. Then it is easy to show that \( df_{x_0} \) is onto.

(2) Suppose \( f(X)=f(Y) \). Denote \( X=[x_1, \ldots, x_{n_2}] \), \( Y=[y_1, \ldots, y_{n_2}] \) where \( x_i, y_i \) in \( F^{n_1} \), then \( t_{-x_i} x_j = t_{-y_i} y_j \) (\( i, j=1, \ldots, n_2 \)). We can choose \( x_h, y_k \) (\( h, k=n_2+1, \ldots, n_1 \)) such that \( t_{x_i} x_h = t_{-y_i} y_h = 0 \) and \( t_{x_i} x_k = t_{-y_i} y_k = \delta_{hk} \). Then \( X'=[x_1, \ldots, x_{n_1}] \), \( Y'=[y_1, \ldots, y_{n_1}] \) have the inverse matrices. For \( A = Y'X'^{-1} \), \( A \) is in \( GF(n_1) \) since \( t_{x_i} X' = t_{-y_i} Y' \). We have \( (A, I_{n_2}) \cdot X = Y \).

(3) If \( F=R \) or \( C \), then \( X''=x'' \cdot \text{diag}[1, \ldots, 1, \det X'^{-1}] \) and \( Y''=y'' \cdot \text{diag}[1, \ldots, 1, \det Y'^{-1}] \) are in \( SL(n_1, F) \). Then \( B=Y''X'^{-1} \) is in \( SF(n_1) \) and \( (B, I_{n_2}) \cdot X = Y \) if \( n_1 > n_2 \). Q.E.D.

The tensor product \( F^{n_1} \alpha \ldots \alpha F^{n_s} \) over \( F \) of \( F^{n_1}, \ldots, F^{n_s} \) is defined if \( F=R \) or \( C \). Naturally \( R^{n_1} \alpha \ldots \alpha R^{n_s} = \{ z \in C^{n_1} \alpha \ldots \alpha C^{n_s} ; R \alpha R \times \alpha R \} \) where \( - \) denotes the complex conjugation extended naturally on \( C^{n_1} \alpha \ldots \alpha C^{n_s} \). If \( F=H \), then we consider the real linear map.

\[
\mathcal{J}: C^{2n_1} \alpha \ldots \alpha C^{2n_s} \longrightarrow C^{2n_1} \alpha \ldots \alpha C^{2n_s} ; \quad \mathcal{J}_i z_i (h(P_{il}) \alpha \ldots \alpha h(P_{is})) \mapsto \mathcal{J}_i z_i (h(P_{il}) \alpha \ldots \alpha h(P_{is})), \quad \text{where } z_i \text{ in } C \text{ and } P_{it} \text{ in } H^{n_t} \text{ (} t=1, \ldots, s \text{). Then } \mathcal{J}^2 = \text{id} \text{ (if } s \text{ is even), or } -\text{id} \text{ (if } s \text{ is odd).}
\]

The tensor product \( H^{n_1} \alpha \ldots \alpha H^{n_s} \) over \( H \) of \( H^{n_1}, \ldots, H^{n_s} \) is defined by \( H^{n_1} \alpha \ldots \alpha H^{n_s} = \{ z \in C^{2n_1} \alpha \ldots \alpha C^{2n_s} ; \quad \mathcal{J} z = z \} \) (if \( s \) is even), or \( C^{2n_1} \alpha \ldots \alpha C^{2n_s} \) with the quaternion structure \( \mathcal{J} \) (if \( s \) is odd).

If \( s=1 \), then \( \mathcal{J} \) is the standard quaternion structure on \( C^{2n_1} = h(H^{n_1}) \).

If \( s=2 \), then \( H^{n_1} \alpha H^{n_2} \) is a real form of \( C^{2n_1} \alpha C^{2n_2} \) with respect to the real structure \( \mathcal{J} \) on \( C^{2n_1} \alpha C^{2n_2} \). For an even \( s \),

\[ H^{n_1} \alpha \ldots \alpha H^{n_s} \text{ is equivalent as real spaces to } H \times \ldots \times H \]
Let $\rho_1, \ldots, \rho_s$ be linear representations of Lie groups $G_1, \ldots, G_s$ on $F^{n_1}, \ldots, F^{n_s}$ over $F$ respectively. If $F = \mathbb{R}$ or $\mathbb{C}$, then the exterior tensor product $\rho_1 \otimes \cdots \otimes \rho_s$ over $F$ is defined as the representation of the direct product group $G_1 \times \cdots \times G_s$ on the tensor product space $F^{n_1} \otimes \cdots \otimes F^{n_s}$ over $F$ such that

$$(\rho_1 \otimes \cdots \otimes \rho_s)(g_1, \ldots, g_s) := \rho_1(g_1) \otimes \cdots \otimes \rho_s(g_s)$$

for $(g_1, \ldots, g_s)$ in $G_1 \times \cdots \times G_s$, where the right hand side is the usual tensor product of linear transformations. If $F = \mathbb{R}$, then note that $\mathfrak{g}$ commutes with the representation $(k \circ \rho_1) \otimes \cdots \otimes (k \circ \rho_s)$ of $G_1 \times \cdots \times G_s$ on $\mathfrak{g}^{n_1} \otimes \cdots \otimes \mathfrak{g}^{n_s}$. The exterior tensor product $\rho_1 \otimes \cdots \otimes \rho_s$ over right $\mathbb{H}$ is defined as the representation of $G_1 \times \cdots \times G_s$ on $\mathbb{H}^{n_1} \otimes \cdots \otimes \mathbb{H}^{n_s}$ such that

$$(\rho_1 \otimes \cdots \otimes \rho_s)(g_1, \ldots, g_s) := ((k \circ \rho_1) \otimes \cdots \otimes (k \circ \rho_s))(g_1, \ldots, g_s)$$

for $s$ is even, then it is equivalent as a real representation of $G_1 \times \cdots \times G_s$ to $(\rho_1 \otimes \rho_2) \otimes \cdots \otimes (\rho_s \otimes \rho_s)$. Next, we study the case of $s = 2$ in more detail. The identity representation of a Lie subgroup $K$ of $GF(n)$ is denoted by $id$. We consider the action (3.1) of $K$ on $pF(n)$. 

**Proposition 3.3** If $K$ is a Lie subgroup of $GF(n_2)$ and $n_1 > n_2$, then (1) $\text{coh}(GF(n_1) \times K, id \otimes id, F^{n_1} \otimes F^{n_2}) = \text{coh}(K, pF(n_2))$, (2) $\text{coh}(SO(n_1) \times K, id \otimes id, R^{n_1} \otimes R^{n_2}) = \text{coh}(K, pR(n_2))$, (3) If $n_1 > n_2$, then $\text{coh}(SU(n_1) \times K, id \otimes id, C^{n_1} \otimes C^{n_2}) = \text{coh}(K, pC(n_2))$, (4) $\text{coh}(K, pF(n_2)) \geq \text{coh}(GF(n_2), pF(n_2)) = n_2$ (= $\text{coh}(SF(n_2), pF(n_2))$ if $F = \mathbb{R}$ or $\mathbb{C}$). 

3-5
Proof: If $F = \mathbb{R}$ or $\mathbb{C}$, the representation space $F^{n_1} \otimes F^{n_2}$ is identified with $F(n_1, n_2)$ by the correspondence $\iota: F^{n_1} \otimes F^{n_2} \rightarrow F(n_1, n_2)$ such that $\iota(e_i \otimes e_j) = E_{ij}$ (i = 1, ..., $n_1$; j = 1, ..., $n_2$) with respect to the standard bases $\{e_i\}$, $\{e_j\}$, $\{E_{ij}\}$ of $F^{n_1}$, $F^{n_2}$, $F(n_1, n_2)$ respectively. Through $\iota$, the action of $GF(n_1) \times K$ on $F(n_1, n_2)$ is induced as

$$(A, B) \cdot X = AX^tB$$

for $X$ in $F(n_1, n_2)$, $(A, B)$ in $GF(n_1) \times K$. The o.t.g. induced from this action is equivalent to one from the similar action of $GF(n_1) \times K$ where $K = \{B; B$ is in $K\}$ is the conjugation of $K$ in $GF(n_2)$. Hence the o.t.g. induced from $\iota$ is equivalent to one from the action (3.2) of $GF(n_1) \times K$. When $F = \mathbb{H}$, we consider $\iota: C^{2n_1} \otimes C^{2n_2} \rightarrow C(2n_1, 2n_2)$ for the standard basis $e_1 = h(e_1')$, ..., $e_{n_1} = h(e_{n_1}')$, $e_{n_1+1} = h(e_1'j)$, ..., $e_{2n_1} = h(e_{n_1}'j)$ of $C^{2n_1}$ where $e_1'$, ..., $e_{n_1}'$ is the standard basis of $\mathbb{H}^n$ (i = 1, 2). Then we have

$$(\iota(H^{n_1} \otimes H^{n_2})) = k(H(n_1, n_2))$$

since $JZ = J_1Z_1$ ($Z_1$ in $C^{2n_1}$), $\iota(Z) = J_1Z_1^tJ_2$ ($Z$ in $C^{2n_1} \otimes C^{2n_2}$) and $k(H(n_1, n_2)) = \{X$ in $C(2n_1, 2n_2); J_1X^tJ_2 = X\}$ where

$$J_i = \begin{pmatrix} 0_{n_i} & -I_{n_i} \\ I_{n_i} & 0_{n_i} \end{pmatrix} \quad (i = 1, 2).$$

Through $\iota$, the action of $Sp(n_1) \times K$ on
\( k(H(n_1,n_2)) \) is induced from the representation \( \text{id} \otimes \text{id} \) on \( H \times H \) by \((A,B) \cdot k(X) = k(A)k(X)t^k(B)\) for \( X \) in \( H(n_1,n_2) \), \((A,B)\) in \( \text{Sp}(n_1) \times K \). The o.t.g. induced from this action is equivalent to the one which is induced from the action (3.2) of \( \text{Sp}(n_1) \times K \) on \( H(n_1,n_2) \), since \( t^k(B) = k(t^B) \) and \( k(A)k(X)k(t^B) = k(A^tB) \).

Then (1) follows from Lemma 3.1 and Lemma 3.2(0),(1),(2), since \( MF(n_1,n_2) \) is open and dense in \( F(n_1,n_2) \). (2) follows from (1) since \( GR(n_1)^\circ = SO(n_1) \). (3) follows from Lemma 3.1 and Lemma 3.2(0),(1),(3). (4) follows from that \( GF(n_2) \) (resp. \( SF(n_2) \) if \( F = R \) or \( C \)) transforms any matrix in \( pF(n_2) \) to a diagonal form.

Q.E.D.

Denote \( r(n_1,n_2,n_3) = \text{co}(SO(n_1) \times \text{SO}(n_2) \times \text{SO}(n_3), \text{id} \otimes \text{id} \otimes \text{id}, R^n_1 \otimes R^n_2 \otimes R^n_3) \), \( c(n_1,n_2,n_3) = \text{co}(U(n_1) \times \text{SU}(n_2) \times \text{SU}(n_3), \text{id} \otimes \text{id} \otimes \text{id}, C^{n_1} \otimes C^{n_2} \otimes C^{n_3}) \), \( q(n_1,n_2,n_3) = \text{co}((\text{Sp}(n_1) \times \text{Sp}(n_2)) \times \text{SO}(n_3), (\text{id} \otimes \text{id}) \otimes \text{id}, H \times R \).

Proposition 3.4

(1) \( r(n_1,n_2,n_3) \geq 18 \) if \( n_1 \geq n_2 \geq n_3 \geq 3 \).

(2) \( c(n_1,n_2,n_3) \geq 6 \) if \( n_1 \geq n_2 \geq n_3 \geq 2 \).

(3) \( q(n_1,n_2,n_3) \geq 3 \) if \( n_3 \geq 3 \), \( n_1 \geq n_2 \geq 1 \).

(4) \( q(n_1,n_2,n_3) \geq 8 \) if \( n_3 \geq 3 \), \( n_1 \geq 2 \), \( n_2 \geq 2 \).

Proof: Denote \( \lambda(n_1,n_2,n_3) = \dim \text{pr}(n_2,n_3) - \dim \text{SO}(n_2) \times \text{SO}(n_3) \) if \( n_1 \geq n_2 \geq n_3 \) or \( \dim R^{n_1} \otimes R^{n_2} \otimes R^{n_3} - \dim \text{SO}(n_1) \times \text{SO}(n_2) \times \text{SO}(n_3) \) (otherwise), \( \kappa(n_1,n_2,n_3) = \dim \text{pr}(n_2,n_3) - \dim \text{SU}(n_2) \times \text{SU}(n_3) \) if \( n_1 \geq n_2 \geq n_3 \) or \( \dim C^{n_1} \otimes C^{n_2} \otimes C^{n_3} - \dim \text{SU}(n_1) \times \text{SU}(n_2) \times \text{SU}(n_3) \) (otherwise), and \( \mu(n_1,n_2,n_3) = \dim \text{pr}(4n_1,n_2) - \dim \text{Sp}(n_1) \times \text{Sp}(n_2) \) if \( n_3 \geq 4n_1 n_2 \), \( \dim \text{pr}(n_2,n_3) - \dim \text{Sp}(n_2) \times \text{SO}(n_3) \) if \( n_3 \geq 4n_1 n_2, n_2 n_3 \leq n_1 \) or...
dim(H_{n_1} \otimes H_{n_2}) \otimes \mathbb{R}^{n_3} - \dim \text{Sp}(n_1) \times \text{Sp}(n_2) \times \text{SO}(n_3) \text{ (otherwise).} \]

Then \( \lambda(n_1, n_2, n_3) \leq r(n_1, n_2, n_3), \ k(n_1, n_2, n_3) \leq c(n_1, n_2, n_3) \) and \( \mu(n_1, n_2, n_3) \leq q(n_1, n_2, n_3) \) by Prop. 3.3 since \( (H_{n_1} \otimes H_{n_2}) \otimes \mathbb{R}^{n_3} \) is equivalent to \( H_{n_1} \otimes (H_{n_2} \otimes \mathbb{R}^{n_3}) \) as \( \text{Sp}(n_1) \times \text{Sp}(n_2) \times \text{SO}(n_3) \)-spaces over \( \mathbb{R} \). Since \( \lambda(x_1, x_2, x_3) = (x_2^2 x_3^2 + x_2 x_3 - x_2^2 x_3 - x_2^2 + x_2 + x_3)/2 \) (if \( x_1 \geq x_2 x_3 \)) or \( x_1 x_2 x_3 + (x_1 + x_2 + x_3 - x_1^2 - x_2^2 - x_3^2)/2 \) (otherwise), \( k(x_1, x_2, x_3) = x_2 x_3^2 - x_2^2 x_3 + x_2^2 - (x_2^2 - x_2 - x_3^2)/2 \) (otherwise), and \( \mu(x_1, x_2, x_3) = 8x_1^2 x_2^2 + 2x_1 x_2 - 2x_1^2 - 2x_2^2 - x_1 - x_2 \) (if \( x_3 = 4x_1 x_2 \)), \( 2x_2^2 x_3^2 - x_2^2 x_3^2 - x_2^2 x_3^2 + 2x_3^2 x_2^2 + 2x_3^2 \) (if \( x_3 \leq 4x_1 x_2, x_3 x_2 \leq x_1 \)) or \( 4x_1 x_2 x_3 - x_1 (2x_1 + 1) - x_2 (2x_2 + 1) - x_3^2/2 + x_3/2 \) (otherwise), they define continuous piecewise polynomial functions on \( \mathbb{R}^3 \) if we take \( x_i (i=1, 2, 3) \) as real numbers.

(1) Since \( \partial \lambda/\partial x_1(x_1, x_2, x_3) \geq 0 \) for \( x_1 \geq x_2 \geq x_3 \geq 1 \) \( (i=1, 2, 3) \), we have \( \lambda(n_1, n_2, n_3) \geq \lambda(n_1, n_2, 3) \geq \lambda(3, 3, 3) = 18 \). (2) Similar to (1), \( k(n_1, n_2, n_3) \geq k(2, 2, 2) = 6 \). (3) Since \( \partial \mu/\partial x_1(x_1, x_2, x_3) \geq 0 \) for \( i=1, 2, 3 \); \( x_1, x_2, x_3 \geq 1 \) (if \( x_3 \geq 4x_1 x_2 \) or \( x_3 x_2 \leq x_1 \)), and \( \partial \mu/\partial x_3(x_1, x_2, x_3) = 4(x_1 x_3 - x_2) - 1 \leq 4x_1(x_3 - 1) \) \( -1 \leq 3 \), \( \partial \mu/\partial x_1(x_1, x_2, x_3) = 4(x_2 x_3 - x_1) - 1 \leq 1 \) for \( x_1 \geq x_2 \geq 1, x_3 \geq 2 \) (if \( x_3 \leq 4x_1 x_2 \) and \( x_3 x_2 \geq x_1 \)), we have \( \mu(n_1, n_2, 3) \geq \mu(n_1, n_2, n_3) \geq \mu(n_1, 1, 3) = \mu(n_1, -1, 1, 3) + \partial \mu/\partial x_1(n_1 - \theta, 1, 3) \) (\( 0 < \theta < 1 \)) \( \geq \mu(n_1 - 1, 1, 3) \) (since \( \mu(n_1, 1, 3) \) and \( \mu(n_1 - 1, 1, 3) \) are integers, and \( -1 < \partial \mu/\partial x_1 \) is also an integer, especially \( \partial \mu/\partial x_1 \geq 0 \)) \( \geq \mu(1, 1, 3) = 3 \). (4) Similar to (3), \( \mu(n_1, n_2, n_3) \geq \mu(n_1, 1, 3) \geq \mu(2, 1, 3) = 8 \). Q.E.D.
Let L be the Lie algebra of a connected Lie group G. We write the same letter for a linear representation of L and the corresponding representation of G. According to Iwahori[12], there is the following relation between real irreducible representations of L(resp. G) and complex irreducible representations of L(resp. G) (cf. Goto-Grosshans[6]). For a complex irreducible representation \( \rho \) on a complex vector space \( V \), we denote the real restriction of \( \rho \) on the real restricted vector space \( V_R \) (abbrev. V since \( V=V_R \) as a set) by \( \rho_R \) (abbrev. \( \rho \)), which is not real irreducible if and only if \( \rho \) is 'real', and so we attach to \( \rho \) a real irreducible representation \( \rho^R \) as follows. \( \rho^R = \sigma \) (if \( \rho \) is the complexification \( \sigma^C \) of a real representation \( \sigma \) on a real form \( W \) of \( V \), i.e., \( \rho \) is 'real'.) or \( \rho_R \) (otherwise). Note that \( \rho_1^R \) and \( \rho_2^R \) are equivalent as real representations if and only if \( \rho_1 \) and \( \rho_2 \) are conjugate or equivalent as complex representations of L(resp. G). Conversely the complexification \( \sigma^C \) on \( W^C \) of a real irreducible representation \( \sigma \) on a real vector space \( W \) is not complex irreducible if and only if \( W \) has a L(resp. G)-invariant complex structure (then it is unique), and so we attach to \( \sigma \) a complex irreducible representation \( \sigma^C \) as follows. \( \sigma^C = \sigma \) (if \( W \) has a L(resp. G)-invariant complex structure) or \( \sigma^C \) (otherwise). Note that \( \rho^{RC} \) and \( \rho \) (resp. \( \sigma^{CR} \) and \( \sigma \)) are equivalent as complex(resp. real) representations.
Let \((G,E^N)\) be an o.t.g. Then the Lie algebra \(L\) of \(G\) is a real reductive Lie algebra and has a form:

\[ L = L_0 \oplus L_1 \oplus \ldots \oplus L_s \]

(3.4)

where \(L_0\) is the center of \(L\), and \(L_i\) \((i=1,\ldots,s)\) are simple ideals of \(L\). Let \(G_0\), \(G_i\) be connected Lie subgroups of \(G\) corresponding to \(L_0\), \(L_i\) respectively and \(\hat{G}_0\), \(\hat{G}_i\) be the universal covering groups of \(G_0\), \(G_i\) respectively, then \(\hat{G}_i\) are compact \((i=1,\ldots,s)\). Let \(id:G\to SO(N)\) be the identity representation and \(i\hat{\cdot}\) be the corresponding representation of \(\hat{G}:=\hat{G}_0 \times \hat{G}_1 \times \ldots \times \hat{G}_s\).

In this paper, we consider \((G,E^N)\) in case that \(id\) is a real irreducible representation of \(G\). Then \(G\) is compact (cf. Kobayashi-Nomizu[14]), and so \(G_0=U(1)\) or the trivial group 1. For \(t\) in \(R^X:=R-\{0\}\), we denote \(\hat{t}:R\to U(1)\) the complex irreducible representation of \(R\) such that \(\hat{t}(x)=e^{2\pi x t i}\) for \(x\) in \(R\). We shall decompose \(i\hat{\alpha}^C\) into an exterior tensor product of complex irreducible representations of \(\hat{G}_i\) \((i=0,\ldots,s)\).

**Case i)** \(i\hat{\alpha}^C=i\hat{\alpha}^C\): Then \(G_0\) is trivial, and \((\hat{G},i\hat{\alpha}^C,C^N)\) is equivalent as complex representations to some

\[
(\hat{G}_1 \times \ldots \times \hat{G}_s, \rho_1 \otimes \ldots \otimes \rho_s, \rho_{N^1} \otimes \ldots \otimes \rho_{N^s})
\]

where \(\rho_i\) is a self-conjugate complex irreducible representation of \(\hat{G}_i\) on \(C^{n_i}\), \(n_i \geq 2\) \((i=1,\ldots,s)\), \(\Pi_i=1 n_i=N\), and \(#\{i; \rho_i\ \text{'quaternion'}\}\) is even. We may assume \(\rho_j\) \((j=1,\ldots,2r)\) are 'quaternion' and \(\rho_k\) \((k=2r+1,\ldots,2r+q; s=2r+q)\) are 'real', and \(\sigma_i\) denotes a real representation of \(\hat{G}_i\) on \(R^{n_i}\) whose
complexification is $\rho_{2r+1}(i=1,\ldots,q)$; where $r$ and $q$ are non-negative integers. Then $n_{2r+1} \geq 3(i=1,\ldots,q)$, and

$$(G, i\hat{\gamma}, R^N)$$
is equivalent as real representation to

$$(G_1 \times \ldots \times G_{2r} \times G_{2r+1} \times \ldots \times G_{2r+q}, (\rho_{2r-1} \rho_2) \ldots (\rho_{2r-1} \rho_2) \hat{\sigma}_{R} \hat{\sigma}_{R} \hat{\sigma}_{R} \hat{\sigma}_{R} (H_{n_{2r-1}/2} H_{n_{2r} n_{2r+1}} \ldots H_{n_{2r+q}}) (R_{n_{2r+1}} \ldots R_{n_{2r}}))$$

(3.5)

Case ii) $id^C = i\hat{\gamma}$, $G_0 = U(1)$: Then $(G, i\hat{\gamma}, C^{N/2})$ is equivalent as complex representations to some

$$(RxG_1 \times \ldots \times G_{2r} \times G_{2r+1} \times \ldots \times G_{2r+q}, \tau \hat{\rho}_{1} \hat{\rho}_{2} \ldots \hat{\rho}_{s} \hat{\rho}_{S}, C^{n_1} \ldots C^{n_s})$$

where $t$ is in $R$, $\rho_i$ is a complex irreducible representation of $G_i$ on $C^{n_i}$, $n_i \geq 2(i=1,\ldots,s)$ and $\Pi_{s=1}^{S} n_i = N/2$. So $(G, i\hat{\gamma}, R^N)$ is equivalent as real representation to

$$(RxG_1 \times \ldots \times G_{2r} \times G_{2r+1} \times \ldots \times G_{2r+q}, (\tau \hat{\rho}_{1} \hat{\rho}_{2} \ldots \hat{\rho}_{s} \hat{\rho}_{S})_R, (C^{n_1} \ldots C^{n_s})_R)$$

(3.6)

Case iii) $id^C = i\hat{\gamma}$, $G_0 = 1$: Then $(G, i\hat{\gamma}, C^{N/2})$ is equivalent as complex representations to some

$$(G_1 \times \ldots \times G_{2r} \times G_{2r+1} \times \ldots \times G_{2r+q}, \rho_{1} \hat{\rho}_{2} \ldots \hat{\rho}_{s} \hat{\rho}_{S}, C^{n_1} \ldots C^{n_s})$$

where $\rho_i$ is a complex irreducible representation of $G_i$ on $C^{n_i}$, $n_i \geq 2 (i=1,\ldots,s)$ and $\Pi_{s=1}^{S} n_i = N/2$. So $(G, i\hat{\gamma}, R^N)$ is equivalent as real representation to

$$(G_1 \times \ldots \times G_{2r} \times G_{2r+1} \times \ldots \times G_{2r+q}, (\rho_{1} \hat{\rho}_{2} \ldots \hat{\rho}_{s} \hat{\rho}_{S})_R, (C^{n_1} \ldots C^{n_s})_R)$$

(3.7)

where $\rho_{1} \hat{\rho}_{2} \ldots \hat{\rho}_{s} \hat{\rho}_{S}$ is not 'real' since $(\rho_{1} \hat{\rho}_{2} \ldots \hat{\rho}_{s} \hat{\rho}_{S})_R$ is real irreducible.
Theorem 3.5 Let \((G,E_N^N)\) be an o.t.g. of cohomogeneity at most 3. If \(\text{id}: G \rightarrow SO(N)\) is real irreducible and \(s \geq 3\) (cf. (3.4)), then \((G,\text{id}, R^N)\) is equivalent as real representation to
\[
(\mathfrak{A}_1 \times \mathfrak{A}_1 \times \mathfrak{A}_1)_{\mathbb{H}} R_{\mathbb{R}}^N, (\Lambda_1 \otimes \Lambda_1)_{\mathbb{H}} (2\Lambda_1)_{\mathbb{R}}, (\mathfrak{H} \mathfrak{m} \mathfrak{H}) \mathfrak{m} R^3_{\mathbb{R}})
\] (3.8)
Especially \(\text{coh}(G,E_N^N) = 3\).

Proof: Suppose \(\text{id}\) is real irreducible and \(s \geq 3\). Then \(O(G,\text{id}, R^N)\) is contained in \((1)\) \(O(Sp(n_1/2) \times Sp(n_2/2)) \times SO(n_3), (\text{id} \otimes \text{id}) \otimes \text{id}, (H^{n_1/2} \mathfrak{m} H^{n_2/2} \mathfrak{m} R^{n_3})\) for some \(n_1, n_2 \geq 2, n_3 \geq 3; N = n_1 n_2 n_3\), \((2)\) \(O(SO(n_1) \times SO(n_2) \times SO(n_3), (\text{id} \otimes \text{id} \otimes \text{id}), R^1 \mathfrak{m} R^{n_1 \times R^{n_2 \times R^{n_3}}}\) for some \(n_1, n_2, n_3 \geq 3; N = n_1 n_2 n_3\), or \((3)\) \(O(U(n_1) \times SU(n_2) \times SU(n_3), (\text{id} \otimes \text{id} \otimes \text{id}), (C^n \mathfrak{m} C^n \mathfrak{m} C^n)_{\mathbb{R}}\) for some \(n_1, n_2, n_3 \geq 2; N = 2n_1 n_2 n_3\) owing to (3.5), (3.6) and (3.7). On the other hand, \(\text{coh}(2) \geq 18, \text{coh}(3) \geq 6, \text{coh}((1)(\text{max}(n_1, n_2) \geq 4)) \geq 8\) by Prop.3.4(1)(2)(4). There \(G_0\) is trivial, and \(O(G,\text{id}, R^N)\) is contained in \(O((Sp(1) \times Sp(1)) \times SO(n_3), (\text{id} \otimes \text{id}) \otimes \text{id}, (H \mathfrak{m} H) \mathfrak{m} R^{n_3})\) which is equivalent to \(O(SO(4) \times SO(n_3), (\text{id} \otimes \text{id}), R^4 \mathfrak{m} R^{n_3})\).

Then \(n_3 = 3\) since \(\text{coh}(G,E_N^N) \leq 3\). So \(O(G,\text{id}, R^N)\) is contained in \(O(\mathfrak{A}_1 \times \mathfrak{A}_1 \times \mathfrak{A}_1)_{\mathbb{H}} R_{\mathbb{R}}^N, (\Lambda_1 \otimes \Lambda_1)_{\mathbb{H}} (2\Lambda_1)_{\mathbb{R}}, (\mathfrak{H} \mathfrak{m} \mathfrak{H}) \mathfrak{m} R^3_{\mathbb{R}}\). Since \(s \geq 3\), \(G\) is isomorphic to \(\mathfrak{A}_1 \times \mathfrak{A}_1 \times \mathfrak{A}_1\), and \(O(G,\text{id}, R^N) = O(\Lambda_1 \otimes \Lambda_1)_{\mathbb{H}} (2\Lambda_1)_{\mathbb{R}}, (\mathfrak{H} \mathfrak{m} \mathfrak{H}) \mathfrak{m} R^3_{\mathbb{R}}\).

Then \((G,\text{id}, R^N)\) and (3.8) are equivalent as real representation since \(\Lambda_1, 2\Lambda_1\) are characterized by degrees of complex irreducible representations of \(\mathfrak{A}_1\), and \(12 = 2^2 \cdot 3\) (cf. Section 2).
And \(\text{coh}(G,E_N^N) = 3\) by Prop.3.3. O.E.D.
Suppose $s=2$: $L=L_0\otimes L_1\otimes L_2$ (cf. (3.4)). Then $(\hat{\sigma},\text{id},R^N)$ is equivalent as real representation to one of the followings:

**Type I**) $(\hat{\sigma}_1 \times \hat{\sigma}_2, \rho_1 \otimes \rho_2, R^{N_1} \otimes R^{N_2})$; $n_1 \geq n_2 \geq 3$, $N=n_1 n_2$, $\rho_i$ is a 'real' complex irreducible representation of $\hat{\sigma}_i$ on $C^{n_i}, R^{n_i}$ is a $\hat{\sigma}_i$-invariant real form of $C^{n_i} (i=1,2)$.

**Type II**) $(\hat{\sigma}_1 \times \hat{\sigma}_2, \rho_1 \otimes \rho_2, H^{n_1} \otimes H^{n_2})$; $n_1 \geq n_2 \geq 1$, $N=n_1 n_2$, $\rho_i$ is a 'quaternion' complex irreducible representation of $G_i$ on $C^{2n_i}$, and $H^{n_i}$ is $C^{2n_i}$ with the $G_i$-invariant quaternionic structure (i.e., the right multiplication of $j(i=1,2)$.

**Type III**) $(Rx \hat{\sigma}_1 \times \hat{\sigma}_2, (\rho_1 \otimes \rho_2)_R, (C^{n_1} \otimes C^{n_2})_R)$; $n_1 \geq n_2 \geq 2$, $N=2n_1 n_2$, $\rho_i$ is a complex irreducible representation of $\hat{\sigma}_i (i=1,2)$, $t$ is in $R^x$.

**Type IV**) $(\hat{\sigma}_1 \times \hat{\sigma}_2, (\rho_1 \otimes \rho_2)_C, (C^{n_1} \otimes C^{n_2})_C)$; $n_1 \geq n_2 \geq 2$, $N=2n_1 n_2$, $\rho_i$ is a complex irreducible representation of $G_i$ on $C^{n_i} (i=1,2)$, and $\rho_1 \otimes \rho_2$ is not 'real'.

**Lemma 3.6** Let $\rho_i$ be a linear representation on $F^{m_i}$ of a compact Lie group $K_i$, and denote $d_i=2^i m_i - \text{dim} K_i$ where $i=0$ (if $F=R$), 1 (if $F=C$), or 2 (if $F=H$). Then

(1) If $1 \leq n \leq m_i$, then $\text{doh}(K_1 \times GF(n), \rho_i \otimes \text{id}, F^{m_i} \otimes F^n) \geq d_i + n(2^i - n + 1) \geq d_i + 3$ if moreover $n \geq 3$.

(2) If $1 \leq n < m_i$, then $\text{doh}(K_1 \times GF(n), \rho_i \otimes \text{id}, F^{m_i} \otimes F^n) \geq d_i + 2^i + 2 - n + (d_i + 2) \geq d_i + 2^i (n-1) - n + 2^{i-1} n$. Replacing $m_i$ by $n$ (resp. $n+1$), we have (1) (resp. (2)). Q.E.D.
Suppose $s=1$: $L=L_0 \circ L_1$ (cf. (3.4)). Then $(\hat{G}, \text{id}, R^N)$ is equivalent as real representation to one of the followings:

**Type V)** $(\hat{G}, \rho_1, R^{n_1})$; $n_1 \geq 3$, $N=n_2$; $\rho_1$ is a 'real' complex irreducible representation of $\hat{G}$ on $C^{n_1}$, and $R^{n_1}$ is a $\hat{G}$-invariant real form of $C^{n_1}$.

**Type VI)** $(R \times \hat{G}, t \otimes \rho_1, (C \otimes C^{n_1})_R)$; $n_1 \geq 2$, $N=2n_2$, and $\rho_1$ is a complex irreducible representation of $\hat{G}$ on $C^{n_1}$.

**Type VII)** $(\hat{G}, \rho_1, C^{n_1})$; $n_1 \geq 2$, $N=2n_1$, $\rho_1$ is a complex irreducible representation of $G$ on $C^{n_1}$, and $\rho_1$ is not 'real'.

**Lemma 3.7** If $n_1 \leq n_2$, then $GF(n_1)$ (or $GF(n_1) \times \{I_{n_2}\}$ in $GF(n_1)$) transforms any matrix $X=[x_{i_1}, \ldots, x_{i_{n_1}}]$ in $F(n_1,n_2)$ (where $x_i$ is in $F$ for $i=1, \ldots, n_1$) to a matrix $Y=[y_{1_1}, \ldots, y_{1_{n_1}}]$ in $F(n_1,n_2)$ (where $y_i$ is in $F(1,n_2)$ for $i=1, \ldots, n_1$) such that $y_i \overline{y_j} = c_i \delta_{ij}$ for some $c_i$ in $R$ ($i,j=1, \ldots, n_1$) by the action (3.2).

**Proof:** There is $A$ in $GF(n_1)$ such that $A$ transforms $X^T X$ in $pF(n_1)$ to a diagonal form $\left[ \begin{array}{ccc} c_1 & & \\ & \ddots & \\ & & c_{n_1} \end{array} \right]$ by the action (3.1).

Then $Y=AX$ satisfied the desired property. Q.E.D.

Suppose $s=0$: $L=L_0 \circ L_1$ (cf. (3.4)). Then $(G, \text{id}, R^N)$ is equivalent as real representation to one of the followings:

**Type VIII)** $(R, t_R, C_R)$; $t$ is in $R^X$.

**Type IX)** $(1, 0, R)$; $1$ is the trivial group, and $0$ is the trivial representation on $R$.

Note that the o.t.g. of type VIII is equivalent to $O(SO(2), \text{id}, R^2)$. 

3-14
For general $s \geq 0$, the estimate of $\text{coh}(G,E^N)$ is given in each cases i), ii), iii), if $\text{id}: G \rightarrow SO(N)$ is real irreducible, by the following theorem. If moreover $s \geq 3$, especially we have $\text{coh}(G,E^N) \geq s$.

**Theorem 3.8**

1. In case i), $\text{coh}(G,E^N) = \text{coh}$ of $(3.5) \geq 4^{r+5q-6r-3q}$,
2. In case ii), $\text{coh}(G,E^N) = \text{coh}$ of $(3.6) \geq 2^{s+1-3s-1}$,
3. In case iii), $\text{coh}(G,E^N) = \text{coh}$ of $(3.7) \geq 2^{s+1-3s-1}$.

**Proof:** (3) follows from (2). For (2), we may assume $n_1 \geq \cdots \geq n_s \geq 2$. If $s < 3$, then (2) is trivial. Suppose $s \geq 3$. If $n_1 \geq n_2 \cdots n_s$, then we denote $f(n_1, \ldots, n_s) = \dim \text{pC}(n_2 \cdots n_s) - \dim \text{SU}(n_2) \times \cdots \times \text{SU}(n_s) = n_2^2 \cdots n_s^2 - n_2^2 \cdots - n_s^2 + s - 1$. Then $\delta f/\delta n_i = 2(n_2^2 \cdots n_i^2 \cdots n_s^2 - n_1^2 \cdots n_i^2 \cdots n_s^2 - 1)$ or $0 \geq 0$. If $n_1 \leq n_2 \cdots n_s$, then we denote $f(n_1, \ldots, n_s) = \dim \text{C}^{n_1} \cdots \text{C}^{n_s} - \dim \text{U}(n_1) \times \text{SU}(n_2) \times \cdots \times \text{SU}(n_s) = 2n_1 \cdots n_s^2 - n_1 \cdots n_s^2 + s - 1$. Then $\delta f/\delta n_i = 2(n_1^2 \cdots n_i^2 \cdots n_s^2 - n_1^2 \cdots n_i^2 \cdots n_s^2) \geq 2(n_2^2 \cdots n_s^2 - n_1^2 \cdots n_s^2) \geq 0$.

Therefore $\text{coh}(3.6) \geq f(n_1, \ldots, n_s) \geq f(2, \ldots, 2) = 2^{s+1-3s-1}$.

1. Suppose $s = 2r + q \leq 2$. If $r, q \leq 1$, then (1) is trivial.

If $r = 0$, $q = s = 2$, then (1) follows from Prop. 3.3. If $s = 3$, then (1) follows from Prop. 3.4. Assume $s \geq 4$. Suppose $r = 0$: Then we may assume $n_1 \geq \cdots \geq n_s \geq 3$. If $n_1 \geq n_2 \cdots n_s$, then denote $f(n_1, \ldots, n_s) = \dim \text{pR}(n_2 \cdots n_s) - \dim \text{SO}(n_2) \times \cdots \times \text{SO}(n_s) = (n_2^2 \cdots n_s^2 + n_2 \cdots n_s^2 - n_2 \cdots n_s^2) / 2$. Then $\delta f/\delta n_i = n_i(n_2^2 \cdots n_i^2 \cdots n_s^2 - 1) + (n_2 \cdots n_i \cdots n_s^2 + 1) / 2$ or $0 \geq 0$. If $n_1 \leq n_2 \cdots n_s$, then denote $f(n_1, \ldots, n_s) = \dim \text{R}^{n_1} \cdots \text{R}^{n_s} - \dim \text{SO}(n_1) \times \cdots \times \text{SO}(n_s) = n_1 \cdots n_s^2 (n_2^2 + \cdots + n_s^2) / 2 + (n_1^2 + \cdots + n_s^2) / 2$. Then $\delta f/\delta n_i = n_1 \cdots n_i \cdots n_s^2 - n_i + 1 / 2 \geq n_2 \cdots n_s - n_i + 1 / 2 \geq 1 / 2$. Therefore $\text{coh}(3.5) \geq f(n_1, \ldots, n_s) \geq$
f(3, ..., 3) = 3^s - 3^s = 3^q - 3^q. Suppose q = 0: Then we may assume

\[ n_1 \geq \ldots \geq n_s \geq 2. \] If \( n_1 n_2 \geq n_3 \ldots n_s \), then denote \( g(n_1, \ldots, n_s) = \]

\[ \dim \text{Pr}(n_3 \ldots n_s) - \dim \text{Sp}(n_3/2)x \ldots x\text{Sp}(n_s/2) = (n_3^2 \ldots n_s^2 + n_3 \ldots n_s - n_3^2 \ldots n_s^2 - n_3^2 \ldots n_s^2)/2. \] Since \( \text{ch}/\text{en}_1 \geq 0 \) (i = 1, ..., s), \( \text{coh}(3.5) \geq g(n_1, n_2, n_3, \ldots, n_s) \geq g(n_1, n_2, 2, \ldots, 2) = 2^{2s-5} + 2^{s-3} - 3(s-2) = 2^{2r-2r-5+2-3} - 6r + 6 = 4^r - 6r. \] If \( n_1 n_2 \leq n_3 \ldots n_s \), then denote \( h(n_1, \ldots, n_s) = \]

\[ \dim H^{n_1/2} \ldots H^{n_s/2} - \dim \text{Sp}(n_1/2)x \ldots x\text{Sp}(n_2/2) = n_1 \ldots n_s - (n_1^2 \ldots n_s^2 + n_1^2 \ldots n_s^2)/2. \] Since \( \text{ch}/\text{en}_1 \geq 0 \) (i = 1, ..., s), \( \text{coh}(3.5) \geq h(n_1, \ldots, n_s) \geq h(n_3, n_3, n_3, n_4, \ldots, n_s) \geq h(n_4, n_4, n_4, n_4, n_5, \ldots, n_s) \geq h(2, \ldots, 2) = 2^s - 3s = 4^r - 6r. \] Finally suppose \( r, q \geq 1 \): Then we may assume \( n_1 \geq \ldots \geq n_2r \geq 2 \) and \( n_2r+1 \geq \ldots \geq n_2r+q \geq 3. \] If \( n_1 n_2 \geq n_3 \ldots n_s \), then denote \( g(n_1, \ldots, n_s) = \]

\[ \dim \text{Pr}(n_3 \ldots n_s) - \dim \text{Sp}(n_3/2)x \ldots x\text{Sp}(n_2/2) \]

\[ x\text{SO}(n_2r+1)x \ldots x\text{SO}(n_2r+q) = (n_3^2 \ldots n_s^2 + n_3 \ldots n_s - n_3^2 \ldots n_s^2 - n_3^2 \ldots n_s^2)/2. \] Since \( \text{ch}/\text{en}_1 \geq 0 \) (i = 1, ..., s), \( \text{coh}(3.5) \geq g(n_1, \ldots, n_s) \geq g(n_1, n_2, 2, \ldots, 2, 3, \ldots, 3) = 2^{2r-2r-5} + 3^q(2^{2r-5} + 3^q + 2^{-5}) + 6 - 6r - 3q \geq 4^r - 3q - 6r - 3q. \] If \( n_1 n_2 \leq n_3 \ldots n_s \), then denote \( h(n_1, \ldots, n_s) = \]

\[ \dim H^{n_1/2} \ldots H^{n_2r/2} - \dim \text{Sp}(n_1/2)x \ldots x\text{Sp}(n_2r/2) \]

\[ x\text{SO}(n_2r+1)x \ldots x\text{SO}(n_2r+q) = n_1 \ldots n_s - (n_1^2 + n_2^2 + n_1^2 + \ldots + n_2^2 + n_2^2 - n_2^2 - n_2^2 + \ldots + n_2^2 + n_2^2 + n_2^2)/2. \] Since \( \text{ch}/\text{en}_1 \geq n_2 \ldots n_s - n_1 - 1/2 \geq n_2 - 1/2 \geq 2(2^2 - 1)/2 > 0 \), \( \text{coh}(3.5) \geq h(n_1, \ldots, n_s) \geq h(n_3, n_3, n_3, n_4, \ldots, n_s) \geq h(n_4, n_4, n_4, n_5, \ldots, n_s) \geq h(2, \ldots, 2, 3, \ldots, 3) = 4^r - 3q - 6r - 3q. \] Q.E.D.
4. Orthogonal transformation groups

of cohomogeneity at most 3

(I) Let \((G, F^N)\) be a real irreducible o.t.g. of type I.

Proposition 4.1 \(\text{coh}(G, F^N) \leq 3\) if and only if \(\left\langle \hat{\alpha} \right\rangle \) is equivalent as real representation to one of the followings:

- \(\text{coh}=1: \) none,
- \(\text{coh}=2: \) none,
- \(\text{coh}=3: \)
  1. \((A_1 \times A_1, (2A_1)^\mathbb{T}, (2A_1)^\mathbb{T}, R^2 \oplus R^2), \)
  2. \((A_2 \times A_1, A_2^\mathbb{T}, (2A_1)^\mathbb{T}, R^6 \oplus R^3), \)
  3. \((C_2 \times A_1, A_2^\mathbb{T}, (2A_1)^\mathbb{T}, R^5 \oplus R^3), \)
  4. \((R^k \times A_1, A_1^\mathbb{T}, (2A_1)^\mathbb{T}, R^{2k+1} \oplus R^3); \) \(k \geq 3, \)
  5. \((D_k \times A_1, A_1^\mathbb{T}, (2A_1)^\mathbb{T}, R^{2k} \oplus R^3); \) \(k \geq 4, \)
  6. \((B_3 \times A_1, A_3^\mathbb{T}, (2A_1)^\mathbb{T}, R^8 \oplus R^3), \)
  7. \((D_4 \times A_1, A_i^\mathbb{T}, (2A_1)^\mathbb{T}, R^8 \oplus R^3); \) \(i=3, 4. \)

Proof: Suppose \(\text{coh}(G, F^N) \leq 3\). Then \(\left\langle \hat{\alpha} \right\rangle \) is equivalent as real representation to (1),..., (6), or (7) owing to Prop.3.3(2) (4), Prop.2.17, Lemma3.6(1) \((F=R, i=0, n=3), 3 \leq \text{doh}(G, F^N) \leq d_0+3,\)
Prop.2.1 \((d_0 \leq 3), \) \(\text{doh}(A_k \times A_1, (A_1+\lambda_k)^\mathbb{T}, (2A_1)^\mathbb{T}, R^\dim A_k \oplus R^3)=2\dim A_k - 3 \geq 13 \)
\((k \geq 2), \) Prop.2.5, \(\text{doh}(C_k \times A_1, (2A_1)^\mathbb{T}, R^\dim C_k \oplus R^3)=2\dim C_k - 17 \)
\((k \geq 2), \) \(\text{doh}(C_k \times A_1, A_2^\mathbb{T}, (2A_1)^\mathbb{T}, R^{2k+1} \oplus R^3)=4k(k-1)-6 \geq 18 (k \geq 3), \) Prop.2.8, \(\text{doh}(B_k \times A_1, A_2^\mathbb{T}, (2A_1)^\mathbb{T}, R^{2k+1} \oplus R^3)=2\dim B_k - 3 \geq 39 (k \geq 3), \) \(\text{doh}(B_4 \times A_1, A_4^\mathbb{T}, (2A_1)^\mathbb{T}, R^{16} \oplus R^3)=9, \) Prop.2.11, \(\text{doh}(D_k \times A_1, A_2^\mathbb{T}, (2A_1)^\mathbb{T}, R^{2k+1} \oplus R^3)=2\dim D_k - 53 (k \geq 4), \) the equivalence of o.t.g.'s \(O(D_4 \times A_1, \)
A_{i}^{T} @ (2A_{1})^{T}, R^{8} \otimes R^{3} \) for i=1,3,4 (cf. Remark 2.13), Prop.2.15,

Remark 2.16, 2dimE_{8} \geq 2dimE_{7} \geq 2dimE_{6} \geq 2dimF_{4} \geq 2dimG_{2} \geq 25,

doh(F_{4} \times \mathbb{A}_{1}, \mathbb{A}_{i}^{T} \otimes (2A_{1})^{T}, R^{26} \otimes R^{3}) = 23, doh(G_{2} \times \mathbb{A}_{1}, \mathbb{A}_{i}^{T} \otimes (2A_{1})^{T}, R^{7} \otimes R^{3}) = 4.

Conversely if \((G,E^{N})\) is induced from (1),..., (5), or (7), then \((G,E^{N})\) can also be induced from \((SO(n_{1}) \times SO(3), \text{id} \otimes \text{id}, R^{n_{1}} \otimes R^{3})\).

for some \(n_{1} \neq 4\). So \(\text{coh}(G,E^{N}) = 3\) (cf. Prop.3.3(2)(4)). An o.t.g. induced from (6) is of coh 3. In fact \(\text{Spin}(7) \times SO(3)\) acts on \(R(8,3)\) through \(\iota\) by the action (3.2) (cf. Prop.3.3 Proof), and the isotropy subgroup at

\[
\begin{pmatrix}
    x_{1} \\
    x_{2} \\
    x_{3}
\end{pmatrix}
\]

where \(|x_{i}| (i=1,2,3)\) are non-zero distinct real numbers, is locally isomorphic to \(SU(2)\) (cf. Yokota [24, Theorem 5.27, Theorem 5.2]). O.F.D.

(II) Let \((G,E^{N})\) be a real irreducible o.t.g. of type II.

Proposition 4.2 \(\text{coh}(G,E^{N}) \leq 3\) if and only if \((\mathbb{A}, \iota, R^{N})\) is equivalent as real representation to one of the followings:

\(\text{coh}=1: (8) \ (A_{1} \times \mathbb{A}_{1}, \mathbb{A}_{1} \otimes \mathbb{A}_{1}, H \otimes H), \)

\(\text{coh}=2: (9) \ (C_{k} \times \mathbb{A}_{1}, \mathbb{A}_{1} \otimes \mathbb{A}_{1}, H^{k} \otimes H), \ k \geq 2, \)

\(\text{coh}=3: (10) \ (C_{k} \times \mathbb{C}_{2}, \mathbb{A}_{1} \otimes \mathbb{A}_{1}, H^{k} \otimes H^{2}), \ k \geq 2, \)

\(\text{coh}=4: (11) \ (A_{1} \times \mathbb{A}_{1}, 2A_{1} \otimes \mathbb{A}_{1}, H^{2} \otimes H), \)

\(\text{coh}=5: (12) \ (C_{k} \times \mathbb{C}_{3}, \mathbb{A}_{1} \otimes \mathbb{A}_{1}, H^{k} \otimes H^{3}), \ k \geq 3, \)

\(\text{coh}=6: (13) \ (C_{k} \times \mathbb{A}_{1}, 3A_{1} \otimes \mathbb{A}_{1}, H^{k} \otimes H^{2}), \ k \geq 2. \)
Proof: Suppose \( \text{co}(G,E^N) \leq 3 \). Then \( n_2 \leq 3 \) (cf. Prop. 3.3(1)(4)).
Assume \( n_2 = 3 \). Then \( (G_2, \rho_2, H^{n_2}) \) is equivalent as complex representation to \( (C_3, A_1, H^3) \) owing to Prop. 2.20 and \( \text{co}(\text{Sp}(n_1) \times A_1, \text{id} \otimes A_1, H^{n_1} \otimes H^3) \geq \text{doh}(A_1, pH(3)) = 12 \) (cf. Prop. 3.3(1)). So \( (G, \text{id}, R^N) \) is equivalent as real representation to \( (G, \text{id}, R^N) \) owe to Lemma 3.6(1) (\( F = H, i = 2, m_2 = n_1, n = n_2 = 3, d_1 + k(2^{i-1}(k-3) + 1) = d_2 + 3, 3 \geq \text{doh}(G, E^N) \geq d_2 + 3 \), Prop.'s 2.4, 2.7, 2.10, 2.14, \( \text{doh}(D_6 \times C_3, \text{sp} A_1, H^{16} \otimes H^3) = 105(i = 5, 6) \), Prop. 2.15, Remark 2.16, \( \text{doh}(E_7 \times C_3, A_6 \otimes A_1, H^{28} \otimes H^3) = 171 \).

Assume \( n_2 = 2 \). Then \( (G_2, \rho_2, H^{n_2}) \) is equivalent as complex representation to \( (C_2, A_1, H^2) \) or \( (A_1, 3A_1, H^2) \) owing to Prop. 2.20, \( \deg \rho_1 = 2n_2 = 4 \) (cf. Prop. 2.20 and \( \text{doh}(A_1 \times A_1, 3A_1 \otimes A_1, H^2 \otimes H^2) = 10 \)). So \( (G, \text{id}, R^N) \) is equivalent as real representation to (10) or (13)

Assume \( n_2 = 1 \). Then \( (G_2, \rho_2, H^{n_2}) \) is equivalent as complex representation to \( (A_1, A_1, H) \) by Prop. 2.20. So \( (G, \text{id}, R^N) \) is equivalent as real representation to (8),(9) or (11) owing to \( \text{doh}(G, E^N) \geq d_2 - 3, \text{Prop. 2.4, \xspace coh}(A_5 \times A_1, A_5 \otimes A_1, H^{10} \otimes H) = 4 \) (cf. The linear isotropy representation of the symmetric pair \( (E_6, SU(6) \cdot \text{Sp}(1)) \) of rank 4 is characterized as a real 40 dimensional irreducible almost faithful representation of \( A_5 \times A_1 \) owing to Section 2), Prop.'s 2.7, 2.10, 2.14, Remark 2.13, \( \text{coh}(D_6 \times A_1, A_6 \otimes A_1, H^{16} \otimes H) = 4(i = 5, 6) \) (cf.}
The linear isotropy representation of the symmetric pair \((E_7, \text{Spin}(12) \cdot \text{Sp}(1))\) of rank 4 is characterized as a real 64-dimensional irreducible almost faithful representation of \(D_6 \times A_1\) owing to Section 2), Prop.2.15, Remark 2.16, \(\text{coh}(E_7 \times A_1, A_6 \otimes A_1, H^2_8, \mathbb{H}) = 4\) (cf. The linear isotropy representation of the symmetric pair \((E_8, E_7 \cdot \text{Sp}(1))\) of rank 4 is characterized as a real 112-dimensional irreducible almost faithful representation of \(E_7 \times A_1\) owing to Section 2).

Conversely an o.t.g. induced from (8) or (9) is of \(\text{coh} 1\) by Prop.3.3(1)(4)\((F=H, n_2=1, K=\text{Sp}(1))\). An o.t.g. induced from (10) is of \(\text{coh} 2\) by Prop.3.3(1)(4)\((F=H, n_2=2, K=\text{Sp}(2))\). An o.t.g. induced from (12) is of \(\text{coh} 3\) by Prop.3.3(1)(4)\((F=H, n_2=3, K=\text{Sp}(3))\). An o.t.g. induced from (11) is of \(\text{coh} 2\)(cf. The linear isotropy representation of the symmetric pair \((G_2, \text{SO}(4))\) of rank 2 is characterized as a real 8-dimensional irreducible almost faithful representation of \(A_1 \times A_1\) owing to Prop.'s 2.1, 2.2, 2.4). If \((G, E^N)\) is induced from (13), then \(\text{coh}(G, E^N) = \text{coh}(A_1, pH(2)) = \text{doh}(A_1, pH(2)) = 3\) (cf. Prop.3.3) and \(\text{coh}(G, E^N) \leq \text{coh}(A_1, hH(2)) = \text{coh}(A_1, \theta(4A_1)^r, R \otimes R^5) = 1 + \text{coh}(A_1, (4A_1)^r, R^5) = 3\)(cf. The linear isotropy representation of the symmetric pair \((\text{SU}(3), \text{SO}(3))\) of rank 2 is characterized as a real 5-dimensional irreducible representation of \(A_1\) owing to Prop.'s 2.1, 2.2, 2.4), where the action of \(A_1\) on \(pH(2)\) is given as Prop.3.3 and Lemma 3.2. Q.E.D.
(III) Let \((G,E^N)\) be a real irreducible o.t.g. of type III.

**Proposition 4.3** \(\text{coh}(G,E^N) \leq 3\) if and only if \((G,id,R^N)\) is equivalent as real representation to one of the followings:

- \(\text{coh}=1:\) none,
- \(\text{coh}=2:\) \(14\) \(\left( \text{Rx}_k x A_1, t \otimes A_1 \otimes A_1, C^k C^{k+1} C^2 \right); \ k \geq 1, t \in R^X.\)
- \(\text{coh}=3:\) \(15\) \(\left( \text{Rx}_k x A_2, t \otimes A_1 \otimes A_1, C^k C^{k+1} C^3 \right); \ k \geq 2, t \in R^X.\)
- \(\text{coh}=4:\) \(16\) \(\left( \text{Rx}_k x A_1, t \otimes A_1 \otimes A_1, C^k C^{k+2} C^3 \right); \ k \geq 2, t \in R^X.\)

**Proof:** Suppose \(\text{coh}(G,E^N) \leq 3\). Then \(n_2 \leq 3\) (cf. Prop. 3.3(1)(4)).

Assume \(n_2 = 3\). Then \((G_2, \rho_2, C^{n_2})\) is equivalent as complex representation to \((A_2, A_1, C^3)\) owing to Prop. 2.18, Remark 2.19 and \(\text{coh}(U(n_1)x A_1, id \otimes A_1; C^n C^3) \geq \text{doh}(A_1, pC(3)) = 6\). If \(\rho_1\) is 'real' and \(n_1 \geq 6\), then \(\text{coh}(G,E^N) = \text{coh}(U(1)x A_2, id \otimes A_1, C^{n_1} C^3) = \text{coh}(G_1 x (U(1)x A_2), \rho_1 \otimes (id \otimes A_1), R^{n_1} C_{(C^3)} R) \geq \text{coh}(SO(n_1)x U(3), \text{id} \otimes \text{id}, R^{n_1} \otimes C_{(C^3)} R) = \text{doh}(U(3), pR(6)) \geq \text{doh}(U(3), pR(6)) = 12\) (cf. Prop. 3.3).

So \((G_1, \rho_1, C^{n_1})\) is not 'real' or \(n_1 \leq 5\). Then \((G, id, R^N)\) is equivalent as real representation to \((15)\) owing to Lemma 3.6(1) \(F=C, i=1, m_1=n_1, n=n_2=3\), \(\geq \text{coh}(G,E^N) \geq d_1 + 3\), Prop. 2.2 \((A_2(k=3)\) is 'real' of degree 6), Remark 2.3, \(\text{doh}(\text{Rx}_k x A_2, t \otimes A_1 \otimes A_1, C^k C^{k+2} C^2 C^3) = (k+1)(2k-1)-8 \geq 27\) \((k \geq 4)\), Prop. 2.6 \((A_2(k=2)\) is 'real' of degree 11), \(\text{doh}(\text{Rx}_k x A_2, t \otimes A_1 \otimes A_1, C^k C^{k+4} C^3) = 5, \text{coh}(\text{Rx}_k x A_2, t \otimes A_1 \otimes A_1, C^k C^{k+2} C^3) \geq C_{(C^3)} C_{(C^3)} C_{(C^3)} \geq C_{(C^3)} C_{(C^3)} C_{(C^3)} \geq C_{(C^3)} C_{(C^3)} C_{(C^3)} \geq\)

\(\dim C_{(C^3)} 2k C^3 - \dim \text{Rx}_k x A_2 + \dim C_{k-3} = 6\) (cf. Any isotropy subgroup contains \(C_{k-3}\)), Prop. 2.9 \((A_1(k=3)\) is 'real' of degree \(\geq 7\), \(\text{doh}(\text{Rx}_k x A_2, t \otimes A_1 \otimes A_1, C^k C^{k+2} C^3) = 3 \cdot k^2 + k^2 - k^2 - k \geq 2\) \((k \geq 3, k \neq 4)\), Prop. 2.12 \((A_1(k=4)\) is 'real' of degree \(\geq 8\), \(\text{doh}(\text{Rx}_k x A_2, t \otimes A_1 \otimes A_1, C^k C^{k+2} C^3) = 3 \cdot 2^k - k(k-1)(k-2) \geq 11\) for \(i=k, k-1\) (if \(k \geq 4\), Prop. 2.15, Remark 2.16,
doh(Rx_{E_6} x A_2, t_\varphi A_1 \otimes A_1, C^2 C^{27} C^{3}) = 75, 
doh(Rx_{E_7} x A_2, t_\varphi A_6 \otimes A_1, C^2 C^{56} C^{3}) = 194.

Assume \( n_2 = 2 \). Then \((G, \rho_2, C^{n_2})\) is equivalent as complex representation to \((A_1, A_1, C^2)\) by Prop.2.18. If \((G, \rho_1, C^{n_1})\) is 'real' of degree \( n_1 \geq 4 \), then \( \text{co}(G, E^N) = \text{co}(U(1) \times G_1 x A_1, id \otimes \rho_1 \otimes A_1, C^2 C^{n_1} C^{27}) = \text{co}(SO(n_1) x R, R^{n_1} C^{27} C^{3} R) \geq \text{doh}(U(2), pR(4)) \geq \text{doh}(U(2), pR(4)) = 6. \) So \((G, \rho_1, C^{n_1})\) is not 'real' or \( n_1 \leq 5 \). Then \((G, \text{id}, R^N)\) is equivalent as real representation to (14) or (16) owing to Prop.2.18, Lemma 3.6(2) (F=C, i=1, \( m_1 = n_1 > n = n_2 = 2 \)), \( \geq \text{co}(G, E^N) \geq d_1 + 2 \). Prop.2.2 \((A_2, k=3)\) is 'real' of degree 6, Remark 2.3, \( \text{doh}(Rx_{A_2} x A_1, t_\varphi \Lambda_1 \otimes A_1, C^2 C^{k+1} C^{27} C^{3}) = k^2 \cdot 4^{12}(k \geq 4) \), \( \text{doh}(Rx_{A_2} x A_1, t_\varphi \Lambda_1 \otimes A_1, C^2 C^{k+2} C^{27} C^{3}) = (k+1)(k+3) \cdot 3 \cdot 5^{(k-1)} \). Prop.2.6 \((A_2, k=2)\) is 'real' of degree 11, Prop.2.9 \((A_1, k=3)\) is 'real' of degree \( k \geq 7 \), \( \text{doh}(Rx_{B_k} x A_1, t_\varphi \Lambda_1 \otimes A_1, C^2 C^{k+2} C^{3}) = 2k^2 - (2k+1) \cdot 4^{12}(k \geq 7) \), Prop.2.12 \((A_1, k=4)\) is 'real' of degree \( \geq 8 \), \( A_1, k=4 \) for \( i=3, 4 \) are 'real' of degree 8, \( \text{doh}(Rx_{D_k} x A_1, t_\varphi \Lambda_1 \otimes A_1, C^2 C^{2k-1} C^{3}) = 2k^2 - k^2 + 1 \cdot 4^{12} \). for \( i=4, k \) (if \( k \geq 5 \)), Prop.2.15, Remark 2.16, \( \text{doh}(Rx_{E_6} x A_1, t_\varphi \Lambda_1 \otimes A_1, C^{27} C^{3}) = 26, \text{doh}(Rx_{E_7} x A_1, t_\varphi A_6 \otimes A_1, C^{56} C^{3}) = 87. \)

Conversely an o.t.g. induced from (14) (resp. (15)) is of \( \text{coh} 2 \) (resp. 3) (cf. Prop.3.3(1)(4)). If \((G, E^N)\) is induced from (16), then \( \text{co}(G, E^N) = \text{co}(U(1) \times C_k x A_1, id \otimes \Lambda_1 \otimes A_1, C^2 C^{2k} C^{3}) = \text{co}(U(1) \times C_k x A_1, id \otimes \Lambda_1 \otimes A_1, R^2 C^2 C^{2k} C^{3} C^2) = \text{co}(SO(2) x C_k x A_1, id \otimes \Lambda_1 \otimes A_1, R^2 C^2 C^{2k} C^{3} C^2) = \text{co}(SO(2) x C_k x A_1, pR(2)) + \text{co}(A_1, C^{2k} C^{3} C^2) = 2k^2 + 1 = 3\) (cf. Prop.3.3). Q.E.D.
(IV) Let \((G, E^N)\) be a real irreducible o.t.g. of type IV.

**Proposition 4.4** \(\text{coh}(G, E^N) \leq 3\) if and only if \((\hat{G}, \hat{\mathbb{I}}, \mathbb{R}^N)\) is equivalent as real representation to one of the followings:

- **coh=1:** none,
- **coh=2:** (17) \((A_1 \times A_1, \Lambda_1 \otimes \Lambda_1, C^{k+1} \mathbb{C}^2)\); \(k \geq 2\),
- **coh=3:** (18) \((A_1 \times A_2, \Lambda_1 \otimes \Lambda_1, C^{k+1} \mathbb{C}^3)\); \(k \geq 3\).

**Proof:** Suppose \(\text{coh}(G, E^N) \leq 3\). Then \((\hat{G}, \hat{\mathbb{I}}, \mathbb{R}^N)\) is equivalent as real representation to (17) or (18) owing to Prop.4.3. In fact

\((C \times A_1, \Lambda_1 \otimes \Lambda_1, C^{k+1} \mathbb{C}^2)\) (\(k \geq 2\)) and \((A_1 \times A_1, \Lambda_1 \otimes \Lambda_1, C^{k+1} \mathbb{C}^2)\) are 'real',

so they are not real irreducible, and \(\text{coh}(A_1 \times A_2, \Lambda_1 \otimes \Lambda_1, C^{k+1} \mathbb{C}^3) = 4\) since \((U(1) \times A_1, \mathbb{C}^3)\) is equivalent to the linear isotropy representation of the Hermitian symmetric pair \((SU(6), S(U(3) \times U(3)))\) of rank 3 whose restricted root system is of type C (cf. Tasaki-Yasukura[22], Helgason[7]).

Conversely an o.t.g. induced from (17) (resp. (18)) is of \(\text{coh} 2\) (resp. 3) since \((U(1) \times A_1 \times A_1, \mathbb{C}^3)\) of \(h \geq 2\) (resp. \((U(1) \times A_1 \times A_2, \mathbb{C}^3)\) of \(h \geq 3\)) is equivalent to the linear isotropy representation of the Hermitian symmetric pair \((SU(k+3), S(U(k+1) \times U(2)))\) of rank 2 (resp. \((SU(h+4), S(U(h+1) \times U(3)))\) of rank 3) whose restricted root system is of type BC (cf.[22], [7]).

Q.E.D.
(V) Let \((G, E^N)\) be a real irreducible o.t.g. of type V.

**Proposition 4.5** coh\((G, E^N)\) \(\leq 3\) if and only if \((\varpi, \text{id}, R^N)\) is equivalent as real representation to one of the followings:

- coh=1: (19) \((A_1, (2A_1)^r, R^3)\),
  - (20) \((A_2, A_2^r, R^6)\),
  - (21) \((C_2, A_2^r, R^5)\),
  - (22) \((B_k, A_1^r, R^{2k+1}); k \geq 3\),
  - (23) \((D_k, A_1^r, R^{2k}); k \geq 4\),
  - (24) \((D_i, A_i^r, R^8); i=3, 4\),
  - (25) \((B_3, A_3^r, R^8)\),
  - (26) \((B_4, A_4^r, R^{16})\),
  - (27) \((G_2, A_2^r, R^7)\),

- coh=2: (28) \((A_2, (A_1+\Lambda_2)^r, R^8)\),
  - (29) \((A_2^r, (4A_1)^r, R^5)\),
  - (30) \((C_3, A_2^r, R^{14})\),
  - (31) \((C_2, (2A_1)^r, R^{10})\),
  - (32) \((G_2, A_1^r, R^{14})\),
  - (33) \((F_4, A_4^r, R^{26})\),

- coh=3: (34) \((A_3, (A_1+\Lambda_3)^r, R^{15})\),
  - (35) \((C_5, (2A_1)^r, R^{21})\),
  - (36) \((C_4, A_2^r, R^{27})\),
  - (37) \((B_3, A_2^r, R^{21})\).

**Proof:** Suppose coh\((G, E^N)\) \(\leq 3\). Then \((\varpi, \text{id}, R^N)\) is equivalent as real representation to one of (19)-(37) owing to Prop.2.1, coh\((A_k, (\Lambda_1+\Lambda_k)^r, R^{\text{dim}A_k})=k\), Prop.2.5, coh\((C_k, (2A_1)^r, R^{\text{dim}C_k})=k\), coh\((C_k, A_2^r, R^{(k-1)(2k+1)})=k-1\)(cf. O\((C_k, A_2^r, R^{(k-1)(2k+1)})\) is equivalent to the linear isotropy representation of the symmetric pair \((SU(2k), \text{Sp}(k))\) of rank \(k-1\), Prop.2.8, coh\((B_k, A_2^r, R^{\text{dim}B_k})=k\), Prop.2.11, coh\((D_k, A_2^r, R^{\text{dim}D_k})=k\), the equivalence of O\((D_4, \Lambda_i^r, R^8)\) for \(i=1, 4, 3\), Prop.2.15, coh\((F_4, \Lambda_1^r, R^{52})=4\), coh\((E_6, A_6^r, R^{78})=6\), coh\((E_7, \Lambda_1^r, R^{144})=7\), coh\((E_8, \Lambda_7^r, R^{248})=8\).

Conversely an o.t.g. induced from one of (19)-(24) is equivalent to \((\text{SO}(n), \text{id}, R^n)\) for some \(n\not=4\), which is of coh 1.

An o.t.g. induced from (25), (26) or (27) is of coh 1(cf. Yokota [24, Theorems 5.27, 5.50, 5.3]). O.t.g.'s (28)-(33) are equivalent
to the linear isotropy representation of the symmetric pairs
(SU(3) x SU(3), SU(3)), (SU(3), SU(2)), (SU(6), Sp(3)), (Sp(2) x Sp(2),
Sp(2)), (G_2 x G_2, G_2), (E_6, F_4) of rank 2 respectively (cf. Prop.'s
2.1, 2.5, 2.15). O.t.g.'s induced from (34)\^\nu(37) are equivalent
to the linear isotropy representations of the symmetric pairs
(SU(4) x SU(4), SU(4)), (Sp(3) x Sp(3), Sp(3)), (SU(8), Sp(4)), (SO(7) x
SO(7), SO(7)) of rank 3 respectively (cf. Prop.'s 2.1, 2.5, 2.8).
They are also characterized by their degrees among 'real' complex
irreducible representations. Q.E.D.
(VI) Let \((G,E^N)\) be a real irreducible o.t.g. of type VI.

Proposition 4.6 coh\((G,E^N)\) \(\leq 3\) if and only if \((G,i^*, R^N)\) is equivalent as real representation to one of the followings:

coh=1: (38) \((RxA_k, tA_1, C_{C}^k C^{k+1})\); \(k \geq 1\), \(t\) in \(R^x\),

(39) \((RxC_k, tA_1, C_{C}^k C^{2k})\); \(k \geq 2\), \(t\) in \(R^x\),

coh=2: (40) \((RxB_k, tA_1, C_{C}^k C^{2k+1})\); \(k \geq 3\), \(t\) in \(R^x\),

(41) \((RxD_k, tA_1, C_{C}^k C^{2k})\); \(k \geq 4\), \(t\) in \(R^x\),

(42) \((RxD_k, tA_1, C_{C}^k C^{2k})\); \(k \geq 4\), \(t\) in \(R^x\),

(43) \((RxA_k, tA_2, C_{C}^k C^{3})\); \(t\) in \(R^x\),

(44) \((RxA_2, tA_2, C_{C}^6)\); \(t\) in \(R^x\),

(45) \((RxC_2, tA_2, C_{C}^5)\); \(t\) in \(R^x\),

(46) \((RxC_2, tA_2, C_{C}^5)\); \(t\) in \(R^x\),

(47) \((RxB_3, tA_3, C_{C}^8)\); \(t\) in \(R^x\),

(48) \((RxD_5, tA_5, C_{C}^{16})\); \(t\) in \(R^x\),

(49) \((RxA_4, tA_2, C_{C}^{10})\); \(t\) in \(R^x\),

coh=3: (50) \((RxA_2, tA_2, C_{C}^6)\); \(t\) in \(R^x\),

(51) \((RxA_5, tA_2, C_{C}^{15})\); \(t\) in \(R^x\),

(52) \((RxA_6, tA_2, C_{C}^{21})\); \(t\) in \(R^x\),

(53) \((RxB_4, tA_4, C_{C}^{16})\); \(t\) in \(R^x\),

(54) \((RxE_6, tA_1, C_{C}^{27})\); \(t\) in \(R^x\).
Proof: Suppose $\text{coh}(G, E^N) \leq 3$. Then $(G, \text{id}, R^N)$ is equivalent as real representation to one of (38)\(\sim\)(54) owing to Lemma 3.6(1)\((F=C, i=1, n=1)\), Prop. 2.2, Remark 2.3, $\text{coh}(U(1) \times A_k, \text{id} \otimes A_2, \mathbb{C} \otimes \mathbb{C}^{k+1/2}) = ((k+1)/2)\) (cf. $(U(1) \times A_k, \text{id} \otimes A_2, \mathbb{C} \otimes \mathbb{C}^{k+1/2})$ is equivalent to the linear isotropy representation of the symmetric pair $(\text{SO}(2k+2), U(k+1))$ of rank $[(k+1)/2]$, $[(k+1)/2] \geq 4(k \geq 7)$, Prop. 2.6, Prop. 2.9, Prop. 2.12, Remark 2.13, $\text{coh}(U(1) \times D_6, \text{id} \otimes A_6, \mathbb{C} \otimes C^{32}) \geq 4$ (cf. $(U(1) \times D_6, \text{id} \otimes A_6, \mathbb{C} \otimes C^{32})$ is contained in the linear isotropy representation of the symmetric pair $(E_7, \text{Spin}(12))$ of rank 4), Prop. 2.15, Prop. 2.16, $\text{coh}(U(1) \times F_4, \text{id} \otimes A_4, \mathbb{C} \otimes C^{26}) \geq 7$ (cf. Each isotropy subgroup contains a group which is isomorphic to $SU(3)$ in $G_2 \subset \text{Spin}(7) \subset \text{Spin}(8)$) $F_4$ by Yokota[24, Prop.'s 5.45, 5.48, Thm.'s 5.33, 5.27, 5.2]), $\text{coh}(U(1) \times E_7, \text{id} \otimes A_6, \mathbb{C} \otimes C^{56}) \geq 4$ (cf. $(U(1) \times E_7, \text{id} \otimes A_6, \mathbb{C} \otimes C^{56})$ is contained in the linear isotropy representation of the symmetric pair $(E_8, \text{Spin}(1))$ of rank 4), $\text{doh}(U(1) \times G_2, \text{id} \otimes A_1, \mathbb{C} \otimes C^{14}) = 13$, $\text{doh}(U(1) \times F_4, \text{id} \otimes A_1, \mathbb{C} \otimes C^{52}) = 51$, $\text{doh}(U(1) \times E_6, \text{id} \otimes A_6, \mathbb{C} \otimes C^{78}) = 77$, $\text{doh}(U(1) \times E_7, \text{id} \otimes A_7, \mathbb{C} \otimes C^{133}) = 132$, $\text{doh}(U(1) \times E_8, \text{id} \otimes A_7, \mathbb{C} \otimes C^{248}) = 247$. Conversely $\text{coh}(38) = \text{coh}(39) = 1$ since $SU(k+1)$ and $\text{Spin}(k)$ are transitive on hyperspheres in the representation spaces. \[(40)\sim(45)\] are equivalent to $(\text{SO}(2) \times \text{SO}(n), \text{id} \otimes \text{id}, R^R \otimes R^R)$ for some $n \neq 4$ of $\text{coh} 2$. The o.t.g. induced from (46) is equivalent to $O(\text{SO}(2) \times G_2, \text{id} \otimes A_2^R, R^R \otimes R^R)$ and the isotropy subgroup at $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ in $R(2,7) = R^R \otimes R^R \otimes R^R \otimes R^R \otimes R^R \otimes R^R \otimes R^R \otimes R^R$ ($\alpha \gt \beta \gt 0$) is isomorphic to $SU(2)$ by Yokota[24, Example 5.1], so $\text{coh} (46) = 2$ (cf. Prop. 3.3(1)(4)). The o.t.g. induced from (48) is equivalent to the linear isotropy representation of the symmetric
pair \((E_6, U(1) \cdot \text{Spin}(10))\) of rank 2 by Prop. 2.12 and Remark 2.13 since it is characterized by its degree up to equivalence. Since 
\([(k+1)/2] = 2\) for \(k=4\), \(\text{coh}(49) = 2\). The o.t.g. induced from (50) is equivalent to the linear isotropy representation of the symmetric pair \((\text{Sp}(3), U(3))\) of rank 3 by Prop. 2.2 and Remark 2.3. Since 
\([(k+1)/2] = 3\) for \(k=5\) or 6, \(\text{coh}(51) = \text{coh}(52) = 3\). The o.t.g. induced from (53) is equivalent to \(O(\text{SO}(2) \times \text{Spin}(9), \text{id} \otimes \Lambda_4^r, R^2 \otimes R^{16})\). Any element of \(R(2, 16) = R^2 \otimes R^{16}\) to the form
\[
\begin{bmatrix}
\alpha & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\
0 & \beta & 0 & \ldots & 0 & \gamma & \delta & \varepsilon & \ldots & 0
\end{bmatrix}
\]

and the isotropy subgroup is isomorphic to \(\text{SU}(3)\) if \(\alpha^2 \neq \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2\) owing to the use of the mapping \(f\) in Lemma 3.2 and Yokota [24, Theorems 5.51, 5.27, 5.2]. So \(\text{coh}(53) = 3\). The o.t.g. induced from (54) is equivalent to the linear isotropy representation of the symmetric pair \((E_7, U(1) \cdot E_6)\) of rank 3 by Prop. 2.15 and Remark 2.16. So \(\text{coh}(54) = 3\). Q.E.D.

(VII) Let \((G, E^N)\) be a real irreducible o.t.g. of type VII.

**Proposition 4.7** \(\text{coh}(G, E^N) \leq 3\) if and only if \((G, \text{id}, R^N)\) is equivalent as real representation to one of the followings:

- \(\text{coh} = 1\): (55) \((A_k, \Lambda_1, C^{k+1})\); \(k \geq 1\),
- \(\text{coh} = 2\): (56) \((C_k, \Lambda_1, C^{2k})\); \(k \geq 2\),
- \(\text{coh} = 2\): (57) \((D_5, A_5, C^{16})\),
- \(\text{coh} = 3\): (58) \((A_4, A_2, C^{10})\),
- \(\text{coh} = 3\): (59) \((A_6, A_2, C^{21})\).
Proof: Suppose $\mathrm{coh}(G, E^N) \leq 3$. Then $(\bar{\varphi}, \bar{i}, R^N)$ is equivalent as real representation to $(55)$ or $(58)$ or $(59)$ by Prop. 4.6. In fact 
$(B_k, \Lambda_1, C^{2k+1})$, $(D_k, \Lambda_1, C^{2k})$, $(A_1, 2\Lambda_1, C^3)$, $(A_2, 2\Lambda_2, C^6)$, $(C_2, 2\Lambda_2, C^5)$, $(G_2, 2\Lambda_2, C^7)$, $(B_3, \Lambda_3, C^8)$, $(B_4, \Lambda_4, C^{16})$ are 'real' and not real irreducible, so they are not of type VII, and $\mathrm{coh}(A_2, 2\Lambda_1, C^6) = \mathrm{coh}(A_5, \Lambda_2, C^{15}) = \mathrm{coh}(E_6, \Lambda_1, C^{27}) = 4$ since the restricted root systems of $(\text{Sp}(3), U(3))$, $(\text{SO}(12), U(6))$, $(E_7, U(1) \cdot E_6)$ are of type $BC$ (cf. [7], [22]).

Conversely $\mathrm{coh}(55) = \mathrm{coh}(56) = 1$ is evident. O.t.g.'s induced from (57), (58) are of coh 2 since the restricted root systems of $(E_6, U(1) \cdot \text{Spin}(10))$ and $(\text{SO}(10), U(5))$ are of type BC. The o.t.g. induced from (59) is of coh 3 since the restricted root system of $(\text{SO}(14), U(7))$ is of type BC (cf. [7] and [22]). Q.E.D.

Now we have the following result.

Theorem 4.8 Let $(G, E^N)$ be an o.t.g. such that the identity representation $\text{id}: G \rightarrow \text{SO}(N)$ is real irreducible. Then $\mathrm{coh}(G, E^N) \leq 3$ if and only if $(\bar{\varphi}, \bar{i}, R^N)$ is equivalent as real representation to one of the followings:

coh=1: (IX), (VIII), (8), (9), (19), (20), (21), (22), (23), (24), (25), (26), (27), (38), (39), (55), (56).

coh=2: (10), (11), (14), (17), (28), (29), (30), (31), (32), (33), (40), (41), (42), (43), (44), (45), (46), (47), (48), (49), (57), (58).

coh=3: (37), (1), (2), (3), (4), (5), (6), (7), (12), (13), (15), (16), (18), (34), (35), (36), (37), (50), (51), (52), (53), (54), (59).
Proof: Unifying (3.7) of Theorem 3.5, Propositions 4.1-4.7 and type VIII, IX in Section 3, we have the result. Q.E.D.

Remark 4.9 O.t.g.'s induced from (25), (26), (27), (39), (55), (56), (17), (46), (47), (57), (58), (6), (18), or (59) are not maximal. O.t.g.'s induced from (13), (16), or (53) are not obtained from the linear isotropy representations of any Riemannian symmetric pairs. Others are equivalent to the linear isotropy representations of some Riemannian symmetric pairs of rank at most 3 if they are maximal. (26) is obtained from the linear isotropy representation of $(F_4, \text{Spin}(9))$. The o.t.g. induced from (24)(resp. (42), (7)) is equivalent to one from (23)(resp. (41), (5)) of $k=4$.

Remark 4.10 O.t.g.'s induced from (13) or (16) are missed in the Theorem 7 of Hsiang-Lawson[11] if $k$ and 3 are relatively prime and $k \geq 4$, since the dimension of the representation spaces of (13) or (16) is $8k$ and the others of cohomogeneity 3 are of dimension $3m$ for some integer $m$ except (53) of dimension 16.
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